STT 843 Key to Homework 2 Spring 2018

Due date: Mar. 26, 2018

5.1. (a) Given the data matrix X, the sample mean is $\bar{X} = (6, 10)'$ and sample covariance

$$S = \begin{pmatrix} 8 & -10/3 \\ -10/3 & 2 \end{pmatrix}.$$

Then, the Hotelling T^2 test statistic is

$$T^2 = n(\bar{X} - \mu)'S^{-1}(\bar{X} - \mu) = 13.63636.$$

(b) The distribution of the above T^2 statistic is $\{2(4-1)/2\}F_{2,2} = 3F_{2,2}$.

(c) The upper α-quantile of $F_{2,2}$ is 19. Therefore, the T^2 test statistic is smaller than the critical value $3F_{2,2} = 57$. The p-value is $P(F_{2,2} > 13.63636/3) = 0.1803$, which is bigger than 0.05. We do not have evidence to reject the null hypothesis.

5.2. The hypothesis of interest is $H_0 : \mu_0 = (9, 5)'$ vs $H_1 : \mu_0 \neq (9, 5)$. After the transformation, the hypothesis of interest becomes $H_0 : C\mu_0 = C(9, 5)' = (4, 14)'$ vs $H_1 : C\mu_0 \neq (4, 14)$.

Given the data matrix after transformation, the sample mean is $\bar{X} = (2, 14)'$ and sample covariance

$$S = \begin{pmatrix} 19 & -5 \\ -5 & 7 \end{pmatrix}.$$

Then, the Hotelling T^2 test statistic is

$$T^2 = n(\bar{X} - \mu)'S^{-1}(\bar{X} - \mu) = 0.7778 = 7/9.$$

Therefore, the T^2 test statistic remains unchanged after the transformation.

5.9. (a) The large sample 95% simultaneous confidence intervals for μ_j's ($j = 1, \cdots, 6$) are

$$(\bar{X}_j - \sqrt{\chi^2_{6;\alpha}}\sqrt{s_{jj}/n}, \bar{X}_j + \sqrt{\chi^2_{6;\alpha}}\sqrt{s_{jj}/n}).$$

where $\bar{X}_j$ is the sample mean of the j-th variable, $\chi_{6;\alpha}$ is the upper α quantile of χ^2_6 and s_{jj} is the sample variance of the j-th variable. Using the sample mean and sample covariance provided in the question, we can obtain the following simultaneous confidence intervals:

simultaneous confidence interval for μ_1:	$(69.55347, 121.48653)$
simultaneous confidence interval for μ_2:	$(152.17278, 176.58722)$
simultaneous confidence interval for μ_3:	$(49.60667, 61.77333)$
simultaneous confidence interval for μ_4:	$(83.48823, 103.29177)$
simultaneous confidence interval for μ_5:	$(16.54687, 19.41313)$
simultaneous confidence interval for μ_6:	$(29.03513, 33.22487)$

The large sample simultaneous confidence intervals are plotted in Figure 1.

(b) The 95% large sample confidence region for $\mu_{14} = (\mu_1, \mu_4)'$ is given by

$$\{\mu_{14} : n(\bar{X} - \mu_{14})' S_{14}^{-1} (\bar{X} - \mu_{14}) \leq \chi^2_{6;\alpha}\},$$

where $\bar{X}_{14} = (95.52, 93.39)$ and

$$S_{14} = \begin{pmatrix} 3266.46 & 1175.50 \\ 1175.50 & 474.98 \end{pmatrix}.$$

The ellipse of the confidence region is given by Figure 1.

(c) The Bonferroni 95% simultaneous confidence intervals for μ_j's $(j = 1, \cdots, 6)$ are

$$(\bar{X}_j - t_{n-1;\alpha/12}\sqrt{s_{jj}/n}, \bar{X}_j + t_{n-1;\alpha/12}\sqrt{s_{jj}/n}).$$

where $\bar{X}_j$ is the sample mean of the j-th variable, $t_{n-1;\alpha/12}$ is the upper $\alpha/12$ quantile of t_{n-1} and s_{jj} is the sample variance of the j-th variable. Using the sample mean and sample covariance provided in the question, we can obtain the following simultaneous confidence intervals:

simultaneous confidence interval for μ_1:	$(75.55331, 115.48669)$
simultaneous confidence interval for μ_2:	$(154.99339, 173.76661)$
simultaneous confidence interval for μ_3:	$(51.01229, 60.36771)$
simultaneous confidence interval for μ_4:	$(85.77614, 101.00386)$
simultaneous confidence interval for μ_5:	$(16.87801, 19.08199)$
simultaneous confidence interval for μ_6:	$(29.51917, 32.74083)$

(d) The plot of the Bonferroni confidence rectangle, large sample simultaneous confidence intervals and confidence region are given by Figure 1.

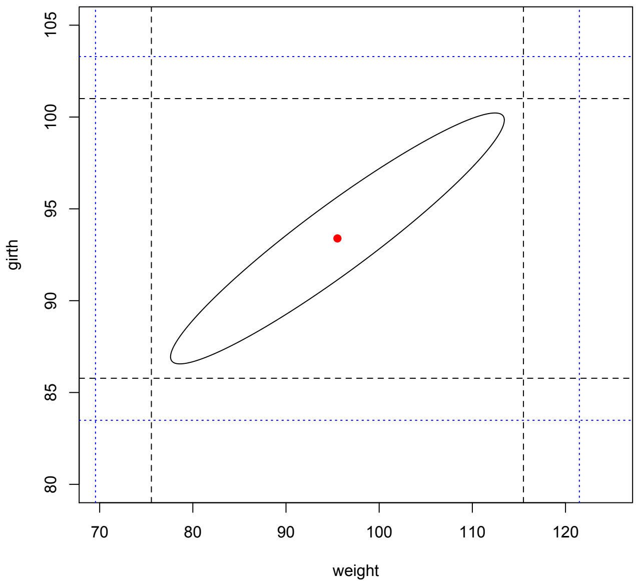

Figure 1: The 95% simultaneous confidence region (solid ellipse), Bonferroni confidence rectangle (black dash rectangle) and large sample simultaneous confidence intervals (blue dash rectangle) for $(\mu_1, \mu_4)'$.

(e) The Bonferroni 95% simultaneous confidence intervals for μ_j's ($j = 1, \cdots, 6$) and $\mu_6 - \mu_5$ are, respectively,

$$(\bar{X}_j - t_{n-1;\alpha/14}\sqrt{s_{jj}/n}, \bar{X}_j + t_{n-1;\alpha/14}\sqrt{s_{jj}/n})$$

and

$$(\bar{X}_6 - \bar{X}_5 - t_{n-1;\alpha/14}\sqrt{a' S_{67} a/n}, \bar{X}_j + t_{n-1;\alpha/14}\sqrt{a' S_{67} a/n}).$$

where $a = (1, -1)$ and

$$S_{67} = \begin{pmatrix} 9.95 & 13.88 \\ 13.88 & 21.26 \end{pmatrix},$$

$\bar{X}_j$ is the sample mean of the j-th variable, $t_{n-1;\alpha/14}$ is the upper $\alpha/14$ quantile of t_{n-1} and s_{jj} is the sample variance of the j-th variable. Using the sample mean and sample covariance provided in the question, we can obtain the following simultaneous confidence intervals:

simultaneous confidence interval for μ_1: $(75.13681, 115.90319)$
simultaneous confidence interval for μ_2: $(154.79758, 173.96242)$
simultaneous confidence interval for μ_3: $(50.91471, 60.46529)$
simultaneous confidence interval for μ_4: $(85.61731, 101.16269)$
simultaneous confidence interval for μ_5: $(16.85502, 19.10498)$
simultaneous confidence interval for μ_6: $(29.48557, 32.77443)$
simultaneous confidence interval for $\mu_6 - \mu_5$: $(12.48757, 13.81243)$

6.8. (a) Data in treatment 1 could be decomposed as the following:

$$\begin{pmatrix} 6 & 5 & 8 & 4 & 7 \\ 7 & 9 & 6 & 9 & 9 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 0 & -1 & 2 & -2 & 1 \\ -1 & 1 & -2 & 1 & 1 \end{pmatrix}$$

Data in treatment 2 could be decomposed as the following:

$$\begin{pmatrix} 3 & 1 & 2 \\ 3 & 6 & 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix} + \begin{pmatrix} -2 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \end{pmatrix}$$

Data in treatment 3 could be decomposed as the following:

$$\begin{pmatrix} 2 & 5 & 3 & 2 \\ 3 & 1 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix} + \begin{pmatrix} -1 \\ -3 \end{pmatrix} + \begin{pmatrix} -1 & 2 & 0 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

(b) Using the information in part (a), we obtain the following one-way MANOVA table

$$
\begin{aligned}
B &= 5\begin{pmatrix} 2 \\ 3 \end{pmatrix}\begin{pmatrix} 2 & 3 \end{pmatrix} + 3\begin{pmatrix} -2 \\ -1 \end{pmatrix}\begin{pmatrix} -2 & -1 \end{pmatrix} + 4\begin{pmatrix} -1 \\ -3 \end{pmatrix}\begin{pmatrix} -1 & -3 \end{pmatrix} \\
&= \begin{pmatrix} 36 & 48 \\ 48 & 84 \end{pmatrix} \\
W &= \begin{pmatrix} 18 & -13 \\ -13 & 18 \end{pmatrix} \\
B + W &= \begin{pmatrix} 54 & 35 \\ 35 & 102 \end{pmatrix}
\end{aligned}
$$

Then, the one-way MANOVA table is given by the following

Source of variation	Matrix sum of squares	Degrees of freedom
Treatment	$B = \begin{pmatrix} 36 & 48 \\ 48 & 84 \end{pmatrix}$	3-1=2
Residual	$W = \begin{pmatrix} 18 & -13 \\ -13 & 18 \end{pmatrix}$	5+3+4-3=9
Corrected Total	$B + W = \begin{pmatrix} 54 & 35 \\ 35 & 102 \end{pmatrix}$	5+3+4-1=11

(c) The Wilks' lambda Λ^* is

$$\Lambda^* = |W|/|B + W| = 155/4283 = 0.03618959.$$

Using Table 6.3,

$$\frac{12 - 3 - 1}{3 - 1} \frac{1 - \sqrt{\Lambda^*}}{\sqrt{\Lambda^*}} \sim F_{4,16},$$

where

$$\frac{12 - 3 - 1}{3 - 1} \frac{1 - \sqrt{\Lambda^*}}{\sqrt{\Lambda^*}} = 17.02656.$$

We reject the null hypothesis when Λ^* is small, correspondingly, we reject the null when the above transformed test statistic is large. Therefore, the p-value of the test is $P(F_{4,16} > 17.02656) = 1.282703e - 05$. We reject the null hypothesis at the nominal level 0.01.

The Bartlett corrected log-likelihood test statistic is

$$-(n - 1 - \frac{p + g}{2}) \log(\Lambda^*) = -(12 - 1 - \frac{2 + 3}{2}) \log(0.0362) = 28.21136.$$

Comparing with the chi-square distribution with degrees of freedom $2(3 - 1) = 4$, the p-value of the test is $P(\chi^2_4 > 28.21136) = 1.130111e - 05$, which is smaller than the nominal level 0.01. Therefore, we reject the null hypothesis at the nominal level 0.01.

The conclusion based on the sampling distribution and Bartlett corrected test statistic are the same. The Wilks' Lambda depends on the normality assumption, while the Bartlett corrected log-likelihood test is an asymptotic test.

6.12. (a) Given the profiles are parallel between μ_1 and μ_2, the sequential increments in p treatments in μ_1 are the same as the sequential increments in p treatments in μ_2. This means that if the profiles are parallel, the following equalities are true

$$\mu_{1i} - \mu_{1(i-1)} = \mu_{2i} - \mu_{2(i-1)} \quad \text{for } i = 2, \cdots, p. \tag{1}$$

If the p treatments in population 1 are linear, it implies that the increments in nearby treatments are equal to each other. Specifically, it means the following equalities are true

$$\{\mu_{1i} - \mu_{1(i-1)}\} - \{\mu_{1(i-1)} - \mu_{1(i-2)}\} = 0 \quad \text{for } i = 3, \cdots, p. \tag{2}$$

Similarly, if the p treatments in population 2 are linear, it implies that the following equalities are true

$$\{\mu_{2i} - \mu_{2(i-1)}\} - \{\mu_{2(i-1)} - \mu_{2(i-2)}\} = 0 \quad \text{for } i = 3, \cdots, p. \tag{3}$$

Because equations in (1) are given (known to be true), examining equations in both (2) and (3) is equivalent to examining following equations

$$\begin{aligned}
&\{\mu_{1i} - \mu_{1(i-1)}\} - \{\mu_{1(i-1)} - \mu_{1(i-2)}\} + \{\mu_{2i} - \mu_{2(i-1)}\} - \{\mu_{2(i-1)} - \mu_{2(i-2)}\} \\
&= \{\mu_{1i} - \mu_{1(i-1)} + \mu_{2i} - \mu_{2(i-1)}\} - \{\mu_{1(i-1)} - \mu_{1(i-2)} + \mu_{2(i-1)} - \mu_{2(i-2)}\} \\
&= 0 \quad \text{for } i = 3, \cdots, p.
\end{aligned} \tag{4}$$

Thus, to test that the profiles are linear, according to equations in (4), the hypothesis could be written as, for $i = 3, \cdots, p$,

$$H_0 : \{\mu_{1i}+\mu_{2i}\}-\{\mu_{1(i-1)}+\mu_{2(i-1)}\} = \{\mu_{1(i-1)}+\mu_{2(i-1)}\}-\{\mu_{1(i-2)}+\mu_{2(i-2)}\}.$$

It is not difficult to check that the above hypothesis could also in a matrix form as specified in the question.

(b) Plugging in the values of $\bar{x}_1, \bar{x}_2, S_{pooled}$ and the matrix C, we obtain that $T^2 = 16.83613$. Meanwhile, the cutoff value is

$$\frac{(30+30-2)(4-2)}{30+30-4+1} F_{2,30+30-4+1;\alpha} = \frac{58 \times 2}{57} 3.1588 = 6.4285.$$

Because $T^2 > 6.4285$, we reject the null hypothesis. This indicates that the profiles are not linear.

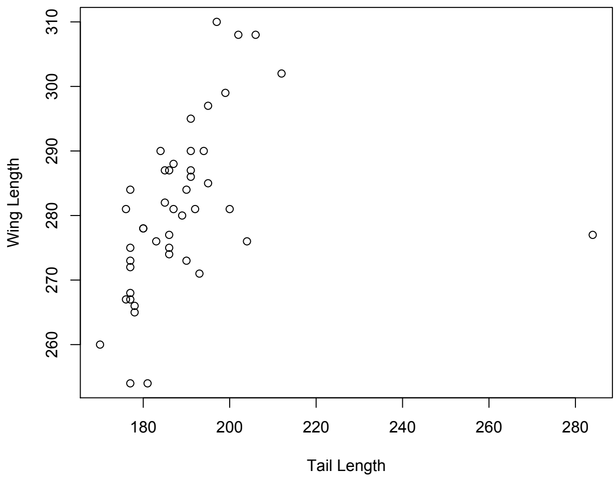

Figure 2: Scatter plot of tail length versus wing length for male hook-billed kites.

6.20. (a) The scatter plot of the tail length versus the wing length is given in the Figure 2. It is clear that one observation with $x_1 = 284$ is very different from the most of observations. This data point could be considered as an outlier.

(b) To test the equality of mean vectors μ_1 (for male) and μ_2 (for female), we applied the Hotelling's T^2 test statistic, which can be computed as following

$$T^2 = (\frac{n_1 n_2}{n_1 + n_2})(\bar{X}_1 - \bar{X}_2)' S^{-1}_{\text{pooled}} (\bar{X}_1 - \bar{X}_2).$$

For these data sets, $\bar{X}_1 = (189.31, 280.87)'$, $\bar{X}_2 = (193.62, 279.78)'$ and

$$\begin{aligned} S_{\text{pooled}} &= \frac{n_1 - 1}{n_1 + n_2 - 2} S_1 + \frac{n_2 - 1}{n_1 + n_2 - 2} S_2 \\ &= \begin{pmatrix} 207.7298 & 100.8987 \\ 100.8987 & 188.8975 \end{pmatrix}. \end{aligned}$$

Then, the Hotelling's T^2 test statistic is $T^2 = 3.642538$. Under the null, the test statistic follows distribution $\{(n_1 + n_2 - 2)p/(n_1 + n_2 - p - 1)\}F_{p, n_1+n_2-p-1}$. The critical value for the test is 6.273886 at the nominal level $\alpha = 0.05$. Thus, we do not have evidence to reject the null hypothesis.

If the outlier #31 is removed from the male data set, the Hotelling's T^2 test statistic is $T^2 = 24.9649$, which should be compared with $\{(n_1 + n_2 -$

$2)p/(n_1+n_2-p-1)\}F_{p,n_1+n_2-p-1;\alpha} = 6.277257$. In this case, we can reject the null hypothesis.

If we replace the x_1 coordinate of #31 observation with $x_1 = 184$, the Hotelling's T^2 test statistic is $T^2 = 25.66253$, which should be compared with $\{(n_1+n_2-2)p/(n_1+n_2-p-1)\}F_{p,n_1+n_2-p-1;\alpha} = 6.273886$. In this case, we can reject the null hypothesis.

The linear combinations $a'(\bar{X}_1 - \bar{X}_2)$ that are most responsible for the rejection is the vector a that maximizes the following quantity

$$\hat{a} = \arg\max_a \frac{a'(\bar{X}_1 - \bar{X}_2)(\bar{X}_1 - \bar{X}_2)'a}{a'S_{\text{pooled}}a},$$

which is the square of the signal-to-noise ratio. Let

$$\mathbf{A} = S_{\text{pooled}}^{-1/2}(\bar{X}_1 - \bar{X}_2)(\bar{X}_1 - \bar{X}_2)'S_{\text{pooled}}^{-1/2}.$$

We note the following

$$\max_a \frac{a'(\bar{X}_1 - \bar{X}_2)(\bar{X}_1 - \bar{X}_2)'a}{a'S_{\text{pooled}}a} = \max_{b=S_{\text{pooled}}^{1/2}a} \frac{b'\mathbf{A}b}{b'b} = \max_{c'c=1} c'\mathbf{A}c$$

By the above derivation, it can be seen that the vector c that maximizes the above objective function is the eigenvector of $\mathbf{A}$ corresponding to the largest eigenvalue of $\mathbf{A}$. It is not difficult to see that $\mathbf{A}$ is a matrix with rank one. Thus, the large eigenvalue of $\mathbf{A}$ is $\lambda_1 = (\bar{X}_1 - \bar{X}_2)'S_{\text{pooled}}^{-1}(\bar{X}_1 - \bar{X}_2)$ with the corresponding eigenvector $c = S_{\text{pooled}}^{-1/2}(\bar{X}_1 - \bar{X}_2)/\sqrt{\lambda_1}$. Then, the value b maximizes the above objective function is $b = S_{\text{pooled}}^{-1/2}(\bar{X}_1 - \bar{X}_2)$. It follows that the value a that maximizes the objective function is $\hat{a} = S_{\text{pooled}}^{-1}(\bar{X}_1 - \bar{X}_2)$.

For the test with outlier #31 removed, the vector $\hat{a}$ is

$$\hat{a} = \begin{pmatrix} 207.7298 & 100.8987 \\ 100.8987 & 188.8975 \end{pmatrix}^{-1} \begin{pmatrix} -6.463131 \\ 1.176768 \end{pmatrix} = \begin{pmatrix} -0.15661407 \\ 0.09342743 \end{pmatrix}.$$

For the test with the outlier #31 replaced, the vector $\hat{a}$ is

$$\hat{a} = \begin{pmatrix} 103.6389 & 105.2927 \\ 105.2927 & 188.8975 \end{pmatrix}^{-1} \begin{pmatrix} -6.533333 \\ 1.088889 \end{pmatrix} = \begin{pmatrix} -0.15885637 \\ 0.09431202 \end{pmatrix}.$$

Based on the above results, we see that there are significant differences among the results with outliers been handled or not. However, the differences among the the results with outlier deleted or replaced are not very significant. Both methods for handling outlier in this data set is comparable.

(c) The 95% confidence region for $\mu_1 - \mu_2$ is given by the following ellipse

$$\{\mu_1 - \mu_2 : (\frac{n_1 n_2}{n_1 + n_2})(\bar{X}_1 - \bar{X}_2)' S_{\text{pooled}}^{-1}(\bar{X}_1 - \bar{X}_2) \leq c_\alpha\}$$

where $c_\alpha = 2(n_1 + n_2 - 2)F_{p,n_1+n_2-p-1;\alpha}/(n_1 + n_2 - 3)$.

The Bonferroni 95% simultaneous confidence intervals for $\mu_{1j} - \mu_{2j}$ ($j = 1, 2$) are given by

$$\bar{X}_{1j} - \bar{X}_{2j} \pm t_{n_1+n_2-2;\alpha/4}\sqrt{(n_1 + n_2)s_{jj}/(n_1 n_2)}.$$

where s_{jj} is the j-th diagonal element of S_{pooled}, and $t_{n_1+n_2-2;\alpha/4}$ is the upper $\alpha/4$ quantile of a t-distribution with degrees of freedom n_1+n_2-2. The simultaneous confidence intervals for $\mu_{11} - \mu_{21}$ and $\mu_{12} - \mu_{22}$ are, respectively, (-11.411601, -1.475198) and (-5.504445, 7.911261).

The confidence region and simultaneous confidence intervals are given in the Figure 3

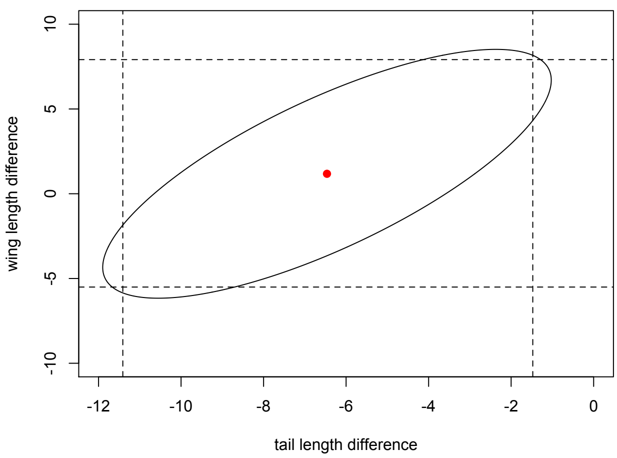

Figure 3: Confidence region and simultaneous confidence intervals for $\mu_1 - \mu_2$.

(d) Based on the confidence region and simultaneous confidence intervals, we see that there is no significant difference in wing length between male and female. However, there exists significant difference in tail length between male and female. Female birds have longer tail length than male birds.

6.24. The one-way MANOVA table is given by the following

Source of variation	Matrix sum of squares	D.F.
Treatment	$B = \begin{pmatrix} 150.20 & 20.30 & -161.83 & 5.03 \\ 20.30 & 20.60 & -38.73 & 6.43 \\ -161.83 & -38.73 & 190.29 & -10.86 \\ 5.03 & 6.43 & -10.86 & 2.02 \end{pmatrix}$	2
Residual	$W = \begin{pmatrix} 1785.40 & 172.5 & 128.97 & 289.63 \\ 172.50 & 1924.3 & 178.80 & 171.90 \\ 128.97 & 178.8 & 2153.00 & -1.70 \\ 289.63 & 171.9 & -1.70 & 840.20 \end{pmatrix}$	87
Corrected Total	$B + W = \begin{pmatrix} 1935.60 & 192.80 & -32.87 & 294.67 \\ 192.80 & 1944.90 & 140.07 & 178.33 \\ -32.87 & 140.07 & 2343.29 & -12.56 \\ 294.67 & 178.33 & -12.56 & 842.22 \end{pmatrix}$	89

Using the above table, the Wilks' lambda test statistic is

$$\Lambda^* = \frac{|W|}{|B + W|} = 0.8301027.$$

Using Table 6.3,

$$\frac{90 - 4 - 2}{4} \frac{1 - \sqrt{\Lambda^*}}{\sqrt{\Lambda^*}} \sim F_{8,168},$$

where

$$\frac{90 - 4 - 2}{4} \frac{1 - \sqrt{\Lambda^*}}{\sqrt{\Lambda^*}} = 2.049069.$$

We reject the null hypothesis when Λ^* is small, correspondingly, we reject the null when the above transformed test statistic is large. Therefore, the p-value of the test is $P(F_{8,168} > 2.049069) = 0.04358254$. We reject the null hypothesis at the nominal level 0.05.

The 95% Bonferroni simultaneous confidence intervals for means differences among three periods, for $i \neq j \in \{1, 2, 3\}$ and $k = 1, \cdots, 4$,

$$\bar{X}_{ik} - \bar{X}_{jk} \pm t_{n-g; \frac{\alpha}{pg(g-1)}} \sqrt{\frac{w_{kk}}{n-g} \frac{n_i + n_j}{n_i n_j}},$$

where w_{kk} is the k-th diagonal element of W given in the above one-way MANOVA table. The simultaneous confidence intervals are given in the following:

$\mu_{11} - \mu_{21}$: (-4.442, 2.442) $\mu_{11} - \mu_{31}$: (-6.542, 0.342) $\mu_{21} - \mu_{31}$: (-5.5423, 1.342)
$\mu_{12} - \mu_{22}$: (-2.673, 4.473) $\mu_{12} - \mu_{32}$: (-3.773, 3.373) $\mu_{22} - \mu_{32}$: (-4.6737, 2.473)
$\mu_{13} - \mu_{23}$: (-3.680, 3.880) $\mu_{13} - \mu_{33}$: (-0.646, 6.913) $\mu_{23} - \mu_{33}$: (-0.7467, 6.813)
$\mu_{14} - \mu_{24}$: (-2.061, 2.661) $\mu_{14} - \mu_{34}$: (-2.394, 2.328) $\mu_{24} - \mu_{34}$: (-2.6947, 2.028)

There are two assumptions we need to check for the usual MANOVA model. One the normality assumption and another is the homogeneity of covariance matrices. For the homogeneity of covariance matrices, we could perform Box's M test for comparing covariances among three periods. The Box's M test statistic is 21.048. Comparing it with the chi-square distribution with degrees of freedom 20, we obtain p-value 0.3943. Thus, it is reasonable to assume covariances are the same for this data set.

To check the joint normality assumption, we can construct chi-square plots using residuals and pooled sample covariance. The chi-square plot is given in 4. Based on the chi-square plot, we may conclude that the joint normality assumption is reasonable for this data set.

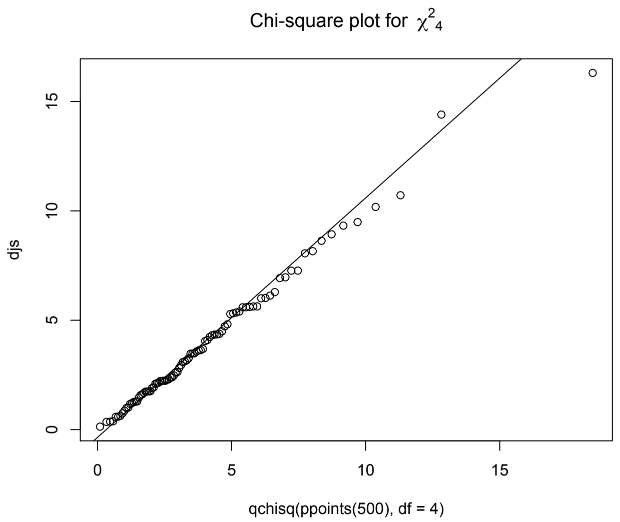

Figure 4: Chi-square plot for checking joint normality assumption.

6.37. Applying Box's M test on the homogeneity of covariance matrices, we obtain the test statistics as 23.405, which can be compared to chi-square distribution with degrees of freedom 6. Then the resulting p-value is 0.0006716. Therefore, we reject the hypothesis and conclude that the female and male groups have different covariance matrices.

6.39. (a) To obtain the Hotelling's T^2 test statistic, we note $\bar{X}_1 = (348.275, 37.26357)'$, $\bar{X}_2 = (228.753, 7.290357)'$ and

$$S_{\text{pooled}} = \begin{pmatrix} 2606.3882 & 667.9435 \\ 667.9435 & 204.2362 \end{pmatrix}.$$

Then the Hotelling's T^2 test statistic is

$$T^2 = (\frac{n_1 n_2}{n_1 + n_2})(\bar{X}_1 - \bar{X}_2)' S_{\text{pooled}}^{-1} (\bar{X}_1 - \bar{X}_2) = 76.91534.$$

Under the null, the test statistic follows distribution $\{(n_1+n_2-2)p/(n_1+n_2-p-1)\}F_{p,n_1+n_2-p-1}$. The critical value for the test is 6.462936 at the nominal level $\alpha = 0.05$. Thus, we have enough evidence to reject the null hypothesis.

(b) To check if it is reasonable to pool variances in this data set, we perform a Box's M test on testing the homogeneity of covariance matrices. The Box'M test statistic is 100.32. We compared the test statistic with a chi-square distribution with degrees of freedom 3. Then the resulting p-value is <2.2e-16. Therefore, we reject the hypothesis and conclude that the female and male groups have different covariance matrices.

If the sampling distribution in part (a) is used for the two sample test, then it is not appropriate to pool two variances together because the sample distribution was derived under the assumption that two covariance matrices are the same. However, if the large sample distribution is used, it is still appropriate to pooled the sample variances together because these two samples have the same sample size.

(c) The Bonferroni 95% simultaneous confidence intervals for $\mu_{1j} - \mu_{2j}$ ($j = 1, 2$) are given by

$$\bar{X}_{1j} - \bar{X}_{2j} \pm t_{n_1+n_2-2;\alpha/4}\sqrt{s_{11}/n_1 + s_{22}/n_2}.$$

where s_{jj} is the j-th diagonal element of S_{pooled}, and $t_{n_1+n_2-2;\alpha/4}$ is the upper $\alpha/4$ quantile of a t-distribution with degrees of freedom n_1+n_2-2. The simultaneous confidence intervals for $\mu_{11} - \mu_{21}$ and $\mu_{12} - \mu_{22}$ are, respectively, (88.062, 150.980) and (21.166, 38.779).