MAHARAM EXTENSION FOR NONSINGULAR GROUP ACTIONS

By Parthanil Roy and Gennady Samorodnitsky [*]

Cornell University

We establish a generalization of the Maharam Extension Theorem to nonsingular $\mathbb{Z}^d$-actions. We also present an extension of Krengel's representation of dissipative transformations to nonsingular actions.

1. Introduction. Maharam extension theorem extends in a natural way an invertible nonsingular conservative transformation of a σ-finite standard measure space to an invertible conservative *measure preserving* transformation on an extended space, the so-called *Maharam skew product*. The result was established in Maharam (1964)) and has been used in a number of ways, allowing, in particular, extensions of certain notions from the measure preserving case to the nonsingular case; see e.g. Silva and Thieullen (1995).

In this paper we generalize Maharam's theorem to nonsingular $\mathbb{Z}^d$-actions. Our approach is different than the one often used in the case $d = 1$, based on the fact that conservativity is equivalent to incompressibility. We use, instead, a result on the maximal value assigned by a group action over an increasing sequence of cubes to a nonnegatve function (Proposition 3.1 below), which may be of an independent interest. In the proof of one of the statements in that proposition we use a recently established extension of Krengel's theorem (see Krengel (1969)) on the structure of dissipative nonsingular transformations to nonsingular $\mathbb{Z}^d$-actions. This result has not, apparently, been stated before. Apart from that, the proof of the main result of this paper is entirely from the first principles.

We state the extensions of both Maharam's theorem and Krengel's theorem in Section 2. The proof of Maharam Extension Theorem is given in Section 3.

2. Maharam's theorem and Krengel's theorem for nonsingular $\mathbb{Z}^d$-actions . Let $\{\phi_t\}_{t \in \mathbb{Z}^d}$ be a nonsingular action on a standard Borel

space $(S, \mathcal{S})$ with a σ-finite measure μ. Then, by Theorem 1 in Maharam (1964),

$$\phi_t^*(s, y) := (\phi_t(s), y \frac{d\mu}{d\mu \circ \phi_t}(s)), \quad t \in \mathbb{Z}^d$$

is a measure preserving group action on the product space $\big(S \times (0, \infty), \mathcal{S} \times \mathcal{B}, \mu \times \mathrm{Leb}\big)$. Here Leb is the Lebesgue measure on $(0, \infty)$.

The following is our main result.

Theorem 2.1. *The group action $\{\phi_t^*\}_{t \in \mathbb{Z}^d}$ is conservative on $\big(S \times (0, \infty), \mathcal{S} \times \mathcal{B}, \mu \times \mathrm{Leb}\big)$ if and only if the group action $\{\phi_t\}_{t \in \mathbb{Z}^d}$ is conservative on $(S, \mathcal{S}, \mu)$.*

In the case $d = 1$ this is the content of Maharam Extension Theorem (Maharam (1964)).

The proof of Theorem 2.1 presented in the next section relies on a result on the maximal value of a function transformed by the group of dual operators, given in Proposition 3.1. The argument for one part of the proposition uses the following extension of Krengel's Theorem (see Krengel (1969)) on the structure of dissipative nonsingular maps to $\mathbb{Z}^d$-actions. It follows immediately from Theorem 2.2 in Rosiński (2000) and Corollary 2.4 in Roy and Samorodnitsky (2006). It appears that the result has not been stated previously. Recall that nonsingular group actions $\{\phi_t\}_{t \in G}$ and $\{\psi_t\}_{t \in G}$, defined on standard measure spaces $(S, \mathcal{S}, \mu)$ and $(T, \mathcal{T}, \nu)$ resp., are equivalent if there is a Borel isomorphism Φ between the measure spaces such that $\nu \sim \mu \circ \Phi^{-1}$ and for each $t \in T$, $\psi_t \circ \Phi = \Phi \circ \phi_t$ μ-a.e.

Theorem 2.2 (Krengel's Theorem for $\mathbb{Z}^d$-actions)**.** *Let $\{\phi_t\}$ be a nonsingular $\mathbb{Z}^d$-action on a σ-finite standard measure space $(S, \mathcal{S}, \mu)$. Then $\{\phi_t\}$ is dissipative if and only if it is equivalent to the $\mathbb{Z}^d$-action*

$$(2.1) \qquad\qquad \psi_t(w, s) := (w, t + s), \quad t \in \mathbb{Z}^d$$

defined on $(W \times \mathbb{Z}^d, \tau \otimes l)$, where $(W, \mathcal{W}, \tau)$ is some σ-finite standard measure space and l is the counting measure on $\mathbb{Z}^d$.

3. Proof of Theorem 2.1. Let $\{\phi_t\}_{t \in \mathbb{Z}^d}$ be as above and $\hat{\phi}_t : L^1(\mu) \to L^1(\mu)$ be the dual to ϕ_{-t} operator (see Section 1.3 in Aaronson (1997))

$$\hat{\phi}_t g(s) = g \circ \phi_t(s) \frac{d\mu \circ \phi_t}{d\mu}(s), \quad s \in S.$$

The following result, which may be of independent interest, is the key step in the proof of Theorem 2.1. The inequalities in the statement of this proposition and elsewhere are understood in the sense of the natural partial order on $\mathbb{Z}^d$.

Proposition 3.1. *(a) If $\{\phi_t\}_{t\in\mathbb{Z}^d}$ is conservative then for all $g \in L^1(\mu)$, $g \geq 0$, we have*

$$(3.1) \qquad \frac{1}{n^d} \int_S \max_{0 \leq t \leq (n-1)\mathbf{1}} \hat{\phi}_t g(s)\mu(ds) \to 0.$$

(b) If $\{\phi_t\}_{t\in\mathbb{Z}^d}$ is dissipative then for all $g \in L^1(\mu)$, $g \geq 0$, $\mu(g > 0) > 0$, we have

$$(3.2) \qquad \frac{1}{n^d} \int_S \max_{0 \leq t \leq (n-1)\mathbf{1}} \hat{\phi}_t g(s)\mu(ds) \to a$$

for some $0 < a < \infty$.

Proof. (a) There is no loss of generality in assuming that μ is a probability measure. We can also assume that the support of the family $\{\hat{\phi}_t g\}_{t\in\mathbb{Z}^d}$ is the entire set S. Let $\{\alpha_u : u \in \mathbb{Z}^d\}$ be a collection of positive numbers summing up to 1. Then applying the group action version of Theorem 1.6.3 in Aaronson (1997) to $f = \sum_{u\in\mathbb{Z}^d} \alpha_u \hat{\phi}_u g$ we have,

$$(3.3) \qquad \sum_{t\in\mathbb{Z}^d} \hat{\phi}_t g(s) = \infty \quad \text{for } \mu\text{-a.a. } s.$$

To prove (3.1) we will show that

$$a_n := \frac{1}{(2n+1)^d} \int_S \max_{t\in J_n} \hat{\phi}_t g(s)\mu(ds) \to 0,$$

where, $J_n := \{(i_1, i_2, \ldots, i_d) : -n \leq i_1, i_2, \ldots, i_d \leq n\}$. Note that

$$a_n \leq \frac{1}{(2n+1)^d} \left(\int_S \max_{t\in J_n} \left[\hat{\phi}_t g(s) I\left(\hat{\phi}_t g(s) \leq \epsilon \sum_{u\in J_n} \hat{\phi}_u g(s) \right) \right] \mu(ds) \right.$$

$$\left. + \int_S \max_{t\in J_n} \left[\hat{\phi}_t g(s) I\left(\hat{\phi}_t g(s) > \epsilon \sum_{u\in J_n} \hat{\phi}_u g(s) \right) \right] \mu(ds) \right)$$

$$= a_n^{(1)} + a_n^{(2)},$$

where $\epsilon > 0$ is arbitrary. Clearly,

$$(3.4) \qquad a_n^{(1)} \leq \frac{\epsilon}{(2n+1)^d} \sum_{u\in J_n} \int_S \hat{\phi}_u g(s)\mu(ds) = \epsilon\|g\|,$$

where, $\|g\| := \int_S g(s)\mu(ds) < \infty$. Also, by duality,

$$(3.5) \qquad a_n^{(2)} \leq \frac{1}{(2n+1)^d} \sum_{t\in J_n} \int_S \hat{\phi}_t g(s) I_{A_{t,n}}(s)\mu(ds)$$

$$= \frac{1}{(2n+1)^d} \sum_{t\in J_n} \int_S g(s) I_{\phi_t^{-1}(A_{t,n})}(s)\mu(ds),$$

where, $A_{t,n} = \{s : \hat{\phi}_t g(s) > \epsilon \sum_{u \in J_n} \hat{\phi}_u g(s)\}$, $n \geq 1$, $t \in J_n$. Define

$$U_n := \{(i_1, i_2, \ldots, i_d) : -n + [\sqrt{n}] \leq i_1, i_2, \ldots, i_d \leq n - [\sqrt{n}]\}.$$

Observe that by the nonnegativity, for every $t \in U_n$,

$$
\begin{aligned}
\phi_t^{-1}(A_{t,n}) &= \left\{s : g(s) > \epsilon \sum_{u \in J_n} \hat{\phi}_{u+t} g(s)\right\} \\
&\subseteq \left\{s : g(s) > \epsilon \sum_{u \in J_{[\sqrt{n}]}} \hat{\phi}_u g(s)\right\}.
\end{aligned}
$$

Therefore, for any $M > 0$

$$\max_{t \in U_n} \mu(\phi_t^{-1}(A_{t,n})) \leq \mu\{s : g(s) > \epsilon M\} + \mu\left(\sum_{t \in J_{[\sqrt{n}]}} \hat{\phi}_t g(s) \leq M\right).$$

Letting first $n \to \infty$, using (3.3), and then letting $M \to \infty$ we see that

$$\lim_{n \to \infty} \max_{t \in U_n} \mu(\phi_t^{-1}(A_{t,n})) = 0.$$

From here we immediately see that

$$
\begin{aligned}
&\frac{1}{(2n+1)^d} \sum_{t \in U_n} \int_S \hat{\phi}_t g(s) I_{A_{t,n}}(s) \mu(ds) \\
\text{(3.6)} \qquad &= \frac{1}{(2n+1)^d} \sum_{t \in U_n} \int_{\phi_t^{-1}(A_{t,n})} g(s) \mu(ds) \to 0.
\end{aligned}
$$

Define $V_n = J_n \setminus U_n$, and note that $\text{Card}(V_n) = o(n^d)$ as $n \to \infty$. Therefore, using (3.5) and (3.6) we have,

$$a_n^{(2)} \leq \frac{1}{(2n+1)^d} \sum_{t \in U_n} \int_S \hat{\phi}_t g(s) I_{A_{t,n}}(s) \mu(ds) + \frac{\text{Card}(V_n)}{(2n+1)^d} \|g\| \to 0,$$

implying that

$$\limsup a_n \leq \limsup a_n^{(1)} + \limsup a_n^{(2)} \leq \epsilon \|g\|.$$

Since $\epsilon > 0$ is arbitrary, the claim follows.

(b) Since the statement is invariant under a passage from one group action to an equivalent one, we will use Theorem 2.2 and check that for any σ-finite

standard measure space $(W, \mathcal{W}, \tau)$ we have for all $f \in L^1(W \times \mathbb{Z}^d, \tau \otimes l)$ and $f \geq 0$ with $\tau \otimes l(f > 0) > 0$,

$$(3.7) \qquad \frac{1}{n^d} \sum_{s \in \mathbb{Z}^d} \int_W \max_{0 \leq t \leq (n-1)\mathbf{1}} f(w, s + t)\, \tau(dw) \to a$$

for some $0 < a < \infty$. In fact, we will show that (3.7) holds with $a = \int_W h(w)\tau(dw) \in (0, \infty)$ where $h(w) := \sup_{s \in \mathbb{Z}^d} f(w, s)$ for all $w \in W$.

We start with the case where f has compact support, that is

$$f(w, s) I_{W \times [-m\mathbf{1}, m\mathbf{1}]^c}(w, s) \equiv 0 \text{ for some } m = 1, 2, \dots,$$

where $[u, v] := \{t \in \mathbb{Z}^d : u \leq t \leq v\}$. In that case, we have, for all $n \geq 2m - 1$,

$$\sum_{s \in \mathbb{Z}^d} \int_W \max_{0 \leq t \leq (n-1)\mathbf{1}} f(w, s + t)\, \tau(dw)$$

$$= \sum_{(-m-n+1)\mathbf{1} \leq s \leq m\mathbf{1}} \int_W \max_{0 \leq t \leq (n-1)\mathbf{1}} f(w, s + t)\, \tau(dw)$$

$$= \sum_{s \in A_n} \int_W \max_{0 \leq t \leq (n-1)\mathbf{1}} f(w, s + t)\, \tau(dw)$$

$$+ \sum_{s \in B_n} \int_W \max_{0 \leq t \leq (n-1)\mathbf{1}} f(w, s + t)\, \tau(dw) =: T_n + R_n,$$

where $A_n = [(m - n - 1)\mathbf{1}, -m\mathbf{1}]$ and $B_n = [(-m - n + 1)\mathbf{1}, m\mathbf{1}] - [(m - n - 1)\mathbf{1}, -m\mathbf{1}]$. Observe that, for $n \geq 2m + 1$ we have for each $s \in A_n$,

$$\max_{0 \leq t \leq (n-1)\mathbf{1}} f(w, s + t)| = h(w)$$

while for each $s \in B_n$

$$\max_{0 \leq t \leq (n-1)\mathbf{1}} f(w, s + t)| \leq h(w),$$

and so

$$T_n = (n - 2m)^d \int_W h(w)\tau(dw),$$

$$R_n \leq [(2m + n)^d - (n - 2m)^d] \int_W h(w)\tau(dw).$$

Therefore (3.7) follows when f has compact support. In the general case, given $\epsilon > 0$, choose a compactly supported f_ϵ such that $f_\epsilon(w, s) \leq f(w, s)$

for all w, s and

$$\sum_{s \in \mathbb{Z}^d} \int_W f(w, s) \, \tau(dw) - \sum_{s \in \mathbb{Z}^d} \int_W f_\epsilon(w, s) \, \tau(dw) \leq \epsilon.$$

Let

$$h_\epsilon(w) = \sup_{s \in \mathbb{Z}^d} f_\epsilon(w, s), \quad w \in W.$$

Then

$$
\begin{aligned}
0 \quad &\leq \quad \int_W h(w) \tau(dw) - \int_W h_\epsilon(w) \tau(dw) \\
&\leq \quad \int_W \sup_{s \in \mathbb{Z}^d} \big(f(w, s) - f_\epsilon(w, s) \big) \, \tau(dw) \\
&\leq \quad \int_W \sum_{s \in \mathbb{Z}^d} \big(f(w, s) - f_\epsilon(w, s) \big) \, \tau(dw) \\
&= \quad \sum_{s \in \mathbb{Z}^d} \int_W f(w, s) \, \tau(dw) - \sum_{s \in \mathbb{Z}^d} \int_W f_\epsilon(w, s) \, \tau(dw) \leq \epsilon.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\left| \frac{1}{n^d} \sum_{s \in \mathbb{Z}^d} \int_W \max_{0 \leq t \leq (n-1)\mathbf{1}} f(w, s + t) \, \tau(dw) - \int_W h(w) \, \tau(dw) \right| \\
\leq \quad &\frac{1}{n^d} \left| \sum_{s \in \mathbb{Z}^d} \int_W \max_{0 \leq t \leq (n-1)\mathbf{1}} f(w, s + t) \, \tau(dw) \right. \\
&\qquad\qquad \left. - \sum_{s \in \mathbb{Z}^d} \int_W \max_{0 \leq t \leq (n-1)\mathbf{1}} f_\epsilon(w, s + t) \, \tau(dw) \right| \\
+ \quad &\left| \frac{1}{n^d} \sum_{s \in \mathbb{Z}^d} \int_W \max_{0 \leq t \leq (n-1)\mathbf{1}} f_\epsilon(w, s + t) \, \tau(dw) - \int_W h_\epsilon(w) \tau(dw) \right| \\
+ \quad &\left| \int_W h_\epsilon(w) \tau(dw) - \int_W h(w) \tau(dw) \right| =: T_n^{(1)} + T_n^{(2)} + T_n^{(3)}.
\end{aligned}
$$

By the above, $T_n^{(3)} \leq \epsilon$, and the same argument shows that $T_n^{(1)} \leq \epsilon$ as well. Furthermore, by the already considered compact support case, $T_n^{(2)} \to 0$ as $n \to \infty$. Hence

$$\limsup_{n \to \infty} \left| \frac{1}{n^d} \sum_{s \in \mathbb{Z}^d} \int_W \max_{0 \leq t \leq (n-1)\mathbf{1}} f(w, s + t) \, \tau(dw) - \int_W h(w) \, \tau(dw) \right| \leq 2\epsilon,$$

and, since $\epsilon > 0$ is arbitrary, the proof is complete.

$\square$

The following corollary is immediate.

Corollary 3.2. *If $g \in L^1(\mu)$, $g \geq 0$, and $\mu(Support(g) \cap \mathcal{D}) > 0$ where $\mathcal{D}$ is the dissipative part of $\{\phi_t\}$, then*

$$\frac{1}{n^d} \int_S \max_{0 \leq t \leq (n-1)\mathbf{1}} \hat{\phi}_t g(s)\mu(ds) \rightarrow a$$

for some $0 < a < \infty$.

Remark 3.3. From Corollary 3.2 it follows that, if (3.1) holds for some $g \in L^1(\mu)$, $g \geq 0$, then

$$Support(g) \subseteq \mathcal{C} \mod \mu,$$

where $\mathcal{C}$ is the conservative part of $\{\phi_t\}$. In other words, if there exists a sequence of functions $g_m \in L^1(\mu)$, $g_m \geq 0$, whose support increases to S, such that (3.1) holds for g_m for all $m \geq 1$, then $\{\phi_t\}$ is conservative.

Proof of Theorem 2.1. If $\{\phi_t^*\}$ is conservative, so is clearly $\{\phi_t\}$. Suppose now that $\{\phi_t\}$ is conservative. To show conservativity of $\{\phi_t^*\}$ we will use Remark 3.3. Since μ is σ-finite, there is a sequence of measurable sets $S_m \uparrow S$, such that, $\mu(S_m) < \infty$ for all $m \geq 1$. Consider a sequence of nonnegative functions $g_m^* := I_{S_m \times (0,m)} \in L^1(\mu \otimes Leb)$, $m \geq 1$. Note that the support of g_m^* is $S_m \times (0,m) \uparrow S \times (0,\infty)$.

Observe that $g_m^*(s,y) = I\{(s,y) : 0 < y < mI_{S_m}(s)\}$. If $w_t := \frac{d\mu \circ \phi_t}{d\mu}$, $t \in \mathbb{Z}^d$, then for all $m \geq 1$ we have,

$$\frac{1}{n^d} \int_0^\infty \int_S \max_{0 \leq t \leq (n-1)\mathbf{1}} \hat{\phi}_t^* \, g_m^*(s,y)\mu(ds)\text{Leb}(dy)$$

$$= \frac{1}{n^d} \int_0^\infty \int_S \max_{0 \leq t \leq (n-1)\mathbf{1}} g_m^* \circ \phi_t^*(s,y)\mu(ds)\text{Leb}(dy)$$

$$= \frac{1}{n^d} \int_S \int_0^\infty I\Big\{(s,y) : 0 < y < \max_{0 \leq t \leq (n-1)\mathbf{1}} mw_t(s)I_{S_m}(\phi_t(s))\Big\}$$

$$\text{Leb}(dy)\mu(ds)$$

$$= \frac{m}{n^d} \int_S \max_{0 \leq t \leq (n-1)\mathbf{1}} \Big[I_{S_m}(\phi_t(s))\frac{d\mu \circ \phi_t}{d\mu}(s)\Big]\mu(ds)$$

$$= \frac{m}{n^d} \int_S \max_{0 \leq t \leq (n-1)\mathbf{1}} \hat{\phi}_t I_{S_m}(s) \rightarrow 0$$

by part (a) of Proposition 3.1. By Remark 3.3 this is enough to prove the theorem. $\square$

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SCHOOL OF OPERATIONS RESEARCH
AND INDUSTRIAL ENGINEERING
CORNELL UNIVERSITY
ITHACA, NY 14853
E-MAIL: pr72@cornell.edu

SCHOOL OF OPERATIONS RESEARCH
AND INDUSTRIAL ENGINEERING
CORNELL UNIVERSITY
ITHACA, NY 14853
E-MAIL: gennady@orie.cornell.edu