STATIONARY SYMMETRIC α-STABLE DISCRETE PARAMETER RANDOM FIELDS

By Parthanil Roy and Gennady Samorodnitsky *

Cornell University

We establish a connection between the structure of a stationary symmetric α-stable random field ($0 < \alpha < 2$) and ergodic theory of non-singular group actions, elaborating on a previous work by Rosiński (2000). With the help of this connection, we study the extreme values of the field over increasing boxes. Depending on the ergodic theoretical and group theoretical structures of the underlying action, we observe different kinds of asymptotic behavior of this sequence of extreme values.

1. Introduction. In this paper we study the structure of stationary symmetric α-stable discrete parameter non-Gaussian random fields. A random field $\{X_t\}_{t\in\mathbb{Z}^d}$ is called a symmetric α-stable ($S\alpha S$) random field if for all $c_1, c_2, \ldots, c_k \in \mathbb{R}$, and, $t_1, t_2, \ldots, t_k \in \mathbb{Z}^d$, $\sum_{j=1}^k c_j X_{t_j}$ follows a symmetric α-stable distribution. In this paper we will concentrate on the non-Gaussian case, and hence, we will assume $0 < \alpha < 2$, unless mentioned otherwise. For further reference on $S\alpha S$ distributions and processes the reader is suggested to read Samorodnitsky and Taqqu (1994). A random field $\{X_t\}_{t\in\mathbb{Z}^d}$ is called stationary if

$$\{X_t\} \stackrel{d}{=} \{X_{t+s}\} \quad \text{for all } s \in \mathbb{Z}^d \,. \tag{1.1}$$

Stationarity means that the law of the random field is invariant under the action of the group of shift transformations on the index-parameter $t \in \mathbb{Z}^d$.

More generally, if $(G, +)$ is a countable abelian group with identity element 0, then a random field $\{X_t\}_{t\in G}$ is called G-stationary if (1.1) holds for all $s \in G$. Most of the structure results in this paper have immediate analogs for G-stationary fields. We will mention these briefly along the way. Even though our main interest lies with $\mathbb{Z}^d$-indexed random fields, at a certain point in the paper a more general group structure will become important.

*Supported in part by NSF grant DMS-0303493, NSA grant MSPF-05G-049 and NSF training grant "Graduate and Postdoctoral Training in Probability and Its Applications" at Cornell University.

AMS 2000 subject classifications: Primary 60G60; secondary 37A40

Keywords and phrases: random field, stable process, ergodic theory, maxima, extreme value theory, group action, non-singular map, dissipative, conservative

Our first task in this paper is to establish a connection between ergodic theory of nonsingular $\mathbb{Z}^d$-actions (see Section 1.6 of Aaronson (1997)) and $S\alpha S$ random fields. Using the language of the Hopf decomposition of nonsingular flows a decomposition of stationary $S\alpha S$ processes was established in Rosiński (1995). For a general $d > 1$ a similar decomposition of $S\alpha S$ random fields into independent components was given in Rosiński (2000). We show the connection between this decomposition and ergodic theory. This is done in Section 3, using an approach different from the one-dimensional case, namely, without referring to the Chacon-Ornstein theorem, which is unavailable in the case $d > 1$.

We use the connection with ergodic theory to study the rate of growth of the partial maxima sequence $\{M_n\}$ of the random field X_t as t runs over a d-dimensional hypercube of size with an increasing edge length n. In the case $d = 1$ it has been shown in Samorodnitsky (2004) that this rate drops from $n^{1/\alpha}$ to something smaller as the flow generating the process changes from dissipative to conservative. One can argue that this phase transition qualifies as a transition between short and long memory. In this paper we establish a similar phase transition result for a general $d \geq 1$.

In Section 4, we first discuss the asymptotic behavior of a certain deterministic sequence which controls the size of the partial maxima sequence $\{M_n\}$. The treatment here is different from the one-dimensional case due to unavailability of Maharam extension theorem (see Theorem 2 in Maharam (1964)) in the case $d > 1$. In this section, we also calculate the rate of growth of partial maxima of the random field. We show that the rate of growth of M_n is equal to $n^{d/\alpha}$ if the group action has a nontrivial dissipative component, and is strictly smaller than that otherwise.

We discuss connections with the group theoretical properties of the action in Section 5. For $S\alpha S$ random fields generated by conservative actions, we view the underlying action as a group of nonsingular transformations and study the algebraic structure of this group to get better estimates on the rate of growth of the partial maxima. Examples illustrating how the maxima of a random field can grow are discussed in Section 6.

2. Some Ergodic Theory. The details on the notions introduced in this section can be found, for example, in Aaronson (1997). Unless stated otherwise, the statements about sets (e.g. equality or disjointness of two sets) are understood as holding up to a set of measure zero with respect to the underlying measure.

Suppose $(S, \mathcal{S}, \mu)$ is a σ-finite standard measure space and $(G, +)$ is a countable group with identity element 0. A collection of measurable maps

$\phi_t : S \to S,\ t \in G$ is called a group action of G on S if

1. ϕ_0 is the identity map on S, and,
2. $\phi_{u+v} = \phi_u \circ \phi_v$ for all $u, v \in G$.

A group action $\{\phi_t\}_{t\in G}$ of G on S is called nonsingular if $\mu \circ \phi_t \sim \mu$ for all $t \in G$.

A set $W \in \mathcal{S}$ is called a wandering set for the action $\{\phi_t\}_{t\in G}$ if $\{\phi_t(W) : t \in G\}$ is a pairwise disjoint collection. The following result (see Proposition 1.6.1 of Aaronson (1997)) gives a decomposition of S into two disjoint and invariant parts.

Proposition 2.1. *Suppose G is a countable group and $\{\phi_t\}$ is a nonsingular action of G on S. Then $S = \mathcal{C} \cup \mathcal{D}$ where $\mathcal{C}$ and $\mathcal{D}$ are disjoint and invariant measurable sets such that*

1. $\mathcal{D} = \bigcup_{t\in G} \phi_t(W_*)$ for some wandering set W_*,

2. $\mathcal{C}$ has no wandering subset of positive measure.

$\mathcal{D}$ is called the dissipative part, and $\mathcal{C}$ the conservative part of the action. The action $\{\phi_t\}$ is called conservative if $S = \mathcal{C}$ and dissipative if $S = \mathcal{D}$.

An action $\{\phi_t\}_{t\in G}$ is free if $\mu\big(\{s \in S : \ \phi_t(s) = s\}\big) = 0$ for all $t \in G - \{0\}$. Note that this definition makes sense because $(S, \mathcal{S})$ is a standard Borel space and hence $\{s \in S : \ \phi_t(s) = s\} \in \mathcal{S}$. The following result is a version of Halmos' Recurrence Theorem for a nonsingular action of a countable group.

Proposition 2.2. *Let $\{\phi_t\}$ be a nonsingular action of a countable group G. If $A \in \mathcal{S}$ and $A \subseteq \mathcal{C}$, then*

$$\sum_{t\in G} I_A \circ \phi_t = \infty \ \text{ a.e. on } A.$$

Proof. Define,

$$F := \{s \in S : \exists\, t \in G, t \neq 0 \text{ such that } \phi_t(s) = s\}.$$

Observe that F is $\{\phi_t\}$-invariant. Restrict $\{\phi_t\}$ to $S - F$. Let $\mathcal{C}_1$ be the conservative part of the restriction. It is easy to observe that $A \cap F^c \subseteq \mathcal{C}_1$ for all $A \subseteq \mathcal{C}$. Since the restricted action is free by Proposition 1.6.2 of

Aaronson (1997) we have,

$$\sum_{t\in G} I_A \circ \phi_t \geq \sum_{t\in G} I_{A\cap F^c} \circ \phi_t = \infty \text{ a.e. on } A \cap F^c.$$

Clearly,

$$\sum_{t\in G} I_A \circ \phi_t = \infty \text{ a.e. on } A \cap F.$$

This completes the proof. □

Recall that the dual operator of a nonsingular transformation T on S is a linear operator $\hat{T}$ on $L^1(S, \mu)$ such that

$$\int_S \hat{T} f.g d\mu = \int_S f.g \circ T d\mu \quad \text{for all } f \in L^1(\mu) \text{ and } g \in L^\infty(\mu)\,.$$

In particular, if T is invertible, then

$$\hat{T} f = \frac{d\mu \circ T^{-1}}{d\mu} f \circ T^{-1} \quad \text{for all } f \in L^1(\mu)\,,$$

see Section 1.3 in Aaronson (1997). The following proposition is an extension of Theorem 1.6.3 of Aaronson (1997) to not necessarily measure-preserving transformations, and can be established using an argument parallel to that of Propositon 1.3.1 in Aaronson (1997).

Proposition 2.3. *If G is a countable group and $\{\phi_t\}$ is a nonsingular action of G on S then for all $f \in L^1(\mu)$, $f > 0$,*

$$\mathcal{C} = \{s \in S : \sum_{t\in G} \hat{\phi}_t f(s) = \infty\}\,.$$

The following is an immediate corollary, particularly suitable for our purposes.

Corollary 2.4. *If G is a countable group and $\{\phi_t\}$ is a nonsingular action of G then*

$$[\sum_{t\in G} \frac{d\mu \circ \phi_t}{d\mu} f \circ \phi_t = \infty] = \mathcal{C} \ \textit{ for all } f \in L^1(\mu),\ f > 0.$$

Note that, as mentioned earlier, the equalities of sets in Proposition 2.3 and Corollary 2.4 above hold up to sets of μ-measure zero.

3. Stationary Symmetric Stable Random Fields. Suppose $\mathbf{X} = \{X_t\}_{t\in\mathbb{Z}^d}$ is a $S\alpha S$ random field, $0 < \alpha < 2$. We know from Theorem 13.1.2 of Samorodnitsky and Taqqu (1994) that it has an integral representation of the from

$$X_t \stackrel{d}{=} \int_S f_t(s)M(ds), \quad t \in \mathbb{Z}^d, \tag{3.1}$$

where M is a $S\alpha S$ random measure on some standard Borel space $(S, \mathcal{S})$ with σ-finite control measure μ and $f_t \in L^\alpha(S,\mu)$ for all $t \in \mathbb{Z}^d$. Note that f_t's are deterministic functions and hence all the randomness of $\mathbf{X}$ is hidden in the random measure M, and, the inter-dependence of the X_t's is captured in $\{f_t\}$. The representation (3.1) is called an integral representation of $\{X_t\}$. Without loss of generality we can also assume that the family $\{f_t\}$ satisfies the full support assumption

$$\text{Support}\Big(f_t,\, t \in \mathbb{Z}^d\Big) = S, \tag{3.2}$$

because, if that is not the case, we can replace S by $S_0 = \text{Support}\big(f_t,\, t \in \mathbb{Z}^d\big)$ in (3.1).

If, further, $\{X_t\}$ is stationary, then the fact that the action of the group $\mathbb{Z}^d$ on $\{X_t\}_{t\in\mathbb{Z}^d}$ by translation of indices preserves the law, and certian rigidity of spaces L^α, $\alpha < 2$ guarantees existence of intergral representations of a special form. This has been established in Rosiński (1995) for $d = 1$ and Rosiński (2000) for a general d. Specifically, there always exists a representation of the form

$$f_t(s) = c_t(s)\left(\frac{d\mu\circ\phi_t}{d\mu}(s)\right)^{1/\alpha} f\circ\phi_t(s), \quad t \in \mathbb{Z}^d, \tag{3.3}$$

where, $f \in L^\alpha(S,\mu)$, $\{\phi_t\}_{t\in\mathbb{Z}^d}$ is a nonsingular $\mathbb{Z}^d$-action on (S,μ) and $\{c_t\}_{t\in\mathbb{Z}^d}$ is a measurable cocycle for $\{\phi_t\}$ taking values in $\{-1,+1\}$ i.e. each c_t is a measurable map $c_t : S \to \{-1,+1\}$ such that $\forall\, u, v \in \mathbb{Z}^d$

$$c_{u+v}(s) = c_v(s)c_u\big(\phi_v(s)\big) \text{ for } \mu\text{-a.a. } s \in S.$$

Conversely, if $\{f_t\}$ is of the form (3.3) then $\{X_t\}$ defined by (3.1) is a stationary $S\alpha S$ random field. In particular, every *minimal* representation of the process (see Hardin Jr. (1982)) turns out to be of the form (3.3).

We will say that a stationary $S\alpha S$ random field $\{X_t\}_{t\in\mathbb{Z}^d}$ is generated by a nonsingular $\mathbb{Z}^d$-action $\{\phi_t\}$ on (S,μ) if it has a integral representation of the form (3.3) satisfying (3.2). With this terminology, we have the following extension of Theorem 4.1 in Rosiński (1995) to random fields.

Proposition 3.1. *Suppose* $\{X_t\}_{t\in\mathbb{Z}^d}$ *is a stationary* $S\alpha S$ *random field generated by a nonsingular* $\mathbb{Z}^d$*-action* $\{\phi_t\}$ *on* (S,μ) *and* $\{f_t\}$ *is given by (3.3). Also let,* $\mathcal{C}$ *and* $\mathcal{D}$ *be the conservative and dissipative parts of* $\{\phi_t\}$*. Then we have,*

$$\begin{aligned}\mathcal{C} &= \{s\in S : \sum_{t\in\mathbb{Z}^d}|f_t(s)|^\alpha = \infty\} \;\; \textit{mod } \mu, \textit{ and},\\ \mathcal{D} &= \{s\in S : \sum_{t\in\mathbb{Z}^d}|f_t(s)|^\alpha < \infty\} \;\; \textit{mod } \mu\,.\end{aligned}$$

In particular, if a stationary $S\alpha S$ *random field* $\{X_t\}_{t\in\mathbb{Z}^d}$ *is generated by a conservative (dissipative, resp.)* $\mathbb{Z}^d$*-action, then in any other integral representation of* $\{X_t\}$ *of the form (3.3) satisfying (3.2), the* $\mathbb{Z}^d$*-action must be conservative (dissipative, resp.). Hence the classes of stationary* $S\alpha S$ *random fields generated by conservative and dissipative actions are disjoint.*

Proof. Define g as

$$g(s) = \sum_{u\in\mathbb{Z}^d}\alpha_u\frac{d\mu\circ\phi_u}{d\mu}(s)|f\circ\phi_u(s)|^\alpha$$

where $\alpha_u > 0$ for all $u\in\mathbb{Z}^d$ and $\sum_{u\in\mathbb{Z}^d}\alpha_u = 1$. Clearly $g\in L^1$ and, by (3.2), $g>0$ a.e. μ. Since

$$\sum_{t\in\mathbb{Z}^d}\frac{d\mu\circ\phi_t}{d\mu}(s)g\circ\phi_t(s) = \sum_{t\in\mathbb{Z}^d}\frac{d\mu\circ\phi_t}{d\mu}(s)|f\circ\phi_t(s)|^\alpha = \sum_{t\in\mathbb{Z}^d}|f_t(s)|^\alpha$$

we can use Corollary 2.4 to establish the first part of the proposition, from which the second part of the proposition follows by the same argument as in the one-dimensional case (see Theorem 4.1 in Rosiński (1995)). □

As in the one-dimensional case, it follows that the test described in the previous proposition can be applied to any full support integral representation of the process, not necessarily that of a specific form.

Corollary 3.2. *The stationary* $S\alpha S$ *random field* $\{X_t\}_{t\in\mathbb{Z}^d}$ *is generated by a conservative (dissipative, resp.)* $\mathbb{Z}^d$*-action if and only if for any (equivalently, some) integral representation (3.1) of* $\{X_t\}$ *satisfying (3.2), the sum*

$$\sum_{t\in\mathbb{Z}^d}|f_t(s)|^\alpha$$

is infinite (finite, resp) μ*-a.e. .*

Proposition 3.1 also enables us to extend the connection between the structure of stationary stable processes and ergodic theory of nonsingular actions (given in Rosiński (1995)) to the case of stationary stable random fields. A decomposition of a stable random field into three independent parts is available in Rosiński (2000). A connection with the conservative-dissipative decomposition is still missing in the case of random fields. Here we provide the missing link. Recall that a stable random field $\mathbf{X}$ is called a mixed moving average if it can be represented in the form

$$\mathbf{X} \stackrel{d}{=} \left\{ \int_{W\times\mathbb{Z}^d} f(v, t+s)\, M(dv, ds) \right\}_{t\in\mathbb{Z}^d}, \tag{3.4}$$

where $f \in L^\alpha(W \times \mathbb{Z}^d, \nu \otimes l)$, l is the counting measure on $\mathbb{Z}^d$, ν is a σ-finite measure on a standard Borel space $(W, \mathcal{W})$, and the control measure μ of M equals $\nu \otimes l$ (see Surgailis et al. (1993) and Rosiński (2000)). The following result gives two equivalent characterizations of stationary $S\alpha S$ random fields generated by dissipative actions.

Theorem 3.3. *Suppose $\{X_t\}_{t\in\mathbb{Z}^d}$ is a stationary $S\alpha S$ random field. Then, the following are equivalent:*

1. *$\{X_t\}$ is generated by a dissipative $\mathbb{Z}^d$-action.*
2. *For any integral representation $\{f_t\}$ of $\{X_t\}$ we have,*

$$\sum_{t\in\mathbb{Z}^d} |f_t(s)|^\alpha < \infty \text{ for } \mu\text{-a.a. } s.$$

3. *$\{X_t\}$ is a mixed moving average.*

Proof. 1 and 2 are equivalent by Corollary 3.2, and, 2 and 3 are equivalent by Theorem 2.1 of Rosiński (2000).

□

Theorem 3.3 allows us to describe the decomposition of a stationary $S\alpha S$ random field given in Theorem 3.7 of Rosiński (2000) in terms of the ergodic-theoretical properties of nonsingular $\mathbb{Z}^d$-actions generating the field. The statement of the following corollary is an extension of the one-dimensional decomposition in Theorem 4.3 in Rosiński (1995) to random fields.

Corollary 3.4. *A stationary $S\alpha S$ random field $\mathbf{X}$ has a unique in law decomposition*

$$X_t \stackrel{d}{=} X_t^{\mathcal{C}} + X_t^{\mathcal{D}}, \tag{3.5}$$

where $\mathbf{X}^{\mathcal{C}}$ *and* $\mathbf{X}^{\mathcal{D}}$ *are two independent stationary* $S\alpha S$ *random fields such that* $\mathbf{X}^{\mathcal{D}}$ *is a mixed moving average, and* $\mathbf{X}^{\mathcal{C}}$ *is generated by a conservative action.*

As mentioned before, all of the structure results of this section extend immediately to G-stationary random fields for countable abelian groups G more general than $\mathbb{Z}^d$. The only place where an additional argument is needed is the equivalence of parts 2 and 3 in Theorem 3.3, with a G-mixed moving average defined by

$$\mathbf{X} \stackrel{d}{=} \left\{ \int_{W \times G} g(v, t+s)\, M(dv, ds) \right\}_{t \in G},$$

in notation parallel to (3.4). This equivalence needs an extension of Theorem 2.1 in Rosiński (2000) to general countable abelian groups. See Roy (2007) for details of this extension which does not require any additional ideas to what is already in the original proof.

As in the one-dimensional case, it is possible to think of stable random fields generated by conservative actions as having longer memory than those generated by dissipative actions, simply because a conservative action "keeps coming back", and so the same values of the random measure M contribute to observations X_t far separated in t. From this point of view, the $\mathbb{Z}^d$-action $\{\phi_t\}$ is a parameter (though highly infinite-dimensional) of the stationary $S\alpha S$ random field $\{X_t\}$ that determines, among others, the length of its memory.

4. Maxima of Stable Random Fields. The length of memory of stable random fields is manifested, in particular, in the rate of growth of its extreme values. If X_t is generated by a conservative action, the extreme values tend to grow at a slower rate because longer memory prevents erratic changes in X_t even when t becomes "large". This has been formalized in Samorodnitsky (2004) for $d = 1$, and it turns out to be the case for stable random fields as well.

For a stationary $S\alpha S$ random field $\{X_t\}_{t \in \mathbb{Z}^d}$, we will study the partial maxima sequence

$$M_n \;:=\; \max_{0 \le t \le (n-1)\mathbf{1}} |X_t|, \quad n = 0, 1, 2, \ldots \tag{4.1}$$

where $u = (u^{(1)}, u^{(2)}, \ldots, u^{(d)}) \le v = (v^{(1)}, v^{(2)}, \ldots, v^{(d)})$ means $u^{(i)} \le v^{(i)}$ for all $i = 1, 2, \ldots, d$ and $\mathbf{1} = (1, 1, \ldots, 1)$. As in the one-dimensional case, the asymptotic behavior of the maximum functional M_n is related to the deterministic sequence

$$b_n \;:=\; \left(\int_S \max_{0\le t\le (n-1)\mathbf{1}} |f_t(s)|^\alpha \mu(ds)\right)^{1/\alpha}, \quad n=0,1,2,\dots\,. \tag{4.2}$$

Note that b_n is completely determined by the process, and does not depend on a particular integral representation (see Corollary 4.4.6 of Samorodnitsky and Taqqu (1994)). We are interested in the features of this sequence that are related to the decomposition of a stable random field in Corollary 3.4. The next result shows that the sequence b_n grows at a slower rate for random fields generated by a conservative action than for random fields generated by a dissipative action.

Proposition 4.1. *Let $\{f_t\}$ be given by (3.3). Assume that (3.2) holds.*

1. *If the action $\{\phi_t\}$ is conservative then:*

$$n^{-d/\alpha} b_n \;\to\; 0 \quad \text{as } n\to\infty. \tag{4.3}$$

2. *If the action $\{\phi_t\}$ is dissipative, and the random field is given in the mixed moving average form (3.4), then:*

$$\lim_{n\to\infty} n^{-d/\alpha} b_n \;=\; \left(\int_W (g(v))^\alpha \nu(dv)\right)^{1/\alpha} \in (0,\infty), \tag{4.4}$$

where

$$g(v) = \sup_{s\in\mathbb{Z}^d} |f(v,s)| \quad \text{for } v\in W. \tag{4.5}$$

Proof. 1. Firstly we observe that without loss of generality we can assume that μ is a probability measure. This is because if ν is a probability measure equivalent to the σ-finite measure μ then instead of (3.1) we will use

$$X_t \stackrel{d}{=} \int_S h_t(s) N(ds)$$

where,

$$h_t(s) = c_t(s)\left(\frac{d\nu\circ\phi_t}{d\nu}(s)\right)^{1/\alpha} h\circ\phi_t(s), \quad t\in\mathbb{Z}^d$$

where $h = f\left(\frac{d\mu}{d\nu}\right)^{1/\alpha} \in L^\alpha(S,\nu)$ and N is a $S\alpha S$ random measure on S with control measure ν.

Since $\{b_n\}$ is an increasing sequence, it is enough to show (4.3) along the odd subsequence. By stationarity of $\{X_t\}$, we need to check that

$$a_n := \frac{1}{(2n+1)^d} \int_S \max_{t \in J_n} |f_t(s)|^\alpha \mu(ds) \to 0\,,$$

where $J_n := \{(i_1, i_2, \ldots, i_d) : -n \le i_1, i_2, \ldots, i_d \le n\}$. Let $g = |f|^\alpha$. Then $\|g\| := \int_S g(s)\mu(ds) < \infty$, and we have for $0 < \epsilon < 1$

$$\begin{aligned} a_n &= \frac{1}{(2n+1)^d} \int_S \max_{t \in J_n} \hat{\phi}_t g(s) \mu(ds) \\ &\le \frac{1}{(2n+1)^d} \Bigg(\int_S \max_{t \in J_n} \Big[\hat{\phi}_t g(s) I\Big(\hat{\phi}_t g(s) \le \epsilon \sum_{u \in J_n} \hat{\phi}_u g(s) \Big) \Big] \mu(ds) \\ &\qquad + \int_S \max_{t \in J_n} \Big[\hat{\phi}_t g(s) I\Big(\hat{\phi}_t g(s) > \epsilon \sum_{u \in J_n} \hat{\phi}_u g(s) \Big) \Big] \mu(ds) \Bigg) \\ &= a_n^{(1)} + a_n^{(2)}\,. \end{aligned}$$

Clearly,

$$a_n^{(1)} \le \frac{\epsilon}{(2n+1)^d} \sum_{u \in J_n} \int_S \hat{\phi}_u g(s) \mu(ds) = \epsilon \|g\|\,, \tag{4.6}$$

and

$$a_n^{(2)} \le \frac{1}{(2n+1)^d} \sum_{t \in J_n} \int_S \hat{\phi}_t g(s) I_{A_{t,n}}(s) \mu(ds)\,, \tag{4.7}$$

where $A_{t,n} = \{s : \hat{\phi}_t g(s) > \epsilon \sum_{u \in J_n} \hat{\phi}_u g(s)\}$, $n \ge 1$, $t \in J_n$. Notice that for all $n \ge 1$, and, for all $t \in J_n$,

$$\int_S \hat{\phi}_t g(s) I_{A_{t,n}}(s) \mu(ds) = \int_S g(s) I_{\phi_t^{-1}(A_{t,n})}(s) \mu(ds)\,. \tag{4.8}$$

The following is the most important step of this proof: if we define

$$U_n := \{(t_1, t_2, \ldots, t_d) : -n + [\sqrt{n}] \le t_1, t_2, \ldots, t_d \le n - [\sqrt{n}]\}$$

then we have,

$$\lim_{n \to \infty} \max_{t \in U_n} \mu(\phi_t^{-1}(A_{t,n})) = 0\,. \tag{4.9}$$

To prove (4.9) observe that for all $t \in U_n$

$$\begin{aligned}
&\phi_t^{-1}(A_{t,n}) \\
&\quad = \Big\{\phi_{-t}(s) : \, g \circ \phi_{-t}(s) \frac{d\mu \circ \phi_{-t}}{d\mu}(s) > \epsilon \sum_{u \in J_n} g \circ \phi_{-u}(s) \frac{d\mu \circ \phi_{-u}}{d\mu}(s)\Big\} \\
&\quad = \Big\{s : \, g(s) > \epsilon \sum_{u \in J_n} g \circ \phi_{u+t}(s) \frac{d\mu \circ \phi_{u+t}}{d\mu}(s)\Big\} \\
&\quad \subseteq \Big\{s : \, g(s) > \epsilon \sum_{\tau \in J_{[\sqrt{n}]}} g \circ \phi_\tau(s) \frac{d\mu \circ \phi_\tau}{d\mu}(s)\Big\} .
\end{aligned}$$

The last inclusion holds because $J_{[\sqrt{n}]} \subseteq t + J_n$. Hence, for any $M > 0$

$$\begin{aligned}
\max_{t \in U_n} \mu(\phi_t^{-1}(A_{t,n})) &\le \mu\{s : \, g(s) > \epsilon M\} + \mu\Big(\sum_{t \in J_{[\sqrt{n}]}} g \circ \phi_t \frac{d\mu \circ \phi_t}{d\mu} \le M\Big) \\
&\le \frac{\|g\|}{\epsilon M} + \mu\Big(\sum_{t \in J_{[\sqrt{n}]}} |f_t|^\alpha \le M\Big) .
\end{aligned}$$

Now (4.9) follows by first using Proposition 3.1 with a fixed M and then letting $M \to \infty$.

From (4.8) and (4.9) it follows that

$$\begin{aligned}
&\frac{1}{(2n+1)^d} \sum_{t \in U_n} \int_S \hat{\phi}_t g(s) I_{A_{t,n}}(s) \mu(ds) \\
&\qquad = \frac{1}{(2n+1)^d} \sum_{t \in U_n} \int_{\phi_t^{-1}(A_{t,n})} g(s) \mu(ds) \to 0 . \qquad (4.10)
\end{aligned}$$

If we define $V_n = J_n - U_n$, then

$$\frac{1}{(2n+1)^d} \sum_{t \in V_n} \int_S \hat{\phi}_t g(s) I_{A_{t,n}}(s) \mu(ds) \le \frac{1}{(2n+1)^d} \sum_{t \in V_n} \int_S \hat{\phi}_t g(s) \mu(ds) \to 0 .$$

Then using (4.7) and (4.10) we see that $a_n^{(2)} \to 0$ as $n \to \infty$. Therefore we get,

$$\limsup a_n \le \limsup a_n^{(1)} + \limsup a_n^{(2)} \le \epsilon \|g\| ,$$

and, since $\epsilon > 0$ is arbitrary, the result follows.

2. The argument here is similar to that used in the one-dimensional case in Theorem 3.1 of Samorodnitsky (2004). One uses a direct computation to check the claim in the case where f has compact support, that is

$$f(v,s)I_{W\times[-m\mathbf{1},m\mathbf{1}]^c}(v,s) \equiv 0 \text{ for some } m = 1,2,\dots$$

where $[u,v] := \{t \in \mathbb{Z}^d : u \le t \le v\}$. The proof in the general case follows then by approximating a general kernel f by a kernel with a compact support.

□

Remark 4.2. The statement of the first part of the proposition clearly extends to G-stationary random fields for any free abelian group G of rank d, since the same is true for Proposition 3.1. See the discussion after Corollary 3.4.

We are now ready to investigate the rate of growth of the sequence $\{M_n\}$ of partial maxima of a stationary symmetric α-stable random field, $0 < \alpha < 2$. We will see that if such a random field has a nonzero component $X^{\mathcal{D}}$ in (3.5) generated by a dissipative action, then the partial maxima grow at the rate $n^{d/\alpha}$, while if the random field is generated by a conservative action, then the partial maxima grow at a slower rate. As we will see in the sequel, the actual rate of growth of the sequence $\{M_n\}$ in the conservative case, depends on a number of factors. The dependence on the group theoretical properties of the action is very prominent. We start with the following result, which extends Theorem 4.1 of Samorodnitsky (2004) to $d > 1$. It is based on Proposition 4.1, and the argument is parallel to the one-dimensional case.

Theorem 4.3. *Let $\mathbf{X} = \{X_t\}_{t\in\mathbb{Z}^d}$ be a stationary $S\alpha S$ random field, with $0 < \alpha < 2$, integral representation (3.1), and functions $\{f_t\}$ given by (3.3).*

1. Suppose that $\mathbf{X}$ is not generated by a conservative action (i.e. the component $X^{\mathcal{D}}$ in (3.5) generated by the dissipative part is nonzero). Then

$$\frac{1}{n^{d/\alpha}} M_n \Rightarrow C_\alpha^{1/\alpha} K_X Z_\alpha \tag{4.11}$$

as $n \to \infty$, where

$$K_X = \left(\int_W (g(v))^\alpha \nu(dv) \right)^{1/\alpha}$$

and g is given by (4.5) for any representation of $X^{\mathcal{D}}$ in the mixed moving average form (3.4), C_α is the stable tail constant (see (1.2.9) in Samorodnitsky and Taqqu

(1994)) and Z_α is the standard Frechet-type extreme value random variable with the distribution

$$P(Z_\alpha \le z) = e^{-z^{-\alpha}}, \quad z > 0.$$

2. Suppose that $\mathbf{X}$ *is generated by a conservative* $\mathbb{Z}^d$*-action. Then*

$$\frac{1}{n^{d/\alpha}} M_n \xrightarrow{p} 0 \tag{4.12}$$

as $n \to \infty$*. Furthermore, with* b_n *given by (4.2),*

$$\left\{ \frac{1}{c_n} M_n \right\} \text{ is not tight for any positive sequence } c_n = o(b_n), \tag{4.13}$$

while

$$\left\{ \frac{1}{b_n \zeta_n} M_n \right\} \text{ is tight, where } \zeta_n = \begin{cases} 1, & \text{if } 0 < \alpha < 1, \\ L_n, & \text{if } \alpha = 1, \\ (\log n)^{1/\alpha'}, & \text{if } 1 < \alpha < 2, \end{cases} \tag{4.14}$$

where $L_n := \max(1, \log \log n)$*, and for* $\alpha > 1$*,* α' *is such that* $1/\alpha + 1/\alpha' = 1$*.*

If, for some $\theta > 0$ *and* $c > 0$*,*

$$b_n \ge c n^\theta \quad \text{for all } n \ge 1, \tag{4.15}$$

then (4.14) holds with $\zeta_n \equiv 1$ *for all* $0 < \alpha < 2$*.*

Finally, for $n = 1, 2, \ldots$*, let* η_n *be a probability measure on* $(S, \mathcal{S})$ *with*

$$\frac{d\eta_n}{d\mu}(s) = b_n^{-\alpha} \max_{0 \le t \le (n-1)\mathbf{1}} |f_t(s)|^\alpha, \quad s \in S, \tag{4.16}$$

and let $U_j^{(n)}$*,* $j = 1, 2$ *be independent* S*-valued random variables with common law* η_n*. Suppose that (4.15) holds and for any* $\epsilon > 0$*,*

$$P\Bigg(\text{for some } t \in [0, (n-1)\mathbf{1}], \quad \frac{|f_t(U_j^{(n)})|}{\max_{0 \le u \le (n-1)\mathbf{1}} |f_u(U_j^{(n)})|} > \epsilon, \; j = 1, 2 \Bigg) \to 0 \tag{4.17}$$

as $n \to \infty$*. Then*

$$\frac{1}{b_n} M_n \Rightarrow C_\alpha^{1/\alpha} Z_\alpha \tag{4.18}$$

as $n \to \infty$*.*

Remark 4.4. An easily verifiable sufficient condition for (4.17) is

$$\lim_{n\to\infty} \frac{b_n}{n^{d/2\alpha}} = \infty. \tag{4.19}$$

Alternatively, (4.17) holds if we assume that μ is a finite measure, $\{\phi_t\}$ is measure preserving, the sequence $\{b_n^{-\alpha} \max_{0\le t\le (n-1)\mathbf{1}} |f_t(s)|^\alpha\}$, $t \in \mathbb{Z}^d$ is uniformly integrable with respect to μ and, for every $\epsilon > 0$

$$\lim_{n\to\infty} n^{d/2}\mu\{s \in S : |f(s)| > \epsilon b_n\} = 0\,. \tag{4.20}$$

The arguments are the same as in the case $d = 1$.

5. Connections with Group Theory. When the underlying action is not conservative Theorem 4.3 yields the exact rate of growth of M_n. For conservative actions, however, the actual rate of growth of the partial maxima depends on further properties of the action. In this section we investigate the effect of the group theoretic structure of the action on the rate of growth of the partial maximum. We start with introducing the appropriate notation.

Consider $A := \{\phi_t : t \in \mathbb{Z}^d\}$ as a subgroup of the group of invertible nonsingular transformations on (S,μ) and define a group homomorphism

$$\Phi : \mathbb{Z}^d \to A$$

by $\Phi(t) = \phi_t$ for all $t \in \mathbb{Z}^d$. Let $K := Ker(\Phi) = \{t \in \mathbb{Z}^d : \phi_t = 1_S\}$, where 1_S denote the identity map on S. Then K is a free abelian group and by first isomorphism theorem of groups (see, for example, Lang (2002)) we have,

$$A \simeq \mathbb{Z}^d/K\,.$$

Hence by Theorem 8.5 in Chapter I of Lang (2002) we get,

$$A = \bar{F} \oplus \bar{N}$$

where $\bar{F}$ is a free abelian group and $\bar{N}$ is a finite group. Assume $rank(\bar{F}) = p \ge 1$ and $|\bar{N}| = l$. Since $\bar{F}$ is free, there exists an injective group homomorphism

$$\Psi : \bar{F} \to \mathbb{Z}^d$$

such that $\Phi \circ \Psi = 1_{\bar{F}}$. Let $F = \Psi(\bar{F})$. Then F is a free subgroup of $\mathbb{Z}^d$ of rank p.

The rank p is the effective dimension of the random field, giving more precise information on the rate of growth of the partial maximum than the nominal dimension d. We start with showing that this is true for the sequence $\{b_n\}$ in (4.2).

Proposition 5.1. *Let $\{f_t\}$ be given by (3.3). Assume that (3.2) holds. Then we have the following:*

1. *If $\{\phi_t\}_{t\in F}$ is conservative then*

$$n^{-p/\alpha} b_n \to 0\,. \tag{5.1}$$

2. *If $\{\phi_t\}_{t\in F}$ is dissipative then*

$$n^{-p/\alpha} b_n \to a \tag{5.2}$$

for some $a \in (0,\infty)$.

Proof. 1. It is easy to check that $F \cap K = \{0\}$ and hence the sum $F + K$ is direct. Suppose $G = F \oplus K$. Using group isomorphism theorems we have,

$$\mathbb{Z}^d/G \simeq (\mathbb{Z}^d/K)/(F \oplus K/K) \simeq A/\bar{F} \simeq \bar{N}\,.$$

Assume that $x_1 + G$, $x_2 + G$, $\dots, x_l + G$ are all the cosets of G in $\mathbb{Z}^d$. Let $rank(K) = q$. Choose a basis $\{u_1, u_2, \dots, u_p\}$ of F and a basis $\{v_1, v_2, \dots, v_q\}$ of K. We need the following

Lemma 5.2. *There are positive integers c, d, and, N such that for every $n \geq N$*

$$\bigcup_{k=1}^{l} (x_k + G_{[n/d]}) \subseteq [-n\mathbf{1}, n\mathbf{1}] \subseteq \bigcup_{k=1}^{l} (x_k + G_{cn}) \tag{5.3}$$

where for $m \geq 1$

$$G_m := \{\sum_{i=1}^{p} \alpha_i u_i + \sum_{j=1}^{q} \beta_j v_j :\ |\alpha_i|, |\beta_j| \leq m \text{ for all } i,j\}\,.$$

Proof. Let $r = p + q$. For ease of notation we define

$$w_i = \begin{cases} u_i & 1 \leq i \leq p\,, \\ v_{i-p} & p+1 \leq i \leq r\,. \end{cases}$$

Then $\{w_1, w_2, \dots, w_r\}$ is a basis for G. The first inclusion in (5.3) is obvious. To establish the second inclusion we first prove

Step 1. There is an integer $c' \geq 1$ such that

$$[-n\mathbf{1}, n\mathbf{1}] \cap G \subseteq G_{c'n} \quad \text{for all } n \geq 1\,.$$

Proof of Step 1. Take $y \in [-n\mathbf{1}, n\mathbf{1}] \cap G$. Then, $y = \eta_1 w_1 + \eta_2 w_2 + \cdots + \eta_r w_r$ for some $\eta_1, \eta_2, \dots, \eta_r \in \mathbb{Z}$. We have to show $|\eta_i| \leq c'n$ for all $1 \leq i \leq r$ for some $c' \geq 1$ that does not depend on n. Let $\widetilde{\eta}^T := (\eta_1, \eta_2, \dots, \eta_r) \in \mathbb{Z}^r$. Then,

$$y = W\widetilde{\eta} \tag{5.4}$$

where, W is the $d \times r$ matrix with w_i as the i^{th} column. The columns of W are linearly independent over $\mathbb{Z}$ and hence over $\mathbb{R}$. Hence there is a $r \times d$ matrix Z such that

$$ZW = I$$

where I is the identity matrix of order r. Hence from (5.4) we have,

$$\widetilde{\eta} = Zy\,.$$

For all $1 \leq i \leq r$ we get,

$$|\eta_i| \leq \|\widetilde{\eta}\| \leq \|Z\|\|y\| \leq \|Z\|n\sqrt{d} \leq c'n$$

where, $c' = \left[\|Z\|\sqrt{d}\right] + 1$. This proves Step 1.

Step 2. Let

$$M = \max_{1 \leq k \leq l} \|x_k\|_\infty + 1$$

where $\|\cdot\|_\infty$ denotes the sup-norm on R^d, and $c = c'M$. Then for all $n \geq 1$ we have,

$$[-n\mathbf{1}, n\mathbf{1}] \subseteq \bigcup_{k=1}^{l} (x_k + G_{cn})\,.$$

Proof of Step 2. Take $y \in [-n\mathbf{1}, n\mathbf{1}]$. Then $y \in x_{k_0} + G$ for some $1 \leq k_0 \leq l$. Clearly, $y' := y - x_{k_o} \in [-(n+M-1)\mathbf{1}, (n+M-1)\mathbf{1}] \cap G$. By Step 1, $y' \in G_{c'(n+M-1)} \subseteq G_{cn}$, and hence, $y \in x_{k_0} + G_{cn} \subseteq \bigcup_{k=1}^{l}(x_k + G_{cn})$, proving Step 2 and the lemma.

□

For $k = 1, \dots, l$ let

$$g_k = f \circ \phi_{x_k} \left(\frac{d\mu \circ \phi_{x_k}}{d\mu} \right)^{1/\alpha}.$$

Then for $t = x_k + \sum_{i=1}^p \alpha_i u_i + \sum_{j=1}^q \beta_j v_j$ we have

$$|f_t(s)| = |g_k \circ \phi_{\sum_{i=1}^p \alpha_i u_i}(s)| \left(\frac{d\mu \circ \phi_{\sum_{i=1}^p \alpha_i u_i}}{d\mu}(s) \right)^{1/\alpha} . \tag{5.5}$$

By stationarity, Lemma 5.2 and (5.5) we have, for all $n \geq N$,

$$\begin{aligned} b_n^\alpha &\leq b_{2n+1}^\alpha = \int_S \max_{-n\mathbf{1} \leq t \leq n\mathbf{1}} |f_t(s)|^\alpha \mu(ds) \\ &\leq \int_S \max_{1 \leq k \leq l} \max_{|\alpha_i| \leq cn} \left(|g_k \circ \phi_{\sum_{i=1}^p \alpha_i u_i}(s)|^\alpha \frac{d\mu \circ \phi_{\sum_{i=1}^p \alpha_i u_i}}{d\mu}(s) \right) \mu(ds) \\ &\leq \sum_{k=1}^l \int_S \max_{|\alpha_i| \leq cn} \left(|g_k \circ \phi_{\sum_{i=1}^p \alpha_i u_i}(s)|^\alpha \frac{d\mu \circ \phi_{\sum_{i=1}^p \alpha_i u_i}}{d\mu}(s) \right) \mu(ds) \\ &= o(n^p) . \end{aligned}$$

The last step follows from Proposition 4.1 and Remark 4.2.

2. Proof of this part is similar to the proof of Theorem 3.1 in Samorodnitsky (2004). We start this proof with the following combinatorial fact:

Lemma 5.3. *For $n \geq 1$ and $k = 1, 2, \ldots, l$, let*

$$F_{k,n} = \left\{ u \in x_k + F : \text{ there exists } v \in K \text{ such that } u + v \in [-n\mathbf{1}, n\mathbf{1}] \right\} .$$

Then there is a positive real number $\mathcal{V}$ such that for all $k = 1, 2, \ldots, l$,

$$\lim_{n \to \infty} \frac{|F_{k,n}|}{n^p} = \mathcal{V} . \tag{5.6}$$

Here $|A|$ stands for the cardinality of a set A.

Proof. One of $F_{k,n}$ is the set

$$F_n = \left\{ y \in F : \text{ there exists } v \in K \text{ such that } y + v \in [-n\mathbf{1}, n\mathbf{1}] \right\} .$$

Firstly, we will show

$$\lim_{n \to \infty} \frac{|F_n|}{n^p} = \mathcal{V} \tag{5.7}$$

for some $\mathcal{V} > 0$. To show this let W be the matrix used in the proof of Lemma 5.2. We can partition W into two submatrices as follows:

$$W = [\, U \,|\, V \,]$$

where, U is the $d \times p$ matrix whose i^{th} column is u_i and, V is the $d \times q$ matrix whose j^{th} column is v_j. Since the columns of U are linearly independent over $\mathbb{Z}$, we have,

$$|F_n| = |\{\alpha \in \mathbb{Z}^p : \text{there exists } \beta \in \mathbb{Z}^q \text{ such that } \|U\alpha + V\beta\|_\infty \le n\}| .$$

Let $P := \{x \in \mathbb{R}^r : \|Wx\|_\infty \le 1\}$ and $\pi : \mathbb{R}^r \to \mathbb{R}^p$ denote the projection map on the first p coordinates:

$$\pi(x_1, x_2, \dots, x_r) = (x_1, x_2, \dots, x_p) .$$

Then we have,

$$\frac{|F_n|}{n^p} = \frac{|\pi(\mathbb{Z}^r \cap nP)|}{n^p} =: a_n .$$

Let

$$b_n := \frac{|\mathbb{Z}^p \cap n\pi(P)|}{n^p} .$$

Clearly, $a_n \le b_n$. Since P is a rational polytope (i.e. a polytope whose vertices have rational coordinates) so is $\pi(P)$. Hence, by Theorem 1 of De Loera (2005), it follows that

$$\limsup_{n\to\infty} a_n \le \lim_{n\to\infty} b_n = \mathcal{V} \tag{5.8}$$

where $\mathcal{V} = Volume(\pi(P))$, the p-dimensional volume of $\pi(P)$. This volume is positive since the latter set, obviously, contains a small ball centered at the origin. For the other inequality we let

$$P_m := \left\{ x \in \mathbb{R}^r : \|Wx\|_\infty \le 1 - \frac{\|W\|_\infty}{m} \right\}$$

where $\|W\|_\infty := \sup_{x\neq 0} \frac{\|Wx\|_\infty}{\|x\|_\infty} \in \mathbb{Z}$ since W is a matrix with integer entries. Hence for all $m > \|W\|_\infty$, P_m is a rational polytope of dimension r. Also, $P_m \uparrow P$. Now fix $m > \|W\|_\infty$. Observe that

$$\left\{ y \in \mathbb{R}^r : \|y - x\|_\infty \le \frac{1}{m} \right\} \subseteq P \quad \text{for all } x \in P_m .$$

Hence, it follows that for all $n > m$,

$$\pi\left(\frac{1}{n}\mathbb{Z}^r \cap P\right) \supseteq \frac{1}{n}\mathbb{Z}^p \cap \pi(P_m) ,$$

which, along with Theorem 1 of De Loera (2005), implies

$$\liminf_{n\to\infty} a_n \ge \lim_{n\to\infty} \frac{|\mathbb{Z}^p \cap n\pi(P_m)|}{n^p} = \mathcal{V}_m \tag{5.9}$$

where $\mathcal{V}_m = Volume(\pi(P_m))$ is, once again, the p-dimensional volume. Since $P_m \uparrow P$, it follows that $\mathcal{V}_m \uparrow \mathcal{V}$. Hence (5.7) follows from (5.8) and (5.9).

Now fix $k = 1, 2, \ldots, l$ and let $M = \|x_k\|$. Observe that for all $n > M$,

$$|F_{n-M}| \leq |F_{k,n}| \leq |F_{n+M}| .$$

Hence (5.6) follows from (5.7).

□

We now return to the proof of the second part of the proposition. We give a group structure to

$$H := \bigcup_{k=1}^{l} (x_k + F) \tag{5.10}$$

as follows. For all $u_1, u_2 \in H$, there exists unique $u \in H$ such that $(u_1 + u_2) - u \in K$. We define this u to be $u_1 \oplus u_2$. It is not hard to check that $(H, \oplus)$ is a countable abelian group. In fact, $H \simeq \mathbb{Z}^d/K$. We can define a nonsingular group action $\{\psi_u\}$ of H on S as

$$\psi_u = \phi_u \quad \text{for all } u \in H .$$

Notice that if $h \in L^1(S, \mu), h > 0$, then, since (5.10) is a disjoint union,

$$\sum_{u \in H} \frac{d\mu \circ \psi_u}{d\mu} h \circ \psi_u = \sum_{t \in F} \frac{d\mu \circ \phi_t}{d\mu} \tilde{h} \circ \phi_t , \tag{5.11}$$

where,

$$\tilde{h} = \sum_{k=1}^{l} \frac{d\mu \circ \phi_{x_k}}{d\mu} h \circ \phi_{x_k} .$$

Clearly $\tilde{h} \in L^1(S, \mu)$ and $\tilde{h} > 0$. Hence using Corollary 2.4 and the dissipativity of $\{\phi_t\}_{t \in F}$, we see that the second sum in (5.11) is finite almost everywhere. Another appeal to Corollary 2.4 shows that $\{\psi_u\}_{u \in H}$ is a dissipative group action.

Define a random field $\{Y_u\}_{u \in H}$ as

$$Y_u = \int_S \tilde{f}_u(s) M(ds), \quad u \in H, \tag{5.12}$$

where,

$$\tilde{f}_u = f \circ \psi_u \left(\frac{d\mu \circ \psi_u}{d\mu} \right)^{1/\alpha} \quad u \in H.$$

Clearly $\{Y_u\}_{u\in H}$ is an H-stationary $S\alpha S$ random field generated by the dissipative action $\{\psi_u\}_{u\in H}$. Hence there is a standard Borel space $(W,\mathcal{W})$ with a σ-finite measure ν on it such that

$$Y_u \stackrel{d}{=} \int_{W\times H} g(w,u\oplus s)\, N(dw,ds) \quad u\in H,$$

for some $g\in L^\alpha(W\times H,\nu\otimes\tau)$, where τ is the counting measure on H, and, N is a $S\alpha S$ random measure on $W\times H$ with control measure $\nu\otimes\tau$ (see the discussion following Corollary 3.4.)

Let, for all $w\in W$,

$$g^*(w) := \sup_{u\in H} |g(w,u)|\,. \tag{5.13}$$

Then, clearly, $g^*\in L^\alpha(W,\nu)$. We will show that (5.2) holds with

$$a := \left(\frac{\mathcal{V}l}{2^p}\int_W (g^*(w))^\alpha d\nu(w)\right)^{1/\alpha} \in (0,\infty)\,. \tag{5.14}$$

Since b_n is an increasing sequence, it is enough to show

$$\lim_{n\to\infty}\frac{b_{2n+1}}{(2n+1)^{p/\alpha}} = a\,. \tag{5.15}$$

Let $H_n := \bigcup_{k=1}^{l} F_{k,n}$. Then by stationarity of $\{X_t\}_{t\in\mathbb{Z}^d}$ we have, for all $n\geq 1$,

$$\begin{aligned} b_{2n+1}^\alpha &= \int_S \max_{-n\mathbf{1}\leq t\leq n\mathbf{1}} |f_t(s)|^\alpha \mu(ds) \\ &= \int_S \max_{u\in H_n} |\tilde{f}_u(s)|^\alpha \mu(ds) \\ &= \sum_{s\in H}\int_W \max_{u\in H_n} |g(w,s\oplus u)|^\alpha \nu(dw)\,. \end{aligned} \tag{5.16}$$

The last equality follows from Corollary 4.4.6 of Samorodnitsky and Taqqu (1994). We define a map $N: H\to\{0,1,\dots\}$ as,

$$N(u) := \min\{\|u+v\|_\infty : v\in K\}\,.$$

Clearly, for all $u\in H$,

$$N(u^{-1}) = N(u)\,, \tag{5.17}$$

where, u^{-1} is the inverse of u in H. Also, $N(\cdot)$ satisfies the following "triangle inequality": for all $u_1, u_2 \in H$,

$$N(u_1 \oplus u_2) \leq N(u_1) + N(u_2)\,. \tag{5.18}$$

Observe that $H_n = \{u \in H : N(u) \leq n\}$. From Lemma 5.2 we have H_n's are finite and Lemma 5.3 yields

$$|H_n| \sim \mathcal{V} l n^p\,. \tag{5.19}$$

Also, clearly, $H_n \uparrow H$. As in the proof of Theorem 3.1 of Samorodnitsky (2004), we first assume g has compact support, i.e. $g(w,u)I_{W\times H_m^c}(w,u) = 0$ for some $m \geq 1$. Then using (5.17) and (5.18), the expression in (5.16) becomes

$$\begin{aligned} b_{2n+1}^\alpha &= \sum_{s\in H_{n+m}} \int_W \max_{u\in H_n} |g(w, s\oplus u)|^\alpha \nu(dw) \\ &= \sum_{s\in H_{n-m}} \int_W \max_{u\in H_n} |g(w, s\oplus u)|^\alpha \nu(dw) \\ &\quad + \sum_{s\in H_{n+m}\cap H_{n-m}^c} \int_W \max_{u\in H_n} |g(w, s\oplus u)|^\alpha \nu(dw) =: A_n + B_n \end{aligned}$$

for all $n > m$. Using (5.17) and (5.18) once again, we have, for all $s \in H_{n-m}$,

$$\max_{u\in H_n} |g(w, s\oplus u)| = g^*(w)\,.$$

Hence, using (5.19), we get,

$$A_n = |H_{n-m}| \int_W (g^*(w))^\alpha \nu(dw) \sim a^\alpha (2n+1)^p\,,$$

while

$$B_n \leq \big(|H_{n+m}| - |H_{n-m}|\big) \int_W (g^*(w))^\alpha \nu(dw) = o(n^p)\,.$$

Hence, (5.15) follows for g having compact support. The proof in the general case follows by approximating a general kernel g by a kernel with a compact support as done in the proof of Theorem 3.1 in Samorodnitsky (2004). This completes the proof of the proposition. □

The following result sharpens the the description of the asymptotic behaviour of the partial maxima of a random field given in Theorem 4.3. It reduces to the latter result if $K = 0$.

Theorem 5.4. *Let* $\mathbf{X} = \{X_t\}_{t\in\mathbb{Z}^d}$ *be a stationary* $S\alpha S$ *random field, with* $0 < \alpha < 2$, *integral representation (3.1), and functions* $\{f_t\}$ *given by (3.3). Then, in the terminology introduced in this section, we have the following:*

1. If $\{\phi_t\}_{t\in F}$ *is not conservative then*

$$\frac{1}{n^{p/\alpha}} M_n \Rightarrow c\, Z_\alpha \tag{5.20}$$

for some $c \in (0,\infty)$, *and* Z_α *as in (4.11). In fact,*

$$c = \left(\frac{\mathcal{V} l C_\alpha}{2^p}\int_W (g^*(w))^\alpha d\nu(w)\right)^{1/\alpha},$$

where $\mathcal{V}$ *is given by (5.6), while* g^* *is given by (5.13) applied to the dissipative part of the random field* (5.12), *and* C_α *is as in (4.11).*

2. If $\{\phi_t\}_{t\in F}$ *is conservative then*

$$\frac{1}{n^{p/\alpha}} M_n \xrightarrow{p} 0\,. \tag{5.21}$$

Proof. 1. Let r_n be the left hand side of (4.17). Then we have,

$$\begin{aligned} r_n &\leq P\left(\text{for some } u \in H_n, \frac{|f_u(U_j^{(n)})|}{\max_{s\in H_n}|f_s(U_j^{(n)})|} > \epsilon,\ j = 1,2\right) \\ &\leq |H_n|\left(\epsilon^{-\alpha} b_n^{-\alpha}\int_S |f(s)|^\alpha \mu(ds)\right)^2 . \qquad (5.22) \end{aligned}$$

The inequality (5.22) follows using the argument given in Remark 4.2 of Samorodnitsky (2004). Since $\{\phi_t\}_{t\in F}$ is not conservative, Proposition 5.1 yields that b_n satisfies (5.2). Hence by (5.19) we get that (4.17) holds in this case. Since b_n satisfies (5.2) with a given by (5.14), we get (5.20) by Theorem 4.3.

2. As in the proof of (4.3) in Samorodnitsky (2004) we can get a stationary $S\alpha S$ random field $\mathbf{Y}$ generated by a conservative $\mathbb{Z}^d$-action such that b_n^Y satisfies (4.15) as well as (5.1) (this is possible, for instance, by Example 6.1 below). Therefore, (5.21) follows using the exact same argument as in the proof of (4.3) in Samorodnitsky (2004). □

Remark 5.5. The previous discussion asssumes that $p \geq 1$. When $p = 0$ (i.e. when $\mathbb{Z}^d/K$ is a finite group) the random field takes only finitely many different values. Therefore, the sequence M_n remains constant after some stage and so converges to the maximum of finitely many X_t's, not an extreme value random variable.

6. Examples. In this section we consider several examples of stationary $S\alpha S$ random fields associated with conservative flows. As in the one-dimensional case considered in Samorodnitsky (2004), the idea is to exhibit a variety of possible in this case behaviour.

The first example is parallel to examples 5.1 and 5.4 in Samorodnitsky (2004).

Example 6.1. Let the random field have an integral representation of the form

$$X_t \stackrel{d}{=} \int_{\mathbb{R}^{\mathbb{Z}^d}} g_t \, dM, \quad t \in \mathbb{Z}^d \tag{6.1}$$

where M is a $S\alpha S$ random measure on $\mathbb{R}^{\mathbb{Z}^d}$ whose control measure μ is a probability measure under which the projections $(g_t,\ t \in \mathbb{Z}^d)$ are i.i.d. random variables, with a finite absolute αth moment.

If $(g_t,\ t \in \mathbb{Z}^d)$ are i.i.d. standard normal random variables under μ, then, as in the one-dimensional case, one sees that

$$b_n^\alpha \sim (2d \log n)^{\alpha/2} \,,$$

the assumption (4.15) in Theorem 4.3 fails, and $b_n^{-1} M_n$ converges to a nonextreme value limit. See also Remark 5.5 above.

On the other hand, if, under μ, $(g_t,\ t \in \mathbb{Z}^d)$ are i.i.d. positive Pareto random variables with

$$\mu(g_0 > x) = x^{-\theta} \quad \text{for } x \geq 1$$

for some $\theta > \alpha$, then as in the one-dimensional case we see that

$$b_n \sim c_{\alpha,\theta}^{1/\alpha} \, n^{d/\theta} \quad \text{as } n \to \infty \,,$$

for some finite positive constant $c_{p,\theta}$, Theorem 4.3 applies, and $n^{-d/\theta} M_n$ converges to an extreme value distribution and hence this example also shows that the rate of growth of M_n can be n^γ for any $\gamma \in (0, d/\alpha)$. Note that existence of such a process was needed in the proof of (5.21) in Theorem 5.4.

Next is an example of an application of Theorem 5.4.

Example 6.2. Suppose $d = 3$, and define the $\mathbb{Z}^3$-action $\{\phi_{(i,j,k)}\}$ on $S = \mathbb{R} \times \{-1, 1\}$ as

$$\phi_{(i,j,k)}(x, y) = (x + i + 2j, (-1)^k y) .$$

An action-invariant measure μ on S is defined as the product of the Lebesgue measure on $\mathbb{R}$ and the counting measure on $\{-1, 1\}$.

Take any $f \in L^\alpha(S)$ and define a stationary $S\alpha S$ random field $\{X_{(i,j,k)}\}$ as follows

$$X_{(i,j,k)} = \int_{\mathbb{R}\times\{-1,1\}} f\big(\phi_{(i,j,k)}(x, y)\big)\, dM(x, y) ,$$

where M is a $S\alpha S$ random measure on $\mathbb{R} \times \{-1, 1\}$ with control measure μ. Note that the above representation of $\{X_{(i,j,k)}\}$ is of the form (3.3) generated by a measure preserving conservative action with $c_{(i,j,k)} \equiv 1$.

In the notation of Section 5 we have

$$K = \{(i, j, k) \in \mathbb{Z}^3 : i + 2j = 0 \text{ and } k \text{ is even}\} ,$$

and so

$$A \simeq \mathbb{Z}^3/K \simeq \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} ,$$

and

$$F = \{(i, 0, 0) : i \in \mathbb{Z}\} .$$

In particular $p = 1$ and $\{\phi_t\}_{t\in F}$ is dissipative. Hence Theorem 5.4 applies and says that $\frac{1}{n^{1/\alpha}} M_n$ converges to an extreme value distribution.

In all the examples we have seen so far, the action has a conservative direction i.e there is $u \in \mathbb{Z}^d - \{0\}$ such that $\{\phi_{nu}\}_{n\in\mathbb{Z}}$ is a conservative $\mathbb{Z}$-action. The following example of a $\mathbb{Z}^2$-action, suggested to us by M.G. Nadkarni, lacks such a conservative direction. In a sense, this example is "less one-dimensional" than the previous examples.

Example 6.3. Suppose that $d = 2$, and define the action $\{\phi_{(i,j)}\}_{i,j\in\mathbb{Z}}$ of $\mathbb{Z}^2$ on $S = \mathbb{R}$ with $\mu = Leb$ by

$$\phi_{(i,j)}(x) = x + i + j\sqrt{2}, \quad \forall\, x \in \mathbb{R} .$$

Clearly, this action is measure preserving and it does not have any conservative direction. It is, however, well known that this action does not admit a wandering set of positive Lebesgue measure, and hence is conservative. In fact, if we take the kernel $f = I_{[0,1]}$ and define $\{X_{(i,j)}\}$ by (3.1) and (3.3) with, say, $c_{(i,j)} \equiv 1$, then we have, for all $n \geq 2$,

$$b_n^\alpha = \mu\Bigg(\bigcup_{0\leq i,j\leq(n-1)} \phi_{(i,j)}\big([0, 1]\big) \Bigg) = \mu\big([0, 1 + (n - 1)(1 + \sqrt{2})]\big) .$$

So, $b_n \sim (1+\sqrt{2})^{1/\alpha} n^{1/\alpha}$ and, a simple calculation shows that left hand side of (4.17) is bounded from above by $b_n^{-2\alpha}(\mu \otimes \mu)(B_n)$ where

$$B_n = \left\{(x,y) \in \mathbb{R}^2 : -(n-1)(1+\sqrt{2}) \le x, y \le 1,\ |x-y| \le 1\right\}.$$

Since $(\mu \otimes \mu)(B_n) = O(n)$, (4.17) holds and hence

$$\frac{1}{n^{1/\alpha}} M_n \Rightarrow \big((1+\sqrt{2})C_\alpha\big)^{1/\alpha} Z_\alpha \,.$$

Acknowledgement. The authors are thankful to Mahendra Ganpatrao Nadkarni and Laurent Saloff-Coste for a number of useful discussions and to the anonymous referees for their comments.

References.

J. Aaronson (1997): *An Introduction to Infinite Ergodic Theory*, volume 50 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence.

J. A. De Loera (2005): The many aspects of counting lattice points in polytopes. *Mathematische Semesterberichte* 52:175–195.

C. Hardin Jr. (1982): On the spectral representation of symmetric stable processes. *Journal of Multivariate Analysis* 12:385–401.

S. Lang (2002): *Algebra*. Graduate Texts in Mathematics. Springer-Verlag, New York, revised 3rd edition.

D. Maharam (1964): Incompressible transformations. *Fundamenta Mathematicae* 56:35–50.

J. Rosiński (1995): On the structure of stationary stable processes. *The Annals of Probability* 23:1163–1187.

J. Rosiński (2000): Decomposition of stationary α–stable random fields. *Annals of Probability* 28:1797–1813.

P. Roy (2007): *Stable random fields*. Ph.D. thesis, School of Operations Research and Industrial Engineering, Cornell University, Ithaca, NY.

G. Samorodnitsky (2004): Extreme value theory, ergodic theory, and the boundary between short memory and long memory for stationary stable processes. *Annals of Probability* 32:1438–1468.

G. Samorodnitsky and M. Taqqu (1994): *Stable Non-Gaussian Random Processes*. Chapman and Hall, New York.

D. Surgailis, J. Rosiński, V. Mandrekar and S. Cambanis (1993): Stable mixed moving averages. *Probab. Theory Related Fields* 97:543–558.

School of Operations Research
and Industrial Engineering
Cornell University
Ithaca, NY 14853
e-mail: pr72@cornell.edu

School of Operations Research
and Industrial Engineering
Cornell University
Ithaca, NY 14853
e-mail: gennady@orie.cornell.edu