Boundary Conditions for Two-Sided Fractional Diffusion

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Abstract

This paper develops appropriate boundary conditions for the two-sided fractional diffusion equation, where the usual second derivative in space is replaced by a weighted average of positive and negative fractional derivatives. Mass preserving, reflecting boundary conditions for two-sided fractional diffusion involve a balance of left and right fractional derivatives at the boundary. Stable, consistent explicit and implicit Euler methods are detailed, and steady state solutions are derived. Steady state solutions for two-sided fractional diffusion equations using both Riemann-Liouville and Caputo flux are computed. For Riemann-Liouville flux and reflecting boundary conditions, the steady-state solution is singular at one or both of the end-points. For Caputo flux and reflecting boundary conditions, the steady-state solution is a constant function. Numerical experiments illustrate the convergence of these numerical methods. Finally, the influence of the reflecting boundary on the steady-state behavior subject to both the Riemann-Liouville and Caputo fluxes is discussed.

Keywords: fractional calculus, boundary conditions, Riesz derivative, stability analysis

1. Introduction

Two-sided fractional diffusion equations replace the second derivative with a weighted average of positive and negative fractional derivatives. Many applications in hydrology [7, 9, 42] and turbulent transport [13] require a combination of negative and positive fractional derivatives (e.g., the fractional Laplacian). Stable and consistent numerical methods for space fractional diffusion equations and wave equations are necessary for solving many practical problems in turbulence transport models [13], hydrology [25, 41], biomedical acoustics [37],
and nonlocal diffusion/peridynamics [12, 14, 35] in bounded domains. Most available numerical schemes assume Dirichlet boundary conditions (BCs) [25, 26, 30]. However, many problems involving space fractional diffusion equations in bounded domains require mass conservation. Dirichlet BCs, which impose a fixed value at the boundary, do not conserve mass. As a result, considerable effort has been spent on developing mass-preserving, reflecting (Neumann) BCs for space fractional diffusion equations [5, 6, 11, 15]. In particular, Baeumer et al. [5] proposed explicit Euler schemes for one-sided space fractional diffusion equations in one dimension using either a positive Riemann-Liouville derivative or a positive Patie-Simon derivative in the unit interval, assuming reflecting BCs.

Fractional diffusion using the Riesz derivative in space and a Caputo derivative in time subject to a reflecting boundary condition was discussed by Krepysheva et al. [20] from both a microscopic (particle) and macroscopic (field) perspective. That paper considered symmetric diffusion on a semi-infinite domain. More general continuous time random walks (CTRWs) in a bounded domain were discussed by Burch and Lehoucq [8], while prescribed fractional flux BCs were considered in Zhang et al. [41] from a hydrology perspective. A nonlocal normal derivative was introduced in Dipierro et al. [15] to model reflecting boundaries associated with the two-sided fractional Laplacian.

In this paper, we develop effective numerical methods for two-sided fractional diffusion equations with Neumann or Dirichlet boundary conditions. In Section 2, we formulate the two-sided Riemann-Liouville and Patie-Simon fractional diffusion equations, write both in a conservation form, and develop reflecting and absorbing boundary conditions for these two diffusion equations. In Section 3, we propose explicit and implicit Euler schemes for these diffusion equations, extending the results of Baeumer et al. [5] for the one-sided equations. In Section 4, we prove that the explicit Euler schemes are conditionally stable, and that the implicit Euler schemes are unconditionally stable, using the Gerschgorin circle theorem. In Section 5, we compute the kernels and steady-state solutions for the fractional diffusion equa-
tions using both the Riemann-Liouville and Patie-Simon fractional derivatives. Numerical
experiments are presented in Section 6, followed by discussion in Section 7 and conclusions
in Section 8.

2. Space-Fractional Diffusion Equations

We consider space-fractional diffusion equations with a combination of positive and neg-
ative Riemann-Liouville fractional derivatives on a bounded domain \([L, R]\):

\[
\frac{\partial}{\partial t} u(x, t) = pC \Delta_L^\alpha u(x, t) + qC \Delta_R^{-\alpha} u(x, t) + s(x, t) \tag{2.1}
\]

where \(1 < \alpha \leq 2\), where \(C > 0\) is the diffusion coefficient, \(p, q \geq 0\), and \(p + q = 1\), while
\(s(x, t)\) is a source term. The positive and negative Riemann-Liouville derivatives are defined
by

\[
\Delta_L^\alpha u(x, t) = \frac{\partial^n}{\partial x^n} I_L^{n-\alpha} f(x) = \frac{1}{\Gamma(n - \alpha)} \frac{\partial^n}{\partial x^n} \int_L^x \frac{u(y, t)}{(x - y)^{\alpha-n+1}} dy \tag{2.2a}
\]

\[
\Delta_R^{-\alpha} u(x, t) = (-1)^n \frac{\partial^n}{\partial x^n} I_R^{n-\alpha} f(x) = \frac{(-1)^n}{\Gamma(n - \alpha)} \frac{\partial^n}{\partial x^n} \int_x^R \frac{u(y, t)}{(y - x)^{\alpha-n+1}} dy \tag{2.2b}
\]

where \(I_L^{n-\alpha}\) and \(I_R^{n-\alpha}\) are the positive and negative Riemann-Liouville fractional integrals
of order \((n - \alpha)\), respectively, and \(n = \lfloor \alpha \rfloor\) and \(\alpha \neq n\). If \(\alpha = 2\), then the positive and
negative Riemann-Liouville derivatives in (2.1) reduce to the ordinary second derivative. In
the symmetric case \((p = q = 1/2)\), the symmetric space-fractional diffusion equation

\[
\frac{\partial}{\partial t} u(x, t) = C \frac{\partial^\alpha}{\partial |x|^\alpha} u(x, t) + s(x, t) \tag{2.3}
\]

is recovered, where

\[
\frac{\partial^\alpha}{\partial |x|^\alpha} u(x, t) = \frac{c^\alpha}{\Gamma(n - \alpha)} \frac{\partial^n}{\partial x^n} \int_L^R \frac{u(y, t)}{|x - y|^{\alpha-n+1}} dy \tag{2.4}
\]
is the Riesz derivative (fractional Laplacian) defined on a bounded interval \([21]\) and \(c_\alpha = 1/(2|\cos(\pi\alpha/2)|)\).

We also consider an alternative space-fractional diffusion equation

\[
\frac{\partial}{\partial t} u(x, t) = p C D_{L+}^\alpha u(x, t) + q C D_{R-}^\alpha u(x, t) + s(x, t)
\]

(2.5)

where

\[
D_{L+}^\alpha u(x, t) = \frac{\partial}{\partial x} \partial_{L+}^{\alpha-1} u(x, t) = \frac{1}{\Gamma(2 - \alpha)} \frac{\partial}{\partial x} \int_L^x \frac{u'(y, t)}{(x-y)^{\alpha-1}} dy
\]

(2.6a)

\[
D_{R-}^\alpha u(x, t) = -\frac{\partial}{\partial x} \partial_{R-}^{\alpha-1} u(x, t) = \frac{1}{\Gamma(2 - \alpha)} \frac{\partial}{\partial x} \int_x^R \frac{u'(y, t)}{(y-x)^{\alpha-1}} dy
\]

(2.6b)

are the Patie-Simon \([28]\) (also called the mixed Caputo \([6, \text{Definition 1}]\)) fractional derivatives and

\[
\partial_{L+}^{\alpha} u(x, t) = \frac{1}{\Gamma(n - \alpha)} \frac{\partial}{\partial x} \int_L^x \frac{u^{(n)}(y, t)}{(x-y)^{\alpha-n+1}} dy
\]

(2.7a)

\[
\partial_{R-}^{\alpha} u(x, t) = \frac{(-1)^n}{\Gamma(n - \alpha)} \frac{\partial}{\partial x} \int_x^R \frac{u^{(n)}(y, t)}{(y-x)^{\alpha-n+1}} dy
\]

(2.7b)

are the positive and negative Caputo derivatives \([18, \text{Theorem 2.1}]\), respectively.

**Remark 2.1.** For \(1 < \alpha < 2\), the Riemann-Liouville and Patie-Simon derivatives are related via

\[
D_{L+}^\alpha f(x) = \mathbb{D}_{L+}^\alpha f(x) - f(L) \frac{(x-L)^{-\alpha}}{\Gamma(1-\alpha)}
\]

(2.8a)

\[
D_{R-}^\alpha f(x) = \mathbb{D}_{R-}^\alpha f(x) + f(R) \frac{(R-x)^{-\alpha}}{\Gamma(1-\alpha)}
\]

(2.8b)

see \([5, \text{Equation (6.6)}]\).
2.1. Conservation Form

From a physical point of view, \( u(x, t) \) may represent the concentration of an ensemble of particles. This concentration is governed by a local mass conservation (continuity) equation

\[
\frac{\partial}{\partial t} u(x, t) + \frac{\partial}{\partial x} F(x, t) = 0
\]  
(2.9)

where \( F(x, t) \) is a flux function (generalized Fick’s law) \([13, 19, 27, 34]\) that accounts for nonlocal diffusion. Comparing (2.9) with (2.1) and (2.5) with no source \((s(x, t) = 0)\), the flux function is given by

\[
F_{RL}(x, t) = qC^1_R u(x, t) - pC^1_L u(x, t)
\]  
(2.10a)

\[
F_{C}(x, t) = qC^1_R u(x, t) - pC^1_L u(x, t)
\]  
(2.10b)

respectively, where \( F_{RL}(x, t) \) is a Riemann-Liouville flux and \( F_{C}(x, t) \) is a Caputo flux. The continuity equation (2.9) complemented with either the Riemann-Liouville flux (2.10a) or Caputo flux (2.10b) is the conservation form. For traditional diffusion \((\alpha = 2)\), both the Riemann-Liouville flux and Caputo flux reduce to the classical Fick’s law. An expression similar to (2.10a), written using a pseudo-differential operator on the entire real line, was given in Paradisi et al. \([27, \text{Equation (2.5)}]\), while the Caputo flux (2.10b) was proposed for hydrology applications in Zhang et al. \([41]\) (we have corrected a minus sign error in that formula).

2.2. Reflecting (no-flux) boundary conditions

Using the flux functions defined in (2.10), we can identify a no-flux BC by setting \( F(x, t) = 0 \) at the boundary. Setting \( F(x, t) = 0 \) at \( x = L \) and \( x = R \) in (2.10) yields
reflecting BCs:

\begin{align}
\text{RL: } pD_L^{\alpha-1}u(x,t) &= qD_R^{\alpha-1}u(x,t) \quad \text{for } x = L \text{ and } x = R \text{ for all } t \geq 0 \\
\text{C: } p\partial_L^{\alpha-1}u(x,t) &= q\partial_R^{\alpha-1}u(x,t) \quad \text{for } x = L \text{ and } x = R \text{ for all } t \geq 0.
\end{align}

These boundary conditions are nonlocal since the BC at \(x = L\) or \(x = R\) depends on all values of \(u(x,t)\) in the interval \([L, R]\).

The special case \(p = 1\) was considered in Baeumer et al. [5], yielding the no-flux BC

\begin{align}
\text{RL: } D_L^{\alpha-1}u(x,t) &= 0 \quad \text{for } x = L \text{ and } x = R \text{ for all } t \geq 0 \\
\text{C: } \partial_L^{\alpha-1}u(x,t) &= 0 \quad \text{for } x = L \text{ and } x = R \text{ for all } t \geq 0.
\end{align}

For the special case \(q = 1\), the positive Riemann-Liouville and Caputo derivatives \(D_L^{\alpha-1}\) and \(\partial_L^{\alpha-1}\) are replaced by the negative Riemann-Liouville and Caputo derivatives \(D_R^{\alpha-1}\) and \(\partial_R^{\alpha-1}\), respectively. In the symmetric (fractional Laplacian) case \(p = q\), we have

\begin{align}
\text{RL: } D_L^{\alpha-1}u(x,t) &= D_R^{\alpha-1}u(x,t) \quad \text{for } x = L \text{ and } x = R \text{ for all } t \geq 0 \\
\text{C: } \partial_L^{\alpha-1}u(x,t) &= \partial_R^{\alpha-1}u(x,t) \quad \text{for } x = L \text{ and } x = R \text{ for all } t \geq 0.
\end{align}

Unlike the one-sided cases, reflecting boundary conditions for the symmetric diffusion equation involves a balance of two fractional derivatives of order \((\alpha - 1)\).
2.3. Reflecting/Absorbing, Absorbing/Reflecting, and Absorbing BCs

We also consider reflecting on the left boundary and absorbing on the right boundary (reflecting/absorbing BCs)

\[ RL: p \delta_{L+}^{\alpha-1} u(x, t) = q \delta_{R+}^{\alpha-1} u(x, t) \text{ for } x = L \text{ and } u(R, t) = 0 \text{ for all } t \geq 0 \quad (2.14a) \]

\[ C: p \partial_{L+}^{\alpha-1} u(x, t) = q \partial_{R+}^{\alpha-1} u(x, t) \text{ for } x = L \text{ and } u(R, t) = 0 \text{ for all } t \geq 0, \quad (2.14b) \]

and absorbing on the left and reflecting on the right (absorbing/reflecting BCs)

\[ RL: u(L, t) = 0 \text{ and } p \delta_{L+}^{\alpha-1} u(x, t) = q \delta_{R+}^{\alpha-1} u(x, t) \text{ for } x = R \text{ for all } t \geq 0 \quad (2.15a) \]

\[ C: u(L, t) = 0 \text{ and } p \partial_{L+}^{\alpha-1} u(x, t) = q \partial_{R+}^{\alpha-1} u(x, t) \text{ for } x = R \text{ for all } t \geq 0. \quad (2.15b) \]

The special case of \( p = 1 \) and \( q = 0 \) of these BCs was considered in Baeumer et al. [5]. Absorbing (Dirichlet) BCs on both boundaries \( u(L, t) = u(R, t) = 0 \) will also be considered.

2.4. Conservation of Mass

The no-flux (reflecting) BCs given by (2.11) imply that the total mass is conserved.

**Proposition 2.2.** Let \( M_0 = \int_L^R u(x, t) \, dx \) be the total mass and let \( \mathcal{D}_{RL}^\alpha = pC \delta_{L+}^{\alpha-1} + qC \delta_{R+}^{\alpha-1} \)

and \( \mathcal{D}_{PS}^\alpha = pC \delta_{L+}^{\alpha-1} + qC \delta_{R+}^{\alpha-1} \) be the fractional operators on the right-hand side of (2.1) and (2.5), respectively, with \( s(x, t) = 0 \) and reflecting boundary condition (2.11a) or (2.11b), respectively. Given a non-negative initial condition \( u(x, 0) = u_0(x) \in \text{Dom}(\mathcal{D}_{RL}^\alpha) \) for (2.1) or \( u(x, 0) = u_0(x) \in \text{Dom}(\mathcal{D}_{PS}^\alpha) \) for (2.5), the total mass is conserved.

**Proof.** Using the definition of the generator for the corresponding \( C_0 \) semigroups on the Banach space \( L^1(L, R) \) [24, Section 3.3], the time derivative may be moved inside the integral

\[ \frac{\partial M_0}{\partial t} = \int_L^R \frac{\partial}{\partial t} u(x, t) \, dx = \int_L^R \mathcal{D}^\alpha u(x, t) \, dx, \]
where $D^a = D^a_{PS}$ or $D^a_C$. Then apply the conservation form (2.9),

$$\frac{\partial M_0}{\partial t} = - \int_L^R \frac{\partial}{\partial x} F(x, t) \, dx = F(L, t) - F(R, t).$$

Since $F(L, t) = F(R, t) = 0$ for all $t$ by (2.12), $\partial M_0/\partial t = 0$ and $M_0 = \int_L^R u_0(x) \, dx$ for all $t \geq 0$.

Remark 2.3. Note that a zero-flux boundary condition is a sufficient, but not necessary condition for mass conservation. A more general condition is $F(L, t) = F(R, t)$, where the flux leaving the right boundary re-enters the domain at the left boundary (and vice versa).

3. Finite-Difference Approximations

To discretize (2.1), we can use either an explicit or implicit Euler scheme combined with the shifted Grünwald estimate [26]:

$$\mathcal{D}^a_{L+} f(x_j) = h^{-\alpha} \sum_{i=0}^{j+1} g_i^\alpha f(x_{j-i+1}) + O(h) \quad (3.1a)$$

$$\mathcal{D}^a_{R-} f(x_j) = h^{-\alpha} \sum_{i=0}^{n-j+1} g_i^\alpha f(x_{j+i-1}) + O(h) \quad (3.1b)$$

where $h = (R - L)/n$ is the grid spacing, $x_j = L + hj$ are the $n + 1$ grid points, and

$$g_i^\alpha = \frac{(-1)^i \Gamma(\alpha + 1)}{\Gamma(i + 1) \Gamma(\alpha - i + 1)} \quad (3.2)$$
are the Grünwald weights [24, Equation (2.4)]. The resulting explicit Euler scheme is given by

\[ u(x_j, t_{k+1}) = u(x_j, t_k) + \frac{pC\Delta t}{h^\alpha} \sum_{i=0}^{j+1} g_i^\alpha u(x_{j-i+1}, t_k) \]

\[ + \frac{qC\Delta t}{h^\alpha} \sum_{i=0}^{n-j+1} g_i^\alpha u(x_{j+i-1}, t_k) + s(x_j, t_k). \]

(3.3)

Defining a row vector containing the solution at time \( t_k = k\Delta t \) via \( u_k = [u(x_i, t_k)] \) along with the source \( s_k = [s(x_i, t_k)] \) yields

\[ u_{k+1} = u_k + \beta_+ u_k B^+ + \beta_- u_k B^- + s_k \]

(3.4)

where \( \beta_+ = pCh^{-\alpha}\Delta t, \beta_- = qCh^{-\alpha}\Delta t, \) and \( B^\pm \) are \((n+1) \times (n+1)\) iteration matrices, which will be written explicitly below. These iteration matrices depend upon both the flux function and the boundary conditions. The explicit scheme (3.4) may be written compactly as

\[ u_{k+1} = u_k A + s_k \]

(3.5)

where \( A = I + \beta_+ B^+ + \beta_- B^- \).

Applying an implicit Euler discretization to (2.1) yields

\[ u_{k+1} = u_k + \beta_+ u_{k+1} B^+ + \beta_- u_{k+1} B^- + s_{k+1}, \]

(3.6)

where \( B^\pm \) are the same iteration matrices utilized in (3.4). This implicit scheme may be written as

\[ u_{k+1} M = u_k + s_{k+1}, \]

(3.7)

where \( M = I - \beta_+ B^+ - \beta_- B^- \). The discretization of (2.5) leads to the same iteration equations (3.5) and (3.7), but with a slightly different iteration matrix, which will be written explicitly below.
3.1. Iteration Matrices: Riemann-Liouville Flux

We first consider the explicit and implicit Euler schemes associated with (2.1) subject to reflecting BCs. The entries of $B^+$ are given by [5, Equation 4.2]

$$b_{i,j} = \begin{cases} 
  g_{j-i+1}^\alpha & \text{if } 0 < j < n \text{ and } i \leq j + 1 \\
  1 & \text{if } i = 1 \text{ and } j = 0 \\
  1 - \alpha & \text{if } i = j = 0 \\
  -g_{n-i}^{\alpha-1} & \text{if } j = n \text{ and } i \leq n \\
  0 & \text{otherwise}.
\end{cases} \quad (3.8)$$

The entries for column $j = 0$ prevent mass from leaving the left boundary $x = L$, while the entries for $j = n$ prevent mass from leaving the right boundary $x = R$. The fraction of mass that would otherwise leave the domain is deposited at the boundary, thereby modeling inelastic collisions at $x = L$ and $x = R$. Comparing the second and third terms in (3.3), we see that the entries of $B^-$ associated with the negative Riemann-Liouville fractional derivative are $[b_{n-i,n-j}]$.

Next, consider the reflecting/absorbing BCs given by (2.14a). The iteration matrix $B^{ra+}$ for the one-sided Riemann-Liouville derivative with these BCs is simply (3.8) with all entries in column $j = n$ set equal to zero. The iteration matrix $B^{ar+}$ for the one-sided Riemann-Liouville derivative with absorbing/reflecting BCs (2.15a) is (3.8) with all entries in column $j = 0$ set equal to zero. Finally, the iteration matrix $B^{aa+}$ for the one-sided Riemann-Liouville derivative with absorbing BCs $u(L,t) = u(R,t) = 0$ is (3.8) with all entries in both columns $j = 0$ and $j = n$ set equal to zero. Comparing the second and third terms in (3.3), we see that replacing $i$ and $j$ by $n - i$ and $n - j$ reverses the roles for $r$ and $a$. Hence, the entries of $B^{ra-}$, $B^{ar-}$, and $B^{aa-}$ are simply $b_{n-i,n-j}^{ra}$, $b_{n-i,n-j}^{ra}$, and $b_{n-i,n-j}^{aa}$, respectively.
3.2. Iteration Matrices: Caputo Flux

In Section 6 of [5], an explicit Euler scheme was proposed to solve (2.5) in the special case \( q = 0 \) subject to Dirichlet (absorbing) and Neumann (reflecting) BCs. Absorbing/reflecting and reflecting/absorbing BCs were also considered. For reflecting BCs (2.11b), the iteration matrix \( B = [b_{i,j}] \) is given by [5, Equation 6.11]

\[
b_{i,j} = \begin{cases} 
g_j^{\alpha} & \text{if } 0 < j < n \text{ and } i \leq j + 1 \\
1 & \text{if } i = 1 \text{ and } j = 0 \\
-1 & \text{if } i = j = 0 \\
g_j^{\alpha - 1} & \text{if } i = 0 \text{ and } 0 < j < n \\
-g_j^{\alpha - 2} & \text{if } i = 0 \text{ and } j = n \\
-g_n^{\alpha - 1} & \text{if } j = n \text{ and } 0 < i \leq n \\
0 & \text{otherwise ,}
\end{cases}
\]  

(3.9)

and then the entries of \( B^{-1} \) are \([b_{n-i,n-j}]\). As in the Riemann-Liouville flux case, the iteration matrices for reflecting/absorbing and absorbing/reflecting BCs are simply (3.9) with all entries in the \( n \)-th column or zeroth column set to zero, respectively [5, Equations 6.15 and 6.17]. Finally, for absorbing BCs \( u(L, t) = u(R, t) = 0 \), the iteration matrix is given by (3.9) with all entries in columns \( j = 0 \) and \( j = n \) set to zero.

4. Stability Analysis

4.1. Riemann-Liouville Flux

To prove conditional stability of the explicit Euler scheme (3.5) and unconditional stability of the implicit Euler scheme (3.7), we estimate the eigenvalues of the matrices \( A \) and \( M \) using the Gerschgorin circle theorem [3, Theorem 9.1]. The following Lemma is used.
Lemma 4.1. The radii of the Gerschgorin circles of the matrix $B^+ = [b_{i,j}]$ given by (3.8)

$$r_i = \sum_{j=0, j \neq i}^{n} |b_{i,j}|$$  \hspace{1cm} (4.1)

are given by

$$r_i = \begin{cases} 
\alpha - 1 & \text{if } i = 0 \\
\alpha & \text{if } 0 < i < n \\
1 & \text{if } i = n,
\end{cases}$$  \hspace{1cm} (4.2)

while the radii of the Gerschgorin circles of the matrix $B^- = [b_{n-i,n-j}]$ are $r_{n-j}$.

Proof. Using (3.2) we can see that $g_0^\alpha = 1$, $g_1^\alpha = -\alpha$, $g_i^\alpha > 0$ for all $i > 1$, $g_0^{\alpha-1} = 1$, and $g_i^{\alpha-1} < 0$ for all $i > 0$. Hence all the off-diagonal entries in both $B^+$ and $B^-$ are non-negative, allowing us to neglect the absolute value in (4.1). Then write

$$r_0 = \sum_{j=1}^{n-1} g_{j+1}^\alpha - g_n^{\alpha-1}$$

$$= \sum_{j=2}^{n} g_j^\alpha - g_n^{\alpha-1}$$

$$= g_n^{\alpha-1} - 1 + \alpha - g_n^{\alpha-1}$$

$$= \alpha - 1$$

where we used [33, Equation 20.4]

$$\sum_{j=0}^{n} g_j^\alpha = g_n^{\alpha-1}.$$  \hspace{1cm} (4.3)
Next, consider rows $0 < i < n$:

$$r_i = 1 + \sum_{j=i+1}^{n-1} g_j^{\alpha} - g_{n-i}^{\alpha-1}$$

$$= 1 + \sum_{j=2}^{n-i} g_j^{\alpha} - g_{n-i}^{\alpha-1}$$

$$= 1 - 1 + \alpha + g_{n-i}^{\alpha-1} - g_{n-i}^{\alpha-1}$$

$$= \alpha.$$

For row $i = n$, we have $r_n = 1$ since there is only one off-diagonal entry. Finally, the radii of the Gerschgorin circles of the matrix $B^-$ are

$$\sum_{j=0,j\neq i}^{n} b_{n-i,n-j} = \sum_{j=0,j\neq n-i}^{n} b_{n-i,j} = r_{n-i},$$

completing the proof.

**Remark 4.2.** The Gerschgorin radii associated with $B^{ar+}$, $B^{ra+}$, and $B^{aa+}$ are less than or equal to the radii of $B^+$ since the entries of $B^{ar+}$, $B^{ra+}$, and $B^{aa+}$ are either those of $B^+$ or zero. The same is true for $B^{ar-}$, $B^{ra-}$, and $B^{aa-}$.

**Proposition 4.3.** The explicit Euler method (3.4) for (2.1) subject to any combination of absorbing and reflecting BCs is stable if $\Delta t/h^\alpha \leq 1/(\alpha C)$ over the region $L \leq x \leq R$ and $0 \leq t \leq T$.

**Proof.** First consider the case of reflecting BCs. By the Gerschgorin circle theorem [3, Theorem 9.1], it suffices to show the eigenvalues of $A$ are inside the closed unit disk. Using
(4.2), the radii of the Gerschgorin circles for the matrix $A$ are given by

$$R_i = \begin{cases} 
\beta_+ (\alpha - 1) + \beta_- & \text{if } i = 0 \\
\beta_+ \alpha + \beta_- \alpha & \text{if } 0 < i < n \\
\beta_+ + \beta_- (\alpha - 1) & \text{if } i = n
\end{cases} \tag{4.4}$$

while the diagonal entries of $A$ are

$$a_{i,i} = \begin{cases} 
1 - \beta_+ (\alpha - 1) - \beta_- & \text{if } i = 0 \\
1 - (\beta_+ + \beta_-) \alpha & \text{if } 0 < i < n \\
1 - \beta_+ - \beta_- (\alpha - 1) & \text{if } i = n
\end{cases} \tag{4.5}$$

Hence $a_{i,i} + R_i = 1$ for all $i$, while $a_{i,i} - R_{i,i} = 1 - 2R_i$. To ensure $|\lambda_i| \leq 1$ and stability, we require $1 - 2R_i \geq -1$, or $R_i \leq 1$. Since the largest $R_i$ is $\alpha (\beta_+ + \beta_-)$, we require

$$\alpha (\beta_+ + \beta_-) \leq 1,$$

which is true by hypothesis. The cases of absorbing/reflecting, reflecting/absorbing, and absorbing BCs are similar, using Remark 4.2.

**Remark 4.4.** The same explicit stability condition was shown for the fractional diffusion equation with Dirichlet BCs in [26]. For the case of $\alpha = 2$ with a diffusion coefficient $C = 1$, we recover the well-known stability constraint for the diffusion equation with both Dirichlet (absorbing) and Neumann (reflecting) BCs [32]:

$$\beta = \frac{\Delta t}{h^2} \leq \frac{1}{2}. \tag{4.6}$$

For the case of $\alpha = 1$ and $C = 1$, we recover the stability constraint for the transport
equation with both Dirichlet (absorbing) and Neumann (reflecting) BCs:

\[ \beta = \frac{\Delta t}{h} \leq 1. \quad (4.7) \]

**Proposition 4.5.** The implicit Euler method for (2.1) subject to any combination of absorbing and reflecting BCs for $1 < \alpha \leq 2$ is unconditionally stable for all $\Delta t$.

**Proof.** As in the explicit scheme proof, we use [3, Theorem 9.1]. First, note that the off-diagonal entries $m_{i,j}$ of $M$ are simply $-\beta_+ b_{i,j} - \beta_- b_{n-i,n-j}$. Hence, the radii of the Gerschgorin circles for the matrix $M$ are also given by (4.4), while the diagonal entries of $M$ are

\[
m_{i,i} = \begin{cases} 
1 + \beta_+ (\alpha - 1) + \beta_- & \text{if } i = 0 \\
1 + (\beta_+ + \beta_-) \alpha & \text{if } 0 < i < n \\
1 + \beta_+ + \beta_- (\alpha - 1) & \text{if } i = n.
\end{cases} \quad (4.8)
\]

The complex absolute values of the eigenvalues $\lambda_i$ of $M$ are bounded by $m_{i,i} - R_i \leq |\lambda_i| \leq m_{i,i} + R_i$. Clearly, $m_{i,i} - R_i = 1$ for all $0 \leq i \leq n$, while $m_{i,i} + R_i = 1 + 2R_{i,i} > 1$. Hence, $|\lambda_i| \geq 1$, implying that every eigenvalue of the inverse matrix $M^{-1}$ has complex absolute values less than or equal to 1. The proof for other combinations of BCs is similar. \(\square\)

4.2. Caputo Flux

In this section, we prove stability of the explicit and implicit Euler schemes for (2.5).

**Lemma 4.6.** The radii of the Gerschgorin circles of the matrices $B^+$ and $B^-$ with entries specified by (3.9) and $[b_{n-i,n-j}]$, respectively, are given by

\[
r_i = \begin{cases} 
1 & \text{if } i = 0 \text{ or } i = n \\
\alpha & \text{if } 0 < i < n.
\end{cases} \quad (4.9)
\]
Proof. Again, note that all the off-diagonal entries are positive. First, consider row $i = 0$:

$$r_0 = - \sum_{j=1}^{n-1} g_j^{\alpha-1} + g_{n-1}^{\alpha-2}$$

$$= 1 - \sum_{j=0}^{n-1} g_j^{\alpha-1} + g_{n-1}^{\alpha-2}$$

$$= 1 - g_{n-1}^{\alpha-2} + g_{n-1}^{\alpha-2} = 1,$$

where (4.3) is used in the third line. Next, consider rows $0 < i < n$:

$$r_i = 1 + \sum_{j=i+1}^{n-1} g_j^{\alpha} - g_{n-i}^{\alpha-1}$$

$$= 1 + \sum_{j=2}^{n-i} g_j^{\alpha} - g_{n-i}^{\alpha-1}$$

$$= 1 - 1 + \alpha + g_{n-i}^{\alpha-1} - g_{n-i}^{\alpha-1} = \alpha.$$

Finally, if $i = n$, then there is a single entry $r_n = -g_0^{\alpha-1} = 1$. Since $B^{-}$ has entries $[b_{n-i,n-j}]$ and $r_0 = r_n = 1$, it follows that $B^{-}$ also has Gerschgorin radii given by (4.9).

Proposition 4.7. The explicit Euler method for (2.5) subject to any combination of absorbing and reflecting BCs is stable if $\Delta t/h^\alpha \leq 1/(\alpha C)$ over the region $L \leq x \leq R$ and $0 \leq t \leq T$.

Proof. As with (4.3), we consider the case of reflecting BCs. Note that

$$a_{i,j} = \begin{cases} 
1 - (\beta_+ + \beta_-) & \text{if } i = 0 \text{ or } i = n \\
1 - (\beta_+ + \beta_-) \alpha & \text{if } 0 < i < n,
\end{cases}$$

with Gerschgorin radii given by (4.10). Hence, $a_{i,i} + R_i = 1$ for all $i$ and we require $R_i \leq 1$ to bound all eigenvalues in the unit disk. Hence, $\alpha (\beta_+ + \beta_-) \leq 1$, which is satisfied by hypothesis. The other three cases are similar since the Gerschgorin radii are bounded.
above by (4.9).

**Proposition 4.8.** The implicit Euler scheme for (2.5) subject to any combination of absorbing and reflecting BCs for $1 < \alpha \leq 2$ is unconditionally stable for all $\Delta t$.

**Proof.** Using (4.9), the radii of the Gershgorin circles for the matrix $M$ with reflecting BCs are given by

$$R_i = \begin{cases} 
(\beta_+ + \beta_-) & \text{if } i = 0 \text{ or } i = n \\
\alpha (\beta_+ + \beta_-) & \text{if } 0 < i < n,
\end{cases} \tag{4.10}$$

while the diagonal entries of $M$ are

$$m_{i,i} = \begin{cases} 
1 + (\beta_+ + \beta_-) & \text{if } i = 0 \text{ or } i = n \\
1 + (\beta_+ + \beta_-) \alpha & \text{if } 0 < i < n.
\end{cases}$$

Hence, $m_{i,i} - R_i = 1$, while $m_{i,i} + R_i = 1 + 2R_i \geq 1$. Application of the Gershgorin theorem places all eigenvalues of $M$ in the set $|\lambda_i| \geq 1$, implying that the spectral radius of $M^{-1}$ is less than or equal to one. The proof with other combinations of BCs is similar. □

5. Steady-State Solutions

In this section, we compute the steady-state solutions $u_\infty(x)$ that satisfy (2.1) and (2.5), and particular steady-state solutions that satisfy reflecting (no-flux) BCs. We first compute the kernel (null-space) of the two-sided Riemann-Liouville and Patie-Simon derivatives, and then construct steady-state solutions that satisfy reflecting BCs using functions in the kernel.
5.1. Riemann-Liouville Flux

In the one-sided case $(p = 1)$, the kernel (null-space) of the Riemann-Liouville derivative on the interval $[-1, 1]$ was computed in Baeumer et al. [6]

$$\ker \left( \mathbb{D}_{-1}^\alpha \right) = c_0 (x + 1)^{\alpha - 2} + c_1 (x + 1)^{\alpha - 1}, \quad (5.1)$$

where $c_0$ and $c_1$ are arbitrary constants. The only steady state solution with a total mass of one that satisfies reflecting BCs is $u_\infty(x) = 2^{1-\alpha}(\alpha - 1)(1 + x)^{\alpha - 2}$, which is singular at the left end-point $x = -1$ and regular at the right end-point $x = 1$. To check (5.1), note that since the Riemann-Liouville derivative (2.2a) is the second derivative of the $2 - \alpha$ Riemann-Liouville integral, the Riemann-Liouville derivative of a function can be identically zero if and only if the $2 - \alpha$ Riemann-Liouville integral is linear. Then apply the Riemann-Liouville integral of order $2 - \alpha$ to both terms, which yields a linear function in $x$. The second derivative of this expression is identically zero. A similar argument holds for the one-sided negative case $(q = 1)$, yielding $u_\infty(x) = 2^{1-\alpha}(\alpha - 1)(1 - x)^{\alpha - 2}$, which is regular at $x = -1$ and singular at $x = 1$.

In this section, we derive the steady-state solution of (2.1) with $s = 0$ on the interval $[-1, 1]$

$$p \mathbb{D}_{-1}^\alpha u_\infty(x) + q \mathbb{D}_{1-}^\alpha u_\infty(x) = 0 \quad (5.2)$$

subject to a reflecting BC at both boundaries:

$$p \mathbb{D}_{-1}^{\alpha - 1} u_\infty(x) = q \mathbb{D}_{1-}^{\alpha - 1} u_\infty(x) \text{ for } x = -1 \text{ and } 1. \quad (5.3)$$

The kernel and steady state solution may be derived using the method of orthogonal polynomials [2, 31], see also [29, Section 6.4].

**Definition 5.1.** The Jacobi polynomials $P_m^{\mu, \nu}(x)$ of order $m \geq 0$ are $m$-th degree polynomials
orthogonal with respect to the weight $(1-x)\mu (1+x)^\nu$ on the interval $[-1,1]$, where $\mu, \nu > -1$. These polynomials may be defined via [1, Equation (22.3.2)]

$$P_m^{\mu,\nu}(x) = \frac{\Gamma(\mu + m + 1)}{m! \Gamma(\mu + \nu + m + 1)} \sum_{k=0}^{m} \binom{m}{k} \frac{\Gamma(\mu + \nu + m + k + 1)}{2^k \Gamma(\mu + k + 1)} (x-1)^k. \quad (5.4)$$

In particular, $P_0^{\mu,\nu}(x) = 1$ and $P_1^{\mu,\nu}(x) = (\mu + \nu + 2)x/2 + (\mu - \nu)/2$.

**Definition 5.2.** The two-sided Jacobi polyfractonomials used by Mao and Karniadakis [23] $Q_m^{\mu,\nu}(x)$ are defined by

$$Q_m^{\mu,\nu}(x) = (1-x)^\mu (1+x)^\nu P_m^{\mu,\nu}(x) \quad (5.5)$$

where $P_m^{\mu,\nu}(x)$ is defined by (5.4).

An identity involving $Q_m^{\mu,\nu}(x)$ and the two-sided fractional integral is given by Podlubny [29, Theorem 6.4] with $r = 0$ and $k = -1$:

$$\int_{-1}^{1} \left( \text{sgn}(x-t) + \frac{\tan(\pi \gamma)}{\tan(\tilde{\nu}\pi/2)} \right) Q_m^{-\gamma+\tilde{\nu}/2,\gamma+\tilde{\nu}/2-1}(t) \frac{P_m^{\gamma+\tilde{\nu}/2-1,\gamma+\tilde{\nu}/2}(x)}{|x-t|^{\tilde{\nu}}} dt = A_m P_m^{\gamma+\tilde{\nu}/2-1,\gamma+\tilde{\nu}/2}(x), \quad (5.6)$$

where $0 < \tilde{\nu} < 1$, $0 < \gamma < 1$, and

$$A_m = \frac{\pi \Gamma(m + \tilde{\nu})}{m! \Gamma(\tilde{\nu}) \sin(\tilde{\nu}\pi/2) \cos(\gamma\pi)}. \quad (5.7)$$

Using (5.6), we will characterize the kernel $\ker(D_{RL}^p)$ of the two-sided Riemann-Liouville derivative. First, we need a technical lemma.

**Lemma 5.3.** Let $1 < \alpha < 2$, $p + q = 1$, and let $m \geq 0$ be an integer. Then

$$p I_{-1+}^{2-\alpha} Q_m^{\mu,\nu}(x) + q I_{1-}^{2-\alpha} Q_m^{\mu,\nu}(x) = C_m P_m^{\nu,\mu}(x) \quad (5.8)$$

where $I_{-1+}^{2-\alpha}$ and $I_{1-}^{2-\alpha}$ are the positive and negative Riemann-Liouville fractional integrals of
order \((2 - \alpha)\), respectively, and

\[
\mu + \nu = \alpha - 2 \tag{5.9a}
\]

\[
p - q = \cot \left( \pi \left( \frac{\alpha - 1}{2} - \mu \right) \right) \tan \left( \frac{\alpha - 1}{2} \pi \right) \tag{5.9b}
\]

\[
C_m = \sin(\pi(\alpha - 1)/2)\Gamma(m + \alpha - 1) \quad \text{and} \quad m! \sin(\pi(\alpha - 1)/2 - \pi\mu). \tag{5.9c}
\]

**Proof.** In (5.6), let \(W_\pm = \tan(\pi\gamma)/\tan(\bar{\nu}\pi/2) \pm 1, \bar{\nu} = \alpha - 1,\) and define

\[
\mu = -\gamma + \frac{\alpha - 1}{2} \tag{5.10a}
\]

\[
\nu = \gamma - 1 + \frac{\alpha - 1}{2}. \tag{5.10b}
\]

Split (5.6) into two terms and divide both sides by \(\Gamma(2 - \alpha)\), yielding

\[
W_+ I_{-1}^{2-\alpha} Q_m^{\mu,\nu}(x) + W_- I_{1}^{2-\alpha} Q_m^{\mu,\nu}(x) = \frac{A_m}{\Gamma(2 - \alpha)} P_m^{\mu,\nu}(x). \tag{5.11}
\]

By the Euler reflection formula \(\Gamma(2 - \alpha)\Gamma(\alpha - 1) = \pi/\sin(\pi(\alpha - 1))\) and the double angle formula for the sine,

\[
\tilde{A}_m = \frac{A_m}{\Gamma(2 - \alpha)} = \frac{2\Gamma(m + \alpha - 1)\cos((\alpha - 1)\pi/2)}{m!\cos(\gamma\pi)}, \tag{5.12}
\]

yielding

\[
W_+ I_{-1}^{2-\alpha} Q_m^{\mu,\nu}(x) + W_- I_{1}^{2-\alpha} Q_m^{\mu,\nu}(x) = \tilde{A}_m P_m^{\mu,\nu}(x) \tag{5.13}
\]

Now divide both sides of (5.13) by \(W_+ + W_-\) and let

\[
p = \frac{W_+}{W_+ + W_-} = \frac{\sin(\pi\gamma) + \cos(\pi\gamma)\tan((\alpha - 1)\pi/2)}{2\sin(\pi\gamma)} \tag{5.14a}
\]
\[ q = \frac{W_0}{W_0 + W_\pm} = \frac{\sin(\pi \gamma) - \cos(\pi \gamma) \tan((\alpha - 1)\pi/2)}{2 \sin(\pi \gamma)} \]  

(5.14b)

which is valid for \(0 < \gamma < 1\). Solve (5.10a) for \(\gamma = (\alpha - 1)/2 - \mu\) and subtract (5.14b) from (5.14a), yielding (5.9b). Add (5.10a) and (5.10b), yielding (5.9a). Finally, divide (5.12) by \(W_0 + W_\pm\), yielding (5.9c).

\[ \text{Remark 5.4. Ervin et al. [16, Lemma 4.3] use the Gauss hypergeometric function } 2F_1(a, b; c; z) \]

to reach a similar result in the special case of \(m = 0\) on the interval \([0,1]\).

Armed with Lemma 5.3, we can now compute the two-sided Riemann-Liouville derivative of the family of functions defined by \(Q_\mu^{\alpha, \nu}(x)\).

**Theorem 5.5.** Let \(D^\alpha_{RL} = pD^{-\alpha-1}_- + qD^{-\alpha}_+\) be the two-sided Riemann-Liouville derivative on \([-1, 1]\), with \(1 < \alpha < 2\), \(p + q = 1\), and let \(m \geq 0\) be an integer. Then

\[ D^\alpha_{RL} Q_\mu^{\alpha, \nu}(x) = C_m \frac{\partial^2}{\partial x^2} P_m^{\alpha, \nu}(x), \]  

(5.15)

where \(Q_\mu^{\alpha, \nu}(x)\) is given by (5.5) and \(\mu, \nu, \text{ and } C_m\) satisfy (5.9).

**Proof.** Apply the second derivative operator to (5.8). By (2.2), the theorem follows.

**Corollary 5.6.** The kernel of the two-sided Riemann-Liouville derivative on \([-1, 1]\) is given by \(\ker(D^\alpha_{RL}) = c_0 Q_0^{\alpha, \nu}(x) + c_1 x Q_0^{\alpha, \nu}(x)\), where \(Q_0^{\alpha, \nu}(x) = (1 - x)^\mu(1 + x)^\nu\).

**Proof.** Let \(m = 0\) in Theorem 5.5. The right-hand side of (5.15) vanishes since \(P_0^{\alpha, \nu}(x)\) is a constant. Likewise, let \(m = 1\) in the same theorem. Since \(P_1^{\alpha, \nu}(x)\) is linear with respect to \(x\), the right hand side of (5.15) also vanishes.

**Theorem 5.7.** The kernel of the two-sided Riemann-Liouville derivative on the interval \([-1, 1]\) is given by

\[ \ker(D^\alpha_{RL}) = c_0(1 - x)^\mu(1 + x)^\nu + c_1 x(1 - x)^\mu(1 + x)^\nu \]  

(5.16)
with parameters $\mu$ and $\nu$ satisfy (5.9a) and (5.9b). In particular, the steady state solution $u_\infty(x)$ subject to reflecting BCs (2.11) with $L = -1$ and $R = 1$ is

$$u_\infty(x) = \frac{M_0(1 - x)^\mu(1 + x)^\nu}{B(\nu + 1, \mu + 1)2^{\mu+\nu}}$$

(5.17)

where $B(\mu + 1, \nu + 1)$ is the beta function [17, Equation 2.5] and $M_0 = \int_{-1}^{1} u_0(x) \, dx$.

Proof. The kernel (5.16) follows immediately from Corollary 5.16. To demonstrate (5.17), the Riemann-Liouville flux $F_{RL}(x, t) = -p\partial/\partial x I_{-1}^{2-\alpha} u_\infty(x) + q\partial/\partial x I_{1}^{2-\alpha} u_\infty(x) = 0$, which is satisfied by $c_0Q_0^{\mu,\nu}(x)$. Letting $M_0 = \int_{-1}^{1} u_0(x) \, dx$, we have

$$M_0 = \int_{-1}^{1} c_0(1 - x)^\mu(1 + x)^\nu \, dx$$

$$= 2^{\mu+\nu}c_0 \int_{0}^{1} y^\nu(1 - y)^\mu, \, dy$$

$$= 2^{\mu+\nu}c_0 B(\mu + 1, \nu + 1),$$

yielding the constant $c_0$. \(\square\)

Remark 5.8. In the one-sided case ($p = 1$ and $q = 0$), (5.9a) and (5.9b) yield $\mu = 0$ and $\nu = 2$ - $2$. Evaluating the kernel (5.16) with these parameters yields

$$\ker(D_{RL}^\alpha) = c_0(1 + x)^{\alpha-2} + c_1 x(1 + x)^{\alpha-2}$$

$$= (c_0 - c_1)(x + 1)^{\alpha-2} + c_1(x + 1)^{\alpha-1},$$

which agrees with the known one-sided solution (5.1). Likewise, if $p = 0$ and $q = 1$, then (5.9a) and (5.9b) yield $\mu = \alpha - 2$ and $\nu = 0$, and (5.16) evaluates to $\ker(D_{RL}^\alpha) = (c_0 + c_1)(1 - x)^{\alpha-2} - c_1(1 - x)^{\alpha-1}$.
Remark 5.9. In the symmetric case \((p = q = 1/2)\), \(\mu = \nu = \alpha/2 - 1\), yielding the kernel

\[
\ker(D_{RL}^\alpha) = c_0(1 - x^2)^{\alpha/2 - 1} + c_1x(1 - x^2)^{\alpha/2 - 1},
\]

which, in general, is singular at both \(x = -1\) and \(x = 1\). Setting \(c_1 = c_0\), yields \(\ker(D_{RL}^\alpha) = c_0(1 + x)^{\alpha/2}(1 - x)^{\alpha/2 - 1}\), which is regular at \(x = -1\) and singular at \(x = 1\). Setting \(c_1 = -c_0\) yields \(\ker(D_{RL}^\alpha) = c_0(1 + x)^{\alpha/2 - 1}(1 - x)^{\alpha/2}\), which is singular at \(x = -1\) and regular at \(x = 1\). The steady-state solution (5.17) with reflecting BCs with \(B(\alpha/2, \alpha/2) = (\Gamma(\alpha/2))^2 / \Gamma(\alpha)\) is

\[
u_\infty(x) = M_0 \frac{\Gamma(\alpha)}{2^{\alpha - 1}(\Gamma(\alpha/2))^2} (1 - x^2)^{\alpha/2 - 1},
\]

which is symmetric about \(x = 0\). If \(\alpha = 1\), then (5.18) is proportional to the arc sine density [17, Equation 4.4] on the interval \([-1, 1]\).

5.2. Caputo Flux

As with Riemann-Liouville flux, the kernel of the one-sided \((p = 1)\) Patie-Simon derivative was computed in Baeumer et al. [6],

\[
\ker(D_{-1+}^\alpha) = c_0 + c_1(x + 1)^{\alpha - 1},
\]

where \(c_0\) and \(c_1\) are arbitrary constants. Unlike (5.1), (5.19) is regular at both end-points. The only steady solution with total mass of one that satisfies reflecting BCs (2.12b) is \(u_\infty(x) = 1/2\). In this section, we compute the kernel of the two-sided Patie-Simon derivative \(\ker(D_{PS}^\alpha)\)

\[
pD_{-1+}^\alpha u_\infty(x) + qD_{1+}^\alpha u_\infty(x) = 0.
\]

This task requires another technical lemma.
Lemma 5.10. Let $\mu$ and $\nu$ satisfy (5.10). Then
\begin{equation}
\int_{-1}^{x} (1 - y)^\mu (1 + y)^\nu \, dy = \frac{2^\mu}{2 + \nu} (1 + x)^{1+\nu} 2^\nu F_1(-\mu, 1 + \nu; 2 + \nu; (1 + x)/2),
\end{equation}
where the Gauss hypergeometric function $2 F_1(a, b; c; w)$ has an integral representation given by [18, Section 1.6]
\begin{equation}
2 F_1(a, b; c; w) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c - b)} \int_0^1 z^{b-1} (1 - z)^{c-b-1}(1 - zw)^{-a} \, dz
\end{equation}
with $0 < b < c$.

Proof. Let $z = (y + 1)/(x + 1)$ in (5.21), yielding
\begin{equation}
\int_{-1}^{x} (1 - y)^\mu (1 + y)^\nu \, dy = (x + 1)^{\nu+1} 2^\nu \int_0^1 z^\nu (1 - zw)^\mu \, dz,
\end{equation}
where $w = (1 + x)/2$. Compare to (5.22) with $a = -\mu$, $b = 1 + \nu$ and $c = 2 + \nu$ and note that $\Gamma(1 + \nu)\Gamma(1)/\Gamma(2 + \nu) = 1/(2 + \nu)$, yielding (5.21).

Theorem 5.11. The kernel of the two-sided Patie-Simon derivative on the interval $[-1, 1]$ is given by
\begin{equation}
\ker(D^\alpha_{PS}) = c_0 + c_1 (1 + x)^{1+\nu} 2 F_1(-\mu, 1 + \nu; 2 + \nu; (1 + x)/2)
\end{equation}
with parameters $\mu$ and $\nu$ satisfy (5.9a) and (5.9b). In particular, the steady state solution $u_\infty(x)$ subject to reflecting BCs (2.11b) with $L = -1$ and $R = 1$ is $u_\infty(x) = M_0/2$, where $M_0$ is the total mass.

Proof. Let $D^\alpha_{PS} = pD^\alpha_{-1+} + qD^\alpha_{1-}$ be the two-sided Patie-Simon derivative and define
\begin{equation}
u(x) = c_0 + c_1 \int_{-1}^{x} Q^{\mu, \nu}_0(y) \, dy.
\end{equation}
By (2.6),

\[
D_{PS}^\alpha u_\infty(x) = (pD_1^\alpha + qD_{-1}^\alpha) u(x) \\
= \frac{\partial}{\partial x} \left( pI^2_{1-} + qI^2_{1+} \right) \frac{\partial}{\partial x} u(x) \\
= \frac{\partial}{\partial x} \left( pI^2_{1+} + qI^2_{1-} \right) c_1 Q_0^{\mu,\nu}(x).
\]

By (5.8), we have

\[
D_{PS}^\alpha u_\infty(x) = c_1 C_0 \frac{\partial}{\partial x} P_0^{\mu,\nu}(x) = 0.
\]

Then invoke (5.21), thus proving (5.23). Of the two terms in (5.23), only \(u_\infty(x) = c_0\) satisfies the reflecting BCs (2.1b). Then \(\int_{-1}^{1} c_0 \, dx = M_0\), yielding \(c_0 = M_0/2\).

\[\square\]

**Remark 5.12.** Ervin et al. [16, Corollary 4.1] also computed the kernel of \(D_{PS}^\alpha\) on \([0, 1]\), yielding similar expressions involving the Gauss hypergeometric function.

**Remark 5.13.** In the one-sided case \((p = 1 \text{ and } q = 0)\), \(\mu = 0\) and \(\nu = \alpha - 2\). Evaluating (5.23) yields

\[
\ker(D_{PS}^\alpha) = c_0 + c_1 (1 + x)^{\alpha - 1} _2F_1(0, \alpha - 1; \alpha; (1 + x)/2) \\
= c_0 + c_1 (x + 1)^{\alpha - 1}
\]

which agrees with (5.19).

### 6. Numerical Experiments

#### 6.1. One-Sided Fractional Derivative

In our first experiment, we use the method of manufactured solutions (MOMS) to test the convergence of the one-sided fractional diffusion equation using the Riemann-Liouville derivative given by (2.1). We assume reflecting boundary conditions at both \(x = -1\) and
To construct an initial condition, we first identify the domain of the positive Riemann-Liouville derivative that satisfies reflecting BCs on the interval $[-1, 1]$ [6, Table 1, Row 6]:

$$\text{Dom } (\mathbb{D}_{-1+}^\alpha, NN) = \left\{ f \in X : f = I^\alpha_{-1+} g - \frac{I^1_{-1+}}{p^1_1(1)} p^+_\alpha(x) + cp^+_{\alpha-2}(x), g \in X \right\}$$

(6.1)

where $NN$ denotes the Neumann BCs in (2.12a), $p^+_\alpha(x) = (1 + x)^\alpha / \Gamma(\alpha + 1)$, $c \in \mathbb{R}$, and $X$ is some suitable space of functions (e.g., $X = L_1[-1, 1]$). By choosing $g(x) = -(1 + x)^\beta$ where $\beta > 0$ in (6.1), we construct an initial condition

$$u_0(x) = \frac{2^\beta}{1 + \beta} p^+_\alpha(x) - \frac{\Gamma(1 + \beta)}{\Gamma(\alpha + \beta + 1)} (1 + x)^{\alpha + \beta}.$$ 

(6.2)

The manufactured source term $s(x, t)$ is given by

$$s(x, t) = -e^{-t} \left( u_0(x) + \frac{2^\beta}{1 + \beta} - (1 + x)^\beta \right),$$

(6.3)

yielding an exact analytical solution

$$u(x, t) = u_0(x)e^{-t}.$$ 

(6.4)

Equation (2.6) is discretized using the explicit Euler scheme (3.4) and simulated with a uniform grid of $n$ points, where $n = 50, 100, 200, 400, \text{ and } 800$ with $\alpha = 1.5$ and $\beta = 2$. A time step satisfying $\Delta t \leq h^\alpha / \alpha$ is chosen and the relative $L^2$ error between the exact and numerical solutions is computed after 2000 timesteps. Figure 1a compares the exact solution (6.4) with the numerical solution using $n = 100$ grid points, while Figure 1b displays the relative $L^2$ error as a function $n$ for both models. In panel a, there is good agreement between the numerical and analytical solution, although there is some disagreement near the right boundary. As the grid is refined ($n$ increases), the two solutions converge, as shown in panel
Figure 1: Panel a) compares the exact solution (6.4) of (2.1) using the Riemann-Liouville derivative with the numerical solution using \( n = 100 \) grid points and \( \alpha = 1.5 \). Panel b) displays the relative \( L^2 \) error of the numerical solution as a function of the number of grid points \( n \).

b. The numerical convergence rate for the Riemann-Liouville diffusion equation, which is given by the negative of the slope of Figure 1b, is approximately one, which agrees with the truncation error of the shifted Grünwald estimate.

6.2. Two-Sided Fractional Derivative

In the following numerical experiments, we used \( \Delta x = 1/500 \) with \( n = 1000 \) grid-points and the implicit scheme (3.7) with a time-step of \( \Delta t = 0.0025 \). We have verified these results by reproducing these results using the explicit Euler scheme (3.5) with a time-step of \( \Delta t = 0.0002 \), which satisfies the stability limit in Proposition 4.3. A tent function initial condition

\[
u_0(x) = \begin{cases} 
5 - 25|x| & \text{for } |x| < 0.2, \\
0 & \text{otherwise,}
\end{cases} \tag{6.5}
\]

with mass \( M_0 = 1 \).

Figure 2 shows numerical solutions for the fractional diffusion equation with Riemann-Liouville flux using \( \alpha = 1.5 \) and reflecting BCs at both end-points. The weight \( p \) varies from
a) $p = 1$, b) $p = 0.75$, c) $p = 0.5$, and d) $p = 0.25$, and solutions are shown at $t = 0$ (solid), $t = 0.05$ (dotted), $t = 0.1$ (dash-dotted) and $t = 2$ (dashed), while the steady-state solution (5.17) is shown with circles. In the one-sided case shown in panel a, the numerical solution is singular at $x = -1$ but regular at $x = 1$. In contrast, the numerical solutions in panels b), c), and d) are singular at both $x = -1$ and $x = 1$.

Figure 3 displays solutions of the fractional diffusion equation (Caputo flux) with $\alpha = 1.5$ and $p = 0.25$ using a) absorbing BCs, b) absorbing-reflecting BCs, c) reflecting-absorbing BCs, and d) reflecting BCs at $t = 0$ (solid), $t = 0.05$ (dotted), $t = 0.1$ (dash-dotted) and $t = 2$ (dashed). In panel d), the steady-state solution $u_\infty(x) = 1/2$ is shown (circles). In panels a, b, and c, the numerical solutions tend toward a steady state of zero. In panel d, which uses reflecting BCs, the numerical solutions tend toward a steady state of $u_\infty(x) = 1/2$. Unlike solutions using the Riemann-Liouville flux, the solutions using Caputo flux are regular at both end-points for all BC choices. Numerical solutions using the Riemann-Liouville flux and absorbing BCs are identical to Fig. 3a for the same choice of $\alpha$ and $p$ since the two derivatives are equal in this case by (2.8).

7. Discussion

For diffusion with Riemann-Liouville flux shown in Figure 2, the presence of the reflecting boundary has a profound impact on the solution as time evolves: there is a build-up of mass near the wall, yielding a steady state solution that exhibits a singularity at each boundary. From a particle point of view, there is a build-up of particles at both boundaries for $0 < p < 1$. In contrast, the diffusion equation with Caputo flux equipped with reflecting BCs in Fig. 3d has a constant steady state solution for all $0 \leq p \leq 1$, agreeing with both the classical diffusion case ($\alpha = 2$) and the one-sided fractional diffusion equation with Caputo flux [5]. Consideration of the steady state behavior can be a useful guide in model selection.

Remark 7.1. The explicit and implicit Euler schemes given in Section 3 are low-order with
Figure 2: Solutions of the fractional diffusion equation (Riemann-Liouville flux) with $\alpha = 1.5$, $C = 1$, and reflecting BCs at both end-points. The parameter $p$ varies from a) $p = 1$, b) $p = 0.75$, c) $p = 0.5$, and d) $p = 0.25$. $t = 0$ (solid), $t = 0.05$ (dotted), $t = 0.1$ (dash-dotted) and $t = 2$ (dashed), and the steady-state solution (circles).
Figure 3: Solutions of the fractional diffusion equation (Caputo flux) with $\alpha = 1.5$, $C = 1$, and $p = 0.25$ using a) absorbing BCs, b) absorbing-reflecting BCs, c) reflecting-absorbing BCs, and d) reflecting BCs at $t = 0$ (solid), $t = 0.05$ (dotted), $t = 0.1$ (dash-dotted) and $t = 2$ (dashed). In panel d), the steady-state solution $u_\infty(x) = 1/2$ is shown (circles).
an error term $O(h)$. High-order, stable schemes for fractional BVPs with absorbing BCs were proposed in [4, 10, 36]. It would be interesting to augment these high-order schemes with reflecting boundary conditions, yielding efficient, high-order methods for problems with a range of boundary conditions. Development of spectral methods for reflecting BCs using orthogonal polynomials (i.e., poly-fractonomials), which are currently limited to Dirichlet (absorbing) BCs [22, 23, 39, 40], would also be interesting. As noted in [23], the two-sided polyfractonomials $Q_{m}^{a,b}(x)$ capture the singular behavior of the Riemann-Liouville operator near the boundary.

**Remark 7.2.** Section 4 has shown stability for the explicit and implicit Euler schemes. Since these methods are also consistent, these schemes are convergent by the Lax equivalence theorem [32]. Unlike the one-sided case [6, 38], well-posedness for Cauchy problems using the two-sided derivative with reflecting BCs has not been proven mathematically. To prove well-posedness, the approach of Baeumer et al. [6] may be fruitful, which requires identification of the domain of the two-sided fractional derivative for each boundary condition. The steady-state solutions computed in Section 5 may be useful to construct these domains.

### 8. Conclusions

This paper has established appropriate absorbing (Dirichlet) and reflecting (Neumann) boundary conditions for two versions of the two-sided, space-fractional diffusion equation, thus extending the scheme developed for the one-sided case in Baeumer et al. [5]. By expressing the fractional diffusion equation in conservation form, two flux functions were identified: the Riemann-Liouville flux and the Caputo flux. A conditionally stable explicit Euler scheme and an unconditionally stable implicit Euler scheme were proposed using the shifted Grünwald estimate from Meerschaert and Tadjeran [25], and stability was demonstrated using the Gerschgorin circle theorem. Steady state solutions subject to reflecting BCs using Riemann-Liouville flux are singular at one or more of the end-points, while steady-state solutions
subject to reflecting BCs using Caputo flux are constant functions. Numerical experiments illustrated the convergence of the explicit and implicit methods. Finally, the influence of the reflecting boundary on the steady-state behavior subject to both the Riemann-Liouville and Caputo fluxes was discussed.

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Declarations

The authors declare no conflict of interests.


