Statistics & Probability Letters 11 (1991) 287–289
North-Holland

April 1991

Regular variation in $\mathbb{R}^k$ and vector-normed domains of attraction

Mark M. Meerschaert

Department of Mathematics, Albion College, Albion, MI 49224, USA

Received May 1990

Abstract: An extension of the argument used by William Feller in the one variable case is applied to obtain a complete characterization of vectornormed domains of attraction in terms of regular variation.

AMS 1980 Subject Classifications: 60F05, 26B35.

Keywords: Regular variation, domains of attraction, stable laws, infinitely divisible laws.

1. Introduction

Given $\{X_n\}$ independent and identically distributed random vectors on $\mathbb{R}^k$ with distribution μ, let $S_n = X_1 + \cdots + X_n$. We say that μ belongs to the vector-normed domain of attraction of ν if there exist $a_n, b_n \in \mathbb{R}^k$ with $a_n^{(i)} > 0$ for all $i = 1, 2, \ldots, k$ such that

$$\left(S_n^{(1)}/a_n^{(1)}, \ldots, S_n^{(k)}/a_n^{(k)}\right) - b_n \Rightarrow Y, \tag{1.1}$$

where Y is a nondegenerate random vector on $\mathbb{R}^k$ with distribution ν. Vector-normed domains of attraction were first considered by Resnick and Greenwood (1979), who obtained a complete characterization in the case $k = 2$. Some connections with regular variation in $\mathbb{R}^k$ were examined in de Haan, Omey and Resnick (1984). In this paper we use regular variation in $\mathbb{R}^k$ to obtain a new characterization of vector-normed domains of attraction, thereby extending the results of Feller (1971) in $\mathbb{R}^1$.

2. Results

Regular variation in $\mathbb{R}^k$ was defined in Meerschaert (1988). If $x, y \in \mathbb{R}^k$ we denote by xy the componentwise product $(x_1 y_1, \ldots, x_k y_k)$. Let $\mathbb{R}_+^k = \{(x_1, \ldots, x_k)\text{: all } x_i > 0\}$ and $\mathbb{R}_-^k = -\mathbb{R}_+^k$. If $\lambda > 0$ and $\alpha \in \mathbb{R}^k$ let $\lambda^\alpha = (\lambda^{\alpha_1}, \ldots, \lambda^{\alpha_k})$.

A function $f : [A, \infty) \to \mathbb{R}_+^k$ will be said to vary regularly with index α if it is Borel measurable and if for all $\lambda > 0$ we have

$$\lim_{r\to\infty} f(\lambda r) f(r)^{-1} = \lambda^\alpha. \tag{2.1}$$

Suppose now that $F : \mathbb{R}^k \to \mathbb{R}^+$ is Borel measurable. We will say that F is regularly varying at infinity (respectively, zero) if there exists $f : \mathbb{R}^+ \to \mathbb{R}_+^k$ regularly varying with index α in $\mathbb{R}_+^k$ (respectively, $\mathbb{R}_-^k$) and $e \neq 0$ such that whenever $x_r \to x \neq 0$ we have

$$\lim_{r\to\infty} F(f(r)x_r)/F(f(r)e) = \gamma(x) \tag{2.2}$$

for some $\gamma : \mathbb{R}^k - \{0\} \to \mathbb{R}^+$. In this case the choice of $e \neq 0$ is arbitrary and effects the limit γ only in terms of a multiplicative constant. It follows from (2.2) that $R(r) = F(f(r)e)$ varies regularly with some index $\beta \in \mathbb{R}$ and that for all $\lambda > 0$, all $x \neq 0$ we have

$$\lambda^\beta \gamma(x) = \gamma(\lambda^\alpha x).$$

While α, β are not uniquely determined by F, their ratio $\rho = \beta\alpha^{-1}$ is uniquely determined, and we call ρ the index of regular variation of F.

Let $\{X_n\}$ be as above and define the truncated second moment function

$$F(y) = E\langle X_n, y\rangle^2 I\{|\langle X_n, y\rangle| < 1\} \tag{2.3}$$

for $y \neq 0$.

Theorem 2.1. *μ is in the vector-normed domain of attraction of a nondegenerate normal law if and only if the function $F(y)$ defined by* (2.3) *varies regularly at zero with index* $(2, 2, \dots, 2)$.

Now let Π denote the class of σ-finite Borel measures on $\mathbb{R}^k - \{0\}$ which are finite on sets bounded away from the origin, and write $\nu_n \to \nu$ if ν_n, $\nu \in \Pi$ and $\nu_n(A) \to \nu(A)$ for all Borel subsets bounded away from the origin such that $\nu(\partial A) = 0$. We will say that $\mu \in \Pi$ is regularly varying at infinite (respectively, zero) if there exists $f: \mathbb{R}^+ \to \mathbb{R}^k_+$ regularly varying with index $\alpha \in \mathbb{R}^k_+$ (respectively, $\mathbb{R}^k_-$) and a Borel set E such that

$$\mu\{f(r)\, \mathrm{d}x\}/\mu\{f(r)E\} \to \phi\{\mathrm{d}x\} \tag{2.4}$$

for some measure $\phi \in \Pi$ which cannot be supported on any proper subspace of $\mathbb{R}^k$. The set E is arbitrary and effects the limit ϕ only in terms of a multiplicative constant. It follows from (2.3) that $\mu\{f(r)E\}$ is a regularly varying function of $r > 0$ with some index $\beta \in \mathbb{R}$, and that for all $\lambda > 0$,

$$\lambda^\beta \phi\{\mathrm{d}x\} = \phi\{\lambda^\alpha\, \mathrm{d}x\}. \tag{2.5}$$

One again we will call $\rho = \beta\alpha^{-1}$ the index of regular variation.

Theorem 2.2. *μ belongs to the vector-normed domain of attraction of a nondegenerate limit law having no normal component if and only if μ varies regularly at infinity with index $\rho = (\rho_1, \dots, \rho_k)$ where all $\rho_i \in (-2, 0)$.*

If the limit distribution has both normal and nonnormal components, then according to Sharpe (1969) we can decompose ν into the product of two marginals, one normal and one strictly nonnormal. Let $L_1 = \mathrm{Span}\{e_i : Y_i$ normal and $L_2 = L_1^\perp$. Denote by π_i the projection map onto L_i. Then $\pi_1(Y)$ is normal, $\pi_2(Y)$ has no normal component, and $\pi_1(Y)$, $\pi_2(Y)$ are independent. The following result allows us to treat the general limit case by reduction to the two cases considered above.

Theorem 2.3. *μ is in the vector-normed domain of attraction of ν if and only if for $i = 1, 2$,*

$$\pi_i\left(a_n^{-1}S_n - b_n\right) \Rightarrow \pi_i(Y). \tag{2.6}$$

3. Proofs

In this section we prove the theorems stated in Section 2 characterizing vector-normed domains of attraction in terms of regular variation. We will proceed by extending the arguments Feller employed in $\mathbb{R}^1$. Using the notation introduced in the beginning of Section 2, we may rewrite (1.1) in the abbreviated form

$$a_n^{-1}S_n - b_n \Rightarrow Y. \tag{3.1}$$

Proof of Theorem 2.1. Without loss of generality $EX_n = 0$. For all $y \in \mathbb{R}^k$ we have

$$n\left[\int_{|\langle x, y\rangle| < \varepsilon} \langle x, y\rangle^2 \mu\{a_n\, \mathrm{d}x\} - \left(\int_{|\langle x, y\rangle| < \varepsilon} \langle x, y\rangle \mu\{a_n\, \mathrm{d}x\}\right)^2\right] \to Q(y). \tag{3.2}$$

for all $\varepsilon > 0$. Since Y is nondegenerate normal, ϕ is the zero measure and Q is positive definite. As $n \to \infty$ the second integral in (3.2) tends to zero, and so (3.2) remains true with this term deleted. Taking $\varepsilon = 1$ we have

$$nF\left(a_n^{-1}y\right) \to Q(y). \tag{3.3}$$

It follows that F varies regularly at zero, and since $Q(ry) = r^2 Q(y)$ the index of F is $(2, \dots, 2)$.

Conversely suppose F varies regularly at zero with index $(2, \dots, 2)$. Letting $a_n^{-1} = f(r_n)$ where $r_n = \sup\{r > 0: nF(f(r)e) \leqslant 1\}$ we arrive at (3.3). Since $EX_n = 0$ this is again equivalent to (3.2), and now we need only show that $n\mu\{a_n\, \mathrm{d}x\} \to 0$. This follows easily from (3.3) by a reduction to the one variable case: $nF(e_i/a_n^{(i)}) \to Q(e_i)$ and so the truncated second moment of $X_n^{(i)}$ varies slowly at infinity. Hence $n\mu\{a_n A\} \to 0$ for sets of the form $A = \{x: |x_i| > \varepsilon\}$. But any subset of $\mathbb{R}^k$ which is bounded away from the origin is contained in the union of a finite number of these. □

Proof of Theorem 2.2. Suppose that (3.1) holds and Y has no normal component. Then $Q \equiv 0$ and ϕ cannot be supported on any proper subspace of $\mathbb{R}^k$. From the standard convergence criteria for triangular arrays of random vectors we obtain immediately that μ varies regularly at infinity. Since ϕ is a Lévy measure, and in particular

$$\int_{0<|x|<1} |x|^2 \phi\{dx\} < \infty \qquad (3.4)$$

we must have $-2 < \rho_i < 0$.

Conversely suppose μ varies regularly at infinity with index ρ, all $\rho_i \in (-2, 0)$. Let $a_n = f(r_n)$ where $r_n = \sup\{r > 0: n\mu\{f(r)E\} \geqslant 1\}$, so that $n\mu\{a_n\, dx\} \to \phi\{dx\}$. To show that (3.1) holds, by an application of Schwartz inequality it will suffice to show that for all y,

$$\lim_{\varepsilon\to 0} \limsup_{n\to\infty} n\int_{|x|<\varepsilon} \langle x, y\rangle^2 \mu\{a_n\, dx\} = 0. \qquad (3.5)$$

Clearly it suffices to show (3.5) for $y = e_i$; $i = 1,\ldots, k$. And once again, this follows directly by a reduction to the one variable case. □

Proof of Theorem 2.3. The direct half is obvious. As to the converse, suppose that (2.6) holds for $i = 1, 2$. Since $\phi\{\pi_i^{-1}(dx)\}$ is the Lévy measure of $\pi_i Y$ we have for $i = 1, 2$,

$$n\mu\{a_n \pi_i^{-1}(dx)\} \to \phi\{\pi_i^{-1}(dx)\}; \qquad (3.6)$$

and since the limit in (3.6) is the zero measure when $i = 1$, this implies that $n\mu\{a_n\, dx\} \to \phi\{dx\}$. We also have

$$\lim_{\varepsilon\to 0} \limsup_{n\to\infty} \left\{ n\left[\int_{|x|<\varepsilon} \langle x, y\rangle^2 \mu\{a_n\, dx\} - \left(\int_{|x|<\varepsilon} \langle x, y\rangle \mu\{a_n\, dx\}\right)^2\right] - Q(y)\right\} = 0 \qquad (3.7)$$

for all $y \in L_1$ and $y \in L_2$. Suppose then that $y = y_1 + y_2$ where both $y_1 \in L_1$ and $y_2 \in L_2$ are nonzero. We need to show that

$$\lim_{\varepsilon\to 0} \limsup_{n\to\infty} n\int_{|x|<\varepsilon} \langle x, y_1\rangle\langle x, y_2\rangle \mu\{a_n\, dx\} = 0,$$

$$\lim_{\varepsilon\to 0} \limsup_{n\to\infty} n\int_{|x|<\varepsilon} \langle x, y_1\rangle \mu\{a_n\, dx\} \cdot \int_{|x|<\varepsilon} \langle x, y_2\rangle \mu\{a_n\, dx\} = 0. \qquad (3.8)$$

Both integral expressions are dominated by

$$n\int_{|x|<\varepsilon} \langle x, y_1\rangle^2 \mu\{a_n\, dx\} \cdot n\int_{|x|<\varepsilon} \langle x, y_2\rangle^2 \mu\{a_n\, dx\}. \qquad (3.9)$$

The proof of Theorem 2.1 shows that the first term is bounded. Apply (3.5). □

References

Aurajo, A. and E. Giné (1980), *The Central Limit Theorem for Real and Banach Valued Random Variables* (Wiley, New York).

Feller, W. (1971), *An Introduction to Probability Theory and its Applications, Vol. II* (Wiley, New York, 2nd ed.).

Haan, L. de, E. Omey and S. Resnick (1984), Domains of attraction and regular variation in $\mathbb{R}^d$, *J. Multivariate Anal.* **14**, 17–33.

Meerschaert, M. (1988), Regular variation in $\mathbb{R}^k$, *Proc. Amer. Math. Soc.* **102**, 341–348.

Resnick, S. and P. Greenwood (1979), A bivariate stable characterization and domains of attraction, *J. Multivariate Anal.* **9**, 206–221.

Sharpe, M. (1969), Operator-stable probability distributions on vector groups, *Trans. Amer. Math. Soc.* **136**, 51–65.