1. A family with two children is chosen at random; we are interested in the gender of the children. A sample space for this experiment is \( S = \{MM, MF, FM, FF\} \). Assuming all 4 outcomes are equally likely, answer the following:

(a) Let \( A \) be the event that there are more male than female children. Compute \( P(A) \).
(b) Let \( B \) be the event that there are no male children. Compute \( P(B) \).
(c) Let \( C \) be the event that there is at least one female child. Compute \( P(C) \).
(d) Describe in words the event \( B^c \).

Solutions.
(a) The event \( A \) consists of the outcome \( MM \). Since this outcome has probability \( 1/4 \), \( P(A) = 1/4 \).
(b) The event \( B \) consists of the outcome \( FF \), so \( P(B) = 1/4 \).
(c) The event \( C \) consists of the outcomes \( FF, FM, \) and \( MF \). Each has probability \( 1/4 \), so \( P(C) = 3/4 \).
(d) The event \( B^c \) can be described as “there is at least one male child.”

2. A person is given three glasses of soda labelled A, B, C, and is asked to rank them in order of preference, best first. Let \( X \) be his ranking. A sample space is \( \{ABC, ACB, BAC, BCA, CAB, CBA\} \). Suppose that in fact, each glass contains the same drink, so that the ranking is random.

(a) What is an appropriate probability distribution?
(b) What is the probability that A is ranked best?
(c) What is the probability that A is ranked better than B?
(d) What is the probability that A is ranked worst?

Solution.
(a) If the ranking is random, then each of the 6 outcomes should be equally likely, so each should have probability \( 1/6 \).
(b) There are two outcomes \( (ABC \) and \( ACB) \) that rank A best, so the probability that A is ranked best is \( 2/6 \).
(c) There are three outcomes \( (ABC, ACB, \) and \( CAB) \) that rank A better than B, so the probability that A is ranked better than B is \( 3/6 \).
(d) There are two outcomes \( (BCA \) and \( CBA) \) that rank A worst, so the probability that A is ranked worse is \( 2/6 \).

3. Suppose that a test for tuberculosis has the following properties: If the person tested has tuberculosis, it returns a positive result 99% of the time; if the person tested does not have tuberculosis, it returns a negative result 95% of the time. The test is given to people thought to be at risk for tuberculosis; of this group, 2% actually have the disease.
(a) What proportion of the tests will return a positive result?
(b) What proportion of those whose test results are positive actually have tuberculosis?

Solution.
Let $A$ represent the event that the test comes back positive. Let $D$ represent the event that the person has tuberculosis. We know immediately that

- $P(A \mid D) = 0.99$.
- $P(A^c \mid D^c) = 0.95$.
- $P(D) = 0.02$.

(a) We want $P(A)$. To get this we’ll compute $P(A \text{ and } D)$ and $P(A \text{ and } D^c)$ and then add.

$$P(A \text{ and } D) = P(A \mid D)P(D) = (0.99)(0.02) = 0.0198;$$
and

$$P(A \text{ and } D^c) = P(A \mid D^c)P(D^c) = (0.05)(0.98) = 0.049.$$

So $P(A) = 0.0198 + 0.049 = 0.0688$.

(b) We want $P(D \mid A)$. To use the conditional probability formula we need $P(A \text{ and } D)$ and $P(A)$. But we already computed these in part (a). So

$$P(D \mid A) = \frac{P(A \text{ and } D)}{P(A)} = \frac{0.0198}{0.0688} \approx 0.288.$$ 

\[\square\]

4. One test for the presence of the HIV virus (the ELISA test) has the following properties: If the person tested is HIV positive, the test will return a positive result with probability .993; if the person is HIV negative, the test will return a negative result with probability .9999. The proportion of people who are HIV positive has been estimated at .000025. Suppose this is true, and suppose that all people are given the ELISA test.

(a) What proportion of those whose tests are positive are actually HIV positive?
(b) What proportion of those whose tests are negative are actually HIV negative?
(c) Explain the big difference in the answers to parts (a) and (b).

Solution.
Let $A$ represent the event that the test returns a positive result. Let $H$ represent the event that a person is HIV positive. We know immediately that

- $P(H) = 0.000025$.
- $P(A \mid H) = 0.993$.
- $P(A^c \mid H^c) = 0.9999$. 

(a) We want $P(H \mid A)$. To use the conditional probability formula we need to know $P(H \text{ and } A)$ and $P(A)$. Now

$$P(H \text{ and } A) = P(A \mid H)P(H) = (0.993)(0.000025) = 0.00002485.$$ 

and

$$P(H^c \text{ and } A) = P(A \mid H^c)P(H^c) = (0.0001)(0.999975) = 0.000099975.$$ 

We add these to get $P(A) = 0.00002485 + 0.000099975 = 0.0001248475$. So

$$P(H \mid A) = \frac{P(H \text{ and } A)}{P(A)} = \frac{0.00002485}{0.0001248475} \approx 0.199.$$ 

(b) We want $P(H^c \mid A^c)$. To use the conditional probability formula we need $P(H^c \text{ and } A^c)$ and $P(A^c)$. Now

$$P(H^c \text{ and } A^c) = P(A^c \mid H^c)P(H^c) = (0.9999)(0.999975) = 0.999875.$$ 

We’ve already computed $P(A)$, so $P(A^c) = 1 - P(A) = 1 - 0.0001248475 = 0.999875175$. So

$$P(H^c \mid A^c) = \frac{P(H^c \text{ and } A^c)}{P(A^c)} = \frac{0.999875}{0.999875175} \approx 0.99999982.$$ 

(c) The reason that the answer to (b) is so much closer to 1 than the answer to (a) is that there is a very low proportion of HIV-positive people (.000025) while there is a very high proportion of HIV-negative people (.999975).

5. A physician proposes to give a throat culture to detect strep bacteria to every patient who comes to his office. Explain why this may not be a good idea.

**Solution.**

The proportion of those people who come into the physician’s office who have strep bacteria will probably be low which, as we have seen, corresponds to a low proportion of strep cases among the positive test results. It would be better to only give the throat culture to those who have symptoms of strep bacteria, since presumably a higher proportion of this group would actually have the bacteria present.