MTH 133 Lecture 2: Solutions to Practice Problems for Exam 3
December 6, 1999 (Vince Melfi)

***NOTE:*** I’ve proofread these solutions several times, but you should still be wary for typo-
graphical (or worse) errors.

1. Compute the integral \( \int_{-8}^{0} x^{-\frac{1}{3}} \, dx \).

   **Solution:**
   \[
   \int_{-8}^{0} x^{-\frac{1}{3}} \, dx = \lim_{a \to -8} \int_{a}^{0} x^{-\frac{1}{3}} \, dx \\
   = \lim_{a \to -8} \left[ \frac{3}{2} x^{\frac{2}{3}} \right]_{a}^{0} \\
   = \lim_{a \to -8} \left[ (3/2)a^{2/3} - (3/2)(-8)^{2/3} \right] = -6.
   \]

2. Compute the integral \( \int_{1}^{\infty} \frac{3}{x^4} \, dx \).

   **Solution:**
   \[
   \int_{1}^{\infty} \frac{3}{x^4} \, dx = \lim_{b \to \infty} \int_{1}^{b} \frac{3}{x^4} \, dx \\
   = \lim_{b \to \infty} -\frac{1}{x^3} \bigg|_{1}^{b} \\
   = \lim_{b \to \infty} \left[ 1 - \frac{1}{b^3} \right] = 1.
   \]

3. Determine if the following series converge or diverge. Justify your answers completely.

   (a) \( \sum_{n=0}^{\infty} (-1)^n \frac{n!}{3^n} \)

   **Solution:** This series diverges because the terms don’t converge to zero.

   (b) \( \sum_{n=1}^{\infty} \frac{n^4 + 1}{3n^8 - 2n} \)

   **Solution:** We can compare this series with the series \( \sum_{n=1}^{\infty} \frac{1}{n^4} \), which we know converges. Since
   \[
   \lim_{n \to \infty} \frac{(n^4 + 1)}{(3n^8 - 2n)(1/n^4)} = \lim_{n \to \infty} \frac{n^8 + n^4}{3n^8 - 2n} = \frac{1}{3},
   \]
   the series converges by the limit comparison test.

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1 Most of these problems were kindly provided by Dr. Richeson.
(c) \( \sum_{n=1}^{\infty} \frac{5}{n^2} \)

**Solution:** This series converges, since the exponent \( p = 3/2 \) of \( n \) in the denominator is greater than 1.

(d) \( \sum_{n=2}^{\infty} \frac{1}{(\ln n)^2} \)

**Solution:** We compare this series with \( \sum_{n=1}^{\infty} \frac{1}{n} \), which we know diverges. Now using l’Hôpital’s rule twice,

\[
\lim_{n \to \infty} \frac{1}{\ln(n)^2} = \lim_{n \to \infty} \frac{n}{\ln(n)^2} = \lim_{n \to \infty} \frac{2 \ln(n)(1/n)}{n} = \lim_{n \to \infty} \frac{2 \ln(n)}{2(1/n)} = \lim_{n \to \infty} n = \infty,
\]

so the series diverges by the limit comparison test.

(e) \( \sum_{n=0}^{\infty} \frac{n + 1}{n!} \)

**Solution:** Computing the ratio of the \((n+1)\)st and \(n\)th terms gives

\[
\frac{(n + 2)/(n + 1)!}{(n + 1)/n!} = \frac{n + 2}{(n + 1)(n + 1)}.
\]

Since this converges to 0 as \( n \to \infty \), the series converges by the ratio test.

4. Find the interval of convergence for the power series \( \sum_{n=1}^{\infty} (-1)^n \frac{(2x - 3)^n}{n} \).

**Solution:** The ratio of the absolute values of the \((n+1)\)st and \(n\)th terms is

\[
\left| \frac{(-1)^{n+1}(2x - 3)^{n+1}}{(n + 1)} \right| \times \frac{n}{(-1)^n(2x - 3)^n} = \frac{n(2x - 3)}{n + 1}.
\]

Since this converges to \( |2x - 3| \) as \( n \to \infty \), the series converges absolutely (by the ratio test) when \( |2x - 3| < 1 \). Solving for \( x \), the series converges absolutely when \( 1 < x < 2 \). Checking the endpoints of this interval, when \( x = 1 \) the series is given by \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n} \), which we know diverges. When \( x = 2 \), the series is given by \( \sum_{n=1}^{\infty} (-1)^n \frac{1}{n} \), the alternating harmonic series, which we know converges. So the interval of convergence is \((1, 2]\).
5. (a) What are the first four terms of the Maclaurin series for \( f(x) = (1 + x)^{1/4} \)? What is the interval of convergence for this series?

**Solution:** First compute the first three derivatives:
\[
f'(x) = \frac{1}{4(1 + x)^{3/4}}, \quad f''(x) = -\frac{3}{16(1 + x)^{7/4}}, \quad f'''(x) = \frac{21}{64(1 + x)^{11/4}}.
\]
Plugging in \( x = 0 \) yields
\[
f'(0) = \frac{1}{4}; \quad f''(0) = -\frac{3}{16}; \quad f'''(0) = \frac{21}{64}.
\]
So the first four terms of the series are given by
\[
f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 = 1 + \frac{1}{4}x - \frac{3}{32}x^2 + \frac{7}{128}x^3.
\]
This series converges for \(-1 < x < 1\).

(b) What are the first four terms of the Maclaurin series for \( g(x) = (1 + 2x^2)^{1/4} \)? What is the interval of convergence for this series?

**Solution:** We can just plug in \( 2x^2 \) for \( x \) in the above series. This yields
\[
1 + \frac{1}{4}(2x^2) - \frac{3}{32}(2x^2)^2 + \frac{7}{128}(2x^2)^3 = 1 + \frac{1}{2}x^2 - \frac{3}{8}x^4 + \frac{7}{16}x^6.
\]
This series converges when \(|2x^2| < 1\). Solving for \( x \), the series converges when \(-1/\sqrt{2} < x < 1/\sqrt{2}\).

6. Compute the sum \( \sum_{n=1}^{\infty} (-1)^n \frac{2^n}{5^n} \).

**Solution:** This is just \( \sum_{n=1}^{\infty} -\frac{2}{5} \left(-\frac{2}{5}\right)^{n-1} \). This is a geometric series with \( a = -2/5 \) and \( r = -2/5 \), and so converges to \( a/(1 - r) = -2/7 \).

7. Write down the first five terms of the Taylor series for \( \ln x \) centered at \( x = 1 \).

**Solution:** The first four derivatives are \( f'(x) = 1/x, \ f''(x) = -1/x^2, \ f'''(x) = 2/x^3, \) and \( f^{(4)}(x) = -6/x^4 \). Plugging in \( x = 1 \) yields \( f'(1) = 1, \ f''(1) = -1, \ f'''(1) = 2, \) and \( f^{(4)}(1) = -6 \). So the first five terms of the Taylor series about 1 are
\[
0 + (1)(x - 1) - (1/2!)(x - 1)^2 + (2/3!)(x - 1)^3 - (6/4!)(x - 1)^4.
\]

8. Give the first four terms of the Maclaurin series for \( f(x) = x^3 \cos(2x) \).

**Solution:** The first four terms of the Maclaurin series for \( \cos(x) \) are
\[
1 - x^2/2! + x^4/4! - x^6/6!.
\]
Plug in \( 2x \) for \( x \) to get the first four terms of the Maclaurin series for \( \cos(2x) \):
\[
1 - 4x^2/2! + 16x^4/4! - 64x^6/6!.
\]
So the answer is
\[
x^3 - 4x^5/2! + 16x^7/4! - 64x^9/6!.
\]
9. Give the decimal expansion of $e$ accurate to 5 decimal places. Hint: use the Maclaurin series for $e^x$ and the fact that $e = e^1$.

**Solution:** We know that the Maclaurin series for $e^x$ is

$$f(x) = e^x = 1 + x + x^2/2! + x^3/3! + x^4/4! + \ldots,$$

so

$$f(1) = e = 1 + 1 + 1/2! + 1/3! + 1/4! + \ldots = 1 + 1 + 1/2! + \ldots + 1/n! + R_n(1),$$

where

$$R_n(1) = \frac{f^{(n+1)}(c)}{(n+1)!} (1^n = \frac{e^c}{(n+1)!})$$

for some $c$ in the interval $[0, 1]$. Since this function is positive and increasing in $c$, it takes its maximum value on the interval $[0, 1]$ at $c = 1$, so that

$$|R_n(1)| \leq \frac{e^1}{(n+1)!} < \frac{3}{(n+1)!}.$$ 

When $n = 8$, this yields $|R_8(1)| < 3/9! \approx 0.00000083$, which is less than .00001. So

$$1 + 1 + 1/2! + 1/3! + 1/4! + 1/5! + 1/6! + 1/7! + 1/8!$$

approximates $e$ to within 5 decimal places.

10. Does the series $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$ converge absolutely? Does the series converge?

**Solution:** It does not converge absolutely, since the series of absolute values is $\sum_{n=1}^{\infty} (1/\sqrt{n})$, which we know diverges. But the alternating series test tells us that the original series converges, since the $1/\sqrt{n}$ is always positive and decreases to 0.

11. For each of the following series convergence problems, determine whether the argument given is correct. If not, explain precisely why not, and if possible, determine the answer (convergence or divergence) to the problem.

(a) CLAIM: The series $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2 + 1}$ converges, because $\lim_{n \to \infty} \frac{n}{n^2 + 1} = 0$.

**Solution:** This argument is faulty. Convergence of the terms of a series to 0 is not sufficient to imply that the series is convergent. In fact, by using the limit comparison test with the divergent series $\sum_{n=1}^{\infty} \frac{1}{n}$, we find that the original series diverges.
(b) CLAIM: The series \( \sum_{n=1}^{\infty} \frac{10n + 1}{n(n + 1)(n + 2)} \) diverges by the limit comparison test, because when we compare the series with the convergent series \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) we find that
\[
\lim_{n \to \infty} \frac{(10n + 1)/(n(n + 1)(n + 2))}{1/n^2} = 10.
\]
Since this is bigger than 1, the series diverges.

Solution: This argument is also faulty. The limit comparison test says that as long as the limit is a positive number, the two series either both converge or both diverge. Since the limit is 10, the two series above either both diverge or both converge. Since \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) is convergent, both series converge.

(c) CLAIM: The series \( \sum_{n=1}^{\infty} \frac{n^2}{e^n} \) converges, because
\[
\lim_{n \to \infty} \frac{(n + 1)^2/e^{n+1}}{n^2/e^n} = \frac{1}{e^1}.
\]
Since this is less than 1, the series converges by the ratio test.

Solution: This is correct.

(d) CLAIM: The series \( \sum_{n=1}^{\infty} (-1)^n \frac{n}{4n^3 + 7} \) is absolutely convergent by the ratio test, because
\[
\lim_{n \to \infty} \frac{(n + 1)/(4(n + 1)^3 + 7)}{n/(4n^3 + 7)} = 1.
\]
Solution: This is a faulty argument. When the ratio of successive terms converges to 1, the ratio test is inconclusive. In this case we can compare the series of absolute values to the convergent series \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) to show via the limit comparison test that the original series is absolutely convergent.