Recitation 11/17 Solutions

1. A = left engine fails $\quad P(A) = 0.0001$

B = right engine fails $\quad P(B) = 0.0002$

* In general and not specific to this question:

keyword: "At least one" = A or B = $A \cup B$

Find $P(\text{At least one}) = P(A \cup B)$

By the addition rule:

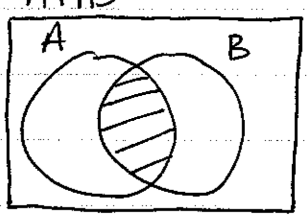
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

We do not know $P(A \cap B)$ so we do not have enough info to find $P(A \cup B)$.

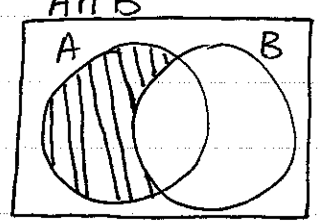
a) $\bar{A}$ = complement of A (in terms of a venn diagram, it's everything outside of A)

The 4 non-overlapping areas can be written in terms of A, B, $\bar{A}$ and $\bar{B}$.

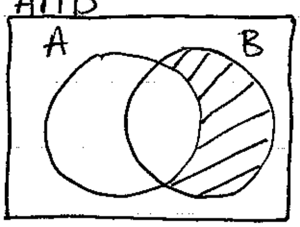
$A \cap B$



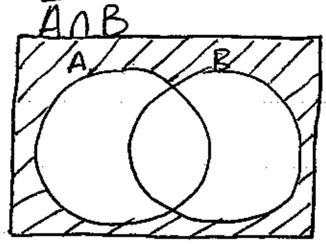
$A \cap \bar{B}$



$\bar{A} \cap B$



$\bar{A} \cap \bar{B}$



so $\bar{A}$ in this context = engine A does <u>not</u> fail

b) We can NOT say anything about independence just by looking at venn diagrams. To check for independence, we check if:

$$P(A \cap B) = P(A)P(A)$$

$$P(B|A) = P(B)$$

So there are 2 different ways of checking and we pick which one to use depending on what information we are given (note: when we "check", we are checking if both sides of the equation are equal)

In (b) we are told that A & B are independent. So we have

$$P(B|A) = P(B)$$

If we rearrange the "conditional probability equation", we have:

$$P(B|A) = \frac{P(B \cap A)}{P(A)}$$

rearranged: $P(B \cap A) = P(B|A)P(A)$ (this is also called the multiplication rule)

so: $P(B \cap A) = P(B)P(A) = 0.0001(0.0002)$ by independence

$$P(A \cap \bar{B}) = P(A) - P(A \cap B) = P(A) - P(A)P(B)$$

$$= 0.0001 - (0.0001)(0.0002)$$

$$= 0.00009998$$

$$P(\bar{A} \cap B) = P(B) - P(A \cap B) = 0.0002 - (0.0001)(0.0002)$$

easiest way is by looking at the venn diagram in (a)

$$= 0.00019998$$

$$P(\bar{A} \cap \bar{B}) = 1 - \left(P(A \cap B) + P(A \cap \bar{B}) + P(\bar{A} \cap B)\right)$$

$$= 1 - \left((0.0001)(0.0002) + 0.00009998 + 0.00019998\right)$$

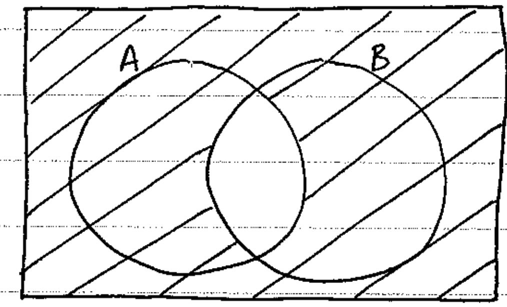
$$= 1 - 0.00029998$$

$$= 0.99970002$$

We can then label these numbers on the venn diagram in (a).

c) P(at least one does <u>not</u> fail) = P(A does <u>not</u> fail OR B does <u>not</u> fail)

$= P(\bar{A} \cup \bar{B})$



everything except for the intersection of A and B

from looking at venn diagram

$$P(\bar{A} \cup \bar{B}) = P(A \cap \bar{B}) + P(\bar{A} \cap B) + P(\bar{A} \cap \bar{B})$$

$$= 0.00009998 + 0.00019998 + 0.99970002$$

$$= 0.99999998$$

d) The easier way to find $P(\bar{A} \cup \bar{B})$ = P(at least one does <u>not</u> fail):

from looking at venn diagram

$$P(\bar{A} \cup \bar{B}) = 1 - P(A \cap B)$$ (because everything must add up to 1)

$$= 1 - 0.00000002$$

$$= 0.99999998$$

2. We are going to assume that A and B are <u>not</u> independent.

so $P(B|A) \neq P(B)$

$P(A \cap B) \neq P(A)P(B)$

We are told:

$P(A) = 0.0001$

$P(B) = 0.0002$

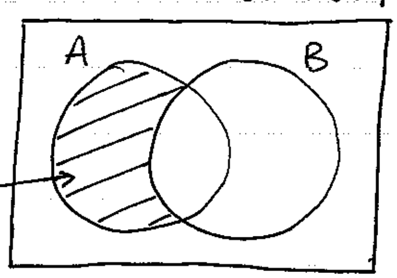
$P(B|A) = 0.07$

a) Both engines fail:

By multiplication rule (conditional prob. eq. rearranged):

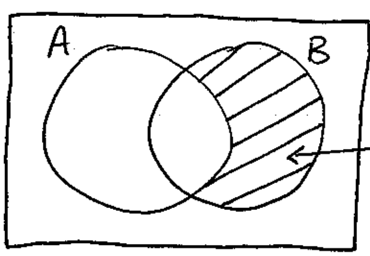
$$P(A \cap B) = P(B|A)P(A) = 0.07(0.0001) = 0.000007$$

b) $$P(A \cap \overline{B}) = P(A) - P(A \cap B) = 0.0001 - 0.000007 = 0.000093$$



0.000093

$$P(\overline{A} \cap B) = P(B) - P(A \cap B) = 0.0002 - 0.000007 = 0.000193$$



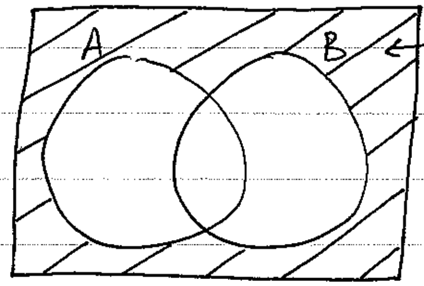
0.000193

sum of the other 3

$$P(\bar{A}\cap\bar{B}) = 1 - \underbrace{\left(P(A\cap B) + P(A\cap\bar{B}) + P(\bar{A}\cap B)\right)}$$

$$= 1 - (0.000007 + 0.000093 + 0.000193)$$

$$= 0.999707$$



0.999707

c) $P(\text{at least one does \underline{not} fail})$

$= P(\bar{A}\cup\bar{B}) = P(A\cap\bar{B}) + P(\bar{A}\cap B) + P(\bar{A}\cap\bar{B})$

or $= 1 - P(A\cap B)$ easier way, "shortcut"

$= 1 - 0.000007$

$= 0.999993$

(for explaination, see 1.(c))

This is smaller ~~that~~ than the independent case (1(c)).

d) $P(\text{engine A fails} \mid \text{B fails}) = P(A \mid B)$

by conditional prob. equation:

$$P(A\mid B) = \frac{P(A\cap B)}{P(B)} = \frac{0.000007}{0.0002} = 0.035$$

e) First we calculate $P(A\mid B)$ and $P(B\mid A)$.

$P(A\mid B) = 0.035$

$$P(B\mid A) = \frac{P(B\cap A)}{P(A)} = \frac{0.000007}{0.0001} = 0.07$$

Failure of engine A (left) concerns me more because that has a much higher chance of causing engine B to fail.

3. A box has : 3 red balls, 4 green balls, 5 yellow balls

We will select 3 balls with "equal prob." and "without replacement".

a) P(red ball on the first pick) = $\frac{\#\text{ red balls}}{\text{total } \#\text{ of balls}}$ ← outcomes satisfying our condition

← total # of outcomes

$= 3/12 = 1/4$

b) $P(R2 \mid R1) = P(\text{red ball on } 2^{nd} \text{ pick} \mid \text{red ball on } 1^{st} \text{ pick})$

An outcome : $\longrightarrow$ (1^{st} pick, 2^{nd} pick, 3^{rd} pick)

We restrict ourselves to outcomes with red balls on the 1^{st} pick :

(ra, rb, ...) 10 balls left to choose from so there are 10 outcomes starting with (ra, rb, ...)

(ra, rc, ...) 10 possible outcomes starting with (ra, rc, ...)

(rb, ra, ...) 10 outcomes

(rb, rc, ...) 10 outcomes

(rc, ra, ...) " "

(rc, rb, ...) " "

So there are 60 outcomes that satisfy our condition: "red ball on 2^{nd} pick given red ball on 1^{st} pick." The total # of outcomes in the restricted set of outcomes is $11 \times 10 \times 3 = 330$

↑ # of choices we have for the 2^{nd} pick ↖ # of choices we have for the 3^{rd} pick

the 3 is the # of choices we have for the 1^{st} pick (must be a red ball)

so $P(R2 \mid R1) = \frac{60}{330} = \frac{2}{11}$

similarly, $P(R2 \mid \overline{R1}) = P(2^{nd} \text{ pick red} \mid 1^{st} \text{ pick } \underline{\text{not}} \text{ red})$

$$= \frac{9 \times 3 \times 10}{9 \times 11 \times 10} = \frac{3}{11}$$

There is a trick/shortcut way of counting outcomes.

$$\begin{aligned}\#\text{ of outcomes} &= \#\text{ of choices for } 1^{st}\text{ pick}\\ &\quad\times \#\text{ of } \text{ " } \text{ " } 2^{nd}\text{ pick}\\ &\quad\times \#\text{ of } \text{ " } \text{ " } 3^{rd}\text{ pick}\end{aligned}$$

eg.) we will caculate $P(R2|R1)$ again

\# of outcomes satisfying our condition "red on 2^{nd} given red on first"

$$= 3 \times 2 \times 10$$

3	$\times 2$	$\times 10$
there are 3 red balls so we have 3 choices for the 1^{st} pick	2 red balls are left so we have 2 choices for the 2^{nd} pick	there are 10 balls left in the box so there are 10 choices for the 3^{rd} pick

total # of outcomes in the restricted set (~~first~~ first ball is red)

$$= 3 \times 11 \times 10$$

3	$\times 11$	$\times 10$
3 choices for 1^{st} pick	11 balls left so 11 choices for 2^{nd} pick	10 balls left so 10 choices for 3^{rd} pick

The next part introduces a NEW equation.

<u>Law of Total Probability</u>:

$$\Rightarrow \quad P(A) = P(A|B)P(B) + P(A|\overline{B})P(\overline{B})$$

We want to find $P(R2)$. Apply the above equation:

$$P(R2) = P(R2|R1)P(R1) + P(R2|\overline{R1})P(\overline{R1})$$

$P(R2 \mid R1) = \frac{2}{11}$

$P(R2 \mid \overline{R1}) = \frac{3}{11}$

$P(R1) = \frac{3 \times 11 \times 10}{12 \times 11 \times 10} = \frac{3}{12} = \frac{1}{4}$

$P(\overline{R1}) = \frac{9 \times 11 \times 10}{12 \times 11 \times 10} = \frac{9}{12} = \frac{3}{4}$

by the shorcut counting method

$P(R2) = \frac{2}{11}\left(\frac{1}{4}\right) + \frac{3}{11}\left(\frac{3}{4}\right) = \frac{11}{44} = \frac{1}{4}$

c) $P(R1 \mid R2) = \frac{P(R1 \cap R2)}{P(R2)}$ ⟵ $P(R1\, R2)$ is a shortform for $\cap$

$P(R1 \cap R2) = P(R2 \mid R1)P(R1)$ ⟸ (multiplication rule)

$= \frac{2}{11}\left(\frac{1}{4}\right) = \frac{1}{22}$

$P(R1 \mid R2) = \frac{\frac{1}{22}}{\frac{1}{4}} = \frac{4}{22} = \frac{2}{11}$

4. Saturday : 80% chance of rain

Sunday : 60% chance of rain

Let A = rain on Sat. $\Rightarrow P(A) = 0.8$

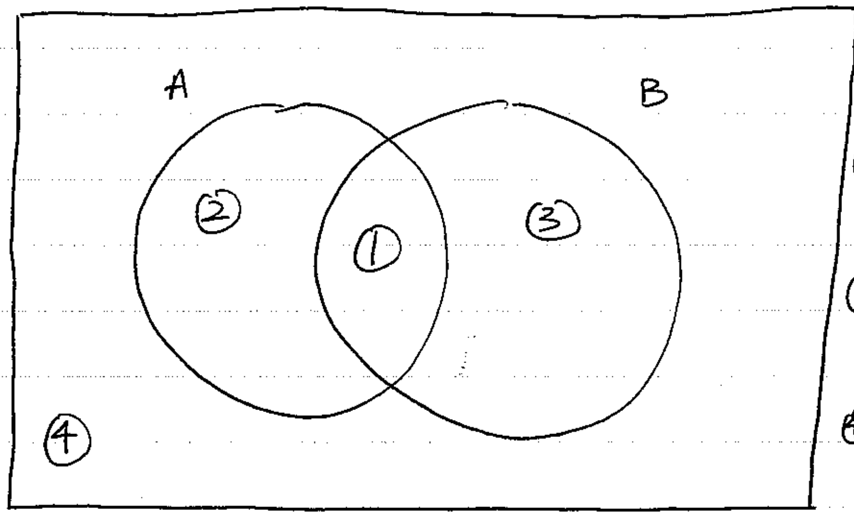
B = rain on Sun. $\Rightarrow P(B) = 0.6$

a) In real life, if it rains today, it is likely that it will keep raining tomorrow. So this is NOT independent (one affects the chances of the other).

b) This is a LOT like question 1 (more practice).

Assuming this is independent:

$$P(A \cap B) = P(A)P(B)$$
$$= 0.8(0.6)$$
$$= 0.48$$



① = rain on both days
② = rain on Sat and no rain on Sun
③ = rain on Sun and no rain on Sat
④ = no rain on both days

① $= P(A \cap B) = 0.48$

② $= P(A) - P(A \cap B) = 0.8 - 0.48 = 0.32$

③ $= P(B) - P(A \cap B) = 0.6 - 0.48 = 0.12$

④ $= 1 - ($① $+$ ② $+$ ③$) = 1 - (0.48 + 0.32 + 0.12)$

$= 1 - 0.92 = 0.08$

c) Assuming this is not independent:

We are given: $P(\text{rain on Sun} \mid \text{rain on Sat})$

$= P(B \mid A) = 0.3$

The areas are then:

① $= P(A \cap B) = P(B \mid A)P(A) = 0.3(0.8) = 0.24$

② $= P(A) - P(A \cap B) = 0.8 - 0.24 = 0.56$

③ $= P(B) - P(A \cap B) = 0.6 - 0.24 = 0.36$

④ $= 1 - (0.24 + 0.56 + 0.36) = 1 - 1.16$ this must ALWAYS be less than 1

The answer is invalid because the number that the question gave for $P(B|A)$ is invalid. ~~A correction to the question will be announced later.~~ The correction to reflect the email is on the last page.

d) Skip.

5. Tree Diagrams and Bayes' Method

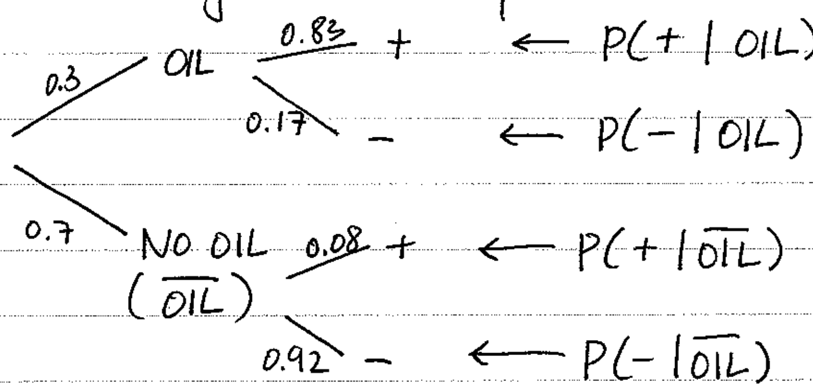
Labels : $+$ = test was positive

$-$ = " " negative

OIL = there actually is oil

$\overline{\text{OIL}}$ = no OIL = there actually is ~~not~~ no oil

a) Tree Diagram (Visual representation for conditional probability)



The way to read this diagram is that each branch (<) represents a probability. ← one branch

eg.) the probability of OIL is 0.3

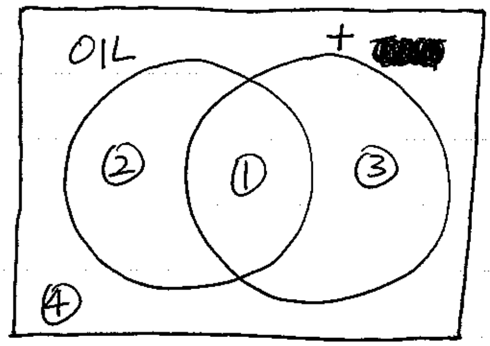
The second tier of branches represent conditional probabilities. I have labeled the conditional probs. above. Think of it as: to get to $P(+ | \text{OIL})$, I had to first get to OIL and then to $+$ so OIL is the given (came first).

To get the AND probabilities ($\cap$), you multiply the branches (you took) traversed.

$P(\text{OIL} \cap +) = 0.3 \times 0.83 = 0.249$

$P(\text{OIL} \cap -) = 0.3 \times 0.17 = 0.051$

$P(\overline{OIL} \cap +) = 0.7 \times 0.08 = 0.056$
$P(\overline{OIL} \cap -) = 0.7 \times 0.92 = 0.644$

b) 

note: the complement of + is −
ie.) $\overline{+} = -$

<u>Using the same method as Q1 & Q4:</u>

① = $P(OIL \cap +) = 0.249$
② = $P(OIL) -$ ① $= 0.3 - 0.249 = 0.051$
③ = $P(+) -$ ①

this is (c) { note: to get $P(+)$, we must add up the branches that end in + (this is the "law of total probability")

$P(+) = \underbrace{0.3 \times 0.83}_{\text{top branch ending in +}} + \underbrace{0.7 \times 0.08}_{\text{2nd last branch ending in +}} = 0.305$

③ = $0.305 - 0.249 = 0.056$
④ = $1 - ($① + ② + ③$) = 1 - 0.356 = 0.644$

Easier way:
① = $P(OIL \cap +) = 0.249$
② = $P(OIL \cap -) = 0.051$
③ = $P(\overline{OIL} \cap +) = 0.056$
④ = $P(\overline{OIL} \cap -) = 0.644$
} values are from (a)

d) $P(OIL \mid +) = \dfrac{P(OIL \cap +)}{P(+)} = \dfrac{0.249}{0.305} = 0.816$

e) $P(OIL \mid -) = \dfrac{P(OIL \cap -)}{P(-)} = \dfrac{0.051}{0.695} = 0.073$

$P(-)$ = sum up the 2 "$-$" branches
$= (0.3 \times 0.17) + (0.7 \times 0.92)$
$= 0.695$

It makes sense that the probability that "there is actually oil given the test is positive" should be higher than "there is actually oil given the test is negative".

4. (c) With correction : $P(B|A) = 0.7$

Then: $① = P(A \cap B) = P(B|A)P(A) = 0.7(0.8) = 0.56$

$② = P(A) - P(A \cap B) = 0.8 - 0.56 = 0.24$

$③ = P(\mathbf{B}) - P(A \cap B) = 0.6 - 0.56 = 0.04$

$④ = 1 - (① + ② + ③) = 1 - (0.56 + 0.24 + 0.04) = 1 - 0.84$
$= 0.16$

d) Find $P(\text{rain on Sat} \mid \text{rain on Sun}) = P(A \mid B)$

Applying the "conditional probability equation":

$P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{0.56}{0.6} = 0.933$ (0.56 from (c))