Recitation Assignment 11 - 24 - 09. This material is drawn from Chapters 15, 16.

Two independent tosses of a "fair" coin. E X = probability weighted average number of heads when two coins are tossed.

1. Consider the possible outcomes of two tosses of a coin. The classical probability model will be assumed. Let (capital) X denote the random variable "number of heads resulting from the two tosses." Below each of the four possible outcomes below give the numerical value (lower case x) equal to the number of heads for that outcome.

outcome	HH	HT	TH	TT
probability	.25	.25	.25	.25
value x of X				

Calculate the "expected number of heads in two tosses" defined by the sum over all four possible outcomes for two tosses of a coin as:
$$E X = \sum_{outcome} p(outcome) \, X(outcome)$$
Does the value of E X make sense to you on an intuitive basis?

2. The value E X can instead be obtained by **grouping the summands** according to **distinct values** x. That is, E X can also be obtained by
$$E X = \sum x \, p(x)$$
where p(x) now refers to the probability of the value x. That is:

$p(2) = P(X = 2) = P(HH) = .25$
$p(1) = P(X = 1) = P(HT \text{ or } TH) = .5$
$p(0) = P(X = 0) = P(TT) = .25$

We arrive at the **probability distribution of random variable X** which is the list of all **distinct values x** together with their probabilities:

value x	2	1	0
p(x)	.25	.5	.25

Calculate $E X = \sum x \, p(x)$ by this second method. It must give the same result as #1 in all cases since is merely groups the terms in #1 method.

X_1 X_1

x_1

$X_1 \quad \sum x_1 \quad x_1$

$$\sum x$$

Expectation E X of a random variable is always the same whether calculated from the probability model as a sum over all possible outcomes or instead calculated from the probability distribution of X as a sum over all distinct values taken by x.

3. Let X_1 be 1 if the first toss is a head, zero otherwise. Calculate E X_1 **from the model for the first toss alone**:

outcome	H1	T1
probability	.5	.5
x_1	1	0

E $X_1 = \sum x_1 \, p(x_1)$. Does this make sense to you?

4. Let X_1 be 1 if the first toss is a head, zero otherwise. Calculate E X_1 from the model for **two tosses**.

outcome	H1 H2	H1 T2	T1 H2	T1 T2
probability	.25	.25	.25	.25
x_1	1	1	0	0

E $X_1 = \sum_{\text{outcome}} p(\text{outcome}) \, X(\text{outcome})$.

Notice that the probability distribution of X_1 in #3 is exactly the same as it is for #4.

The model for two tosses subsumes the model for a single toss.

5. (Additivity of expectation). Refer to # 4. In addition to the random variable X_1 define also random variable $X_2 = 1$ if the second toss is a head and 0 otherwise. Clearly E $X_2 = 0.5$ just as you found for E X_1.

$$X_1 \quad X_2$$

x_1
x_2

$$X_1 \quad X_2$$

Verify that in every outcome for two tosses $X = X_1 + X_2$.

outcome	H1 H2	H1 T2	T1 H2	T1 T2
probability	.25	.25	.25	.25
x_1	1	1	0	0
x_2		0		
x		1		

Observe that your earlier calculation #1 of $E\,X = 1$ agrees with
$$E\,X_1 + E\,X_2 = .5 + .5 = 1.$$

Remarkably, regardless of any possible dependencies among random variables X, Y, Z, etc., we always have
$$E\,(X + Y + Z + ..) = E\,X + E\,Y + E\,Z + ..$$

A simple instance of this idea is that the average money held by students of the class is always the sum of the average coin money and the average paper money: $E(X + Y) = E\,X + E\,Y$.

This "additivity of expectation is a result of grouping terms:

$E\,(X + Y) = \sum_{x,y} (x + y)\,P(X = x \text{ and } Y = y)$ (calculating LHS in a model)

$= \sum_x x \sum_y P(X = x \text{ and } Y = y) + \sum_y y \sum_x P(X = x \text{ and } Y = y)$ (arrange sums)

$= \sum_x x\,P(X = x) + \sum_y y\,P(Y = y)$ (law of total probability)

$= E\,X + E\,Y$ (calculating expectations from the definition)

Moreover, for any constant "c" we have

$E\,(c\,X) = \sum_x (cx)\,P(X = x)$ (calculating LHS in a model)

$= c \sum_x x\,P(X = x) = c\,E\,X$

and

$E\,c = 1\,(c) = c.$

The relations above may be summarized in
  E(a X + bY + c) = a E X + b E Y + c
since the result extends to any number of random variables dependent or not by induction, as for example:
  E(X + Y + Z) = E X + E (Y + Z) = E X + E Y + E Z
using two applications of the relation above.

6. (Uses additivity of E) A corporation has two divisions. A probability model is prepared for the possible gross revenues X, Y from the two divisions in relation to the probabilities thereof. The model finds E X (in Euros) = 23.69 million and also finds E Y = (in dollars) 27.94 million. There are most assuredly dependencies between the random variables X, Y of the probability model (e.g. one division might be Coke and the other their bottled water). You are asked **what is the expected net return defined by**
    net return = 1.3 X + Y - 1.3 .28 X - .17 Y + 1.3 1.4 + 3.2
in which the 1.3 represents a euro to dollar conversion, .28 being the tax rate in E.U., .17 being the tax rate in U. S., 1.4 being stimulus money in E.U., 3.2 being stimulus money in U. S. It is simple to find E (net) because of the properties set forth just above in #5.

7-10 Biased coin.

7. Suppose that the coin has been tampered with so that the chance of a head on any given toss is .54 (scarcely noticed in actual plays but larger than the casino advantage in many games). Tosses are still independent so:

outcome	HH	HT	TH	TT
probability	$.54^2$	.54 .46	.46 .54	$.46^2$ (uses independence)
value x				

Calculate E X directly from the probability experiment
        E X = $\sum_{\text{outcome}}$ p(outcome) X(outcome)
Does the value of E X make intuitive sense to you?

$\sum x$

X_1

$$\sum_{\text{outcome}}$$

8. Work up the probability distribution of random variable X:

value x	2	1	0
p(x)			

Instead of #3, which calculated E X directly as a weighted average over all outcomes possible, calculate E X from the probability distribution of X as:
$$E X = \sum x \, p(x).$$

9. Let X_1 be 1 if the first toss is H and 0 if the first toss is T. Determine the expectation E X_1 using the distribution of X_1 :

x_1	1	0
probability		

Does the answer make sense to you?

10. Using additivity of E determine the expected number of Heads in 100 tosses of the biased coin.

Fundamental property of E (XY) for the case when X, Y are *independent*:

$E(XY) = (E\,X)(E\,Y)$ **if** X, Y are **independent random variables** since then:

$\quad E(XY) = \sum_{x,y} x\,y\,P(X = x \text{ and } Y = y)$ (calculation from a model)

$\quad\quad = \sum_{x,y} x\,y\,P(X = x)\,P(Y = y)$ (if X, Y are *independent*)

$\quad\quad = (\sum_x x\,P(X = x))\,(\sum_y y\,P(Y = y))$ (rearrange terms)

$\quad\quad = (E\,X)(E\,Y).$

$$X^2 \quad (E\,X)^2 \qquad\qquad\qquad \sqrt{\text{Var } X}$$

$$)^2$$

$$(X + Y)^2$$
$$X^2 \qquad\qquad Y^2$$

$$\sum_{x,y} x\, y$$

$$\left(\sum_x x \qquad\qquad \sum_y y\right.$$

Variance and SD of random variable:

Var X is defined to be $E\,X^2 - (E\,X)^2$ and standard deviation by $SD\,X = \sqrt{Var\,X}$.

Variance of a sum of *independent* random variables:

First for a sum of two independent random variables X, Y we have
$$Var\,(X + Y) = Var\,((X - E\,X) + (Y - E\,Y))^2$$
the constants $E\,X$, $E\,Y$ do not change variance. To ease notation we may as well assume that $E\,X = 0$ and $E\,Y = 0$ in what now follows.

$Var\,(X + Y) = E\,(X + Y)^2$ (expectations being assume to be zero)
$= E(\,X^2 + 2\,XY + Y^2\,)$ (the square of X+Y is only expanded)
$= E\,X^2 + E\,(2\,XY) + E\,Y^2$ (additivity of expectation)
$\mathbf{= E\,X^2 + E\,Y^2}$ ($E\,(XY) = (E\,X)\,(E\,Y) = 0$ by independence)
$= Var\,X + Var\,Y$ (remember, we assumed $E\,X$, $E\,Y$ were zero)

11. Toss the biased coin once. Let X_1 be 1 if we get H and 0 otherwise. Again, determine $E\,X_1$.

12. Toss the biased coin once. Let X_1 be 1 if we get H and 0 otherwise. Determine $E\,X_1^2$. Notice that $1^2 = 1$ and $0^2 = 0$ which implies that $X_1^2 = X_1$. So $E\,X_1^2 = E\,X_1$.

13. Toss the biased coin once. Let X_1 be 1 if we get H and 0 otherwise. Determine Var X_1 using the definition above.

$$X_1 \quad X_2 \qquad\qquad X_{100}$$

$$X_1 \quad X_2$$

$$X_{100}$$

$$X_1$$

14. Toss the biased coin (see #7) 100 times. How many heads X do we expect over 100 tosses? Use additivity of E and $X = X_1 + X_2 + ... + X_{100}$ which is the sum of 100 random variables each having the distribution of #9. The tosses are independent also but we do not need that.

15. Toss the biased coin (see #7) 100 times. What is the variance of $X = X_1 + X_2 + ... + X_{100}$ which is the sum of 100 *independent* random variables each having the distribution of #9. So Var $X = 100$ Var X_1. The tosses are *independent* and we need that!

16. Sketch the bell curve having mean E $X = 100$ E X_1 and standard deviation SD $X = \sqrt{\text{Var } X} = \sqrt{100 \text{ Var } X_1}$. Notice that the expected number of heads is constant times the number of tosses but SD X is another constant times the square root of the number of tosses. So the spread of the bell curve is small *relative to the mean* if the number of tosses is large! Label everything in your curve. What is the 68% interval for the number of heads in 100 tosses of the biased coin?