Statistical tests make a decision: either reject or fail to reject H0.

(Note: Errors/changes are highlighted with this color).

Important skills:

- **Design a one or two-sided test having a given alpha.**
- **Plot and interpret P(reject H0 | mu (or p)) vs mu (or p).**
- **Design one or two-sided tests having given alpha and beta.**
- **Execute tests based on sample data.**

Read Chapter 7. The key points are these:

a. The null hypothesis H_0 and alternative hypothesis H_1 describe contrasting sets of parameters. Examples include

a1.	$H_0: \mu \leq 16$ oz.	$H_1: \mu > 16$ oz.	one-sided test
a2.	$H_0: \mu \geq 28$ mpg.	$H_1: \mu < 28$ mpg.	one-sided test
a3.	$H_0: p \leq 0.6$	$H_1: p > 0.6$	one-sided test
a4.	$H_0: p \geq 0.3$	$H_1: p < 0.3$	one-sided test
a5.	$H_0: \mu = 16$ oz.	$H_1: \mu \neq 16$ oz.	two-sided test
a6.	$H_0: p = 0.6$	$H_1: p \neq 0.6$	two-sided test

b. Test statistics are quantities used to measure evidence against the null hypothesis H_0. Let P(T > t_α) = α and P(Z > z_α) = α. Be reminded that z_α is *a* special case of t_α for df = ∞. Except for very large df, we require a normal population if Student's t is used.

b1. reject H_0 if (test statistic) (xBAR - 16) / (s / $\sqrt{n}$) $\geq t_\alpha$

b2. reject H_0 if (xBAR - 28) / (s / $\sqrt{n}$) $\leq -t_\alpha$

b3. reject H_0 if (pHAT - 0.6) / ($\sqrt{0.6\ 0.4}$ / $\sqrt{n}$) $\geq z_\alpha$

b4. reject H_0 if (pHAT - 0.3) / ($\sqrt{0.3\ 0.7}$ / $\sqrt{n}$) $\leq -z_\alpha$

b5. reject H_0 if | (xBAR - 16) / (s / $\sqrt{n}$) | $\geq t_{\alpha/2}$

b6. reject H_0 if | (pHAT - 0.6) / ($\sqrt{0.6\ 0.4}$ / $\sqrt{n}$) | $\geq z_{\alpha/2}$

NOTE CHANGE: We will ONLY use root(p0 q0) NOT root(pHAT qHAT) in the denominator of tests about p since this must be done anyway when designing tests with given beta.

c. The test descriptions of (b) above can be re-phrased in terms of significance value p_{sig} (the probability of more evidence **against** H_0 than has been observed in the data).

In terms of p_{sig} always: reject H_0 if $p_{\text{sig}} \leq \alpha$.

c1. p = P(T > test statistic) or p = P(Z > test statistic)
c2. p = P(T < test statistic) or p = P(Z < test statistic)
c3. p = P(Z > test statistic)
c4. p = P(Z < test statistic)
c5. p = P(|T| > |test statistic|) or P(|Z| > |test statistic|)
c6. p = P(|Z| > |test statistic|)

d. Every test must balance the errors of type I (rejecting null hypothesis when it is true) and type II (failing to reject null hypothesis when it is false). This balance is revealed by plotting

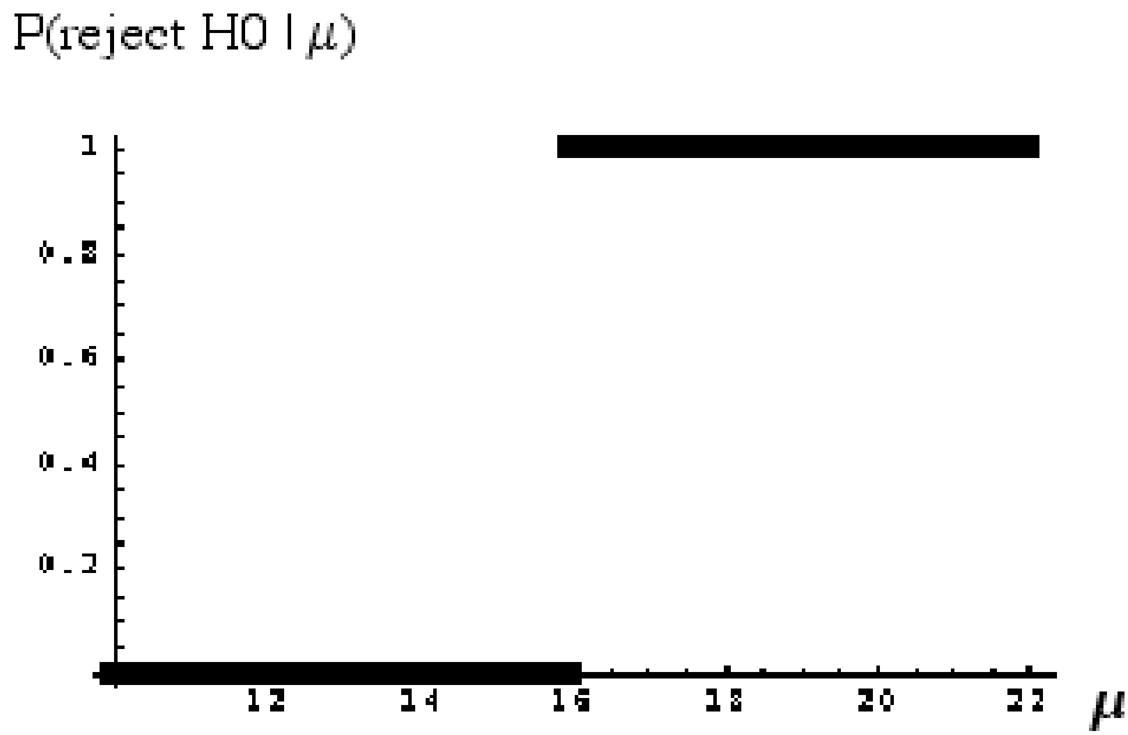
$$P(\text{reject } H_0 \mid \mu) \text{ as a function of } \mu$$

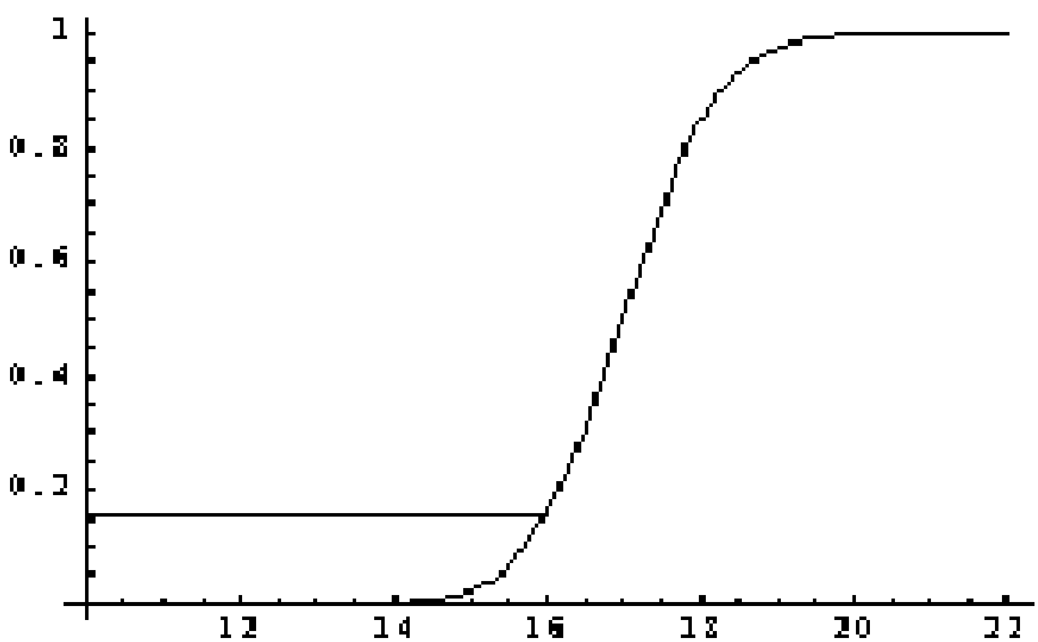
or

$$P(\text{reject } H_0 \mid p) \text{ as a function of } p$$

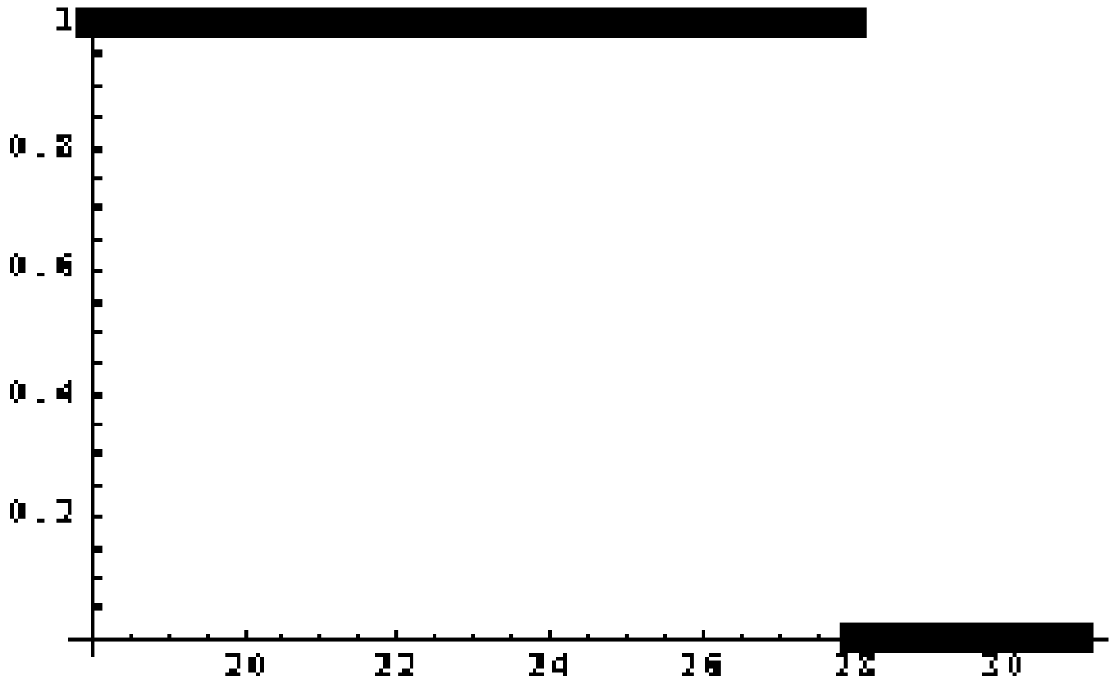
The following are typical shapes of these plots with α identified by a horizontal line above the boundary value (μ0 or p0) where H0 touches H1. Ideal plots are also shown.

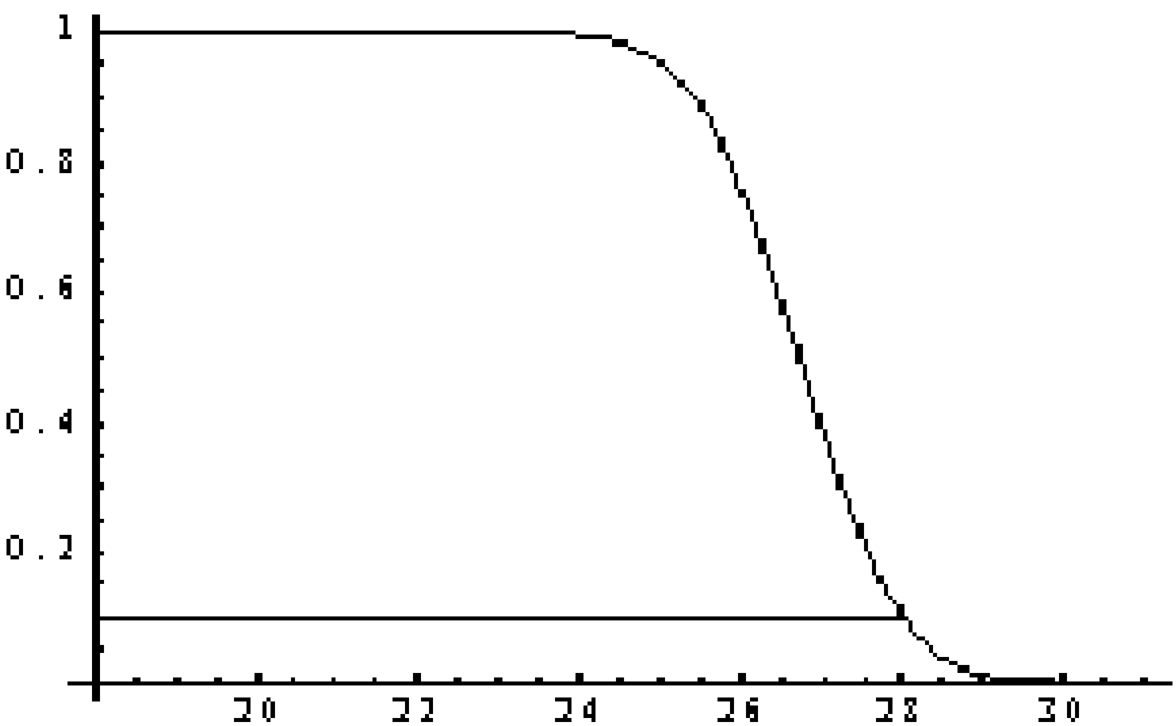
da1. (Ideal first).



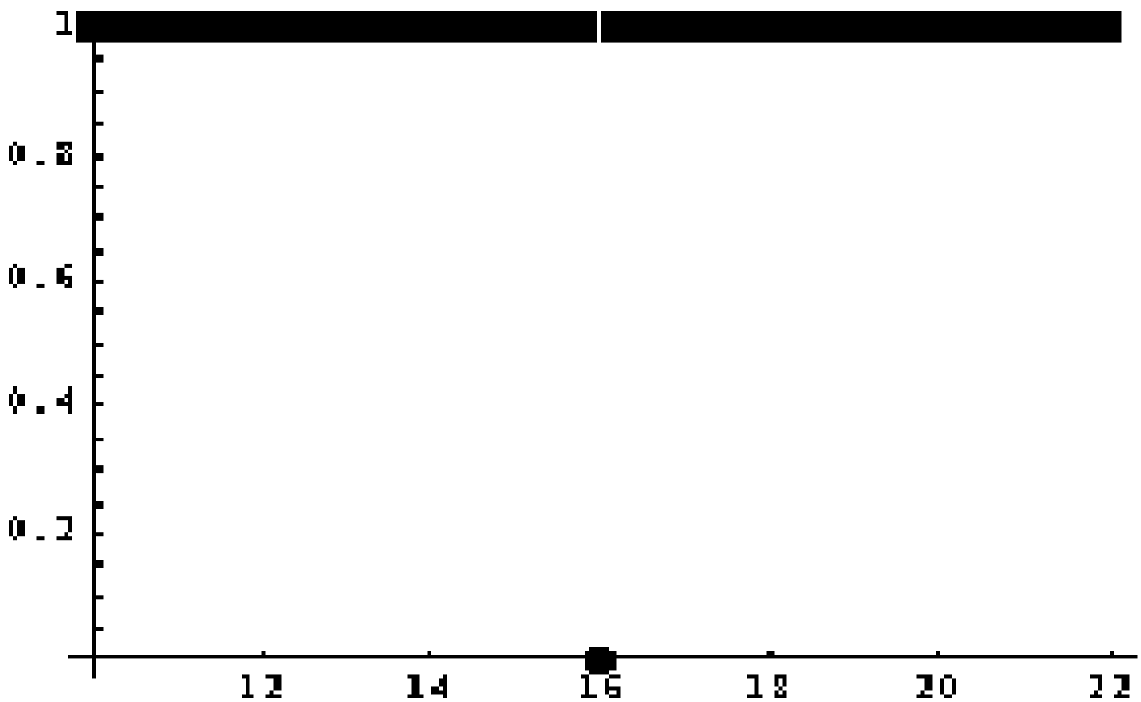


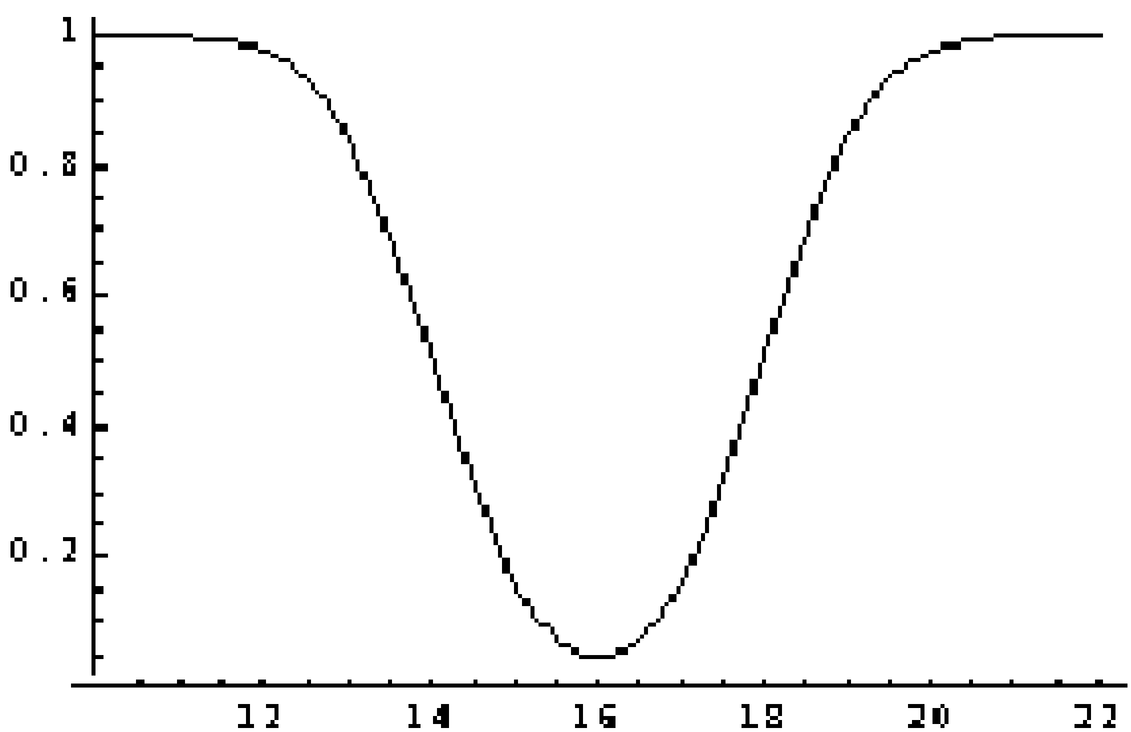
da2.





da5.

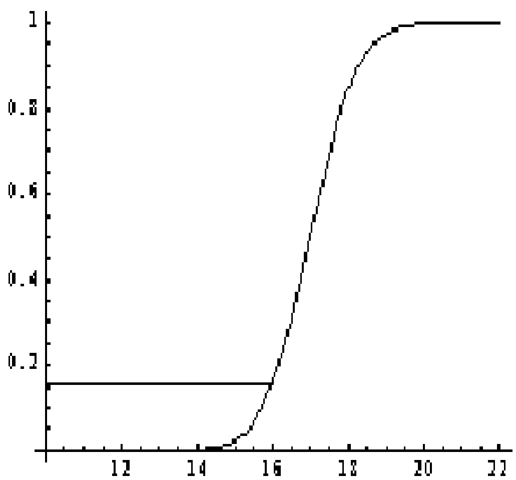


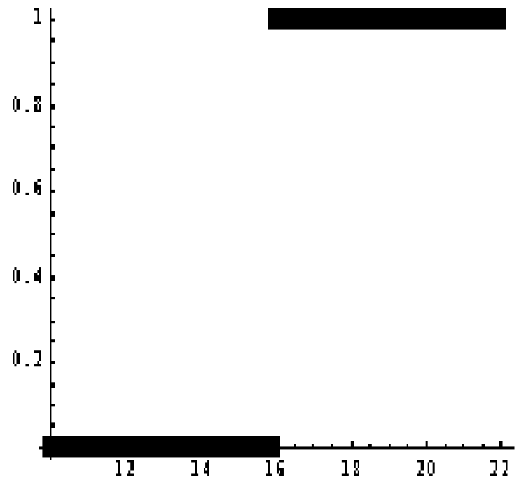


The following examples will be covered in the lectures for the week of 10-30-06. A key will be posted Tuesday.

Exam 3 is Thursday, 11-2-06.

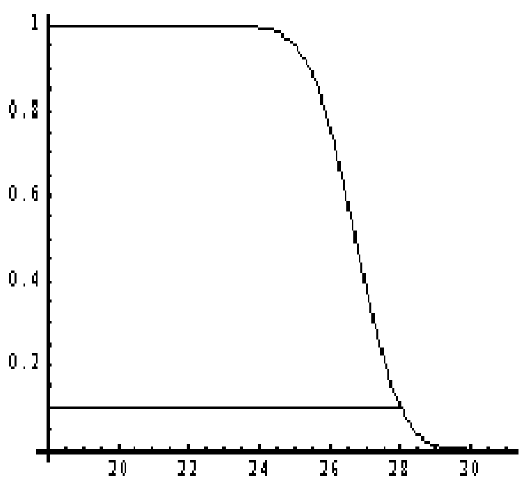
1. Identify the null hypothesis, significance level alpha, power and beta at mu = 18, ideal power at mu = 18 in the figures below. Label axes mu and P(rej H0 | mu) respectively.

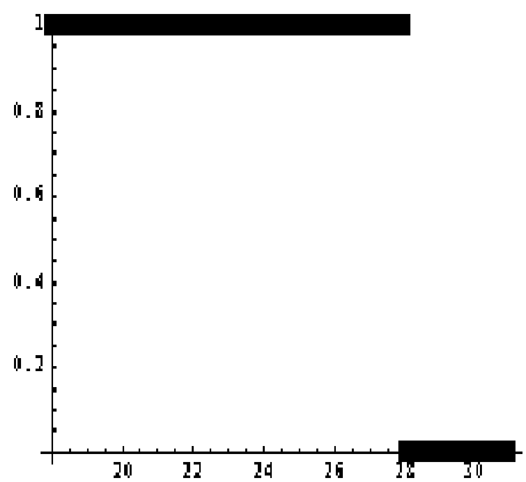




ANS. H0: mu less or equal 16, H1: mu > 16. Significance level alpha = 0.16. Power at mu = 18 is around 0.9 and beta = 0.1. Ideal power is 1 since 18 is not in H0.

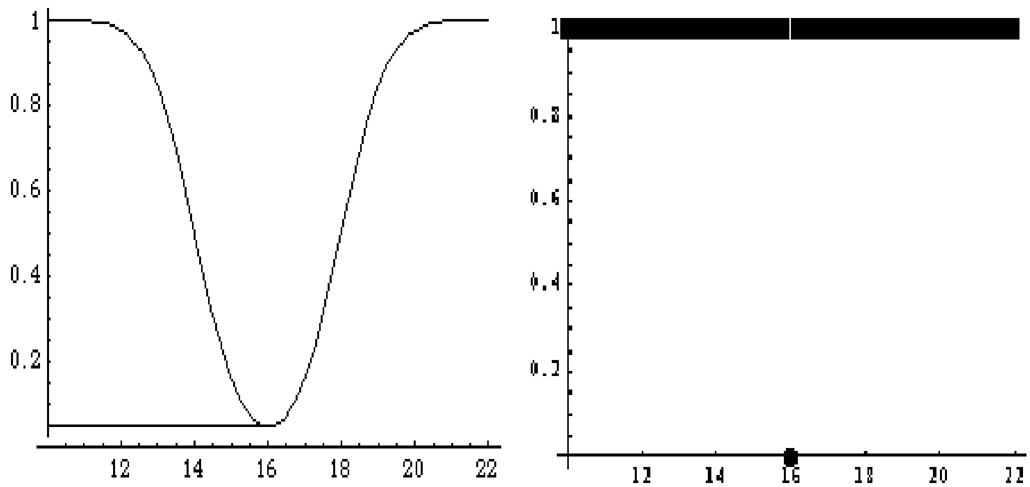
2. Identify the null hypothesis, significance level alpha, power and beta at mu = 26, ideal power at mu = 26 in the figures below. Label axes mu and P(rej H0 | mu) respectively.





ANS. H0: mu greater or equal 28. H1: mu < 28. Alpha is around 0.1 while power at mu = 26 is around 0.76 with beta = 0.24. Ideal power is 1 since 26 is not in H0.

3. Identify the null hypothesis, significance level alpha, power at mu = 13, power at mu = 18, ideal power at mu = 13, ideal power at mu =18 in the figures below. Label axes.



ANS. H0: mu = 16. H1: mu not 16. Alpha ~ 0.05, power at 13 ~ 0.85, beta ~ 0.15. Ideal power is 1 since 13 is not in H0.

4. For the null hypothesis H0 that mu is greater or equal IQ = 100 versus alternative mu < 100 and significance level alpha = 0.02, freehand sketch the **general** shape of the curve P(rej null | mu) versus mu, over the range 70 to 130. Identify alpha clearly and label axes properly. Impose upon your sketch the ideal curve P(rej H0 | mu) over the same range of mu. Also impose upon your sketch another P(rej null | mu) curve for a better test, based upon **more** data, that shares the same alpha with your first one. Indicate which one of your two curves (the ones achievable without a census) is for more data.
ANS. Take d2 as your model.

5. Do (4) but for null hypothesis mu = 100 versus alternative mu not equal to 100.
ANS. Take d5 as your model.

6. Refer to (4). A random with-replacement sample of 400 persons from the population has sample mean IQ = 112.6. Without any calculation, what action is taken by the test? Hint: The score112.6 belongs to H0.
ANS. Since 112.6 belongs to H0 there is no possibility of rejecting H0. The test statistic will be positive, z0 = -z(alpha) is negative, and this test rejects H0 if the test statistic is less than z0.

7. Refer to (4). A random with-replacement sample of 400 persons from the population has sample mean IQ = 98.2 with sample standard deviation s = 14.6.

a. Give the form of the test statistic and state under which circumstances the z-test will reject the null hypothesis. You will need to use the z-table to determine the appropriate z0 value and you will have to reason as to whether should be positive or negative.
ANS. (xBAR-mu0)/(s/root(n)) = (98.2-100)/(14.6/20) = -2.46575. Since H1 is left of H0, reject H0 if this TS is less than z0 = -zALPHA = -z(.02) = -2.05.

b. Calculate the numerical value of the test statistic. Compare it with the threshold z0 of part (a) and state the action taken (reject null or fail to reject null).
ANS. We reject H0 since TS = -2.47 < -2.05 = z0.

c. Determine the **statistical significance** pSIG for this data. It is the probability that xBAR would have been more **unfavorable** to the null that it is for this data (calculated when mu is 100, the boundary between null and alternative). So if your test statistic is equal to –2.47 (for the key I may as well use the actual value) you would report statistical significance equal to the area under the z-curve to the left of –2.47 (since more negative xBAR = more negative TS = more unfavorable to this H0).
ANS. pSIG = area left of –2.47 = P(Z < -2.47) = 0.0068.

d. Use your statistical significance pSIG to conduct the test. That is, reject H0 if the significance pSIG < alpha = 0.02. State the action taken. This should agree with (b), which conducted the test in the other, but equivalent, way.
ANS. **By the pSIG approach to testing, the test (7ab) rejects H0 if the observed significance level pSIG < alpha. Since 0.0068 < 0.02 we (as above) reject H0.**

e. Suppose, instead, that the sample was only n = 61 (not n = 400) and the sample mean was xBAR = 98.2 with sample sd equal to s = 14.6. Also I will change alpha to 0.025 since 0.02 is not on the t-table (what using the closest t-table entry is really achieving). IF THE POPULATION IS KNOWN TO BE IN CONTROL (which is typical for IQ scores) we are entitled to conduct a t-test. Perform the t-test, stating the DF and consulting the t-table to determine the appropriate rejection threshold t0 for this test.
ANS. DF = 61-1 = 60. For this one-sided t-test z0 = -z(.025) = -2.0. The TS is (xBAR-mu0)/(s/root(n)) = (98.2-100)/(14.6/root(61)) = -0.96. Since this is not less than z0 = -2.0 we fail to reject H0. Having the smaller sample size 61 makes it harder to reject for data with the same xBAR and s as the sample of 400 gave us. Tests tend to stay with H0 unless there is strong evidence to the contrary. A smaller sample does not have the same force of evidence as would a larger sample.

8. Let p denote the fraction of voters favoring a Republican candidate. For the null hypothesis that Republicans have at least 50% of the vote versus alternative $p < 0.5$, and significance level alpha = 0.04, freehand sketch the **general** shape of the curve P(rej null | p) versus p in the range 0 to 1 for a possible test of this null hypothesis. Identify alpha clearly and label axes properly. Impose on your sketch the ideal curve P(rej null | p) over the same range of p. Also impose upon your sketch another P(rej null | p) curve for a better test, based upon **more** data, that shares the same alpha with your first one. Indicate which one of your curves is for more data.

ANS. Take d2 as your model. The better curve is lower on H0 and higher on H1 while passing through the point (0.5, 0.04).

9. Let p denote the fraction of customers buying a "red label" product. Suppose that historically this has been at p = 0.38. For the null hypothesis p = 0.38, versus the alternative p not equal to 0.38, and significance level alpha = 0.05, freehand sketch the **general** shape of the curve P(rej H0 | p) versus p in the range 0 to 1. Identify alpha clearly and label axes properly. Impose on your sketch the ideal curve P(rej null | p) over the same range of p in your sketch. Also impose upon your sketch another P(rej null | p) curve for a better test, based upon **more** data, that shares the same alpha with your first one. Indicate which one of your curves is for more data.

ANS. Take d5 as your model.

10. Refers to (9). Suppose a random with-replacement sample of n = 100 customers yields the sample fraction pHAT = 0.47 who favor the red label product.

a. Calculate the test statistic for the z-test of H0: p = 0.38 vs H1: p is not 0.38. **Note: When calculating the test statistic it is common practice to use root(.38 .62)/root(100), not the value root(.47 .53)/root(100) as would be used for the CI. This is because doing so leads to a test having slightly more desirable performance against alternatives close to p = 0.38.**
ANS. (pHAT – p0)/(root(p0 q0)/root(n)) = (0.47 – 0.38)/(root(0.38 0.62)/root(100)) = +1.85419. Be sure to use p0 and not pHAT in the denominator. I actually did that in the 3 p.m. lecture Monday! For various reasons we are going to stick with p0 in the denominator of the test statistic.

b. Use the z-table to determine a threshold z0 for rejection of the null. Make sure to take into account that this is a two-sided test that will reject the null if the test statistic is either too large or too small.
ANS. z(ALPHA/2) = z(0.025) = +1.96. The two-sided z-test rejects H0 if |TS | > z0.

c. Conduct the test stating the action taken (reject H0 or fail to reject H0).
ANS. The test rejects H0 if |TS | = 1.85 > 1.96. So we fail to reject H0. The same action (fail to reject) would have been taken if the test statistic had been –1.85 (it is not) because the two-sided test utilizes the criterion "reject H0 if the absolute value of the TS exceeds z0."

d. For the given data, calculate the statistical significance pSIG. **Note: this is a two-sided test so, for a z-test statistic value of +1.85 we would report significance level pSIG equal to the combined z-areas left of –1.85 and right of 1.85, or (simply put) twice the area right of 1.87 as "more extreme than observed."**
ANS. pSIG = 2 P(Z > 1.85) = 0.0643135.

e. Use your statistical significance pSIG from (d) to (again) conduct the test, this time by comparing pSIG with alpha = 0.05. State the result of this comparison and the action taken (reject H0 or fail to reject H0). It will be the same action taken in (c) but the test is being performed this other way.
ANS. Reject H0 if pSIG < alpha. Since 0.0643135 > 0.05 we fail to reject H0.

11. Refers to problem (10). Suppose we desire a test of H0: p = 0.38 vs H1: p is not 0.38, having alpha = 0.05 but also having power P(rej null | p = 0.42) = 0.96. That is, the probability of type 2 error is equal to beta = 0.04 (at p = 0.42). If this is achieved we then have a test whose chance of falsely rejecting p = 0.38 (if it is true) is only 0.05 but also one whose chance of rejecting p = 0.38 is 0.96 if truly p = 0.42.

a. Free hand sketch P(rej null | p) as p varies between 0 and 1 for such a test.
ANS. Use p5 as your model.

b. The formula at the top of page 319 may be used to determine a sample size that will support this desired test. The formula tells us the required sample size n for these specifications. If we agree to use this recommended n then the test for alpha = 0.05 will automatically have the desired power 0.96 at p = 0.42 also. Evaluate the formula of pg. 319 to determine this n.

ANS. **$p_0 = 0.38$ $p_1 = 0.47$**

$Z_0 = Z_{alpha/2} = Z_{0.025}$ = 1.96 (since the test is two-sided)

$Z_1 = Z_{beta} = Z_{0.04}$ = 1.75 **(even though the test is two-sided)**

Note: For p_1 = 0.47, the chance of false rejection of H0 can safely ignore the relatively small contribution of doing so by accidentally producing a pHAT far below 0.38. That is why, even though the test is two sided, z_1 the formula for z1 uses beta and not beta/2.

$$n \sim \left(\frac{|z_0| \sqrt{p_0 q_0} + |z_1| \sqrt{p_1 q_1}}{p_0 - p_1} \right)^2$$

Then n ~ (1.96 root(0.38 0.62) + 1.75 root(0.42 0.58))^2 / (0.38 – 0.42)^2 ~ 2060 from the formula of pg. 319.

c. For the "n" of (b) suppose we find that pHAT = (# buying red label)/n = 0.44. Calculate the test statistic for the alpha = 0.05 test of H0: p = 0.38 vs H1: "p is not 0.38" for this n.

ANS. TS = (pHAT – p0) / root(p0 q0 / n) = (0.44 – 0.38) / root(0.38 0.62 / 2060) = 5.61.

d. Using the answer (c) conduct the test. Be sure to show the test statistic, its comparison with z0, and the decision taken by the test (either to reject H0 or not).

ANS. For the two-sided test, reject H0 if | TS | = 5.61 > 1.96 = z0. So we reject H0. By specifying beta = 0.04 at p1 = 0.42 we force the very large sample size n = 2060. For such a large n, the sample proportion pHAT is a very reliable estimate of p. When the sample shows an estimate pHAT = 0.44 this easily leads to rejection of H0: p < 0.38.

12. A company ships packages all over the world and its shipping costs can take unexpected swings due to economic changes, especially changes in its international customer base. Rather than attempt to model or monitor all of these factors it is decided to monitor shipping costs of individual packages directly on a continuing basis. Let's say that, as of today, we believe we average 4.77 shipping cost per package. The following day a sample of packages will be selected and a z-test of H0: "mu = is less or equal to 4.77" vs H1: mu > 4.77 will be conducted at level alpha = 0.025.

a. Suppose we decide to use nINIT = 50 packages (with replacement). Evaluate the z-test statistic, z0, and determine the decision taken by the z-test if this initial sample of 50 finds xBAR = 5.31 with sample sd sINIT = 2.44 (shipping costs seem to vary a lot by package).

ANS. For this one-sided test z0 = z(alpha) = z(0.025) = 1.96. The test rejects H0 if the test statisticexceeds z0 = 1.96.

TS = (xBAR – mu0)/(s/root(n) = (5.31 – 4.77) / (2.44 / root(50) = 1.5649 < 1.96 the test fails to reject H0: mu is less or equal to 4.7.

b. Concerned that a test sample of 50 may not be large enough to allow reliable detection of a change to (say) mu = 5, which would seriously affect total cost for the many thousands of packages shipped daily, it is decided to use a possibly larger n to achieve power 0.9 at mu = 5. That is, 90% of the time we would expect to reject H0 if the mean mu has shifted to 5. Use the formula of pg. 314, with sigma replaced by our estimate sINIT of (a), to determine nFINAL.

ANS. For this one-sided test z0 = z(alpha) = z(0.025) = 1.96 (as above), z1 = z(beta) = z(0.1) = 1.282, mu0 = 4.77 and mu1 = 5. From the formula of pg. 319

$$\mathbf{n} \sim \left(\frac{(|z_0| + |z_1|)\, s_0}{\mu_0 - \mu_1} \right)^2 = \left(\frac{(|1.96| + |1.282|)\, 2.44}{4.77 - 5} \right)^2 \sim 1183$$

c. Suppose we then continue our sample-of-the-day to nFINAL = 1183, finding that xBARfinal = 5.22. Give the form of, and evaluate, the test statistic for the new test (the hybrid test statistic continues to use sINIT but uses root(nFINAL) and xBARfinal).

ANS. Hybrid test statistic is HTS = (xBARfinal – mu0) / (sINIT / root(nFINAL) = 6.34.

d. Comparing (c) with z0 from (a) determine the action taken by the hybrid test (c).

ANS. One-sided test rejects H0 if HTS = 6.34 > 1.96 = z0. So reject H0. By specifying a test with beta = 0.1 at mu = 5 we have upped the sample size to a very large 1183. This has the consequence that xBAR = 5.22 is to be taken seriously, leading to rejection of H0.

Exam 3 will have around 12 questions of which 3 will be dedicated to determining confidence intervals for mu or p, or difference of mu's or p's, or CI with given precision (requiring determination of needed sample size). These bonus questions will be very close in content to some questions from exam 2.

Formulas given on exam 2 will also appear on exam 3.
Formulas of pages 314, 319 will appear on exam 3.
Formulas for test statistics will NOT appear on exam 3.

Tables of t and z will appear on exam 3.

Some questions may have a few parts, particularly with questions about a plot of P(reject H0 | mu (or p)) vs mu (or p).