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WHITE NOISE ANALYSIS FOR THE CANONICAL LÉVY PROCESS

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Abstract. We construct a white noise theory for the Canonical Lévy Process introduced by Solé, Utzet, and Vives. The construction is based on the alternative construction of the chaos expansion of square integrable random variables. Then we establish a white noise characterization in the space of generalized distributions, and prove a Wick-Skorohod identity. Finally, we prove a Clark-Ocone theorem in $L^2(P)$ for the Canonical Lévy space.

1. Introduction

The classical Canonical space for a Lévy process is constucted from the σ-field of cylinder sets and a probaility measure using the Kolmogorov extension theorem [21], [3]. However, Solé, Utzet and Vives [22] have formulated another construction of the Canonical space for the Lévy process to be able to interpret the Malliavin derivative $D_{t,z}$ for the Lévy process. The derivative $D_{t,0}$ is the classical Malliavin derivative with respect to the Wiener process while $D_{t,z}$, $z \neq 0$ is the Malliavin derivative with respect to the pure jump process defined as an increment quotient in the generalized distribution space. We shall refer to the Canonical space constructed by Solé, Utzet, and Vives [22] as *Canonical Lévy process.*

White noise theory was first introduced by Hida for Wiener process, from origins in quantum physics [10]. Subsequently, the theory was extended to the pure jump Lévy process as in [1], [7], [17], by incorporating generalized function spaces related to $L^2(P)$ [11]. We extend this theory for the Canonical Lévy space by deriving an alternative chaos expansion of square integrable random variables; then we prove some important characterizations such as a Wick-Skorohod identity, and a Clark-Ocone theorem in $L^2(P)$.

The Clark-Ocone theorem is the explicit representation of the Itô representation theorem in terms of the Malliavin derivative. The Clark-Ocone theorem in $\mathbb{D}^{1,2}$ for the Canonical Lévy process can be stated as follows [23]. Let $F \in \mathbb{D}^{1,2}$ be $\mathcal{F}_T$-measurable, then

$$F = E[F] + \int_{[0,T]\times\mathbb{R}} E[D_{t,z}F|\mathcal{F}_{t^-}]M(dt,dz), \tag{1.1}$$

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where M is the independently scattered measure given in (2.6).

The Clark-Ocone representation can be weakened to a representation for $F \in L^2(P)$ using white noise analysis with the same form (1.1). However, the Malliavin derivative $D_{t,z}$ and the expectation E will be generalized to a *stochastic gradient* and a *generalized expectation* respectively.

This paper is organized as follows. Section 2 provides background review of the Malliavin calculus for the Canonical Lévy processes and the chaos expansion with respect to the measure M. We present the construction of the Canonical Lévy white noise process [14] in Section 3. Then we construct the alternative chaos expansion of Canonical Lévy process in Section 4. The proof of this alternative chaos expansion uses the chaotic representation property in Theorem 4.1 by Nualart and Schoutens [16] and the results of Solé, Utzet, and Vives in Theorems 4.3 of [23].

Our alternative chaos expansion for the Canonical Lévy process is new. From this expansion, we characterize the white noise theory using some family of function spaces of stochastic test functions and distribution functions. This characterization is an extension of the Wiener case [9] and the Poisson case [17], [8]. White noise analysis in the Canonical Lévy space is presented in Section 5. These concepts have analogues in the Wiener and Poisson cases: [8], [7], [18], [19]. We have shown a Wick-Skorohod identity (Proposition 5.12). Finally, we established a Clark-Ocone theorem for Wick polynomials (Proposition 6.6) and for $L^2(P)$ (Proposition 6.8).

2. Canonical Lévy Space

We present a brief background on Lévy processes, following [3], [8], [21], and introduce preliminary concepts on the Malliavin calculus in the Canonical Lévy space, as in [22], [23].

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. A Lévy process $X = \{X(t) : t \geq 0\}$ is a stochastic process that has following properties: $X(t)$ has independent and stationary increments with $X(0) = 0$, and it is continuous in probability. The Poisson random measure $N : \Omega \times [0,T] \times \mathbb{R}_0 \to \mathbb{N}_0$ is a counting measure defined as

$$N(A) = \sum_{s\in(0,t]} \mathbf{1}_{\{s:(s,\Delta X(s))\in A\}}, \quad A \in \mathfrak{B}([0,T] \times \mathbb{R}_0), \tag{2.1}$$

where $\mathbb{R}_0 = \mathbb{R}\backslash 0$ and $\Delta X(t) = X(t) - X(t^-)$ is the jump of X at time t. The Lévy measure ν of X is defined as the expectation of N as follows:

$$\nu(B) = E[N((0,1] \times B)] = E\left[\sum_{s\in(0,1]} \mathbf{1}_{\{s:\Delta X(s)\in B\}}\right], \quad B \in \mathfrak{B}(\mathbb{R}_0). \tag{2.2}$$

The Lévy measure is σ-finite and satisfies $\nu(\{0\}) = 0$, $\int_{\mathbb{R}_0}(1 \wedge z^2)\nu(dz) < \infty$. The compensated Poisson random measure $\tilde{N} : \Omega \times [0,T] \times \mathbb{R}_0 \to \mathbb{R}$ is given by $\tilde{N}(dt,dz) = N(dt,dz) - dt\nu(dz)$. The characteristic function of the Lévy process is given by the Lévy Khintchine formula, see [8].

Consider the Canonical Lévy space

$$(\Omega, \mathcal{F}, P) = (\Omega_W \times \Omega_J, \mathcal{F}_W \otimes \mathcal{F}_J, P_W \otimes P_J), \tag{2.3}$$

where $(\Omega_W, \mathcal{F}_W, P_W)$ is the Canonical Wiener space and $(\Omega_J, \mathcal{F}_J, P_J)$ is the Canonical pure jump Lévy space.

Let $X(t)$ be a centered, square-integrable Lévy process, then $X(t)$ can be written as follows:

$$X(t) = \sigma W(t) + \int_{[0,t]\times\mathbb{R}_0} z\tilde{N}(ds, dz). \tag{2.4}$$

From the Lévy Khinchine formula, see [3], its characteristic function is given by

$$E\left(\exp(iuX(t))\right] = \exp\left[\left(-\frac{1}{2}\sigma^2u^2 + \int_{\mathbb{R}_0}(\exp(iuz) - 1 - iuz)\nu(dz)\right)t\right]. \tag{2.5}$$

The square-integrable assumption on $X(t)$ implies $\int_{\mathbb{R}_0} z^2\nu(dz) < \infty$.

Itô [13] extended the centered square-integrable Lévy process X to an independent measure M on $(\mathbb{R}_+ \times \mathbb{R}, \mathfrak{B}(\mathbb{R}_+ \times \mathbb{R}))$ which can be expressed as

$$M(E) = \sigma\int_{E_0} dW(t) + \int_{E'} zd\tilde{N}(dt, dz), \tag{2.6}$$

where $E \in \mathfrak{B}(\mathbb{R}_+ \times \mathbb{R})$, $E_0 = \{t \in \mathbb{R}_+ : (t, 0) \in E\}$ and $E' = E \setminus E_0$. Then for $E_1, E_2 \in \mathfrak{B}(\mathbb{R}_+ \times \mathbb{R})$ such that $\mu(E_1) < \infty, \mu(E_2) < \infty$

$$E[M(E_1)M(E_2)] = \mu(E_1 \cap E_2), \tag{2.7}$$

where μ is a measure on $([0, T] \times \mathbb{R}, \mathfrak{B}([0, T] \times \mathbb{R})$ and

$$\mu(E) = \sigma^2\int_{E_0} dt + \int_{E'} z^2 d\nu(z)dt, \qquad E \in \mathfrak{B}([0, T] \times \mathbb{R}). \tag{2.8}$$

In differential form, we have

$$\mu(dtdz) = \sigma^2 d\delta_0(z)dt + z^2(1 - \delta_0(z))d\nu(z)dt = \lambda(dt)\eta(dz), \tag{2.9}$$

where $\lambda(dt) = dt$ is the Lebesgue measure and

$$\eta(dz) = \sigma^2 d\delta_0(z)dt + z^2(1 - \delta_0(x))d\nu(z). \tag{2.10}$$

2.1. Iterated Lévy-Itô Integral. Let $f \in L^2(([0, T] \times \mathbb{R})^n)$ be a deterministic function such that

$$||f||^2_{L^2(\mu^n)} = \int_{([0,T]\times\mathbb{R})^n} |f((t_1, z_1)\cdots(t_n, z_n))|^2\mu(dt_1, dz_1)\cdots\mu(dt_n, dz_n) < \infty. \tag{2.11}$$

The symmetrization of f denoted by $f^\wedge$ over $(t_1, x_1), \cdots, (t_n, x_n)$ is given by

$$f^\wedge((t_1, z_1), \cdots (t_n, z_n)) = \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n} f((t_{\sigma(1)}, z_{\sigma(1)}), \cdots (t_{\sigma(n)}, z_{\sigma(n)}), \tag{2.12}$$

where $\sigma = (\sigma(1), \cdots, \sigma(n))$ is a permutation of $\{1, \cdots, n\}$ and $\mathfrak{S}_n$ is the set of permutations of $\{1, \cdots, n\}$. Denote $S_n = \{(t_1, z_1), \cdots (t_n, z_n) : 0 < t_1 < \cdots t_n < T, x_i \in \mathbb{R}, i \in \{1, \cdots n\}\}$. For $f \in L^2(\mu^n)$ let $J_n(f)$ be the n-fold iterated integral over S_n and $I_n(f)$ be the n-fold multiple integral over $([0, T] \times \mathbb{R})^n$.

Denote by $L^2_s(\mu^n)$ the subspace of symmetric functions in $L^2(\mu^n)$. Then, for $f \in L^2_s(\mu^n)$, we have the following identity: $I_n(f) = n!J_n(f)$. The multiple integral I_n has the following properties [23]:

(1) Symmetry: $\quad I_n(f) = I_n(f^\wedge), \qquad f \in L^2(\mu^n),$

(2) Linearity: $I_n(af+bg) = aI_n(f) + bI_n(g), \quad f,g \in L^2(\mu^n), \quad a,b \in \mathbb{R}$,
(3) Isometry: $E[I_n(f)I_m(g)] = n!\,\langle f^\wedge, g^\wedge \rangle_{L^2(\mu^n)}\,\delta_{mn}, f \in L^2(\mu^n), g \in L^2(\mu^m)$.

Itô has shown the following chaos expansion for the Lévy space.

Theorem 2.1. [13] *Let $F \in L^2(P)$, then F has chaos expansion given by*

$$F = \sum_{n=0}^{\infty} I_n(f_n), \tag{2.13}$$

where we set $I_0(f_0) = E[F]$. The chaos expansion is unique if $f_n \in L_s^2(\mu^n)$ for all $n \in \mathbb{N}$. Furthermore, we have the following isometry relation:

$$||F||_{L^2(P)}^2 = \sum_{n=0}^{\infty} n!||f_n||_{L^2(\mu^n)}^2, \qquad f_n \in L_s^2(\mu^n). \tag{2.14}$$

3. Construction of the Canonical Lévy White Noise Process

Followig [14], we construct the Canonical Lévy white noise process using a procedure which parallels the derivation of Wiener and Poisson white noise processes in [11]. Let $\mathcal{S} \equiv S(\mathbb{R})$ be the Schwartz space of test functions which consists of rapidly decreasing smooth functions $f \in C^\infty(\mathbb{R})$ such that

$$||f||_{\alpha,\beta} = \sup_{x\in\mathbb{R}} |x^\alpha f^{(\beta)}(x)| < \infty. \tag{3.1}$$

In addition, $S(\mathbb{R})$ is a Fréchet space with respect to the seminorm $\|f\|_{\alpha,\beta}$. Its dual $\mathcal{S}' \equiv S'(\mathbb{R})$ is the Schwartz space of tempered distribution functions endowed with a weak* topology. The action of $\omega \in \mathcal{S}'(\mathbb{R})$ on $\phi \in \mathcal{S}(\mathbb{R})$ is given by the mapping $w : \mathcal{S}(\mathbb{R}) \times \mathcal{S}'(\mathbb{R}) \to \mathbb{R}$

$$w(\phi, \omega) =< \omega, \phi > . \tag{3.2}$$

Moreover, we have the following inclusions: $S(\mathbb{R}) \subset L^2(P) \subset S'(\mathbb{R})$.

To construct the Canonical Lévy white noise process on the $\Omega = S'(\mathbb{R})$, we only need to appeal to the Bochner-Minlos theorem which is stated as follows:

Theorem 3.1. *A necessary and sufficient condition for the existence of a probability measure P on $S'(\mathbb{R})$ such that*

$$g(\phi) = E[e^{i<\omega,\phi>}] = \int_{S'(\mathbb{R})} e^{i<\omega,\phi>} dP(\omega) \tag{3.3}$$

is defined on $S'(\mathbb{R})$, is that g satisfies the following conditions:

a.) $g(0) = 1$,
b.) *g is positive definite*
c.) *g is continuous in the Fréchet Topology.*

In our construction, we let

$$g(\phi) = \exp\left(\int_{\mathbb{R}} \Psi(\phi(y))dy\right), \quad \Psi(u) = -\frac{\sigma^2 u^2}{2} + \int_{\mathbb{R}_0} (e^{iuz} - iuz - 1)\nu(dz). \tag{3.4}$$

We can express g as the product $g(\phi) = f(\phi)h(\phi)$ where

$$f(\phi) = \exp\left(-\frac{\sigma^2}{2}\int_{\mathbb{R}} |\phi(y)|^2 dy\right),$$
$$h(\phi) = \exp\left(\int_{\mathbb{R}}\int_{\mathbb{R}_0} (e^{i\phi(y)z} - i\phi(y)z - 1)\nu(dz)dy\right). \tag{3.5}$$

Then, f and h satisfies the Bochner-Minlos theorem corresponding to the Wiener and the compensated Poisson case respectively [11]. Clearly, g satisfies conditions (a) and (c) of the theorem. It is suffice to check (b) to prove that for g defined above, P as in the Bochner-Minlos theorem exists. Define the following $n \times n$ matrices: $G_n = \{g(\phi_i - \phi_j)\}_{ij}$, $F_n = \{f(\phi_i - \phi_j)\}_{ij}$, $H_n = \{h(\phi_i - \phi_j)\}_{ij}$, then $G_n = F_n \odot H_n$ where $\odot$ denotes the Hadamard product. Since f and h are positive definite, so does the matrices F_n and H_n is also positive definite for all $n \in \mathbb{N}$. By the Schur's product theorem [12], G_n is positive definite. Since this holds for all $n \in \mathbb{N}$, then g is positive definite. The functional g is thus as in the Bochner-Minlos theorem, and P therein exists.

By taking $\phi(y) = t\varphi(y)$ with $t \in \mathbb{R}$ fixed, then we obtain

$$E[e^{it<\omega,\varphi>}]$$
$$= \exp\left(-\frac{\sigma^2 t^2}{2}\int_{\mathbb{R}} |\varphi(y)|^2 dy + \int_{\mathbb{R}}\int_{\mathbb{R}_0} (\exp(itz\varphi(y)) - itz\varphi(y) - 1)\nu(dz)dy\right). \tag{3.6}$$

Let $\varphi \in S(\mathbb{R})$, then we claim that

$$E[<\omega,\varphi>] = 0, \qquad E[<\omega,\varphi>^2] = \zeta\int_{\mathbb{R}} \varphi(y)dy, \tag{3.7}$$

where

$$\zeta = \sigma^2 + \int_{\mathbb{R}_0} z^2\nu(dz). \tag{3.8}$$

By a density argument, it suffice to show the identity for $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$. Let the Lévy density $\nu \in [-r,r]\setminus\{0\}$ for some $r > 0$. Then by expanding the terms in (3.6) in a Taylor series, we obtain:

$$\sum_{n=0}^{\infty}\frac{i^n t^n}{n!}E[<\omega,\varphi>^n]$$
$$= \sum_{n=0}^{\infty}\frac{1}{n!}\left(\int_{\mathbb{R}}\left(-\frac{\sigma^2 t^2\varphi^2(y)}{2} + \int_{\mathbb{R}_0}\sum_{k=2}^{\infty}\frac{i^k t^k z^k \varphi^k(y)}{k!}\nu(dz)\right)dy\right)^n. \tag{3.9}$$

Collecting the t and t^2 coefficients asserts our claim.

We extend the definition of $<\omega,\phi>$ from $\phi \in \mathcal{S}(\mathbb{R})$ to $L^2(\mathbb{R})$. Since $\mathcal{S}(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, then for $\varphi \in L^2(\mathbb{R})$ arbitrary, there exists $\varphi_n \in \mathcal{S}(\mathbb{R})$ such that $\varphi_n \to \varphi$ in $L^2(\mathbb{R})$. By completeness of $L^2(\mathbb{R})$, as $m, n \to \infty$

$$|<\omega,\varphi_n> - <\omega,\varphi_m>| = |<\omega,\varphi_n - \varphi_m>| \to 0. \tag{3.10}$$

Hence, $\{<\omega,\varphi_n>: n\in\mathbb{N}\}$ is a Cauchy sequence in $\mathbb{R}$ and its limit is $<\omega,\varphi>$. Then, define $\tilde{X}(t,\omega)\equiv<\omega,\chi_{[0,t]}>$ where $\chi_{[0,t]}\in L^2(\mathbb{R})$ is given as follows:

$$\chi_{[0,t]}=\begin{cases}1, & s\in[0,t], t\geq 0\\ -1, & s\in[-t,0), t<0\\ 0, & otherwise.\end{cases} \tag{3.11}$$

Computing the characteristic function of $\tilde{X}(t)$ yields

$$\begin{aligned}&E[\exp(iu\tilde{X}(t))]=E[\exp(iu<\omega,\chi_{[0,t]}>)]\\ &=\exp\left[\int_{\mathbb{R}}\left(-\frac{\sigma^2u^2\chi^2_{[0,t]}(y)}{2}+\int_{\mathbb{R}_0}\left(e^{iuz}\chi_{[0,t]}(y))-iuz\chi_{[0,t]}(y)-1\right)\nu(dz)\right)dy\right]\\ &=\exp\left[\left(-\frac{\sigma^2u^2}{2}+\int_{\mathbb{R}_0}\left(e^{iuz}-iuz-1\right)\nu(dz)\right)t\right].\end{aligned} \tag{3.12}$$

By the Lévy-Khinchine theorem, $\tilde{X}(t)$ is a Lévy process and there exists a càdlàg modification of $\tilde{X}(t)$, say $X(t)$ which is a Lévy process [3]. The smoothed white noise process for the Canonical Lévy process is given by:

$$<\omega,\phi>=\int_{\mathbb{R}}\phi(t)dX(t,\omega),\qquad \omega\in\Omega,\qquad \phi\in L^2(\mathbb{R}), \tag{3.13}$$

where $X(t)$ has the following representation:

$$X(t)=\sigma\int_0^t dW(t)+\int_{[0,t]\times\mathbb{R}_0}z\tilde{N}(ds,dz). \tag{3.14}$$

We define the filtered probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},P)$ for the white noise Canonical Lévy process where $\mathcal{F}=\mathfrak{B}(\mathcal{S}'(\mathbb{R}))$ and $\mathcal{F}_t=\mathcal{F}_t^X\vee\mathcal{N}$ where $\mathcal{F}_t^X=\sigma\{X(s):s\in[0,t]\}$ is the σ-field generated by X up to time t and $\mathcal{N}$ is the collection of P-null sets.

4. Construction of Alternative Chaos Expansion for Canonical Lévy Processes

We assume that the Lévy measure ν satisfies the so-called Nualart-Schoutens assumption [16]: for all $\varepsilon>0$ there exists $\lambda>0$ such that

$$\int_{\mathbb{R}_0\setminus(-\varepsilon,\varepsilon)}\exp(\lambda|z|)\nu(dz)<\infty. \tag{4.1}$$

This assumption has the following implications:

(1) The absolute moments of order greater than or equal to 2 with respect to ν are finite, that is, for all $p\geq 2$, $\int_{\mathbb{R}_0}|z|^p\nu(dz)<\infty$ and thus, $X(t)$ has moments of all orders $p\geq 2$.

(2) The characteristic function $E[\exp(iuX(t))]$ is analytic in the neighborhood of zero and the polynomials are dense in $L^2(\mathbb{R},P\circ X(t)^{-1})$.

We denote the power jump process $X^{(i)} = \{X^{(i)}(t) : t \geq 0\}$, $i \in \mathbb{N}$ defined as

$$X^{(i)}(t) = \begin{cases} \sum_{s\in(0,t]} \Delta(X(t))^i, & i > 1, \\ X(t), & i = 1, \end{cases} \tag{4.2}$$

and the compensated power jump process $Y^{(i)} = \{Y^{(i)}(t) : t \geq 0\}, i \in \mathbb{N}$ by $Y^{(i)}(t) = X^{(i)}(t) - E\left[X^{(i)}(t)\right]$. The process $Y^{(i)}$ is referred to as the Teugels martingale of order i, and it is a normal martingale.

Let S_1 be the space of real polynomials in $\mathbb{R}_+$, that is,

$$S_1 = \left\{ \sum_{k=0}^{n} c_k z^{k-1} : c_k \in \mathbb{R}, z \in \mathbb{R}_+, k \in \{1, \cdots n\}, n \in \mathbb{N} \right\} \tag{4.3}$$

endowed with the inner product $<< \cdot, \cdot >>_1$ given by

$$<< P, Q >>_1 = \sigma^2 P(0)Q(0) + \int_{\mathbb{R}_0} P(z)Q(z)z^2 \nu(dz) = < P, Q >_{L^2(\eta)}, \tag{4.4}$$

where $P, Q \in S_1$. Let $\{p_i(z)\}_{i\in\mathbb{N}}$ be the orthogonalization of $\{1, z, z^2, \cdots\}$ in S_1. From the Gram-Schmidt orthogonality procedure, we have the following:

$$p_i(z) = \sum_{j=1}^{i} a_{ij} z^{j-1}, \tag{4.5}$$

where

$$a_{ij} = \begin{cases} -\dfrac{<< p_j(z), z^{i-1} >>_1}{\|p_j(z)\|_1^2} = \dfrac{\int_{\mathbb{R}} p_j(z) z^{i-1} \eta(dz)}{\int_{\mathbb{R}} p_j^2(z)\eta(dz)}, & j \in \{1, \cdots, i-1\}, \\ 1, & j = i. \end{cases} \tag{4.6}$$

Likewise, we have the following orthogonality relation [15]

$$< p_i(z), p_j(z) >_{L^2(\eta)} = \int_{\mathbb{R}} p_i(z) p_j(z) \eta(dz) = q_i \delta_{ij}, \tag{4.7}$$

where $q_i = \|p_i\|_{L^2(\eta)}^2 = \sigma^2 + \sum_{k=1}^{i} \sum_{l=1}^{i} a_{ik}^* a_{il}^* m_{k+l}$. On the other hand, let S_2 be the space of linear transformations of Teugels martingales of the Lévy Processes, that is,

$$S_2 = \left\{ \sum_{k=0}^{n} c_k Y^{(k)} : c_k \in \mathbb{R}, k \in \{1, \cdots n\}, n \in \mathbb{N} \right\} \tag{4.8}$$

endowed with the inner product $<< \cdot, \cdot >>_2$ given by

$$<< Y^{(i)}, Y^{(j)} >>_2 = E[[Y^{(i)}, Y^{(j)}]_1] = \sigma^2 \mathbf{1}_{\{i=j=1\}} + m_{i+j}. \tag{4.9}$$

Let $\{H^{(i)}\}_{i\in\mathbb{N}}$ be the orthogonalization of $\{Y^{(1)}, Y^{(2)}, Y^{(3)}, \cdots\}$ in S_2. Then, $\{H^{(i)}\}_{i\in\mathbb{N}}$ are strongly orthogonal martingales. From the Gram-Schmidt orthogonality procedure, we have the following:

$$H^{(i)} = \sum_{j=1}^{i-1} a_{ij}^* Y^{(i)}, \tag{4.10}$$

where

$$a^*_{ij} = \begin{cases} -\frac{<< H^{(j)}, Y^{(i)} >>_2}{||H^{(j)}||_2^2} = -\frac{E[[H^{(j)}, Y^{(i)}]_1]}{E[[H^{(j)}]_1]}, & j \in \{1, \cdots, i-1\} \\ 1, & j = i. \end{cases} \tag{4.11}$$

Then, $a_{ij} = a^*_{ij}$, $j \in \{1, \cdots, i\}, i \in \mathbb{N}$, and therefore $x^{i-1} \leftrightarrow Y^{(i)}$ is an isometry between S_1 and S_2 [16]. Likewise, we have the following quadratic covariation process:

$$< H^{(i)}, H^{(j)} >_t = q_i t \delta_{ij}. \tag{4.12}$$

Denote the following multiple integral for $f \in L^2([0,T]^n)$ with respect to the orthogonal martinagles $H^{(i)}$'s:

$$\begin{aligned} &J_n^{(i_1,\cdots,i_n)}(f) \\ &= \int_0^T \int_0^{t_n^-} \cdots \int_0^{t_2^-} f(t_1, \cdots t_{n-1}, t_n) dH^{(i_1)}(t_1) \cdots dH^{(i_{n-1})}(t_{n-1}) dH^{(i_n)}(t_n). \end{aligned} \tag{4.13}$$

Leon et al. [15] have shown an orthogonality property between different multi-indices $(i_1, \cdots, i_n)$ stated as follows for $f \in L^2([0,T]^n)$ and $g \in L^2([0,T]^m)$:

$$E[J_n^{(i_1,\cdots,i_n)}(f) J_n^{(j_1,\cdots,j_m)}(g)] = q_{i_1} \cdots q_{i_n} \int_{\Sigma_n} f(t_1, \cdots, t_n) g(t_1, \cdots, t_n) dt_1 \cdots dt_n, \tag{4.14}$$

where $(i_1, \cdots, i_n) = (j_1, \cdots, j_n), m = n$ and 0 otherwise and $\Sigma_n = \{(t_1, \cdots, t_n) : 0 < t_1 < \cdots t_n \le T\}$ is the positive simplex of $[0,T]^n$.

Nualart and Schoutens [16] have shown that every $F \in L^2(P)$ can be represented in terms of the iterated integrals with respect to the $H^{(i)}$'s.

Theorem 4.1. *Chaotic Representation Property (CRP): Every random variable $F \in L^2(P)$ has a representation of the form of*

$$F = E[F] + \sum_{n=1}^{\infty} \sum_{j_1,\cdots,j_n \ge 1} J_n^{(j_1,\cdots,j_n)}(f_{j_1,\cdots,j_n}) \tag{4.15}$$

As a corollary to the CRP, Nualart and Schoutens [16] have shown a predictable representation in terms of in terms of $H^{(i)}$.

Corollary 4.2. *Predictable Representation Property (PRP)*
Every random variable $F \in L^2(P)$ has a representation of the form of

$$F = E[F] + \sum_{n=1}^{\infty} \int_0^T \phi^{(n)}(s) dH^{(n)}(s), \tag{4.16}$$

where $\phi^{(j)}(s)$ is a predictable process.

We present some important results, all based in Solé et al. [23], which are crucial in finding the alternative chaos expansion for the Canonical Lévy space.

Theorem 4.3. (i) *Let $g = \{g(t) : t \in [0,T]\}$ be a predictable process such that*

$$E\left[\int_0^T g^2(t)dt\right] < \infty. \tag{4.17}$$

Then, $g(t)p_i(x)$ is integrable with respect to M and

$$\int_0^T g(t)dH^{(i)}(t) = \int_{[0,T]\times\mathbb{R}} g(t)p_i(x)M(dt,dx). \tag{4.18}$$

Also, $H^{(i)}(t)$ can be expressed as follows:

$$H^{(i)}(t) = \int_{[0,T]\times\mathbb{R}} p_i(z)M(dt,dz). \tag{4.19}$$

(ii) *Let $f \in L^2([0,T]^n)$, then*

$$J_n^{(j_1,\cdots,j_n)}(f) = I_n(f(t_1,\cdots,t_n)\mathbf{1}_{\Sigma_n}(t_1,\cdots,t_n)p_{j_1}(z_1)\cdots p_{j_n}(z_n)). \tag{4.20}$$

We follow the approach in Benth et al. [4] in comparing the relationship between Itô's chaos expansion and the CRP. Their approach was limited to chaos expansion with respect to the iterated integral of the compensated Poisson random measure $\tilde{N}$. With their result, Di Nunno was able to derive the alternative expansion in the Poisson case [7]. With the results of the preceding theorem, we are able to establish an alternative chaos expansion for the general Canonical Lévy space.

From Nualart-Schoutens CRP and from the previous theorem, we obtain

$$F-E[F] = \sum_{n=1}^{\infty} I_n\left(\sum_{j_1,\cdots,j_n\geq 1} f_{j_1,\cdots,j_n}(t_1,\cdots,t_n)p_{j_1}(z_1)\cdots p_{j_n}(z_n)\mathbf{1}_{\Sigma_n}(t_1,\cdots,t_n)\right). \tag{4.21}$$

We let

$$g_n((t_1,z_1),\cdots(t_n,z_n)) = \sum_{j_1,\cdots,j_n\geq 1} f_{j_1,\cdots,j_n}(t_1,\cdots,t_n)p_{j_1}(z_1)\cdots p_{j_n}(z_n)\mathbf{1}_{\Sigma_n}(t_1,\cdots,t_n). \tag{4.22}$$

Then, from the uniqueness of Ito's chaos expansion for the Canonical Lévy process, we obtain

$$f_n = g_n^{\wedge}, \qquad \forall n \in \mathbb{N}. \tag{4.23}$$

We define Hermite functions $\{e_n\}_{n\in\mathbb{N}}$ as

$$e_n(x) = \pi^{-1/4}((n-1)!)^{-1/2}e^{-x^2/2}h_{n-1}(\sqrt{2}x), \qquad n \in \mathbb{N}, \tag{4.24}$$

where $\{h_n\}_{n\in\mathbb{N}_0}$ are the classical Hermite polynomials given by

$$h_n(x) = (-1)^n e^{x^2/2}\frac{d^n}{dx^n}e^{-x^2/2}. \tag{4.25}$$

Then $\{e_n\}_{n\in\mathbb{N}}$ forms an orthonormal basis in $L^2(\lambda)$ [11]. Since

$$f_{j_1,\cdots,j_n}(t_1,\cdots,t_n)\mathbf{1}_{\Sigma_n}(t_1,\cdots,t_n) \in L^2(\lambda^n),$$

then we have following orthonormal expansion

$$f_{j_1,\cdots,j_n}(t_1,\cdots,t_n) = \sum_{i_1,\cdots,i_n\geq 1} \gamma^{(j_1,\cdots,j_n)}_{i_1,\cdots,i_n} e_{i_1}(t_1)\cdots e_{i_n}(t_n). \tag{4.26}$$

We let

$$\pi_i(z) = \frac{p_i(z)}{\|p_i\|_{L^2(\eta)}}, \qquad i \in \mathbb{N} \tag{4.27}$$

then, from (4.7), $\{\pi_i\}_{i\in\mathbb{N}}$ are orthonormal basis functions in $L^2(\eta)$. Denote

$$c^{(j_1,\cdots,j_n)}_{i_1,\cdots,i_n} = \|p_{i_1}\|_{L^2(\eta)}\cdots\|p_{i_n}\|_{L^2(\eta)}\,\gamma^{(j_1,\cdots,j_n)}_{i_1,\cdots,i_n}. \tag{4.28}$$

Hence, we can express (4.22) in terms of orthonormal basis functions in $L^2(\mu^n)$ as follows:

$$\begin{aligned} &g_n((t_1,z_1),\cdots,(t_n,z_n)) \\ &= \sum_{i_1,\cdots,i_n\geq 1}\sum_{j_1,\cdots,j_n\geq 1} c^{(j_1,\cdots,j_n)}_{i_1,\cdots,i_n} e_{i_1}(t_1)\pi_{j_1}(z_1)\cdots e_{i_n}(t_n)\pi_{j_n}(z_n). \end{aligned} \tag{4.29}$$

Since the symmetrization operator is linear, then

$$\begin{aligned} &g_n^{\wedge}((t_1,z_1),\cdots(t_n,z_n)) \\ &= \sum_{i_1,\cdots,i_n\geq 1}\sum_{j_1,\cdots,j_n\geq 1} c^{(j_1,\cdots,j_n)}_{i_1,\cdots,i_n} \left(e_{i_1}(t_1)\pi_{j_1}(z_1)\cdots e_{i_n}(t_n)\pi_{j_n}(z_n)\right)^{\wedge}. \end{aligned} \tag{4.30}$$

We want to express (4.30) in terms of orthogonal functions in $L^2_s(\mu^n)$. Denote the Cantor diagonalization mapping $\kappa : \mathbb{N}\times\mathbb{N}$ by $\kappa(i,j) = j + \frac{(i+j-2)(i+j-1)}{2}$. Let $k = \kappa(i,j)$ and

$$\delta_k(t,z) = e_i(t)\pi_j(z), \tag{4.31}$$

then $\{\delta_k\}_{k\in\mathbb{N}}$ is an orthonormal basis in $L^2(\mu)$. Then from (4.30) and (4.31)

$$\begin{aligned} &g_n^{\wedge}((t_1,z_1),\cdots,(t_n,z_n)) \\ &= \sum_{i_1,\cdots,i_n\geq 1}\sum_{j_1,\cdots,j_n\geq 1} c^{(j_1,\cdots,j_n)}_{i_1,\cdots,i_n} \left(\delta_{\kappa(i_1,j_1)}(t_1,z_1)\cdots\delta_{\kappa(i_n,j_n)}(t_n,z_n)\right)^{\wedge}. \end{aligned} \tag{4.32}$$

Denote the following multi-indices given by $\alpha = (\alpha_1,\alpha_2,\cdots), \alpha_i \in \mathbb{N}_0, i \in \mathbb{N}$ with compact support and $\mathcal{I}$ by the set of all such α. Also, we denote the following: $Index(\alpha) = max\{i : \alpha_i \neq 0\}$, $|\alpha| = \sum_{i=1}^m \alpha_i$, $\alpha! = \prod_{i=1}^m \alpha_i, m = Index(\alpha)$. Suppose that $m = Index(\alpha)$ and $n = |\alpha|$, define the following tensor product as:

$$\begin{aligned} &\delta^{\otimes\alpha}((t_1,z_1)\cdots(t_n,z_n)) \\ =&\delta_1^{\otimes\alpha_1}\otimes\cdots\otimes\delta_m^{\otimes\alpha_m}((t_1,z_1)\cdots(t_n,z_n)) \\ =&\delta_1(t_1,z_1)\cdots\delta_1(t_{\alpha_1},z_{\alpha_1})\delta_2(t_{\alpha_1+1},z_{\alpha_1+1})\cdots\delta_2(t_{\alpha_1+\alpha_2},z_{\alpha_1+\alpha_2}) \\ &\cdots \qquad \delta_m(t_{n-\alpha_m+1},z_{n-\alpha_m+1})\cdots\delta_m(t_n,x_n) \end{aligned} \tag{4.33}$$

with the convention $\delta_i^{\otimes 0} = 1$, $i \in \{1, \cdots, m\}$. Also, we denote the symmetrized tensor product as follows:

$$\begin{aligned}\delta^{\hat{\otimes}\alpha}((t_1, z_1)\cdots(t_n, z_n)) = &\left(\delta^{\otimes\alpha}((t_1, z_1)\cdots(t_n, z_n))\right)^{\wedge}\\ =&\delta_1^{\hat{\otimes}\alpha_1}\hat{\otimes}\cdots\hat{\otimes}\delta_m^{\hat{\otimes}\alpha_m}((t_1, z_1)\cdots(t_n, z_n)).\end{aligned} \tag{4.34}$$

Now, since $g_n^{\wedge} \in \overline{span\{\delta^{\hat{\otimes}\alpha} : |\alpha| = n, \alpha \in \mathcal{I}\}}$ then $g_n^{\wedge}$ has of the form

$$f_n = g_n^{\wedge} = \sum_{|\alpha|=n} c_\alpha \delta^{\hat{\otimes}\alpha}. \tag{4.35}$$

By taking $I_0(\delta^{\hat{\otimes}\alpha}) = 1$ and $c_0 = E[F]$, then from (2.13), (4.23), and (4.35) we obtain

$$F = \sum_{\alpha\in\mathcal{I}} c_\alpha I_{|\alpha|}\left(\delta^{\hat{\otimes}\alpha}\right). \tag{4.36}$$

We denote

$$\mathbb{K}_\alpha = I_{|\alpha|}\left(\delta^{\hat{\otimes}\alpha}\right) \tag{4.37}$$

then we have the following chaos expansion.

Proposition 4.4. *Let $F \in L^2(P)$, then it has a unique chaos expansion of the form*

$$F = \sum_{\alpha\in\mathcal{I}} c_\alpha \mathbb{K}_\alpha. \tag{4.38}$$

Proposition 4.5. *(Isometry) For $F \in L^2(P)$ with a chaos expansion of the form* (4.38)*, we have*

$$\|F\|^2_{L^2(P)} = \sum_{\alpha\in\mathcal{I}} c_\alpha^2 \alpha!. \tag{4.39}$$

Proof. We let $m_\alpha = Index(\alpha)$, $m_\beta = Index(\beta)$, $n_\alpha = |\alpha|$, $n_\beta = |\beta|$. Then, by isometry,

$$\begin{aligned}E[\mathbb{K}_\alpha\mathbb{K}_\beta] =&E\left[I_{n_\alpha}\left(\delta^{\hat{\otimes}\alpha}\right) I_{n_\beta}\left(\delta^{\hat{\otimes}\beta}\right)\right]\\ =&n_\alpha! \int_{([0,T]\times\mathbb{R})^{n_\alpha}} \delta^{\hat{\otimes}\alpha}\delta^{\hat{\otimes}\beta} d\mu^{\otimes n}\delta_{n_\alpha n_\beta}.\end{aligned} \tag{4.40}$$

For $n_\alpha \neq n_\beta$, the (4.40) vanishes. Throughout the remainder of the proof, it suffice to evaluate for the case $n = n_\alpha = n_\beta$. Denote $m = m_\alpha$, and consider the tensor product in (4.33). There are $n!$ terms in the symmetrization of $\delta^{\hat{\otimes}\alpha}$ as well as of $\delta^{\hat{\otimes}\beta}$ while each term of these symmetrized tensor product has a factor of $1/n!$. Since $\{\delta_k\}_{k\in\mathbb{N}}$ forms an orthonomal basis in $L^2(\mu)$, then for $\alpha \neq \beta$, (4.40) vanishes.

Consider the case $\alpha = \beta$, for each $n!$ terms in $\mathbb{K}_\alpha$, one can get a non-zero expectation term with a product on a term in $\mathbb{K}_\beta$ by permuting the terms in (4.33) by permuting the first α_1 terms, then permuting the next α_2 terms, and so forth and finally, permuting the last α_m terms. There are $\alpha! = \alpha_1!\cdots\alpha_m!$ possible combinations in this procedure each with the weight of one by orthonormality of $\delta_{k\{k\in\mathbb{N}\}}$ in $L^2(\mu)$. Thus, we obtain

$$E[\mathbb{K}_\alpha\mathbb{K}_\beta] = n! \cdot \frac{1}{(n!)^2} \cdot n! \cdot \alpha! \cdot \mathbf{1}_{\{\alpha=\beta\}} = \alpha! \cdot \mathbf{1}_{\{\alpha=\beta\}}, \tag{4.41}$$

and therefore,

$$\|F\|^2_{L^2(P)} = \sum_{\alpha\in\mathcal{I}}\sum_{\beta\in\mathcal{I}} c_\alpha c_\beta E[\mathbb{K}_\alpha\mathbb{K}_\beta] = \sum_{\alpha\in\mathcal{I}} c_\alpha^2 \alpha!. \tag{4.42}$$

□

5. White Noise Analysis

Consider the following formal expansion

$$F = \sum_{\alpha\in\mathcal{I}} c_\alpha \mathbb{K}_\alpha. \tag{5.1}$$

If the following growth condition holds,

$$\sum_{\alpha\in\mathcal{I}} c_\alpha^2 \alpha! < \infty, \tag{5.2}$$

then $F \in L^2(P)$. We relax this growth condition to obtain a family of generalized function spaces of stochastic test functions and stochastic distribution functions which relate to $L^2(P)$ naturally [11].

5.1. The spaces $\mathcal{G}$ and $\mathcal{G}^*$. The spaces of stochastic test functions $\mathcal{G}$ and of stochastic distribution functions $\mathcal{G}^*$ were first investigated by Pothoff and Timpel in the Wiener case [20]. A parallel definition was carried out by Di Nunno [7] in the Poisson case. We extend these definitions for the Canonical Lévy space.

Definition 5.1. Let $\mathcal{G}_q$, $q \in \mathbb{R}$ be the space of formal expansion

$$F = \sum_{n\in\mathbb{N}_0} I_n(f_n) \tag{5.3}$$

such that

$$\|F\|_{\mathcal{G}_q} = \left(\sum_{n\in\mathbb{N}_0} n! \, \|f_n\|^2_{L^2(\mu^n)} \, e^{2qn}\right)^{1/2} < \infty. \tag{5.4}$$

For every $q \in \mathbb{R}$, $\mathcal{G}_q$ is a Hilbert space with inner product

$$< F, G >_{\mathcal{G}_r} = \sum_{n\in\mathbb{N}_0} n! < f_n, g_n >_{L^2(\mu^n)} e^{2qn}, \tag{5.5}$$

where F and G have the following formal sum:

$$F = \sum_{n\in\mathbb{N}_0} I_n(f_n), \qquad G = \sum_{n\in\mathbb{N}_0} I_n(g_n). \tag{5.6}$$

We define the space of *stochastic test functions* $\mathcal{G}$ as $\mathcal{G} = \bigcap_{q>0} \mathcal{G}_q$ endowed with the projective topology and the stochastic distribution function $\mathcal{G}^*$ as $\mathcal{G}^* = \bigcup_{q>0} \mathcal{G}_{-q}$ endowed with the inductive topology.

Note that $\mathcal{G}^*$ is a dual of $\mathcal{G}$. Let $F \in \mathcal{G}$ and $G \in \mathcal{G}^*$ with the formal expansion of F and G of the form (5.6). The action of G on F is given by:

$$< G, F >_{\mathcal{G},\mathcal{G}^*} = \sum_{n\in\mathbb{N}_0} n! < f_n, g_n >_{L^2(\mu^n)} . \tag{5.7}$$

Also, we can express the $\mathcal{G}_q$-norm, $q \in \mathbb{R}$ in terms of the chaos expansion (4.38) as follows:

$$\|F\|^2_{\mathcal{G}_q} = \sum_{\alpha\in\mathcal{I}} c_\alpha \alpha! e^{2qn}. \tag{5.8}$$

5.2. Kondratiev and Hida spaces. We let $\alpha \in \mathcal{I}$ and suppose that $Index(\alpha) = m$, then we denote $(2\mathbb{N})^{\alpha k} = \prod_{j=1}^m (2j)^{\alpha_j k}$ where $k \in \mathbb{Z}$. In particular, if $\alpha = \varepsilon^{(m)} = (0, \cdots, 0, 1, 0, \cdots)$, that is, $\varepsilon^{(m)}$ is a multi-index with all zeros except for the m-th component which contains one, then $(2\mathbb{N})^{\varepsilon^{(m)}k} = (2m)^k$.

Definition 5.2. Let $p \in [0, 1]$. Suppose that F has a formal expansion of the form (5.6). Then, F belongs to the space $(\mathcal{S})_{p,q}$, $q \in \mathbb{R}$ if

$$\|F\|^2_{p,q} = \sum_{\alpha\in\mathcal{I}} a^2_\alpha (\alpha!)^{1+p} (2\mathbb{N})^{\alpha q} < \infty. \tag{5.9}$$

The *Kondratiev test function* $(\mathcal{S})_p$ is defined as $(\mathcal{S})_p = \bigcap_{q>0} (\mathcal{S})_{p,q}$ endowed with the projective topology. Suppose that G has a formal expansion of the form (5.6). Then, G belongs to the space $(\mathcal{S})_{-p,-q}$, $q \in \mathbb{R}$ if

$$\|G\|^2_{-p,-q} = \sum_{\alpha\in\mathcal{I}} b^2_\alpha (\alpha!)^{1-p} (2\mathbb{N})^{-\alpha q} < \infty. \tag{5.10}$$

The space defined as $(\mathcal{S})_{-p} = \bigcup_{q>0} (\mathcal{S})_{-p,-q}$ endowed with the inductive topology is known as the space of *Kondratiev distribution functions*.

Note that $(\mathcal{S})_{-p}$ is a dual of $(\mathcal{S})_p$. The action of $G \in (\mathcal{S})_{-p}$ on $F \in (\mathcal{S})_p$, with the formal expansion of F and G of the form (5.6) is given by

$$< G, F > = \sum_{\alpha\in\mathcal{I}} \alpha! a_\alpha b_\alpha. \tag{5.11}$$

The Hida spaces are the special cases of the Kondratiev spaces. The Hida test functions $(\mathcal{S})$ and Hida distribution functions $(\mathcal{S})^*$ are given by $(\mathcal{S}) = (\mathcal{S})_0$ and $(\mathcal{S})^* = (\mathcal{S})_{-0}$ respectively. From the above definitions, we have the following inclusions for $p \in [0, 1]$:

$$(\mathcal{S})_1 \subset (\mathcal{S})_p \subset (\mathcal{S})_0 \subset \mathcal{G} \subset L^2(P) \subset \mathcal{G}^* \subset (\mathcal{S})_{-0} \subset (\mathcal{S})_{-p} \subset (\mathcal{S})_{-1}. \tag{5.12}$$

5.3. White Noise Lévy Processes and Lévy White Noise Field. We extend the concept of white noise processes in the Canonical Lévy space. Consider the chaos expansion of $X(t)$ in (2.4)

$$\begin{aligned} X(t) =& I_1(1) = I_1\left(\sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}} \langle 1, e_i\rangle_{L^2(\lambda)} \langle 1, \pi_j\rangle_{L^2(\nu)} e_i(s)\pi_j(z)\right) \\ =& \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}} \int_0^t e_i(s)ds \int_{\mathbb{R}} \pi_j(s)\eta(dz) I_1\left(e_i(s)\pi_j(z)\right). \end{aligned} \tag{5.13}$$

Now since

$$\mathbb{K}_{\varepsilon^{\kappa(i,j)}} = I_1\left(\delta^{\otimes\varepsilon^{\kappa(i,j)}}\right) = I_1\left(e_i(s)\pi_j(z)\right) = \int_0^t \int_{\mathbb{R}} e_i(s)\pi_j(z) ds\eta(dz) \tag{5.14}$$

and $\{\pi_j\}_{j\in\mathbb{N}}$ is orthonormal with respect to $L^2(\eta)$, hence

$$X(t) = \zeta \sum_{i\in\mathbb{N}} \int_0^t e_i(s)ds\mathbb{K}_{\varepsilon^{\kappa(i,1)}}, \tag{5.15}$$

where ζ is given by (3.8).

Definition 5.3. The *white noise Lévy process* $\dot{X}(t)$ is defined by

$$\dot{X}(t) = \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}} e_i(t)\int_{\mathbb{R}} \pi_j(z)\eta(dz)\mathbb{K}_{\varepsilon^{\kappa(i,j)}} = \zeta\sum_{i\in\mathbb{N}} e_i(t)\mathbb{K}_{\varepsilon^{\kappa(i,1)}}. \tag{5.16}$$

Proposition 5.4. *Characterization of* $\dot{X}(t)$

(i) $\dot{X}(t) \in (\mathcal{S})^*$
(ii) $\dot{X}(t) = \frac{dX(t)}{dt}$ *in* $(\mathcal{S})^*$

Proof. The proof is similar to that in [8].

(i) Since $\kappa(i,1) \geq i$ and $\sup_{t\in\mathbb{R}} |e_n(t)| = O(n^{-1/12})$ [11], then,

$$\begin{aligned} \left\|\dot{X}(t)\right\|_{-q}^2 =& \zeta^2 \sum_{i\in\mathbb{N}} \varepsilon^{\kappa(i,1)}! e_i^2(t)(2\mathbb{N})^{-\varepsilon^{\kappa(i,1)}q} \\ =& \zeta^2 \sum_{i\in\mathbb{N}} e_i^2(t)(2\kappa(i,1))^{-q} \\ \leq& \zeta^2 \sum_{i\in\mathbb{N}} e_i^2(t)(2i)^{-q}. \end{aligned} \tag{5.17}$$

The series converges for $q \geq 2$ and thus proves our claim.

(ii) Note that from (5.15) and (5.16), we have the following:

$$\begin{aligned} \frac{X(t+h)-X(t)}{h} - \dot{X}(t) =& \zeta \sum_{i\in\mathbb{N}} \frac{1}{h}\int_t^{t+h} (e_i(s)-e_i(t))ds\mathbb{K}_{\varepsilon^{\kappa(i,1)}} \\ =& \zeta \sum_{i\in\mathbb{N}} a_i(h)\mathbb{K}_{\varepsilon^{\kappa(i,1)}}, \end{aligned} \tag{5.18}$$

where

$$a_i(h) = \frac{1}{h}\int_t^{t+h} (e_i(s)-e_i(t))ds. \tag{5.19}$$

Since $\sup_{t\in\mathbb{R}} |e_n(t)| = O(n^{-1/12})$ then, $\sup_{i\in\mathbb{N}} |a_i(h)| < \infty$ for all $h \in [0,1]$. Furthermore, since $\kappa(i,1) \geq 1$ then,

$$\begin{aligned}\left\|\frac{X(t+h)-X(t)}{h} - \dot{X}(t)\right\|_{-q}^2 =&\zeta^2 \sum_{i\in\mathbb{N}} \varepsilon^{(i,1)}!|a_i(h)|^2(2\kappa(i,1))^{-q} \\ \leq&\zeta^2 \sum_{i\in\mathbb{N}} |a_i(h)|^2(2i)^{-q}. \end{aligned} \tag{5.20}$$

Now, since $a_i(h) \to 0$ as $h \to 0$ for all $i \in \mathbb{N}$ then, for all $q \geq 2$ from the dominated convergence theorem,

$$\sum_{i\in\mathbb{N}} |a_i(h)|^2(2i)^{-q} \to 0 \quad \text{as} \quad h \to 0. \tag{5.21}$$

From the bounded convergence theorem, we obtain

$$\left\|\frac{X(t+h)-X(t)}{h} - \dot{X}(t)\right\|_{-q}^2 \to 0 \quad \text{as} \quad h \to 0. \tag{5.22}$$

□

Definition 5.5. The *Lévy white noise field* $\dot{M}(t,z)$ is defined by

$$\dot{M}(t,z) = \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}} e_i(t)\pi_j(z)\mathbb{K}_{\varepsilon^{\kappa(i,j)}}. \tag{5.23}$$

Lemma 5.6. *$\dot{M}(t,z) \in (\mathcal{S})^*$, $\mu - a.e.$*

Proof. Since

$$\left\|\dot{M}(t,z)\right\|_{-q}^2 = \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}} e_i^2(t)\pi_j^2(z)(2\kappa(i,j))^{-q}. \tag{5.24}$$

Since $\sqrt{ij} \leq \kappa(i,j)$ and by orthonormality, $\{e_i\}_{i\in\mathbb{N}}$ and $\{\pi_j\}_{j\in\mathbb{N}}$ are orthonormal with respect to $L^2(\lambda)$ and $L^2(\eta)$ respectively. Thus

$$\begin{aligned}\int_{\mathbb{R}_+\times\mathbb{R}} \left\|\dot{M}(t,z)\right\|_{-q}^2 \mu(dt,dz) &\leq \int_{\mathbb{R}_+\times\mathbb{R}} \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}} e_i^2(t)\pi_j^2(z)(2\sqrt{ij})^{-q} dt\eta(dz) \\ &= \sum_{i\in\mathbb{N}} (\sqrt{2}i)^{-q} \int_{\mathbb{R}_+} e_i^2(t)dt \sum_{j\in\mathbb{N}} (\sqrt{2j})^{-q} \int_{\mathbb{R}} \pi_j^2(z)\eta(dz) \\ &= \sum_{i\in\mathbb{N}} (\sqrt{2}i)^{-q} \sum_{j\in\mathbb{N}} (\sqrt{2j})^{-q}. \end{aligned} \tag{5.25}$$

The above series converges for $q > 2$, thus proving our claim. □

Remark 5.7. Radon-Nikodym Interpretation of the Lévy white noise field: Let $t \in \mathbb{R}_+$ and $A \in \mathfrak{B}(\mathbb{R})$. Then

$$M(t,A) = \int_0^t \int_A M(ds,dz) = \sigma \int_0^t dW(s) + \int_0^t \int_A z\tilde{N}(ds,dz). \tag{5.26}$$

Likewise, we can express $M(t,A)$ as follows:

$$\begin{aligned} M(t,A) =& I_1(\mathbf{1}_{[0,t]}(s)\mathbf{1}_A(z)) \\ =& I_1\left(\sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}}\left\langle \mathbf{1}_{[0,t]}, e_i\right\rangle_{L^2(\lambda)}\left\langle \mathbf{1}_A, \pi_j\right\rangle_{L^2(\nu)} e_i(s)\pi_j(z)\right) \\ =& \int_0^t\int_A \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}} e_i(s)\pi_j(z)\mu(ds,dz)\mathbb{K}_{\varepsilon^{\kappa(i,j)}} \\ =& \int_0^t\int_A \dot{M}(t,z)\mu(ds,dz) \end{aligned} \tag{5.27}$$

by definition of $\dot{M}$. Hence, from (5.26) and (5.27), $\dot{M}(s,z)$ is interpreted as a Radon-Nikodym derivative in $(S)^*$ as follows:

$$M(dt,dz) = \dot{M}(t,z)\mu(dt,dz). \tag{5.28}$$

5.4. Wick Product. The Wick Product was first introduced by Wick in 1950 as a renormalization tool in quantum field theory. Its application in stochastic analysis was introduced by Hida and Ikeda in 1965 [11]. We state some of its properties which are similar to the Wiener and Poisson white noise theory.

Let $F = \sum_{\alpha\in\mathcal{I}} a_\alpha\mathbb{K}_\alpha \in (\mathcal{S})_{-1}$ and $G = \sum_{\beta\in\mathcal{I}} b_\beta\mathbb{K}_\beta \in (\mathcal{S})_{-1}$, then the Wick Product of X and Y denoted by $X\diamond Y$ is defined as

$$X\diamond Y = \sum_{\alpha\in\mathcal{I}}\sum_{\beta\in\mathcal{I}} a_\alpha b_\beta\mathbb{K}_{\alpha+\beta} = \sum_{\gamma\in\mathcal{I}}\left(\sum_{\alpha+\beta=\gamma} a_\alpha b_\beta\right)\mathbb{K}_\gamma. \tag{5.29}$$

We define the Wick powers of $X \in (\mathcal{S})_{-1}$ as follows: $X^{\diamond n} = X^{\diamond(n-1)}\diamond X, \quad n\in\mathbb{N}$, $X^{\diamond 0} = 1$. If $f:\mathbb{C}\to\mathbb{C}$ is entire, given by the Taylor series expansion $f(z) = \sum_{n=0}^\infty a_n z^n$, then, we define the following Wick version $f^\diamond(X)$, $X\in(\mathcal{S})_{-1}$ by setting $f^\diamond(X) = \sum_{n=0}^\infty a_n X^{\diamond n}$. Also, we define the the Wick exponential of $X\in(\mathcal{S})_{-1}$ denoted as $\exp^\diamond(X) = \sum_{n=0}^\infty \frac{X^{\diamond n}}{n!}$. whenever it is convergent in $(\mathcal{S})_{-1}$.

5.5. Stochastic Derivative. Consider the formal sum

$$F = \sum_{n\in\mathbb{N}_0} I_n(f_n) = \sum_{\alpha\in\mathcal{I}} c_\alpha\mathbb{K}_\alpha, \tag{5.30}$$

where

$$\mathbb{K}_\alpha = I_{|\alpha|}\left(\delta^{\hat{\otimes}\alpha}\right), \quad f_n = \sum_{|\alpha|=n} c_\alpha\delta^{\hat{\otimes}\alpha}. \tag{5.31}$$

As is well-known (see [22]), if $F\in\mathbb{D}^{1,2}$, the Malliavin derivative in $\mathbb{D}^{1,2}$ is as follows,

$$D_{t,z}F = \sum_{n=1}^\infty nI_{n-1}(f_n(.,(t,z))). \tag{5.32}$$

Let us relax the requirement of belonging to $\mathbb{D}^{1,2}$, by defining a stochastic derivative in F with the same form as (5.32) whenever $D_{t,z}F$ converges in a generalized function space. In the Wiener case, the stochastic derivative corresponds to the Hida-Malliavin derivative whenever $D_{t,0}F$ converges in $(\mathcal{S})^*$ [8].

From (5.31), we have:

$$f_n(.,(t,z)) = \sum_{|\alpha|=n} c_\alpha \delta^{\hat{\otimes}\alpha}(.,(t,z)). \tag{5.33}$$

Let $p = Index(\alpha)$, then $\alpha_i = 0$ for $i > p$ and let$\varepsilon_i = (0,\cdots,1,\cdots,0)^T$, a unit vector with a 1 in the i^{th} component and zero otherwise. Then, $\delta^{\hat{\otimes}\alpha}(.,t,z)$ can be computed as follows:

$$\delta^{\hat{\otimes}\alpha}(.,t,z) = \frac{1}{|\alpha|}\sum_{i\in\mathbb{N}} \alpha_i \delta^{\hat{\otimes}(\alpha-\varepsilon_i)}\delta^{\hat{\otimes}\varepsilon_i}(t,z). \tag{5.34}$$

Then, from (5.32), (5.33), and (5.34), we obtain the stochastic derivative

$$D_{t,z}F = \sum_{\alpha\in\mathcal{I}}\sum_{i\in\mathbb{N}} c_\alpha\alpha_i\mathbb{K}_{\alpha-\varepsilon_i}\delta^{\hat{\otimes}\varepsilon_i}(t,z). \tag{5.35}$$

Note that if $F \in \mathbb{D}^{1,2}$, then the Malliavin derivative in (5.32) and the stochastic derivative in (5.35) coincide. Since κ is bijective, then for any $i \in \mathbb{N}$, $\exists(k,m) \in \mathbb{N}\times\mathbb{N}$ such that $i = \kappa(k,m)$. Hence, we can also express (5.35) as follows:

$$D_{t,z}F = \sum_{\alpha\in\mathcal{I}}\sum_{k\in\mathbb{N}}\sum_{m\in\mathbb{N}} c_\alpha\alpha_{\kappa(k,m)}\mathbb{K}_{\alpha-\varepsilon_{\kappa(k,m)}}\delta^{\hat{\otimes}\varepsilon_{\kappa(k,m)}}(t,z) \tag{5.36}$$

$$= \sum_{\alpha\in\mathcal{I}}\sum_{k\in\mathbb{N}}\sum_{m\in\mathbb{N}} c_\alpha\alpha_{\kappa(k,m)}\mathbb{K}_{\alpha-\varepsilon_{\kappa(k,m)}}e_k(t)\pi_m(z) \tag{5.37}$$

$$= \sum_{\beta\in\mathcal{I}}\sum_{k\in\mathbb{N}}\sum_{m\in\mathbb{N}} c_{\beta+\varepsilon_{\kappa(k,m)}}(\beta_{\kappa(k,m)}+1)\mathbb{K}_\beta e_k(t)\pi_m(z). \tag{5.38}$$

Proposition 5.8. *Closability of Stochastic Derivatives. Let $F_m, F \in \mathcal{G}^*$ such that as $m \to \infty$*

(i) *$F_m \to F$ in $\mathcal{G}^*$,*
(ii) *$D_{t,z}F_m$ converges in $\mathcal{G}^*$*

Then, $D_{t,z}F_m \to D_{t,z}F$ in $\mathcal{G}^*$.

Proof. We follow an argument similar to [8]in showing closability in $\mathbb{D}^{1,2}$. Consider the formal expansion

$$F = \sum_{\alpha\in\mathcal{I}} c_\alpha\mathbb{K}_\alpha, \quad F_m = \sum_{\alpha\in\mathcal{I}} c_\alpha^m\mathbb{K}_\alpha \tag{5.39}$$

such that $F_m \to F$ in $\mathcal{G}^*$, then there exists $q > 0$ such that

$$||F_m - F||^2_{\mathcal{G}^*} = \sum_{\alpha\in\mathcal{I}} \alpha!|c_\alpha^m - c_\alpha|^2 e^{-2q|\alpha|} \to 0. \tag{5.40}$$

Hence, $c_\alpha^m \to c_\alpha$. Since the stochastic derivative of $D_{j,t,z}F$ is given as

$$D_{t,z}F_m = \sum_{\alpha\in\mathcal{I}}\sum_{k\in\mathbb{N}}\sum_{l\in\mathbb{N}} c_\alpha^m\alpha_{\kappa(k,l)}\mathbb{K}_{\alpha-\varepsilon_{\kappa(k,l)}}e_k(t)\pi_l(z). \tag{5.41}$$

and since $D_{t,z}F_m$ converges in $\mathcal{G}^*$ then, there exists $r > 0$ such that

$$\begin{aligned} &||D_{t,z}F_m - D_{t,z}F_n||^2_{\mathcal{G}_{-r}} \\ &= \sum_{\alpha\in\mathcal{I}}\sum_{k\in\mathbb{N}}\sum_{l\in\mathbb{N}} |c^m_\alpha - c^n_\alpha|^2 \alpha^2_{\kappa(k,l)}(\alpha - \epsilon_{\kappa(k,l)})! e^{-2r|\alpha-\epsilon_{\kappa(k,l)}|!} \to 0. \end{aligned} \tag{5.42}$$

From Fatou's lemma,

$$\begin{aligned} &\lim_{m\to\infty} \sum_{\alpha\in\mathcal{I}}\sum_{k\in\mathbb{N}}\sum_{l\in\mathbb{N}} |c^m_\alpha - c_\alpha|^2 \alpha^2_{\kappa(k,l)}(\alpha - \epsilon_{\kappa(k,l)})! e^{-2r|\alpha-\epsilon_{\kappa(k,l)}!|} \\ &= \lim_{m\to\infty} \sum_{\alpha\in\mathcal{I}}\sum_{k\in\mathbb{N}}\sum_{l\in\mathbb{N}} \liminf_{n\to\infty} |c^m_\alpha - c^n_\alpha|^2 \alpha^2_{\kappa(k,l)}(\alpha - \epsilon_{\kappa(k,l)})! e^{-2r|\alpha-\epsilon_{\kappa(k,l)}|!} \\ &\le \lim_{m\to\infty} \liminf_{n\to\infty} \sum_{\alpha\in\mathcal{I}}\sum_{k\in\mathbb{N}}\sum_{l\in\mathbb{N}} |c^m_\alpha - c^n_\alpha|^2 \alpha^2_{\kappa(k,l)}(\alpha - \epsilon_{\kappa(k,l)})! e^{-2r|\alpha-\epsilon_{\kappa(k,l)}|!} = 0. \end{aligned} \tag{5.43}$$

Hence, as $m \to \infty$, $||D_{t,z}F_m - D_{t,z}F||^2_{\mathcal{G}_{-r}} \to 0$. Therefore, $D_{t,z}F_m \to D_{t,z}F \in \mathcal{G}_{-r} \subset \mathcal{G}^*$. □

Proposition 5.9. *Let*

$$F = \sum_{n=0}^{\infty} I_n(f_n) \in \mathcal{G}^*, \tag{5.44}$$

where $f_n \in L^2_s(\mu^n)$. Then, $D_{t,z}F \in \mathcal{G}^*$, μ a.e. and is given by

$$D_{t,z}F = \sum_{n=1} nI_n(f_{n-1}(\cdot,(t,z))). \tag{5.45}$$

Proof. We follow arguments which are parallel to [19] in the Poisson case. Since $F \in L^2(P)$, defining its partial sum as

$$F_m = \sum_{n=0}^{m} I_n(f_n) \tag{5.46}$$

then $F_m \to F$ in $\mathcal{G}^*$ as $m \to \infty$. Pick $q > 0$ arbitrarily, then

$$||F_m - F||^2_{\mathcal{G}_{-q}} = \sum_{n=m+1}^{\infty} n!||f_n||^2_{L^2(\mu^n)} e^{-2qn} \to 0. \tag{5.47}$$

Since $q > 0$ is arbitrary, then $F \in \mathcal{G}^*$. Note that

$$||D_{t,z}F_m - D_{t,z}F||^2_{\mathcal{G}_{-q}} = \sum_{n=m+1}^{\infty} nn!||f_n(\cdot,(t,z))||^2_{L^2(\mu^{n-1})} e^{-2q(n-1)}. \tag{5.48}$$

Integrating both sides and letting $m \to \infty$ yields

$$\begin{aligned}\int_{[0,t]\times\mathbb{R}} &||D_{t,z}F_m - D_{t,z}F||^2_{\mathcal{G}_{-q}}\mu(dt,dz)\\ &= \sum_{n=m+1}^{\infty} nn! \int_{[0,t]\times\mathbb{R}} ||f_n(\cdot,(t,z))||^2_{L^2(\mu^{n-1})} e^{-2q(n-1)}\mu(dt,dz)\\ &\leq K \sum_{n=m+1}^{\infty} n!||f_n||^2_{L^2(\mu^n)} \to 0 \end{aligned} \tag{5.49}$$

for some $K > 0$. Thus, our claim is established. □

Lastly, we state the chain rule under Wick-polynomial action for entire functions.

Lemma 5.10. [6] *Let $F \in (\mathcal{S})^*$ and $g : \mathbb{C} \to \mathbb{C}$ be entire, then*

$$D_{t,z} g^{\diamond}(F) = (g')^{\diamond}(F) \diamond D_{t,z}F. \tag{5.50}$$

5.6. Wick Skorohod Identity. We extend the concept of $(S)^*$ integrability [10], [17] to the Canonical Lévy space.

Definition 5.11. ($(\mathcal{S})^*$ integrability). The random field $u : \mathbb{R}_+ \times \mathbb{R}$ is said to be $(S)^*$-*integrable* if the action of u for all $F \in (S)^*$ satisfies $< u, F > \in L^1(\mu)$. The $(S)^*$-integral denoted by $\int_{\mathbb{R}_+\times\mathbb{R}} u(t,z)\mu(dt,dz)$ is the unique element in $(S)^*$ such that

$$\left\langle \int_{\mathbb{R}_+\times\mathbb{R}} u(t,z)\mu(dt,dz), F \right\rangle = \int_{\mathbb{R}_+\times\mathbb{R}} \langle u(t,z), F\rangle \, \mu(dt,dz). \tag{5.51}$$

Proposition 5.12. *Wick-Skorohod Identity.*

Let u be Skorohod-integrable with respect to M, then, for all $(t,z) \in \mathbb{R}_+ \times \mathbb{R}$, we have that $u(t,z) \diamond \dot{M}(t,x)$ is $(S)^*$-integrable and

$$\int_{\mathbb{R}_+\times\mathbb{R}} u(t,x)M(\delta t,dz) = \int_{\mathbb{R}_+\times\mathbb{R}} u(t,z) \diamond \dot{M}(t,z)\mu(dt,dz). \tag{5.52}$$

Proof. The steps of the proof are parallel to [7]. Since u is Skorohod-integrable with respect to M, then it has a representation of the form

$$u(t,z) = \sum_{\alpha\in\mathcal{I}} c_\alpha(t,z)\mathbb{K}_\alpha = \sum_{n=0}^{\infty} I_n(f_n(\cdot,(t,z))), \tag{5.53}$$

where $f_n(\cdot,(t,z)) \in L^2_s(\mu^n)$. The right-hand side of (5.52) yields the following:

$$\int_{\mathbb{R}_+\times\mathbb{R}} u(t,z) \diamond \dot{M}(dt,dz)\mu(dt,dz) = \sum_{\alpha\in\mathcal{I}}\sum_{k\in\mathbb{N}}\sum_{m\in\mathbb{N}} < c_\alpha, e_k p_m >_{L^2(\mu)} \mathbb{K}_{\alpha+\epsilon^{\kappa(k,m)}}. \tag{5.54}$$

Now since

$$f_n(\cdot,(t,z)) = \sum_{|\alpha|=n} c_\alpha(t,z)\delta^{\hat{\otimes}\alpha}, \tag{5.55}$$

then, $f_n(\cdot,(t,z))$ has the following orthonormal expansion

$$f_n(\cdot,(t,z)) = \sum_{k\in\mathbb{N}}\sum_{m\in\mathbb{N}}\sum_{|\alpha|=n} < c_\alpha, e_k\pi_m >_{L^2(\mu)} \delta^{\hat{\otimes}\alpha} e_k(t)\pi_m(z). \tag{5.56}$$

Hence, the left-hand side of (5.52) yields the following:

$$\int_{\mathbb{R}_+\times\mathbb{R}} u(t,z)M(\delta t, dz) = \sum_{\alpha\in\mathcal{I}}\sum_{k\in\mathbb{N}}\sum_{m\in\mathbb{N}} < c_\alpha, e_k p_m >_{L^2(\mu)} \mathbb{K}_{\alpha+\epsilon^{\kappa(k,m)}}. \tag{5.57}$$

Finally, (5.54) and (5.57) give us the desired identity. □

6. Clark-Ocone Theorem in $L^2(P)$

Now that we have presented a framework of concepts for Canonical Lévy processes' white noise theory, our goal is to show a Clark-Ocone theorem in $L^2(P)$ with respect to the independent random measure M. The steps in proving the Clark-Ocone theorem in $L^2(P)$ are similar to the Wiener and Poisson white noise cases [8]: first one shows the Clark-Ocone theorem for a Wick polynomial, and then one proves an auxiliary lemma (Lemma 6.7) to obtain the Clark-Ocone theorem in $L^2(P)$.

We let $P(x)$ be a polynomial in $\mathbb{R}^n$, that is, $P(x)$ can be written as follows:

$$P(x) = \sum_{\alpha\in\mathcal{I}} c_\alpha x^\alpha, \qquad x\in\mathbb{R}^n, \quad c_\alpha\in\mathbb{R}, \quad \mathcal{I}=\mathbb{N}^n. \tag{6.1}$$

Throughout this section, we assume that a process $u:\mathbb{R}_+\to(S)^*$ is differentiable in the $(S)^*$ sense. Define the following processes in $(S)^*$:

$$\begin{aligned} X_{k,m} &= \int_{\mathbb{R}_+\times\mathbb{R}} e_k(x)\pi_m(s)M(ds,dx) = \mathbb{K}_{\epsilon^{\kappa(k,m)}}, \\ X_{k,m}^{(t)} &= \int_{[0,t]\times\mathbb{R}} e_k(x)\pi_m(s)M(ds,dx) \doteq \mathbb{K}^{(t)}_{\epsilon^{\kappa(k,m)}}. \end{aligned} \tag{6.2}$$

From the Wick-Skorohod identity, we have the following derivative in $(S)^*$

$$\frac{d}{dt}X_{k,m}^{(t)} = e_k(x)L_m(t), \tag{6.3}$$

where

$$L_m(t) = \int_{\mathbb{R}} \pi_m(t)\dot{M}(ds,dx)\eta(dx). \tag{6.4}$$

We let

$$\begin{aligned} X = (X_{k_1,m_1},\cdots,X_{k_n,m_n})^T, \quad X^{(t)} = (X^{(t)}_{k_1,m_1},\cdots,X^{(t)}_{k_n,m_n})^T, \\ \alpha = (\alpha_{\kappa(k_1,m_1)},\cdots,\alpha_{\kappa(k_n,m_n)})^T, \end{aligned} \tag{6.5}$$

where $k_i, m_i\in\mathbb{N}$, for all $i\in\{1,\cdots,n\}$. The Wick polynomial of X is given by

$$P^\diamond(X) = \sum_{\alpha\in\mathcal{I}} c_\alpha X^{\diamond\alpha}. \tag{6.6}$$

Moreover, we have the following identities:

$$X^{\diamond\alpha} = (X_{k_1,m_1})^{\diamond\kappa(k_1,m_1)} \diamond \cdots \diamond \left(X_{k,m_n}\right)^{\diamond\kappa(k_n,m_n)} = \mathbb{K}_\alpha,$$
$$\left(X^{(t)}\right)^{\diamond\alpha} = \left(X^{(t)}_{k_1,m_1}\right)^{\diamond\kappa(k_1,m_1)} \diamond \cdots \diamond \left(X^{(t)}_{k,m_n}\right)^{\diamond\kappa(k_n,m_n)} = \mathbb{K}^{(t)}_\alpha. \tag{6.7}$$

Lemma 6.1. *Differentiation of the Wick Polynomial*

(i)

$$D_{t,z}P(X) = \sum_{i=1}^{n}\sum_{\alpha\in\mathcal{I}} c_\alpha \alpha_{\kappa(k_i,m_i)} X^{\alpha-\epsilon^{\kappa(k_i,m_i)}} e_{k_i}(t)\pi_{m_i}(z), \tag{6.8}$$

$$D_{t,z}P^\diamond(X) = \sum_{i=1}^{n}\sum_{\alpha\in\mathcal{I}} c_\alpha \alpha_{\kappa(k_i,m_i)} X^{\diamond(\alpha-\epsilon^{\kappa(k_i,m_i)})} e_{k_i}(t)\pi_{m_i}(z). \tag{6.9}$$

(ii)

$$\frac{d}{dt}P^\diamond(X^{(t)}) = \sum_{i=1}^{n}\left(\frac{\partial P}{\partial x_i}\right)^\diamond \left(X^{(t)}\right) \diamond e_{k_i}(t)L_{m_i}(z). \tag{6.10}$$

Proof. (i) From the chain rule,

$$D_{t,z}P(X) = \sum_{i=1}^{n} \frac{\partial P(X)}{\partial x_i} D_{t,z}X_{k_i,m_i}, \tag{6.11}$$

$$D_{t,z}P^\diamond(X) = \sum_{i=1}^{n} \frac{\partial P^\diamond(X)}{\partial x_i} D_{t,z}X_{k_i,m_i}. \tag{6.12}$$

Since

$$\frac{\partial P(X)}{\partial x_i} = \sum_{\alpha\in\mathcal{I}} c_\alpha \alpha_{\kappa(k_i,m_i)} X^{\alpha-\epsilon^{\kappa(k_i,m_i)}}, \tag{6.13}$$

$$\frac{\partial P^\diamond(X)}{\partial x_i} = \sum_{\alpha\in\mathcal{I}} c_\alpha \alpha_{\kappa(k_i,m_i)} X^{\diamond(\alpha-\epsilon^{\kappa(k_i,m_i)})} \tag{6.14}$$

and

$$X_{k_i,m_i} = \int_{\mathbb{R}_+\times\mathbb{R}} e_{k_i}(s)\pi_{m_i}(x)M(ds,dx) = I_1(e_{k_i}\pi_{m_i}). \tag{6.15}$$

Then,

$$D_{t,z}X_{k_i,m_i} = e_{k_i}(t)\pi_{m_i}(x). \tag{6.16}$$

Plugging (6.13) – (6.16) into (6.11) – (6.12), yields the desired result.

(ii) From the Wick chain rule and (6.15), we obtain

$$\begin{aligned}\frac{d}{dt}P^\diamond(X^{(t)}) &= \sum_{i=1}^{n}\left(\frac{\partial P}{\partial x_i}\right)^\diamond \left(X^{(t)}\right) \diamond \frac{d}{dt}X^{(t)}_{k_i,m_i}\\ &= \sum_{i=1}^{n}\left(\frac{\partial P}{\partial x_i}\right)^\diamond \left(X^{(t)}\right) \diamond e_{k_i}(t)L_{m_i}(z).\end{aligned} \tag{6.17}$$

□

Definition 6.2. [17] Let $F = \sum_{n=0}^{\infty} I_n(f_n) \in \mathcal{G}^*$. We define the *generalized expectation* $E[F]$ in $\mathcal{G}^*$ by

$$E[F] = I_0(f_0) \tag{6.18}$$

and we define the *conditional expectation* of F with respect to $A \in \mathfrak{B}([0,T])$ by

$$E[F|\mathcal{F}_A] = \sum_{n=0}^{\infty} I_n(f_n \mathbf{1}_A^{\otimes n}). \tag{6.19}$$

Let $T > 0$ be a constant, we say that $F \in \mathcal{G}^*$ is $\mathcal{F}_T$ measurable if $E[F|\mathcal{F}_T] = F$.

Remark 6.3. If $F \in L^2(P) \subset (\mathcal{G})^*$, then the generalized expectation coincides with the usual conditional expectation. Let $F \in \mathcal{G}^*$ and $A \in \mathfrak{B}([0,T])$, then $E[F|\mathcal{F}_A] \in \mathcal{G}^*$ and for some $q > 0$, $\|E[F|\mathcal{F}_A]\|_{\mathcal{G}_{-q}} \leq ||F||_{\mathcal{G}_{-q}}$. Hence, $E[F|\mathcal{F}_A] \in (\mathcal{G})^*$

Lemma 6.4. [17] *Let $F, G \in \mathcal{G}^*$, and $A \in \mathfrak{B}([0,T])$, then*

$$E[F \diamond G|\mathcal{F}_A] = E[F|\mathcal{F}_A] \diamond E[G|\mathcal{F}_A]. \tag{6.20}$$

Lemma 6.5. [8] *If $F = P^{\diamond}(X) \in \mathcal{G}^*$, then its generalized conditional expectation in $\mathcal{G}^*$ is given by*

$$E[F|\mathcal{F}_t] = \sum_{\alpha \in \mathcal{I}} c_\alpha \left(X^{(t)}\right)^{\diamond\alpha} = P^{\diamond}\left(X^{(t)}\right). \tag{6.21}$$

$F \in \mathcal{G}^*$ is $\mathcal{F}_T$ measurable iff F can be written as

$$F = \sum_{\alpha \in \mathcal{I}} c_\alpha \left(X^{(T)}\right)^{\diamond\alpha}. \tag{6.22}$$

To show the Clark-Ocone theorem in $L^2(P)$, we first establish a Clark-Ocone theorem for polynomials.

Proposition 6.6. *(Clark-Ocone Theorem for Polynomials). Let $F \in \mathcal{G}^*$ be an $\mathcal{F}_T$ measurable Wick polynomial of degree n. Then*

$$F = E[F] + \int_{[0,T]\times\mathbb{R}} E[D_{t,z}F|\mathcal{F}_{t-}]M(dt,dz). \tag{6.23}$$

Proof. Since F Wick polynomial of degree n, then it has of the form

$$F = P^{\diamond}(X) = \sum_{\alpha \in \mathcal{I}} c_\alpha X^{\diamond\alpha}, \tag{6.24}$$

where $P(x)$ is a polynomial in $\mathbb{R}^n$. Moreover, since F is an an $\mathcal{F}_T$ measurable, then

$$\begin{aligned} F =& E[F|\mathcal{F}_T] = E\left[\sum_{\alpha \in \mathcal{I}} c_\alpha \left(X^{(T)}\right)^{\diamond\alpha} \middle| \mathcal{F}_T\right] \\ =& \sum_{\alpha \in \mathcal{I}} c_\alpha E\left[\left(X^{(T)}\right)^{\diamond\alpha} \middle| \mathcal{F}_T\right] \\ =& \sum_{\alpha \in \mathcal{I}} c_\alpha \left(X^{(T)}\right)^{\diamond\alpha}. \end{aligned} \tag{6.25}$$

The expansion F and $E[D_{t,z}F|\mathcal{F}_{t-}]$ consists of a finite number of terms. Hence, both processes are Skorohod integrable. Then from the Wick-Skorohod identity and from the preceding lemma,

$$\int_{[0,T]\times\mathbb{R}} E[D_{t,z}F|\mathcal{F}_{t-}]M(dt,dz) = \int_{[0,T]} \sum_{i=1}^{n} \left(\frac{\partial P}{\partial x_i}\right)^{\diamond} \left(X^{(t)}\right) \diamond e_{k_i}(t) \int_{\mathbb{R}} \pi_{m_i}(z)\dot{M}(t,z)\eta(dz)dt. \tag{6.26}$$

Now, since

$$e_{k_i}(t)\int_{\mathbb{R}} \pi_{m_i}(z)\dot{M}(t,z)\eta(dz)dt = e_{k_i}(t)L_{m_i}(t) = \frac{d}{dt}X^{(t)}_{k_i,m_i}, \tag{6.27}$$

then, from the Wick chain rule and since F is be $\mathcal{F}_T-$ measurable, we finally obtain

$$\begin{aligned}\int_{[0,T]\times\mathbb{R}} & E[D_{t,z}F|\mathcal{F}_{t-}]M(dt,dz) \\ &= \int_{[0,T]} \sum_{i=1}^{n} \left(\frac{\partial P}{\partial x_i}\right)^{\diamond} \left(X^{(t)}\right) \diamond \frac{d}{dt}X^{(t)}_{k_i,m_i}dt \\ &= \int_{[0,T]} \frac{d}{dt}P^{\diamond}\left(X^{(t)}\right) \\ &= P^{\diamond}\left(X^{(T)}\right) - P^{\diamond}\left(X^{(0)}\right) \\ &= F - E[F].\end{aligned} \tag{6.28}$$

□

We need the following auxiliary lemma in establishing Clark-Ocone in $L^2(P)$.

Lemma 6.7. *Let $F \in \mathcal{G}^*$, then we have the following:*

(i) $D_{t,z}F \in \mathcal{G}^*$, $\mathcal{G}^*, \mu$ *a.e.*,

(ii) *Let $F_n \in \mathcal{G}^*$, $\forall\, n \in \mathbb{N}$ such that $F_n \to F$ in $\mathcal{G}^*$ as $n \to \infty$, then there exists a sub-sequence F_{n_k}, $k \in \mathbb{N}$ such that $D_{t,z}F_{n_k} \to D_{t,z}F \in \mathcal{G}^*$ as $k \to \infty$, $\mathcal{G}^*, \mu$ a.e.*

Proof. The proof is similar to the proof of Okur [18] in the Wiener case. □

Proposition 6.8. *Clark-Ocone Theorem in $L^2(P)$*
Let $F \in L^2(P)$ be $\mathcal{F}_T$measurable, then

$$F = E[F] + \int_{[0,T]\times\mathbb{R}} E[D_{t,z}F|\mathcal{F}_{t-}]M(dt,dz), \tag{6.29}$$

where $E[D_{t,z}F|\mathcal{F}_{t-}] \in L^2(P\times\mu)$, $(t,z)\in[0,T]\times\mathbb{R}$.

Proof. Since F is $\mathcal{F}_T$-measurable, then it has a chaos expansion of the form

$$F = \sum_{\alpha\in\mathcal{I}} c_\alpha \mathbb{K}_\alpha. \tag{6.30}$$

Let F_n be the truncation of F such that

$$F_n = \sum_{\alpha \in \mathcal{I}_n} c_\alpha \mathbb{K}_\alpha, \tag{6.31}$$

where $\mathcal{I}_n = \{\alpha \in \mathcal{I} : |\alpha| \leq n, Index(\alpha) \leq n\}$. Then, from the Clark-Ocone theorem for polynomials, for at $n \in \mathbb{N}$,

$$F_n = E[F_n] + \int_{[0,T]\times\mathbb{R}} E[D_{t,z}F|\mathcal{F}_{t-}]M(dt, dz). \tag{6.32}$$

From Itô's representation theorem, there exists a unique predictable process $u(t, z)$, $(t, z) \in [0, T] \times \mathbb{R}$ such that $E\left[\int_{[0,T]\times\mathbb{R}} u^2(t, z)\mu(t, z)\right] < \infty$,

$$F = E[F] + \int_{[0,T]\times\mathbb{R}} u(t, z)M(dt, dz). \tag{6.33}$$

From the isometry relation with respect to the independent random measure M (Section 1.5 of [23]), we obtain

$$\begin{aligned} &E\left[|(F_n - E[F_n]) - (F - E[F])|^2\right] \\ =&E\left[\left|\int_{[0,T]\times\mathbb{R}} (E[D_{t,z}F_n|\mathcal{F}_{t-}] - u(t, z))M(dt, dz)\right|^2\right] \\ =&E\left[\int_{[0,T]\times\mathbb{R}} |E[D_{t,z}F_n|\mathcal{F}_{t-}] - u(t, z)|^2\mu(dt, dz)\right]. \end{aligned} \tag{6.34}$$

Then, since $F_n \to F$ in $L^2(P)$, the right hand side approaches zero as $n \to \infty$. Thus, we have the following convergence:

$$E[D_{t,z}F_n|\mathcal{F}_{t^-}] \to u(t, z), \qquad L^2(P \times \mu). \tag{6.35}$$

Now since $F_n \to F \in L^2(P) \subset \mathcal{G}^*$, then from Lemma 6.7 there exists a sub-sequence F_{n_k}, $k \in \mathbb{N}$ such that

$$E[D_{t,x}F_{n_k}|\mathcal{F}_{t-}] \to E[D_{t,x}F|\mathcal{F}_{t-}] \in \mathcal{G}^*, \qquad k \to \infty \quad \mathcal{G}^*, \mu \quad a.e. \tag{6.36}$$

Taking a further sub-sequence, we have

$$E[D_{t,x}F_{n_k}|\mathcal{F}_{t-}] \to u(t, z), \qquad k \to \infty \qquad L^2(P), \mu \quad a.e. \tag{6.37}$$

Thus, it follows that

$$u(t, z) = E[D_{t,x}F|\mathcal{F}_{t-}], \qquad L^2(P), \mu \quad a.e. \tag{6.38}$$

□

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