Portfolio Optimization with Discrete Proportional Transaction Costs under Stochastic Volatility

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December 4, 2009

Abstract

This paper is devoted to evaluating the optimal self-financing portfolio and the optimal trading frequency on a risky and risk-free asset to maximize the expected future utility of the terminal wealth in a stochastic volatility setting, when proportional transaction costs are incurred at each discrete trading time. The HARA utility function is used, allowing a simple approximation of the optimization problem, which is implementable forward in time. For each of various transaction cost rates, we find the optimal trading frequency, i.e. the one that attains the maximum of the expected utility at time zero. We study the relation between transaction cost rate and optimal trading frequency. The numerical method used is based on a stochastic volatility particle filtering algorithm, combined with a Monte-Carlo method.

Keywords and phrases: portfolio optimization, stochastic volatility, mean-reverting, particle filtering, Monte-Carlo method, expected utility, diffusion process, numerical implementation, discrete trade, transaction costs, trading frequency.

1 Introduction

The Black-Scholes model is an essential tool to understand stock-market movements, and rightly continues to be celebrated by many authors. It is equally well-accepted that its principal drawback is the constance of its coefficients, most notably its volatility. There are many ways of taking into account the fact that stock market volatility is far from being constant. This article situates itself in the popular continuous-time framework of stochastic volatility (SV). Such models are mathematically convenient because, at the cost of an analysis of multidimensional stochastic differential equations even for single stocks, many of the stochastic calculus tools from the standard Black-Scholes theory for option pricing and portfolio optimization can be extended to handle SV. See the excellent treatment in [7] along these lines. The book [8] contains a collection of recent articles with the same motivations; one can also consult the book [6] for a presentation of how discrete and continuous-time modeling of SV are related.

1.1 Stochastic volatility framework and estimation

For a stock price diffusion process model S given in continuous time by $dS(t) = \alpha S(t)\,dt + V(t)\,S(t)\,dW(t)$, where V is the unobserved volatility process, and W is a Brownian motion, continuous-time observation would imply that one can observe the quadratic variation process $[S]$ of S; since stochastic calculus tells us that $[S](t) = \int_0^t V(r)^2 S(r)^2\,dr$, one immediately has access, in continuous time, to the squared volatility $V(t)^2 = S(t)^{-2}\,d[S](t)/dt$. However, the main practical problem with SV models is that volatility itself is not directly observable, and must be somehow estimated. This proves that, even if one believes or assumes from a modeling point of view (as we do) that stock prices are continuous-time stochastic processes, we can only assume that they are observed in discrete time. Even high-frequency (e.g. tick-by-tick) data for highly traded assets and indices cannot be considered as continuous-time data for the purpose of understanding volatility.

In this paper, we estimate stochastic volatility in what one might call a Bayesian statistical framework. More specifically, in the language of stochastic calculus, we consider the pair of processes (S, V) as above, and seek the stochastic filter of the unobserved process V given the observed process S; both processes are defined in continuous time, but S is observed in discrete time only, and therefore it is consistent to only require an estimation of V at the same discrete instants, given the information contained in all past observations. For simplicity of notation in this introduction, and often in the remainder of this paper, observation times are denoted by the set of integers $i = 0, 1, 2, \cdots$. In other words, we seek the conditional probability law

$$\mathbf{P}\left[V_i \in dy | S_1 = s_1; S_2 = s_2; \cdots; S_i = s_i\right] \tag{1}$$

where $s_1, s_2, \cdots, s_i$ are observed values of S up to time i. We adopt an approach which was introduced in [9] to actually compute an approximation of this probability distribution, by way of a so-called *stochastic volatility particle filter*, which is adapted from the generic method of del Moral et. al in [3]. More details on this method are given in Section 2

1.2 Motivations

With this framework being established, one can ask a variety of quantitative finance questions, such as option pricing, or portfolio optimization, under our discretely observed stochastic volatility stock price. The former topic was considered using a special quadrinomial recombining tree in [5]; part of the issue there was to handle the fact that the market is incomplete under an SV model. See the references therein for other approaches to the problem, not based on the Bayesian estimation of volatility. In this article, we concentrate on the latter topic: portfolio selection in order to optimize a given utility function; in this case, incompleteness of the model is not relevant, since we do not attempt to hedge any positions. This optimization was the topic originally studied in [9], but the numerical method therein was to cumbersome to be implemented beyond a highly simplified binomial version of the model in [4]. A practical breakthrough was achieved in the paper of Batalova et al. [1], where it was noticed that in the case of power utility, the cumbersome portion of the algorithm, which requires numerically solving a discretized Bellman problem with high-dimensional state space in reverse time, can be short-circuited in practice in a purely forward-time algorithm which nonetheless updates its Bayesian estimation of the stochastic volatility distribution at every time step.

In this paper, we propose to take up the program in the paper of Batalova et al. [1], and study its implementation and analyze its performance under the "real-market" assumption that there are transaction costs in discrete time, each time a stock allocation is changed (each time an individual traders buys or sells a stock). Our work contains proportional transaction cost in each time interval. Even if transaction costs are proportional to the trades, which is the case in this article, they cannot be approximated by the continuous-time method of reducing the stock's mean rate of return by a continuously compounded transaction rate. Indeed, in practice, trades incur costs which are not proportional to the frequency of trading, unlike the use of a continuously compounded constant trading rate.

Having transaction costs which are proportional to the trade sizes regardless of the trading frequency is more challenging to implement in practice, and is the main subject of this article. More specifically, for a given risk aversion parameter (for a given power in our Hyperbolic Absolute Risk Averse (HARA) power utility) we find the optimal self-financing portfolio that maximize the wealth's expected future utility at time zero, with proportional transaction costs at each trading time i, under our stochastic volatility model, assuming that S is observed only at the integer trading times.

The main quantitative issue at hand then becomes that of trading frequency. If no transaction costs are incurred, the practitioners may buy or sell stocks as many times as they choose, and theirs becomes only an issue of gathering information. In this case, the gain increases as the number of transactions per unit of time increases. In the best of cases, where they can handle trading at the tick-by-tick frequency, their problem is still discrete, but they take advantage of all the available information. However, when discrete transaction costs are considered, we have the limitation of not letting the transaction costs destroy all our profits just for the sake of taking advantage of as much information as possible, and adjusting our portfolio every time the stock moves. In other words, while increasing the frequency of information usage causes the expected utility to increase, increases the trading frequency also increases the number of transactions and their costs, causing the expected utility to decrease.

1.3 Summary of results

The main goal of this paper is to understand the trade-off between these two opposing forces by determining the trading frequency which maximizes the expected utility of the terminal wealth. We call this the optimal trading frequency and denote it by N^*. Evidently, N^* will also vary depending on the size of the transaction costs, which in our case is measured by their proportionality constant λ to the trade size; the relation between N^* and λ is also studied in this paper. For illustration purposes, we choose a one-year time horizon, and let N be the number of trades, so that N also denotes the frequency, in trades per year, and N^* is also the optimal number of trades per year. We take the point of view of an individual "day trader", who is not likely to trade more frequently than once a day, i.e. likely to have a trading frequency that is bounded above by 250. We will see that typical parameters for day trades, this frequency is usually far higher than N^*, and we will see that our results can be reinterpreted for the case of fixed (non-proportional) transaction costs once per trading day, yielding recommendations for such day traders.

We adapt the solution of the portfolio optimization problem proposed by Batalova et. al [1], using their time-forward algorithm with HARA utility function, and incorporate fixed positive transaction costs whether trades are buys or sells.

Specifically, in this article, we show the following. At any time i, we allow allocations of our wealth w into arbitrary quantities of SV stock S_i and risk-free asset $B_i = e^{ri}$. Let $\bar{s}_i = (s_1, s_2, \cdots, s_i)$ be the sequence of observed stock prices up to time i, and let $\hat{U}(i, \bar{s}_i, w)$ be the simulated expected HARA terminal utility (at time N), given the observations $\bar{s}_i$ up to time i, with wealth at time i equal to w.

1). We show that $\hat{U}$ preserves the HARA utility structure: if for a fixed risk aversion parameter p, we assume $\hat{U}(N, \bar{s}_N, w) = w^p/p$, then the expected utility of the terminal wealth at time i, $\hat{U}(i, \bar{s}_i, w)$ will be the product of the HARA function w^p/p and a time-dependent function the observed stock prices up to time i only. Specifically, we identify a function Γ such that

$$\hat{U}(i, \bar{s}_i, w) = \frac{w^p}{p}\Gamma(i, \bar{s}_i);$$

we prove this using backwards induction, and our proof results in a forward-time algorithm to compute Γ. We state this with full proof in Proposition 3.1 in Section 3.2.

2). Let ξ_i be the quantity of stock in our portfolio; this ξ_i is allowed to depend only on the initial wealth w_0 and the observations $\bar{s}_i$ up to time i. By a self-financing condition, the sequence ξ determines the entire portfolio strategy. Our algorithm to compute Γ also produces an algorithm for the optimal strategy $\xi^* = we^r\kappa^*$, where r is the rate of interest and w is the current wealth. Specifically, κ^* uniquely solves the algebraic equation

$$\sum_{k=1}^{n}(\kappa^*\hat{\beta}_k(\bar{s}_i) + 1)^{p-1}\hat{\beta}_k(\bar{s}_i) = 0,$$

where $\hat{\beta}_k(\bar{s}_i) := (1 + \lambda \text{sign}\,(\xi_{i-1} - \xi_i))\hat{S}_{i+1,k} - s_i e^r$,where n is the number of simulation particles, and $\hat{S}_{i+1,k}$ is the k-th particle of the simulated future stock prices given the observed s_i. We simulate the future stock prices $\hat{S}_{i+1,k}$ by using the stochastic volatility particle filter to initialize the volatility distribution at time i. This is explained in Section 2.2. We state the above result in Summary 3.2.

3). We run the above algorithm on sets of simulated data $\bar{s}$ for many different trading frequencies N and several values of the transaction cost rate λ. We identify the optimal trading frequency N^* for each λ, and find that it increases as λ decreases. This is given in figure **??** and table 2 in Section 5. We also provide recommendations for day traders who incurr fixed transaction costs.

2 Model and framework

2.1 Stochastic volatility Model

Let $\mathbf{P}$ be an "objective market" probability measure, under which the stock price process S, the risk-free asset process B, and a stochastic volatility driving process Y have the following stochastic dynamics: for all $t \geq 0$

$$\begin{cases} dB_t = rB_t dt, B_0 = 1 \\ dS_t = \alpha S_t dt + \sigma(Y_t)S_t dW_t \\ dY_t = \mu(\nu - Y_t)dt + \sqrt{\mu}dZ_t \end{cases}$$

where r is the deterministic constant short rate of interest, α is the deterministic constant mean rate of return of the stock price S, and σ is a deterministic function of the stochastic process Y_t. As is typically done in SV models (see [7] or [1] for instance), we assume $\sigma(x) = \exp(x)$, and our Y above is the mean-reverting Ornstein-Uhlenbeck process with a large mean-reversion rate parameter μ (Y is often known as a fast mean-reverting process); the positive constant ν is the mean level around which the process Y tends to revert. Note that our entire study can be repeated with any number of distributions for the diffusion process Y, such as $dY_t = \mu(Y_t)dt + \theta(Y_t)dZ_t$, where μ and θ satisfy typical Lipshitz and boundedness conditions; we make no further comments on such extensions. In our study W and Z are Brownian motions which may be correlated in order to account for complex leverage effects: we denote $\rho_{w,z} \in (-1, 1)$ their correlation coefficient.

2.2 Stochastic volatility filtering method

We refer to the article of Florescu and Viens [5] and Del Moral, Jacod, and Protter [3] for the interacting particle algorithm.

The main task of the stochastic volatility particle filter is to find the distribution of the volatility process Y when the discrete time observations of stock prices are given, and to do this dynamically in time, as the observations become available. This theoretical problem, which is to estimate the probability distribution of the volatility given information of the stock prices up to time i, as stated for instance in the introduction in (1), can be rewritten here specifically for the driving process Y as

$$p_i^{\bar{s}}(dy) := \mathbf{P}[Y_i \in dy | S_0 = s_0, S_1 = s_1, \cdots, S_i = s_i], \tag{2}$$

where $s_0, s_1, \cdots, s_i$ are the observed stock prices up to time i. This conditional time-dependent probability law, which we call the *stochastic volatility filter* given discrete observations, cannot be computed explicitly, and its numerical approximation is non-trivial. it is unlikely to know the exact distribution of Y.

To estimate $p_i^{\bar{s}}(dy)$, we adopt the particle method inspired by [3], introduced in [9], and explained in detail in [5]. We now briefly explain this procedure.

It uses n particles $(Y_{i,k})_{k=1}^n$ which evolve in discrete time. At time 0, we initialize the $Y_{0,k}$ in order to approximate our best unconditional "guess" for the distribution of Y_0. This "guess" can be seeded systematically by starting the filtering procedure in the distant past, long before trading occurs, which will yield an initial empirical distribution of the system $(Y_{0,k})_{k=0}^n$ which is close to the stationary measure of Y. We do not give further details on this point, assuming only that $(Y_{0,k})_{k=0}^n$ are chosen.

At any given time $i-1$, the past stock prices up to this time are given: $S_0 = s_0, S_1 = s_1, \cdots, S_{i-1} = s_{i-1}$. Assume by induction that $(Y_{i-1,k})_{k=1}^n$ have been computed. Then, as soon as the ith observation $S_i = s_i$ becomes available, we implement a "mutation" step by running an m-sub-time-step Euler scheme to give, for each $k = 1, \cdots, n$, a simulated value of the pair of particles $(\hat{S}_{i,k}, \hat{Y}_{i,k})$, started at time $i-1$ at $(s_{i-1}, Y_{i-1,k})$.

Then using the function $\psi(x) := 1 - |x|, -1 < x < 1$, we let

$$\psi_n(x) := \sqrt[3]{n}\psi(x\sqrt[3]{n}), \tag{3}$$

and measure how close our each of our simulated particles $\hat{S}_{i,k}$ is to the actual observed value s_i by calculating for each k

$$F_{i,k} := \psi_n(\hat{S}_{i,k} - s_i). \tag{4}$$

If the simulated particle $\hat{S}_{i,k}$ is close to the observed value s_i, then $F_{i,k}$ will be very large. If $\hat{S}_{i,k}$ is not close to s_i, then $F_{i,k}$ will be close to zero. This $F_{i,k}$ is the "weight" or "fitness" of $\hat{S}_{i,k}$. We normalize all particle fitnesses by computing $C_i = \sum_{k=1}^{n} F_{i,k}$. Then $\hat{p}_{i,k} := \frac{F_{i,k}}{C_i}$ represents the approximate probability $\hat{S}_{i,k}$ is a good approximation to s_i, and we transfer this likelihood concept to the $\hat{Y}$ particles as follows. We rearrange the particles $\hat{Y}_{i,k}$ according to these probabilities, by picking n new particles $\{Y_{i,k} : k = 1, \cdots, n\}$ independently of each other according to the distribution $P\left[Y = \hat{Y}_{i,k}\right] = \hat{p}_{i,k} : k = 1, \cdots, n$. This is the "selection" step: it thus results in n particles $(Y_{i,k})$ with the estimated probabilities $\hat{p}_{i,k}$; their empirical distribution is the approximate stochastic volatility particle filter at time i.

2.3 Self-Financing portfolio strategies with the proportional transaction costs

A portfolio strategy is a pair

$$\{(\xi_i, \eta_i), i = 0, \cdots, N\}, \tag{5}$$

which is an adapted stochastic process such that $(\xi_i S_i, \eta_i)$ in $\mathbf{P}$-square-integreable for all $i = 0, \cdots, N$. The ξ_i and η_i represent the number of units of stock S and the number of units of risk-free asset B held at time i, respectively.

When we sell or buy the risky assets such as stocks, we pay transaction costs. Proportional transaction costs in discrete time means are cost that are proportional to the dollar amount of stock that is traded at each specific time. Algebraically, this cost is the abolute value of risky asset that exchange hands at time i, times the transaction cost rate λ for trading $(\lambda \in \mathbf{R}^+)$: thus the proportional transaction cost at time i is

$$\lambda|\xi_i - \xi_{i-1}|S_i \tag{6}$$

,where $|\xi_i - \xi_{i-1}|$ means the number of units of the traded stock at time i. In real market, the transaction cost rates for purchasing are different from the ones for selling. But we use the common transaction cost rates for both purchasing and selling cases to make algorithms simple.

In order to manage our portfolio, we consider the simple situation where the initial dollar amount w_0 is determined, and thereafter allocation changes are financed only by stock movements and accrued interest; in other words we assume our portfolio is self-financing: the wealth right before the transaction occurs equals to the one right after the transaction occurs.

The initial wealth $W_0 = w_0$ is given. The wealth at time i is:

$$W_i = \xi_i S_i + \eta_i B_i.$$

For $t \in [i, i+1]$, $W_t = \xi_i S_t + \eta_i B_t$. Therefore elementary algebra yields $\eta_i = (W_i - \xi_i S_i)e^{-ri}$, that is to say, we can eliminate the use of the risk-free account allocation by keeping track of the wealth. We call $W_{(i)-}$ the wealth right after the transaction occurs; calculated using the old portfolio allocation, after transaction costs are deducted, this is

$$W_{(i)-} = \xi_{i-1} S_i + \eta_{i-1} B_i - \lambda |\xi_i - \xi_{i-1}| S_i.$$

For our portfolio strategy (ξ, η) to be self-financing, this $W_{(i)-}$ has to agree with the value W_i of the portfolio under the new allocation at time i. Therefore, the self-financing condition for all $i = 1, ..., N-1$ reads as

$$\xi_i S_i + \eta_i B_i = \xi_{i-1} S_i + (W_{i-1} - \xi_{i-1} S_{i-1}) e^r - \lambda |\xi_i - \xi_{i-1}| S_i. \tag{7}$$

We notice that for any $t \in [i, i+1)$, i.e. before the next transaction,

$$W_t = \xi_{i-1} S_t + (W_{i-1} - \xi_{i-1} S_{i-1}) e^{r(t-i)} - \lambda |\xi_i - \xi_{i-1}| S_t. \tag{8}$$

3 Theoretical Analysis

3.1 Goal

The main computational goal of this study is to find a predictable self-financing portfolio to maximize the expected utility of terminal wealth for a given initial wealth when proportional transaction costs are incurred. For a portfolio to be admissible, for each time i, ξ_i has to depend only on the initial wealth w_0 and the past observed stock values $s_1, s_2, \cdots, s_i$. We should find an admissible self-financing portfolio $\xi^* = (\xi_1^*, \cdots, \xi_N^*)$ that attains the following supremum over the set of all admissible self-financing portfolios

$$U(0) = U(0, s_0, w_0) := \sup_{\xi} \mathbf{E}^{\mathbf{P}} \left[u(W_N^{\xi}) | S_0 = s_0, W_0 = w_0 \right], \tag{9}$$

where W^{ξ} is the wealth process following strategy ξ, and u is a nondecreasing concave utility function. We use $u(x) = x^p/p$, for some positive constant $p < 1$, which is called the power utility function for a risk-averse investor or HARA utility function. Our study can also handle $u(x) = \log x$, the log-utility function, but this is left to the reader to check. Recall that η is determined by the ξ from the equation $\eta_i = (W_i - \xi_i S_i)e^{-ri}$, and we use the notation $\bar{s}_i := \{s_0, \cdots, s_i\}$. Then we can write $\xi_i = \xi_i(\bar{s}_i, w_0)$.

3.2 Mathematical results

We immerse the portfolio optimization problem (9) in the following time-dependent problem:

$$U(i, \bar{s}_i, w_i) := \sup_{\xi} \mathbf{E}^{\mathbf{P}} \left[u(W_N^{\xi}) | \bar{S}_i = \bar{s}_i, W_i = w_i \right]. \tag{10}$$

This could be solved by an iteration of HJB equations backwards in time as in the article [9], but this algorithm is far too cumbersome to be implemented: its state space includes the trajectorial variable $\bar{s}_i$, whose dimension increases unwieldily in time.

Under the HARA utility function, a Monte-Carlo method can be used to estimate the problem (10) via a time-forward recursion, as in [1]. We call this estimation $\hat{U}$. Specifically, we define, for $i = 1, \cdots, N$, and $k = 1, \cdots, n$

$$\hat{U}(i, \bar{s}_i, w_i) = \max_{\xi \in \mathbf{R}} \frac{1}{n} \sum_{k=1}^{n} \hat{U}(i+1, \bar{s}_i, \hat{S}_{i+1,k}, \hat{W}_{i+1,k}) \tag{11}$$

where the simulated value $\hat{S}_{i+1,k}$ is computed using an Euler approximation as in the SV particle filter of Section 2.2, and we set, in agreement with equation (8),

$$\hat{W}_{i+1,k} = \xi_i \hat{S}_{i+1,k} + (w_i - \xi_i s_i) e^r - \lambda |\xi_i - \xi_{i-1}| \hat{S}_{i+1,k}. \tag{12}$$

We notice that the maximum in the definition of $\hat{U}$ in (**??**) is over constant reals only, since this iteration is over a single time interval, during which allocation changes are not allowed.

So far, there is nothing to guarantee that the definition of $\hat{U}$ in (**??**) allows a forward time recursion. To resolve this issue, and find a dynamic portfilio $(\xi_i^*)_{i=1}^n$ that attains the supremum in (10), we need to prove the following proposition.

Proposition 3.1 *Let $\hat{U}$ be defined by (11). Then there exists a function Γ depending only on i, $\bar{s}_i$, and the simulated values used in (12), but not on w, such that for $i = 1, \cdots, N-1$, we have*

$$\hat{U}(i+1, \bar{s}_{i+1}, w) = \frac{w^p}{p} \Gamma(i+1, \bar{s}_{i+1}). \tag{13}$$

Proof. We prove this by using the backward induction. When $i = N-1$, since $\hat{U}(N, \bar{s}_N, w) = w^p/p$, this is obviously proved with $\Gamma = 1$. Then we assume that

$$\hat{U}(i+1, \bar{s}_{i+1}, w) = \frac{w^p}{p} \Gamma(i+1, \bar{s}_{i+1}). \tag{14}$$

And let $b_{i-1} = \frac{\xi_{i-1} \hat{S}_{i+1,k}}{\hat{W}_{i+1,k}}$. First we prove this proposition if $\xi_{i-1} - \xi_i > 0$. The proof of the case of $\xi_{i-1} - \xi_i < 0$ is similarly proved. As we substitute all these to the equation (12), we have

$$\hat{W}_{i+1,k} = \xi_i \hat{S}_{i+1,k} + (w_i - \xi_i s_i) e^r - \lambda (\xi_{i-1} - \xi_i) \hat{S}_{i+1,k} \tag{15}$$

$$= \xi_i \left((1+\lambda) \hat{S}_{i+1,k} - s_i e^r \right) + w_i e^r - \lambda \xi_{i-1} \hat{S}_{i+1,k} \tag{16}$$

$$= \xi_i \left((1+\lambda) \hat{S}_{i+1,k} - s_i e^r \right) + w_i e^r - \lambda b_{i-1} \hat{W}_{i+1,k}. \tag{17}$$

Now we make the following notation.

$$\hat{\beta}_k^+(\bar{s}_i) := (1+\lambda)\,\hat{S}_{i+1,k} - s_i e^r \tag{18}$$

Then finally we have

$$\hat{W}_{i+1,k} = \frac{1}{(1+\lambda b_{i-1})}(\xi_i \hat{\beta}_k^+(\bar{s}_i) + w_i e^r). \tag{19}$$

Then by the assumption (14), we have

$$\hat{U}(i,\bar{s}_i,w_i) = \max_{\xi\in\mathbf{R}} \frac{1}{n}\sum_{k=1}^{n} \frac{\left(\xi\hat{\beta}_k^+(\bar{s}_i) + w_i e^r\right)^p}{p(1+\lambda b_{i-1})^p}\Gamma(i+1,\bar{s}_i,\hat{S}_{i+1,k}) \tag{20}$$

By the definition of b_{i-1}, we can rewrite $\frac{\Gamma(i+1,\bar{s}_i,\hat{S}_{i+1,k})}{(1+\lambda b_{i-1})^p} := \Gamma'(i+1,\bar{s}_i,\hat{S}_{i+1,k})$ for some function Γ'. To simplify the notation, we use the Γ instead of Γ'. Then now we have

$$\hat{U}(i,\bar{s}_i,w_i) = \max_{\xi\in\mathbf{R}} \frac{1}{n}\sum_{k=1}^{n} \frac{\left(\xi\hat{\beta}_k^+(\bar{s}_i) + w_i e^r\right)^p}{p}\Gamma(i+1,\bar{s}_i,\hat{S}_{i+1,k}). \tag{21}$$

To evaluate the extremum, it is enough to find the zeros of the derivative of the above function with respect to ξ. Then ξ solves the following equation

$$\sum_{k=1}^{n}\left(\xi\hat{\beta}_k^+(\bar{s}_i) + w_i e^r\right)^{p-1}\hat{\beta}_k^+(\bar{s}_i)\Gamma(i+1,\bar{s}_i,\hat{S}_{i+1,k}) = 0. \tag{22}$$

Let $\kappa = \frac{\xi}{we^r}$. Then κ solves

$$\sum_{k=1}^{n}\left(\kappa\hat{\beta}_k^+(\bar{s}_i) + 1\right)^{p-1}\hat{\beta}_k^+(\bar{s}_i)\Gamma(i+1,\bar{s}_i,\hat{S}_{i+1,k}) = 0. \tag{23}$$

Now we consider the derivative of the above equation (23) with respect to κ. Then since $0 < p < 1$, we see that

$$\sum_{k=1}^{n}(p-1)\left(\kappa\hat{\beta}_k^+(\bar{s}_i) + 1\right)^{p-2}(\hat{\beta}_k^+(\bar{s}_i))^2\Gamma(i+1,\bar{s}_i,\hat{S}_{i+1,k}) < 0. \tag{24}$$

This proves that κ is the maximum in the equation (21); we denote it by κ^*. So the maximum in the expression in (21) is attained at $\xi := \xi^* = \kappa^* w e^r$. Substitute this ξ^* to the equation (21), then we have

$$\hat{U}(i,\bar{s}_i,w) = \frac{w^p}{p}\frac{1}{n}\sum_{k=1}^{n} e^{rp}\left(\kappa^*\hat{\beta}_k^+(\bar{s}_i) + 1\right)^p\Gamma(i+1,\bar{s}_i,\hat{S}_{i+1,k}) \tag{25}$$

$$= \frac{w^p}{p}\Gamma(i,\bar{s}_i). \tag{26}$$

Similarly, if $\xi_{i-1} - \xi_i < 0$, then

$$\hat{W}_{i+1,k} = \frac{1}{(1-\lambda b_{i-1})}(\xi_i \hat{\beta}_k^-(\bar{s}_i) + w_i e^r) \tag{27}$$

,where

$$\hat{\beta}_k^-(\bar{s}_i) := (1-\lambda)\hat{S}_{i+1,k} - s_i e^r. \tag{28}$$

Then the rest of the proof is the same as the proof of the case $\xi_{i-1} - \xi_i > 0$ except simply replacing $\hat{\beta}_k^+(\bar{s}_i)$ by $\hat{\beta}_k^-(\bar{s}_i)$. This completes the proof of the proposition. ■

Now we use the same method in the article of the Batalova et al. [1] to make our algorithm forward in time and simple.

A further approximation is taken by assuming that the quantity $\Gamma(i+1, \bar{s}_{i+1}, \hat{S}_{i+1,k})$ in the equation (23) does not depend on k. See [1] for an explanation of what this approximation entails. We see that if $\xi_{i-1} - \xi_i > 0$, then κ_+^* is the unique solution of the equation

$$\sum_{k=1}^{n} (\kappa_+^* \hat{\beta}_k^+(\bar{s}_i) + 1)^{p-1} \hat{\beta}_k^+(\bar{s}_i) = 0. \tag{29}$$

Similarly, if $\xi_{i-1} - \xi_i < 0$, then κ_-^* is the unique solution of the equation

$$\sum_{k=1}^{n} (\kappa_-^* \hat{\beta}_k^-(\bar{s}_i) + 1)^{p-1} \hat{\beta}_k^-(\bar{s}_i) = 0. \tag{30}$$

Thus we see that equation (29) and equation (30) can be computed forward in time thanks to the values $\hat{\beta}_k^+(\bar{s}_i)$ and $\hat{\beta}_k^-(\bar{s}_i)$, defined in (18) and (28). We summarize these considerations here.

Summary 3.2 *Let $\hat{U}$ be as given (11). The maximum in this expected utility for $i = 1, \cdots, N-1$, is attained at the approximate value $\xi_i^*(\bar{s}_i) = w_i e^r \kappa^*$, where κ^* is the unique solution to the following algebraic equation.*

Corollary 3.3 *(i) If $\xi_{i-1} - \xi_i > 0$, κ^* uniquely solves the equation*

$$\sum_{k=1}^{n} (\kappa^* \hat{\beta}_k^+(\bar{s}_i) + 1)^{p-1} \hat{\beta}_k^+(\bar{s}_i) = 0, \tag{31}$$

where

$$\hat{\beta}_k^+(\bar{s}_i) := (1+\lambda)\hat{S}_{i+1,k} - s_i e^r. \tag{32}$$

(ii) If $\xi_{i-1} - \xi_i < 0$, κ^* uniquely solves the equation

$$\sum_{k=1}^{n} (\kappa^* \hat{\beta}_k^-(\bar{s}_i) + 1)^{p-1} \hat{\beta}_k^-(\bar{s}_i) = 0 \tag{33}$$

,where

$$\hat{\beta}_k^-(\bar{s}_i) := (1-\lambda)\hat{S}_{i+1,k} - s_i e^r. \tag{34}$$

Remark 3.4 *We notice that if the transaction cost rate λ is zero, in other words,if we assume that there is no the proportional transaction costs, then our results in Proposition 3.1 and Summary 3.2 are exactly the same as the ones in ([1], Section 4). We state these briefly with our notation here, to highlight the similarity: they show that the approximation of the expected utility at time $i+1$ is*

$$\hat{U}(i+1, \bar{s}_{i+1}, w) = \frac{w^p}{p} K(i+1, \bar{s}_{i+1}) \tag{35}$$

for some function K which does not depend on w, and the optimal portfolio is given by $\xi^* = we^r\kappa^*$ where κ^* is the unique solution of

$$\sum_{k=1}^{n} \hat{\beta}_k(\bar{s}_i) \left(\kappa^* \left(\hat{S}_{i+1,k} - s_i e^r \right) + 1 \right)^{p-1}. \tag{36}$$

4 Algorithm

For the practitioners' benefit, we restate all the above considerations in the form of a complete algorithm. It is similar to that which was used (but not fully stated) in Batalova et al. [1]; in our case, the maximization step calculates κ differently than theirs, depending on the sign of $\xi_{i-1} - \xi_i$: if we need to buy (resp. sell) stocks at time i, we use equation (31) (resp. equation (33)) to calculate κ.

Let us briefly summarize the algorithm below. To provide an approximate solution to our portfolio optimization problem (9), based on past observed prices up to time i, using the Euler and Bayesian Monte-Carlo methods yielding stochastic volatility particles $Y_{i,k}$and their corresponding probabilities $\hat{p}^{\bar{s}}_{i,k}$, we simluate the future stock prices $\hat{S}_{i,k}$ one unit of time into the future, using the SV dynamics on $[i, i+1]$. Then with these, we calculate κ_i^* and the optimal portfolio $\xi_i^* = w_i e^r \kappa_i^*$. In addition, we calculate a Monte-Carlo version the initial expected utility $U(0)$ based on the evolution of the optimal portfolio ξ^* for a number of different simulated scenarios $\bar{s}$; this step is not part of the optimization scheme, but allows us to estimate $U(0)$, which will be crucial to our analysis of the optimal trading rate in the next section.

Now we present our forward-in-time algorithm in more detail; to lighten the notation, we omit writing the functional dependence of all quantities on the fixed sequence of observations $\bar{s}$.

1). **Initialization** : Let the k-th initial stock price in the Monte-Carlo step $S_{0,k} = s_0$. Let $Y_{0,k} = y_0$, where $Y_{0,k}$ is the k-th particle of the filter $\hat{p}^{\bar{s}}_0(\cdot) = \frac{1}{n}\sum_{k=1}^{n} \delta_{Y_{0,k}}(\cdot)$. Let the initial wealth $W_{0,k} = w_0$ for all $k = 1, \cdots, n$. Let $\hat{U}(N, w) = w^p/p$. And decide the number of steps in the Euler scheme m and the number of particles in the stochastic volatility filter n. We choose n such that the particle filtering error order $n^{-1/2}$ (see

[3], [9]) is sufficiently small, and similarly for m and the Monte-Carlo error order $\sqrt{m}$. Let $h = \Delta t = t_i - t_{i-1}$.

2). **Calculation of the Stochastic Volatility Particle Filter** : it has two steps, a mutation step and a selection step. For $i = 1, \cdots, N-1$.

(i) (Mutation Step)
For $k = 1, \cdots, n$, we start with $(s_{i-1}, Y_{t_{i-1},k})$
For $0 \le j \le m-1$, calculate

$$\begin{cases} \hat{S}_{t_{i-1},j+1} = \hat{S}_{t_{i-1},j} + \alpha \hat{S}_{t_{i-1},j}\frac{h}{m} + \exp(\hat{Y}_{t_{i-1},j})\hat{S}_{t_{i-1},j}\sqrt{\frac{h}{m}}Z_{i-1,j} \\ \hat{Y}_{t_{i-1},j+1} = \hat{Y}_{t_{i-1},j} + \mu(\nu - \hat{Y}_{t_{i-1},j})\frac{h}{m} + \sqrt{\mu}\sqrt{\frac{h}{m}}Z'_{i-1,j}, \end{cases} \tag{37}$$

where $Z_{i-1,j}$ and $Z'_{i-1,j}$ are *i.i.d.* standard Normal random variables. This is the Euler scheme. Then let $\hat{S}_{t_i} = \hat{S}_{t_{i-1},m}$ and $\hat{Y}_{t_i,k} = \hat{Y}_{t_{i-1},m}$ and repeat this procedure n times to have n particles $\{\hat{S}_{t_i,k}, \hat{Y}_{t_i,k}\}_{k=1}^n$.

(ii) (Selection Step)
For $k = 1, \cdots, n$, we start with $(Y_{t_i,k})$ and set

$$C_i = \sum_{k=1}^{n} \psi_n(\hat{S}_{t_i,k} - s_i) \tag{38}$$

where ψ_n is the function given in (3). Then calculate :

$$\Psi_i^n = \begin{cases} \frac{1}{C_i}\sum_{k=1}^{n} \psi_n(\hat{S}_{t_i,k} - s_i)\delta_{\hat{Y}_{t_i,k}} & \text{if } C_i > 0 \\ \delta_0 & \text{otherwise} \end{cases} \tag{39}$$

Then sample n IID particles from the law Ψ_i^n (in other words, sample n independent times from the discrete distribution with atoms $\hat{Y}_{t_i,k}$ and corresponding weights $\hat{p}_{i,k} := \psi_n(\hat{S}_{t_i,k} - s_i)/C_i$). The resulting IID sample is $\{Y_{t_i,k}\}_{k=1}^n$ and together with its corresponding probabilities $\hat{p}_{i,k}$, is the stochastic volatility particle filter at time i.

3). **Calculation of optimal portfolio strategy** ξ_i : Let the initial optimal strategy $\xi_0^* = 0$ and the initial wealth be w_0, and specify the transaction cost rate λ.
For each $i = 1, 2, \cdots, N$, we assume that w_{i-1} and ξ_{i-1} have been determined, and do the following.

(i) Simulate $\hat{S}_{i+1,k}$ by the Euler scheme with time step $\frac{h}{m}$ for the pair (S, Y) starting from $(s_i, Y_{i,k})$.

(ii) (Maximization step) Set $\hat{\beta}_k^+ = (1+\lambda)\hat{S}_{i+1,k} - s_i e^r$. Find the unique solution κ_i of the equation

$$\sum_{k=1}^{n} (\kappa_i \hat{\beta}_k^+ + 1)^{p-1} \hat{\beta}_k^+ = 0. \tag{40}$$

With this κ_i, now we calculate w_i, where w_i is the current wealth before the portfolio manager changes the allocation of stock and risk-free account. In the Summary 3.2, we know $\xi_i = \kappa_i w_i e^r$ and

$$\hat{W}_{i,k} = \xi_{i-1}\hat{S}_{i,k} + (w_{i-1} - \xi_{i-1}s_{i-1})e^r - \lambda(\xi_i - \xi_{i-1})\hat{S}_{i,k}. \tag{41}$$

We rewrite this equation with $\xi_i = \kappa_i w_i e^r$; then we have

$$\hat{W}_{i,k}(1 + \lambda\kappa_i e^r \hat{S}_{i,k}) = \xi_{i-1}[(1+\lambda)\hat{S}_{i,k} - s_{i-1}e^r] + w_{i-1}e^r. \tag{42}$$

Thus we have

$$\hat{W}_{i,k} = \frac{1}{(1 + \lambda\kappa_i e^r \hat{S}_{i,k})}\{\xi_{i-1}[(1+\lambda)\hat{S}_{i,k} - s_{i-1}e^r] + w_{i-1}e^r\}. \tag{43}$$

We note that we can calculate the right side of the above equation since at time i, we have all information up to time $i-1$, the simulated stock price at time i, which is $\hat{S}_{i,k}$ and κ_i. Then we have $w_i = \frac{1}{n}\sum_{k=1}^{n}\hat{W}_{i,k}$. Then with this w_i and κ_i, we calculate $\xi_i = \kappa_i w_i e^r$. Calculate $\xi_i - \xi_{i-1}$.
If $\xi_i - \xi_{i-1} > 0$, then set the i-th step optimal strategy $\xi_i^* = \xi_i$.
If $\xi_i - \xi_{i-1} < 0$, then find the solution κ_i' of the equation

$$\sum_{k=1}^{n} (\kappa_i' \hat{\beta}_k' + 1)^{p-1} \hat{\beta}_k' = 0, \tag{44}$$

where $\hat{\beta}_k' = (1-\lambda)\hat{S}_{i+1,k} - s_i e^r$. To calculate $\xi_i = \kappa_i' w_i e^r$, we should calculate $w_i = \frac{1}{n}\sum_{k=1}^{n}\hat{W}_{i,k}$. Similarly to the procedure of finding $\hat{W}_{i,k}$ in the case of $\xi_i - \xi_{i-1} > 0$, we see

$$\hat{W}_{i,k} = \frac{1}{(1 - \lambda\kappa_i' e^r \hat{S}_{i,k})}\{\xi_{i-1}[(1-\lambda)\hat{S}_{i,k} - s_{i-1}e^r] + w_{i-1}e^r\}. \tag{45}$$

So now we have w_i and then we calculate $\xi_i = \kappa_i' w_i e^r$. We accept this as our optimal strategy in our algorithm, i.e. $\xi_i^* = \xi_i$.

4.) **Calculation of $\hat{U}(0)$** : Now we calculate the initial maximal expected future utilities $U(0, s_0, w_0)$. In Proposition 3.1, we saw that a good approximation for $U(0)$ is the average of the terminal wealth utilities i.e.

$$\frac{1}{n}\sum_{k=1}^{n}\frac{(w_{N,k})^p}{p}, \tag{46}$$

where n is the number of scenarios. First, we simulate a large number of scenarios $\bar{s}$ by using the algorithm in Step 1 and Step 2. Then we use the algorithm in Step 3 to calculate optimal strategies for each scenario $\bar{s}$. With these optimal strategies, we calculate the terminal wealth utilities for each scenario, and then compute the average of these utilities as in (46).

5 Optimization, recommendations, and conclusion

We run our algorithms above with the following typical parameter choices: $m = 50$, $n = 5000$, $p = 0.4$, $r = 0.02$, the number of scenarios $s = 300$, the initial wealth $w_0 = \$100,000$, $\alpha = 0.05$, $\mu = 5$, $\sigma(Y_0) = 0.25$ for a variety of N's and λ's. The time horizon is 1 year.

The results are summarized in figure 1 and figure 2, table 1, table 3, and table 2. The vertical axis in figure 1 and figure 2 is $\hat{U}(0)$, the expected utility of the terminal wealth, and the horizontal axis is trading frequency N. First, we notice that for each transaction cost rate, λ, each graph has optimal transaction numbers N^*, which maximizes the $\hat{U}(0)$ in figure 1 and figure 2. More specifically, looking at table 3, which corresponds to the case $\lambda = 0.008$, we see that $\hat{U}(0)$ is increasing from $N = 2$ to about $N = 32$, and decreases from then on. This means that if the transaction cost rate is 0.8%, the optimal trading frequency is about 32 trades per year, they get the highest return on their investment in this case. Second, the tail of each graph eventually becomes zero as N increases, as seen in figure 1 for all but the very smallest λ. In table 1, which corresponds to 10% transaction cost rate, an unreasonably high λ, the expected future utility is seen to get to 0 much faster than in the other cases, and the trader can evidently not make any profit at all unless she trades very infrequently.

The optimal trading frequency N^*, estimate for each λ as the point where the corresponding graph in figure 1 or figure 2, is seen to increases as λ decreases, an effect that is to be expected, and which we have quantified in table 2: this table 2 shows the optimal trading frequency N^* as a function of λ, which increases quickly as the transaction cost rate decreases to 0.

The notation $\text{avg}(W_N)$ in the tables is the average of the terminal wealth over the number of scenarios. The corresponding return for each frequency, is calculated over one year as (Terminal Wealth - Principal)/Principal, and is reported in table 3 for $\lambda = 0.008$, as well as in table 2 for all λ and their corresponding optimal rates. We notice the very important fact that there can be a wide range of N's which has approximately the same return as the return under the optimal N^* frequency. For example, if $\lambda = 0.008$ then for $10 \leq N \leq 36$, the returns in this range are around $3.5\% \sim 3.6\%$ which is almost same as the return of optimal N^* which is 3.66%. There is only a slight difference between the return for N^* and for the midpoint $\bar{N}$ of the stated range. This can be interpreted as a kind of robustness result for our portfolio: the trader may choose a frequency which is roughly of the same magnitude as N^*, and still hope to get an optimal return. Our tables also allow one to determine

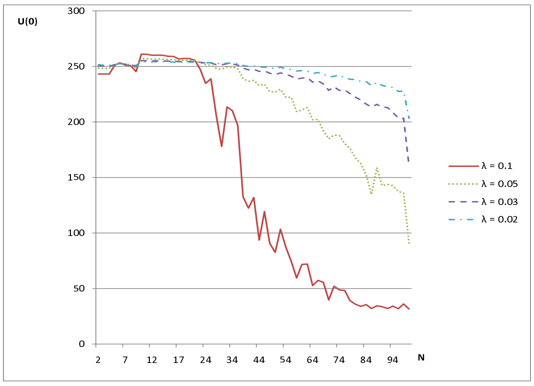

Figure 1: Expected Utility of W_N.

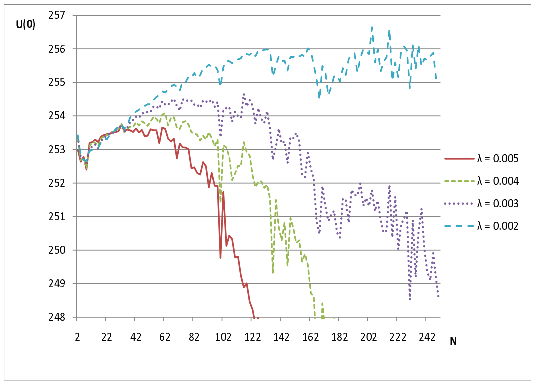

Figure 2: Expected Utility of W_N.

which frequencies will cause the portfolio to be less profitable on average than the risk free account: since r is 2%, those N which have less than \$102,000 as their terminal wealth are to be avoided. In the case of $\lambda = 0.008$, we see that this corresponds to $N > N^{**} = 64$.

Lastly, let us consider the interesting question of the online daily trader with fixed \$10 transaction cost per trade, and $N = 252$ trades per year (one trade per day). In order to get that 252 corresponds to an optimal trading frequency, our algorithm yields that it should come from a proportional transaction cost rate of $\lambda^* = 0.0018$. We can rephrase this information into fixed transaction costs. If each trade is approximately for $P := \frac{\$10}{\lambda^*} \simeq$ \$5,500, then the proportional transaction cost corresponding to every such trade will be $\lambda^*(\frac{10}{\lambda^*}) = \$10$. Therefore we conclude that a frequency of 252 per trades year (once per day), with \$10 transaction costs per trade, can be optimal, so long as each trade is sufficiently large, in this case at least \$5,500.

Table 1: $\hat{U}(0)$ for $\lambda = 0.1$

N	$\hat{U}(0)$	N	$\hat{U}(0)$	N	$\hat{U}(0)$	N	$\hat{U}(0)$	N	$\hat{U}(0)$
2	245.85	22	247.39	42	131.95	62	72.05	82	33.99
4	247.62	24	234.76	44	93.71	64	52.69	84	35.60
6	257.05	26	238.81	46	119.28	66	57.34	86	32.05
8	253.99	28	205.41	48	90.74	68	55.62	88	34.46
10	260.23	30	178.01	50	82.66	70	39.76	90	33.63
12	258.67	32	213.55	52	103.45	72	52.09	92	32.09
14	258.43	34	210.21	54	87.13	74	48.73	94	34.20
16	256.78	36	196.76	56	74.71	76	48.24	96	31.85
18	253.26	38	132.58	58	59.49	78	39.04	98	36.06
20	248.21	40	122.38	60	71.73	80	35.84	100	31.58

Table 2: N^*

λ	N^*	$\hat{U}(0)$	avg(W_N)	$\bar{N}$	return	N^{**}
0.1	10	260.96	111324	10	11.32%	20
0.05	11	256.94	107084	12	7.08%	24
0.03	11	255.18	105261	12	5.26%	28
0.02	14	254.28	104338	14	4.34%	30
0.01	16	253.17	103200	22	3.20%	40
0.008	32	253.62	103663	24	3.66%	54
0.005	32	253.74	103784	43	3.78%	79
0.004	62	254.09	104135	62	4.14%	122
0.003	116	254.66	104725	118	4.73%	162
0.002	204	256.65	106778	234	6.78%	358

Table 3: $\hat{U}(0)$ for $\lambda = 0.008$

N	$\hat{U}(0)$	$avg(W_N)$	return	N	$\hat{U}(0)$	$avg(W_N)$	return	N	$\hat{U}(0)$	$avg(W_N)$	return	N	$\hat{U}(0)$	$avg(W_N)$	return
2	252.87	102894.96	2.89%	28	253.26	103293.13	3.29%	54	252.47	102489.25	2.48%	80	249.61	99610.13	-0.39%
4	252.29	102306.38	2.31%	30	253.36	103393.52	3.39%	56	252.30	102311.87	2.31%	82	249.82	99824.66	-0.18%
6	252.95	102973.81	2.97%	32	253.62	103663.79	3.66%	58	251.96	101971.67	1.97%	84	249.84	99840.05	-0.16%
8	252.49	102510.98	2.51%	34	253.49	103527.24	3.53%	60	252.27	102289.13	2.29%	86	249.08	99086.43	-0.91%
10	253.48	103511.35	3.51%	36	253.50	103541.19	3.54%	62	252.26	102278.86	2.28%	88	249.21	99208.66	-0.79%
12	253.47	103508.19	3.51%	38	252.94	102970.19	2.97%	64	251.61	101618.78	1.62%	90	249.20	99201.32	-0.80%
14	253.50	103531.79	3.53%	40	252.79	102810.36	2.97%	66	251.68	101689.27	1.69%	92	248.54	98543.61	-1.46%
16	253.45	103490.67	3.49%	42	253.02	103051.51	3.05%	68	251.38	101387.07	1.39%	94	248.81	98816.69	-1.18%
18	253.53	103571.45	3.57%	44	252.76	102778.24	2.78%	70	250.75	100755.63	0.76%	96	247.83	97848.95	-2.15%
20	253.49	103524.61	3.52%	46	252.83	102854.02	2.85%	72	250.98	100987.70	0.99%	98	248.58	98590.53	-1.41%
22	253.51	103546.55	3.55%	48	252.54	102561.45	2.56%	74	250.63	100637.22	0.64%	100	243.57	93691.04	-6.31%
24	253.44	103479.92	3.48%	50	252.32	102340.37	2.34%	76	250.97	100975.55	0.98%				
26	253.50	103533.49	3.53%	52	252.75	102773.63	2.77%	78	250.75	100754.01	0.75%				

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