arXiv:0807.1208v2 [math.PR] 18 Jun 2010

SELF-SIMILARITY PARAMETER ESTIMATION AND REPRODUCTION PROPERTY FOR NON-GAUSSIAN HERMITE PROCESSES

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Abstract. Let $(Z_t^{(q,H)})_{t\in[0,1]}$ be a Hermite processes of order q and with Hurst parameter $H \in (\frac{1}{2}, 1)$. This process is H-self-similar, it has stationary increments and it exhibits long-range dependence. This class contains the fractional Brownian motion (for $q = 1$) and the Rosenblatt process (for $q = 2$). We study in this paper the variations of $Z^{(q,H)}$ by using multiple Wiener -Itô stochastic integrals and Malliavin calculus. We prove a reproduction property for this class of processes in the sense that the terms appearing in the chaotic decomposition of the their variations give birth to other Hermite processes of different orders and with different Hurst parameters. We apply our results to construct a consistent estimator for the self-similarity parameter from discrete observations of a Hermite process.

1. Introduction

1.1. Background and motivation. The variations of a stochastic process play a crucial role in its probabilistic and statistical analysis. Best known is the quadratic variation of a semimartingale, whose role is crucial in Itô's formula for semimartingales; this example also has a direct utility in practice, in estimating unknown parameters, such as volatility in financial models, in the so-called "historical" context. For self-similar stochastic processes the study of their variations constitutes a fundamental tool to construct good estimators for the self-similarity parameter. These processes are well suited to model various phenomena where long memory is an important factor (internet traffic, hydrology, econometrics, among others). The most important modeling task is then to determine or estimate the self-similarity parameter, because it is also typically responsible for the process's long memory and other regularity properties. Consequently, estimating this parameter represents an important research direction in theory and practice. Several approaches, such as wavelets, variations, maximum likelihood methods, have been proposed. We refer to the monograph [1] for a complete exposition. The family of Gaussian processes known as fractional Brownian motion (fBm) is particularly interesting, and most popular among self-similar processes, because

2000 *Mathematics Subject Classification.* Primary 60F05; Secondary 60H05, 60G18.
Key words and phrases. multiple stochastic integrals, Malliavin calculus, Hermite process, fractional Brownian motion, non-central limit theorems, quadratic variations, bootstrap.
* This author's research partially supported by NSF grant DMS 0606615.
* This author's research partially supported by NSF grants DMS 0606615 and 0907321.
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of fBm's stationary increments, its clear similarities and differences with standard Brownian motion, and the fact that its self-similarity parameter H, known as the Hurst parameter, is also clearly interpreted as the memory length parameter (the correlation of unit length increments n time units apart decays slowly at the speed n^{2H-2}), and the regularity parameter (fBm is α-Hölder continuous on any bounded time interval for any $\alpha < H$). Soon after fBm's inception, the study of its variations began in the 1970's and 1980's; of interest to us in the present article are several such studies of variations which uncovered a generalization of fBm to non-Gaussian processes known as the Rosenblatt process and other Hermite processes: [2], [5], [7], [17] or [18]. We briefly recall some relevant basic facts. We consider $(B^H_t)_{t\in[0,1]}$ a fractional Brownian motion with Hurst parameter $H \in (0,1)$. As such, B^H is the continuous centered Gaussian process with covariance function

$$R_H(t,s) = \mathbf{E}\left[B^H_t B^H_s\right] = \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right), \quad s,t \in [0,1].$$

Equivalently, $B^H_0 = 0$ and $\mathbf{E}\left[\left(B^H_s - B^H_t\right)^2\right] = |t-s|^{2H}$. It is the only Gaussian process H which is self-similar with stationary increments. Consider $0 = t_0 < t_1 < \ldots < t_N = 1$ a partition of the interval $[0,1]$ with $t_i = \frac{i}{N}$ for $i = 0, \ldots, N$ and define the following q variations

$$v^{(q)}_N = \sum_{i=0}^{N-1} H_q\left(\frac{B^H_{t_{i+1}} - B^H_{t_i}}{\left(\mathbf{E}\left[\left(B^H_{t_{i+1}} - B^H_{t_i}\right)^2\right]\right)^{\frac{1}{2}}}\right) = \sum_{i=0}^{N-1} H_q\left(\frac{\left(B^H_{t_{i+1}} - B^H_{t_i}\right)}{\left(t_{i+1} - t_i\right)^H}\right)$$

where H_q with $q \geq 2$ represents the Hermite polynomial of degree q. Then it follows from the above references that for:

- for $0 \leq H < 1 - \frac{1}{2q}$ the limit in distribution of $N^{-\frac{1}{2}} v^{(q)}_N$ is a centered Gaussian random variable,
- for $H = 1 - \frac{1}{2q}$ the limit in distribution of $(N \log N)^{-\frac{1}{2}} v^{(q)}_N$ is a centered Gaussian random variable,
- for $1 - \frac{1}{2q} < H < 1$ the limit of $N^{q(1-H)-1} v^{(q)}_N$ is a Hermite random variable of order q with self-similarity parameter $q(H-1)+1$.

This latter random variable is non-Gaussian; it equals the value at time 1 of a Hermite process, which is a stochastic process in the qth Wiener chaos with the same covariance structure as fBm; as such, it has stationary increments and shares the same self-similarity, regularity, and long memory properties as fBm; see Definition 2.1. We also mention that very recently, various interesting results have been proven for weighted power variations of stochastic processes such as fractional Brownian motion (see [11]), fractional Brownian sheets (see [15]), iterated Brownian motion (see [12]) or the solution of the stochastic heat equation (see [16] or [3]). Because of a natural coupling, the last limit above also holds in $L^2(\Omega)$ (see [13]). In the critical case $H = 1 - \frac{1}{2q}$ the limit is still Gaussian but the normalization involves a logarithm. These results are widely applied to estimation problems; to avoid the barrier $H = \frac{3}{4}$ that occurs in the case $q = 2$, one can use "higher-order filters", which means that the increments of the fBm are replaced by higher-order increments, such as discrete versions of higher-order derivatives, in order to obtain a Gaussian limit for any H (see [9], [8], [4]). The appearance of Hermite random variables in the above non-central limit theorems begs the study of Hermite processes as such. Their practical aspects are striking: they provide a wide class of processes from which to model long memory, self-similarity, and Hölder-regularity, allowing significant deviation from fBm and other Gaussian processes, without having to invoke non-linear stochastic differential equations based on fBm, and the notorious issues associated with them (see [14]). Just as in the case of fBm, the estimation of the Hermite process's parameter H is crucial for proper modeling; to our knowledge it has not been treated in the literature. We choose to tackle this issue by using variations methods, to find out how the above central and non-central limit theorems fit in a larger picture.

1.2. Main results: summary and discussion. In this article, we show results that are interesting from a theoretical viewpoint, such as the reproduction properties (the variations of Hermite processes give birth to other Hermite processes); and we provide an application to parameter estimation, in which care is taken to show that the estimators can be evaluated practically. Let $Z^{(q,H)}$ be a Hermite process of order q with selfsimilarity parameter $H \in (\frac{1}{2}, 1)$ as defined in Definition 2.1. Define the *centered quadratic variation statistic*

$$V_N := \frac{1}{N} \sum_{i=0}^{N-1} \left[\frac{\left(Z^{(q,H)}_{\frac{i+1}{N}} - Z^{(q,H)}_{\frac{i}{N}} \right)^2}{\mathbf{E}\left[\left(Z^{(q,H)}_{\frac{i+1}{N}} - Z^{(q,H)}_{\frac{i}{N}} \right)^2 \right]} - 1 \right]. \tag{1.1}$$

Also for $H \in (1/2, 1)$, and $q \in \mathbf{N} \setminus \{0\}$, we define a constant which will recur throughout this article:

$$d(H,q) \quad : \quad = \frac{\left(H(2H-1)\right)^{1/2}}{\left(q!(H'(2H'-1))^q\right)^{1/2}}, \quad H' = 1 + \frac{(H-1)}{q}. \tag{1.2}$$

We prove that, under suitable normalization, this sequence converges in $L^2(\Omega)$ to a Rosenblatt random variable.

Theorem 1.1. *Let $H \in (1/2, 1)$ and $q \in \mathbf{N} \setminus \{0\}$. Let $Z^{(q,H)}$ be a Hermite process of order q and self-similarity parameter H (see Definition 2.1). Let c be an explicit positive constant (it is defined in Proposition 3.1). Then $cN^{(2-2H)/q} V_N$ converges in $L^2(\Omega)$ to a standard Rosenblatt random variable with parameter $H'' := \frac{2(H-1)}{q} + 1$, that is, to the value at time 1 of a Hermite process of order 2 and self-similarity parameter H''.*

The Rosenblatt random variable is a double integral with respect to the same Wiener process used to define the Hermite process; it is thus an element of the second Wiener chaos. Our result shows that fBm is the only Hermite process for which there exists a range of parameters allowing normal convergence of the quadratic variation, while for all other Hermite processes, convergence to a second chaos random variable is universal. Our proofs are based on chaos expansions into multiple Wiener integrals and Malliavin calculus. The main line of the proof is as follows: since the variable $Z_t^{(q,H)}$ is an element of the qth Wiener chaos, the product formula for multiple integrals implies that the statistics V_N can be decomposed into a sum of multiple integrals from the order 2 to the order $2q$. The dominant term in this decomposition, which gives the final renormalization order $N^{(2-2H)/q}$, is the term which is a double Wiener integral, and one proves it *always* converges to a Rosenblatt random variable; all other terms are of much lower orders, which is why the only remaining term, after renormalization, converges to a second chaos random variable. The difference with the fBm case comes from the limit of the term of order 2, which in that case is sometimes Gaussian and sometimes Rosenblatt-distributed, depending on the value of H. We also study the limits of the other terms in the decomposition of V_N, those of order higher than 2, and we obtain interesting facts: all these terms, except the term of highest order $2q$, have limits which are Hermite random variables of various orders and self-similarity parameters. We call this *the reproduction property* for Hermite processes, because from one Hermite process of order q, one can reconstruct other Hermite processes of all lower orders. The exception to this rule is that the normalized term of highest order $2q$ converges to a Hermite r.v. of order $2q$ if $H > 3/4$, but converges to Gaussian limit if $H \in (1/2, 3/4]$. Summarizing, we have the following.

Theorem 1.2. *Let $Z^{(q,H)}$ be again a Hermite process, as in the previous theorem. Let T_{2n} be the term of order $2n$ in the Wiener chaos expansion of V_N: this is a multiple Wiener integral of order $2n$, and we write $V_N = \sum_{n=1}^{q} c_{2n} T_{2n}$ where $c_{2q-2k} = k!\binom{q}{k}^2$.*

- *For every $H \in (1/2, 1)$ and for every $k = 1, \ldots, q-1$ we have convergence of the expression $(z_{k,H})^{-1} N^{(2-2H')k} T_{2k}$ in $L^2(\Omega)$ to $Z^{(r,K)}$, a Hermite random variable of order $r = 2k$ with self-similarity parameter $K = 2k(H'-1)+1$, where $z_{k,H}$ is a constant.*
- *For every $H \in (1/2, 3/4)$, with $k = q$, we have convergence of $x_{1,H,q}^{-1/2}\sqrt{N} T_{2q}$ to a standard normal distribution, with $x_{1,H,q}$ a positive constant depending on H and q.*
- *For every $H \in (3/4, 1)$, with $k = q$, we have convergence of $x_{2,H,q}^{-1/2} N^{2-2H} T_{2q}$ in $L^2(\Omega)$ to $Z^{2q,2H-1}$, a Hermite r.v. of order $2q$ with parameter $2H-1$; with $x_{2,H,q}$ a positive constant depending on H and q.*
- *For $H = 3/4$, with $k = q$, we have convergence of $\sqrt{N/\log N} x_{3,H,q}^{-1/2} T_{2q}$ to a standard normal distribution, with $x_{3,H}$ a positive constant depending on H and q.*

Some of the aspects of this theorem had been discovered in the case of $q = 2$ (Rosenblatt process) in [20]. In that paper, the existence of a higher-chaos-order term with normal convergence had been discovered for the Rosenblatt process with $H < 3/4$, while the case of $H \geq 3/4$ had not been studied. The entire spectrum of convergences in Theorem 1.2 was not apparent in [20], however, because it was unclear whether the term T_2's convergence to a Rosenblatt r.v. was due to the fact that we were dealing with input coming from a Rosenblatt process, or whether it was a more general function of the structure of the second Wiener chaos; here we see that the second alternative is true. The paper [20] also exhibited a remarkable structure of the Rosenblatt data when $H < 3/4$. In that case, as we see in Theorem 1.2, there are only two terms in the expansion of V_N, T_2 and T_4; moreover, and this is the remarkable feature, the proper normalization of the term T_2 converges to none other than the Rosenblatt r.v. at time 1. Since this value is part of the observed data, one can subtract it to take advantage of the Gaussian limit of the renormalized T_4, including an application to parameter estimation in [20]. In Theorem 1.2 above, if $q > 2$, even if $H \leq 3/4$, by which a Gaussian limit can be constructed from the renormalized T_{2q}, we still have at least two other terms $T_2, T_4, \cdots T_{2q-2}$, and all but at most one of these will converge in $L^2(\Omega)$ to Hermite processes with different orders, all different from q, which implies that they are not directly observed. This means our Theorem 1.2 proves that the operation performed with Rosenblatt data, subtracting an observed quantity to isolate T_{2q} and its Gaussian asymptotics, is not possible with any higher-order Hermite processes. The last aspect of this paper applies Theorem 1.1 to estimating the parameter H. Let S_N be the empirical mean of the individual squared increments

$$S_N = \frac{1}{N} \sum_{i=0}^{N-1} \left(Z^{(q,H)}_{\frac{i+1}{N}} - Z^{(q,H)}_{\frac{i}{N}} \right)^2$$

and let

$$\hat{H}_N = (\log S_N) / (2 \log N).$$

We show that $\hat{H}_N$ is a strongly consistent estimator of H, and we show asymptotic Rosenblatt distribution for $N^{2(1-H)/q} \left(H - \hat{H} \right) \log N$. The fact that this estimator fails to be asymptotically normal is not a problem in itself. What is more problematic is the fact that if one tries to check ones assumptions on the data by comparing the asymptotics of $\hat{H}$ with a Rosenblatt distribution, one has to know something about the normalization constant $N^{2(1-H)/q}$. Here, we prove in addition that one may replace H in this formula by $\hat{H}_N$, so that the asymptotic properties of $\hat{H}_N$ can actually be checked.

Theorem 1.3. *The estimator $\hat{H}_N$ is strongly consistent, i.e.* $\lim_{N\to\infty} \hat{H}_N = H$ *almost surely. Moreover there exists a standard Rosenblatt random variable R with self-similarity parameter* $1 + 2(H-1)/q$ *such that*

$$\lim_{N\to\infty} \mathbf{E} \left[\left| 2N^{2\left(1-\hat{H}_N\right)/q} \left(H - \hat{H}_N \right) \log N - c_2 c_{1,H}^{1/2} R \right| \right] = 0,$$

Here $a(H') = (1 + (H-1)/q)(1 + 2(H-1)/q)$.

Replacing the constant $c_{1,H}$ by its value in terms of $\hat{H}_N$ instead of H also seems to lead to the above convergence. However, this article does not present any numerical results illustrating model validation based on the above theorem; moreover such applications would be much more sensitive to the convergence speed than to the actual constants; therefore we omit the proof of this further improvement on $c_{1,H}$.

2. Preliminaries

2.1. Multiplication in Wiener Chaos. Let $(W_t)_{t\in[0,1]}$ be a classical Wiener process on a standard Wiener space $(\Omega, \mathcal{F}, \mathbf{P})$. If $f \in L^2([0,1]^m)$ with $m \geq 1$ integer, we introduce the multiple Wiener-Itô integral of f with respect to W. We refer to [14] for a detailed exposition of the construction and the properties of multiple Wiener-Itô integrals. Let $f \in \mathcal{S}$, i.e. f is an elementary function, meaning that

$$f = \sum_{i_1,\ldots,i_m} c_{i_1,\ldots i_m} 1_{A_{i_i}\times\ldots\times A_{i_m}}$$

where $i_1, \ldots, i_m$ describes a finite set and the coefficients satisfy $c_{i_1,\ldots i_m} = 0$ if two indices i_k and i_l are equal and the sets $A_i \in \mathcal{B}([0,1])$ are disjoints. For such a step function f we define

$$I_m(f) = \sum_{i_1,\ldots,i_m} c_{i_1,\ldots i_m} W(A_{i_1}) \ldots W(A_{i_m})$$

where we put $W([a,b]) = W_b - W_a$. It can be seen that the application I_m constructed above from $\mathcal{S}$ to $L^2(\Omega)$ satisfies

$$\mathbf{E}\left[I_n(f)I_m(g)\right] = n!\langle f, g\rangle_{L^2([0,1]^n)} \text{ if } m = n \tag{2.1}$$

and

$$\mathbf{E}\left[I_n(f)I_m(g)\right] = 0 \text{ if } m \neq n.$$

It also holds that $I_n(f) = I_n\left(\tilde{f}\right)$ where $\tilde{f}$ denotes the symmetrization of f defined by $\tilde{f}(x_1, \ldots, x_n) = \frac{1}{n!}\sum_{\sigma\in_n} f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. Since the set $\mathcal{S}$ is dense in $L^2([0,1]^n)$ the mapping I_n can be extended to a linear continuous operator from from $L^2([0,1]^n)$ to $L^2(\Omega)$ and the above properties hold true for this extension. Note also that $I_n(f)$ with f symmetric can be viewed as an iterated stochastic integral

$$I_n(f) = n!\int_0^1\int_0^{t_n}\ldots\int_0^{t_2} f(t_1, \ldots, t_n)dW_{t_1}\ldots dW_{t_n};$$

here the integrals are of Itô type; this formula is easy to show for elementary f's, and follows for general symmetric function $f \in L^2([0,1]^n)$ by a density argument. We recall the product for two multiple integrals (see [14]): if $f \in L^2([0,1]^n)$ and $g \in L^2([0,1]^m)$ are symmetric then it holds that

$$I_n(f)I_m(g) = \sum_{l=0}^{m\wedge n} l!\binom{m}{l}\binom{n}{l} I_{m+n-2l}(f \otimes_l g) \tag{2.2}$$

where the contraction $f \otimes_l g$ belongs to $L^2([0,1]^{m+n-2l})$ for $l = 0, 1, \ldots, m \wedge n$ and it is given by

$$(f \otimes_l g)(s_1, \ldots, s_{n-l}, t_1, \ldots, t_{m-l}) = \int_{[0,1]^l} f(s_1, \ldots, s_{n-l}, u_1, \ldots, u_l)g(t_1, \ldots, t_{m-l}, u_1, \ldots, u_l)du_1 \ldots du_l. \tag{2.3}$$

2.2. The Hermite process. Recall that the fractional Brownian motion process $(B_t^H)_{t\in[0,1]}$ with Hurst parameter $H \in (0,1)$ can be written as

$$B_t^H = \int_0^t K^H(t,s)dW_s, \quad t \in [0,1]$$

where $(W_t, t \in [0,T])$ is a standard Wiener process, $K^H(t,s) = c_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du$ if $t > s$ (and it is zero otherwise), $c_H = \left(\frac{H(2H-1)}{\beta(2-2H, H-\frac{1}{2})}\right)^{\frac{1}{2}}$ and $\beta(\cdot,\cdot)$ is the Beta function. For $t > s$, the kernel's derivative is $\frac{\partial K^H}{\partial t}(t,s) = c_H \left(\frac{s}{t}\right)^{\frac{1}{2}-H} (t-s)^{H-\frac{3}{2}}$. Fortunately we will not need to use these expressions explicitly, since they will be involved below only in integrals whose expressions are known.

We will denote by $(Z_t^{(q,H)})_{t\in[0,1]}$ the Hermite process *with self-similarity parameter* $H \in (1/2, 1)$. Here $q \geq 1$ is an integer. The Hermite process can be defined in two ways: as a multiple integral with respect to the standard Wiener process $(W_t)_{t\in[0,1]}$; or as a multiple integral with respect to a fractional Brownian motion with suitable Hurst parameter. We adopt the first approach throughout the paper. We refer to [13] of [19] for the following integral representation of Hermite processes.

Definition 2.1. The Hermite process $(Z_t^{(q,H)})_{t\in[0,1]}$ of order $q \geq 1$ and with self-similarity parameter $H \in (\frac{1}{2}, 1)$ is given by

$$Z_t^{(q,H)} = d(H,q) \int_{[0,t]^q} dW_{y_1} \dots dW_{y_q} \left(\int_{y_1 \vee \dots \vee y_q}^t \partial_1 K^{H'}(u, y_1) \dots \partial_1 K^{H'}(u, y_q) du \right) \tag{2.4}$$

for $t \in [0,1]$, where $K^{H'}$ is the usual kernel of the fractional Brownian motion and

$$H' = 1 + \frac{H-1}{q} \iff (2H'-2)q = 2H-2. \tag{2.5}$$

Of fundamental importance is the fact that the covariance of $Z^{(q,H)}$ is identical to that of fBm, namely

$$\mathbf{E}\left[Z_s^{(q,H)} Z_t^{(q,H)}\right] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

The constant $d(H,q)$, given in (1.2) on page 3, is chosen to avoid any additional multiplicative constants. We stress that $Z^{(q,H)}$ is far from Gaussian for $q > 1$, since it is formed of multiple Wiener integrals of order q.

The basic properties of the Hermite process are listed below: i) the Hermite process $Z^{(q,H)}$ is H-selfsimilar and it has stationary increments; ii) the mean square of the increment is given by

$$\mathbf{E}\left[\left|Z_t^{(q,H)} - Z_s^{(q,H)}\right|^2\right] = |t-s|^{2H}; \tag{2.6}$$

as a consequence, it follows from Kolmogorov's continuity criterion that $Z^{(q,H)}$ has Hölder-continuous paths of any order $\delta < H$; iii) it exhibits long-range dependence in the sense that $\sum_{n\geq 1}\mathbf{E}\left[Z_1^{(q,H)}(Z_{n+1}^{(q,H)}-Z_n^{(q,H)})\right]=\infty$. In fact, the summand in this series is of order n^{2H-2}. This property is identical to that of fBm since the processes share the same covariance structure, and the property is well-known for fBm with $H>1/2$; iv) if $q=1$ then $Z^{(1,H)}$ is a standard Brownian motion with Hurst parameter H while for $q\geq 2$ this stochastic process is not Gaussian. In the case $q=2$ this stochastic process is known as *the Rosenblatt process*.

3. Variations of the Hermite process

Since $\mathbf{E}\left(Z_{\frac{i+1}{N}}^{(q,H)}-Z_{\frac{i}{N}}^{(q,H)}\right)^2=N^{-2H}$ and by (2.6), the centered quadratic variation statistic V_N given in the introduction can be written as

$$V_N=N^{2H-1}\sum_{i=0}^{N-1}\left[\left(Z_{\frac{i+1}{N}}^{(q,H)}-Z_{\frac{i}{N}}^{(q,H)}\right)^2-N^{-2H}\right].$$

Let $I_i:=[\frac{i}{N},\frac{i+1}{N}]$. In preparation for calculating the variance of V_N we will find an explicit expansion of V_N in Wiener chaos. We have $Z_{\frac{i+1}{N}}^{(q,H)}-Z_{\frac{i}{N}}^{(q,H)}=I_q\left(f_{i,N}\right)$ where we denoted by $f_{i,N}(y_1,\ldots,y_q)$ the expression

$$\begin{aligned}&1_{[0,\frac{i+1}{N}]}(y_1\vee\ldots\vee y_q)d(H,q)\int_{y_1\vee\ldots\vee y_q}^{\frac{i+1}{N}}\partial_1K^{H'}(u,y_1)\ldots\partial_1K^{H'}(u,y_q)du\\&-1_{[0,\frac{i}{N}]}(y_1\vee\ldots\vee y_q)d(H,q)\int_{y_1\vee\ldots\vee y_q}^{\frac{i}{N}}\partial_1K^{H'}(u,y_1)\ldots\partial_1K^{H'}(u,y_q)du.\end{aligned}$$

Using the product formula for multiple integrals (2.2), we obtain

$$I_q(f_{i,N})I_q(f_{i,N})=\sum_{l=0}^{q}l!\binom{q}{k}^2 I_{2q-2l}\left(f_{i,N}\otimes_l f_{i,N}\right)$$

where the $f\otimes_l g$ denotes the l-contraction of the functions f and g given by (2.3). Let us compute these contractions; for $l=q$ we have

$$\langle f_{i,N}\otimes_q f_{i,N}\rangle=q!\langle f_{i,N},f_{i,N}\rangle_{L^2([0,1])^{\otimes q}}=\mathbf{E}\left[\left(Z_{\frac{i+1}{N}}^{(q,H)}-Z_{\frac{i}{N}}^{(q,H)}\right)^2\right]=N^{-2H}.$$

Throughout the paper the notation $\partial_1K(t,s)$ will be used for $\partial_1K^{H'}(t,s)$. For $l=0$ we have

$$\begin{aligned}\langle f_{i,N}\otimes_0 f_{i,N})(y_1,\ldots y_q,z_1,\ldots,z_q\rangle&=(f_{i,N}\otimes f_{i,N})(y_1,\ldots y_q,z_1,\ldots,z_q)\\&=f_{i,N}(y_1,\ldots,y_q)f_{i,N}(z_1,\ldots,z_q)\end{aligned}$$

while for $1 \le k \le q-1$

$$\langle f_{i,N} \otimes_k f_{i,N})(y_1, \dots y_{q-k}, z_1, \dots, z_{q-k}\rangle = d(H,q)^2 \int_{[0,1]^k} d\alpha_1 \dots d\alpha_k$$

$$\Bigg(\mathbf{1}^{y_i}_{i+1,q-k} \mathbf{1}^{\alpha_i}_{i+1,k} \int_{\mathbf{I}^y_{i+1,k}} du \partial_1 K(u,y_1) \dots \partial_1 K(u, y_{q-k}) \partial_1 K(u,\alpha_1) \dots \partial_1 K(u,\alpha_k)$$

$$- \mathbf{1}^{y_i}_{i,q-k} \mathbf{1}^{\alpha_i}_{i,k} \int_{\mathbf{I}^y_{i,k}} du \partial_1 K(u,y_1) \dots \partial_1 K(u, y_{q-k}) \partial_1 K(u,\alpha_1) \dots \partial_1 K(u,\alpha_k) \Bigg)$$

$$\Bigg(\mathbf{1}^{z_i}_{i+1,q-k} \mathbf{1}^{\alpha_i}_{i+1,k} \int_{\mathbf{I}^z_{i+1,k}}^{\frac{i+1}{N}} dv \partial_1 K(v,z_1) \dots \partial_1 K(v, z_{q-k}) \partial_1 K(v,\alpha_1) \dots \partial_1 K(v,\alpha_k)$$

$$- \mathbf{1}^{y_i}_{i,q-k} \mathbf{1}^{\alpha_i}_{i,k} \int_{\mathbf{I}^z_{i,k}}^{\frac{i}{N}} dv \partial_1 K(v,z_1) \dots \partial_1 K(v, z_{q-k}) \partial_1 K(v,\alpha_1) \dots \partial_1 K(v,\alpha_k) \Bigg)$$

where $\mathbf{1}^{x_j}_{i,k}$ denotes the indicator function $1_{[0,\frac{i}{N}]^k}(x_j)$ with x being y, z, or α, and $\mathbf{I}^x_{i,k}$ denotes the interval $[x_1 \vee \dots \vee x_{q-k} \vee \alpha_1 \dots \vee \alpha_k; i/N]$, with x being y or z. By interchanging the order of the integration we get

$$\begin{aligned}
& \langle f_{i,N} \otimes_k f_{i,N})(y_1, \dots y_{q-k}, z_1, \dots, z_{q-k}\rangle \\
= \quad & d(H,q)^2 \Bigg\{ 1_{[0,\frac{i+1}{N}]^{2q-2k}}(y_i, z_i) \int_{y_1 \vee \dots y_{q-k}}^{\frac{i+1}{N}} du \partial_1 K(u,y_1) \dots \partial_1 K(u,y_{q-k}) \\
& \int_{z_1 \vee \dots z_{q-k}}^{\frac{i+1}{N}} dv \partial_1 K(v,z_1) \dots \partial_1 K(v,z_{q-k}) \left(\int_0^{u \wedge v} \partial_1 K(u,\alpha) \partial_1 K(v,\alpha) d\alpha \right)^k \\
& -1_{[0,\frac{i+1}{N}]^{q-k}}(y_i) 1_{[0,\frac{i}{N}]^{q-k}}(z_i) \int_{y_1 \vee \dots y_{q-k}}^{\frac{i+1}{N}} du \partial_1 K(u,y_1) \dots \partial_1 K(u,y_{q-k}) \\
& \int_{z_1 \vee \dots z_{q-k}}^{\frac{i}{N}} dv \partial_1 K(v,z_1) \dots \partial_1 K(v,z_{q-k}) \left(\int_0^{u \wedge v} \partial_1 K(u,\alpha) \partial_1 K(v,\alpha) d\alpha \right)^k \\
& -1_{[0,\frac{i}{N}]^{q-k}}(y_i) 1_{[0,\frac{i+1}{N}]^{q-k}}(z_i) \int_{y_1 \vee \dots y_{q-k}}^{\frac{i}{N}} du \partial_1 K(u,y_1) \dots \partial_1 K(u,y_{q-k}) \\
& \int_{z_1 \vee \dots z_{q-k}}^{\frac{i+1}{N}} dv \partial_1 K(v,z_1) \dots \partial_1 K(v,z_{q-k}) \left(\int_0^{u \wedge v} \partial_1 K(u,\alpha) \partial_1 K(v,\alpha) d\alpha \right)^k \\
& +1_{[0,\frac{i}{N}]^{q-k}}(y_i) 1_{[0,\frac{i}{N}]^{q-k}}(z_i) \int_{y_1 \vee \dots y_{q-k}}^{\frac{i}{N}} du \partial_1 K(u,y_1) \dots \partial_1 K(u,y_{q-k}) \\
& \int_{z_1 \vee \dots z_{q-k}}^{\frac{i}{N}} dv \partial_1 K(v,z_1) \dots \partial_1 K(v,z_{q-k}) \left(\int_0^{u \wedge v} \partial_1 K(u,\alpha) \partial_1 K(v,\alpha) d\alpha \right)^k \Bigg\}
\end{aligned}$$

and since

$$\int_0^{u \wedge v} \partial_1 K(u,\alpha) \partial_1 K(v,\alpha) d\alpha = a(H') |u-v|^{2H'-2}$$

with $a(H') = H'(2H'-1)$, we obtain

$$
\begin{aligned}
&\langle f_{i,N} \otimes_k f_{i,N} \rangle (y_1, \dots y_{q-k}, z_1, \dots, z_{q-k}) = d(H,q)^2 a(H')^k \\
&\times \Bigg\{ 1_{[0,\frac{i+1}{N}]^{q-k}}(y_i) 1_{[0,\frac{i+1}{N}]^{q-k}}(z_i) \int_{y_1 \vee \dots y_{q-k}}^{\frac{i+1}{N}} du \partial_1 K(u, y_1) \dots \partial_1 K(u, y_{q-k}) \\
&\quad \times \int_{z_1 \vee \dots z_{q-k}}^{\frac{i+1}{N}} dv \partial_1 K(v, z_1) \dots \partial_1 K(v, z_{q-k}) |u-v|^{(2H'-2)k} \\
&\quad - 1_{[0,\frac{i+1}{N}]^{q-k}}(y_i) 1_{[0,\frac{i}{N}]^{q-k}}(z_i) \int_{y_1 \vee \dots y_{q-k}}^{\frac{i+1}{N}} du \partial_1 K(u, y_1) \dots \partial_1 K(u, y_{q-k}) \\
&\quad \times \int_{z_1 \vee \dots z_{q-k}}^{\frac{i}{N}} dv \partial_1 K(v, z_1) \dots \partial_1 K(v, z_{q-k}) |u-v|^{(2H'-2)k} \\
&\quad - 1_{[0,\frac{i}{N}]^{q-k}}(y_i) 1_{[0,\frac{i+1}{N}]^{q-k}}(z_i) \int_{y_1 \vee \dots y_{q-k}}^{\frac{i}{N}} du \partial_1 K(u, y_1) \dots \partial_1 K(u, y_{q-k}) \\
&\quad \times \int_{z_1 \vee \dots z_{q-k}}^{\frac{i+1}{N}} dv \partial_1 K(v, z_1) \dots \partial_1 K(v, z_{q-k}) |u-v|^{(2H'-2)k} \\
&\quad + 1_{[0,\frac{i}{N}]^{q-k}}(y_i) 1_{[0,\frac{i}{N}]^{q-k}}(z_i) \int_{y_1 \vee \dots y_{q-k}}^{\frac{i}{N}} du \partial_1 K(u, y_1) \dots \partial_1 K(u, y_{q-k}) \\
&\quad \times \int_{z_1 \vee \dots z_{q-k}}^{\frac{i}{N}} dv \partial_1 K(v, z_1) \dots \partial_1 K(v, z_{q-k}) |u-v|^{(2H'-2)k} \Bigg\}.
\end{aligned}
\tag{3.1}
$$

As a consequence, we can write

$$V_N = T_{2q} + c_{2q-2} T_{2q-2} + \dots + c_4 T_4 + c_2 T_2 \tag{3.2}$$

where

$$c_{2q-2k} := k! \binom{q}{k}^2 \tag{3.3}$$

are the combinatorial constants from the product formula for $0 \le k \le q-1$, and

$$T_{2q-2k} := N^{2H-1} I_{2q-2k} \left(\sum_{i=0}^{N-1} f_{i,N} \otimes_k f_{i,N} \right), \tag{3.4}$$

where the integrands in the last formula above are given explicitly in (3.1). This Wiener chaos decomposition of V_N allows us to find V_N's precise order of magnitude via its variance's asymptotics.

Proposition 3.1. *With*

$$c_{1,H} = \frac{4d(H,q)^4 (H'(2H'-1))^{2q}}{(4H'-3)(4H'-2)[(2H'-2)(q-1)+1]^2[(H'-1)(q-1)+1]^2}, \tag{3.5}$$

it holds that

$$\lim_{N \to \infty} \mathbf{E} \left[c_{1,H}^{-1} N^{(2-2H')2} c_2^{-2} V_N^2 \right] = 1.$$

Proof. We only need to estimate the L^2 norm of each term appearing in the chaos decomposition (3.2) of V_N, since these terms are orthogonal in L^2. We can write, for $0 \le k \le q-1$,

$$\begin{aligned}
\mathbf{E}\left[T_{2q-2k}^2\right] &= N^{4H-2}(2q-2k)! \left\| \left(\sum_{i=0}^{N-1} f_{i,N} \otimes_k f_{i,N} \right)^s \right\|^2_{L^2([0,1]^{2q-2k})} \\
&= N^{4H-2}(2q-2k)! \sum_{i,j=0}^{N-1} \langle f_{i,N} \tilde{\otimes}_k f_{i,N}, f_{j,N} \tilde{\otimes}_k f_{j,N} \rangle_{L^2([0,1]^{2q-2k})}
\end{aligned}$$

where $(g)^s = \tilde{g}$ and $f_{i,N}\tilde{\otimes}_k f_{i,N}$ denotes the symmetrization of the function $f_{i,N} \otimes_k f_{i,N}$. We will consider first the term T_2 obtained for $k = q-1$. In this case, the kernel $\sum_{i=0}^{N-1} f_{i,N} \otimes_{q-1} f_{i,N}$ is symmetric and we can avoid its symmetrization. Therefore

$$\begin{aligned}
\mathbf{E}\left[T_2^2\right] &= 2!\, N^{4H-2} \| \sum_{i=0}^{N-1} f_{i,N} \otimes_{q-1} f_{i,N} \|^2_{L^2([0,1]^2)} \\
&= 2!\, N^{4H-2} \sum_{i,j=0}^{N-1} \langle f_{i,N} \otimes_{q-1} f_{i,N}, f_{j,N} \otimes_{q-1} f_{j,N} \rangle_{L^2([0,1]^2)}.
\end{aligned}$$

We compute now the scalar product in the above expression. By using Fubini theorem, we end up with the following easier expression

$$\begin{aligned}
&\langle f_{i,N} \otimes_{q-1} f_{i,N}, f_{j,N} \otimes_{q-1} f_{j,N} \rangle_{L^2([0,1]^2)} = a(H')^{2q} d(H,q)^4 \int_{I_i}\int_{I_i}\int_{I_j}\int_{I_j} \\
&\times |u-v|^{(2H'-2)(q-1)} |u'-v'|^{(2H'-2)(q-1)} |u-u'|^{2H'-2} |v-v'|^{2H'-2} dv' du' dv du
\end{aligned}$$

Using the change of variables $y = (u - \frac{i}{N})N$ and similarly for the other variables, we now obtain

$$\begin{aligned}
&\mathbf{E}\left[T_2^2\right] = 2d(H,q)^4 (H'(2H'-1))^{2q} N^{4H-2} N^{-4} N^{-(2H'-2)2q} \\
&\times \sum_{i,j=0}^{N-1} \int_0^1\int_0^1\int_0^1\int_0^1 dy dz dy' dz' |y-z|^{(2H'-2)(q-1)} |y'-z'|^{(2H'-2)(q-1)} \\
&\times |y-y'+i-j|^{(2H'-2)} |z-z'+i-j|^{(2H'-2)}.
\end{aligned}$$

This can be viewed as the sum of a diagonal part ($i = j$) and a non-diagonal part ($i \neq j$), where the non-diagonal part is dominant, as the reader will readily check.

Therefore, the behavior of $\mathbf{E}\left[T_2^2\right]$ will be given by

$$\begin{aligned}
&\mathbf{E}\left[T_2'^2\right] \\
:= \quad & 2!d(H,q)^4(H'(2H'-1))^{2q}N^{-2}2\sum_{i>j}\int_0^1\int_0^1\int_0^1\int_0^1 dydzdy'dz' \\
&\times(|y-z||y'-z'|)^{(2H'-2)(q-1)}(|y-y'+i-j||z-z'+i-j|)^{(2H'-2)} \\
= \quad & 2!d(H,q)^4(H'(2H'-1))^{2q}N^{-2}2\sum_{i=0}^{N-2}\sum_{\ell=2}^{N-i}\int_0^1\int_0^1\int_0^1\int_0^1 dydzdy'dz' \\
&\times(|y-z||y'-z'|)^{(2H'-2)(q-1)}(|y-y'+\ell-1||z-z'+\ell-1|)^{(2H'-2)} \\
= \quad & 2!d(H,q)^4(H'(2H'-1))^{2q}N^{-2}2\sum_{\ell=2}^{N}(N-\ell+1)\int_0^1\int_0^1\int_0^1\int_0^1 dydzdy'dz' \\
&\times(|y-z||y'-z'|)^{(2H'-2)(q-1)}(|y-y'+\ell-1||z-z'+\ell-1|)^{(2H'-2)}.
\end{aligned}$$

As in [20] note that

$$\begin{aligned}
&\frac{1}{N^2}\sum_{\ell=2}^{N}(N-\ell+1)|y-y'+\ell-1|^{(2H'-2)}|z-z'+\ell-1|^{(2H'-2)} \\
= \quad & N^{2(2H'-2)}\frac{1}{N}\sum_{\ell=2}^{N}(1-\frac{\ell-1}{N})|\frac{y-y'}{N}+\frac{\ell-1}{N}|^{2H'-2}|\frac{z-z'}{N}+\frac{\ell-1}{N}|^{2H'-2}.
\end{aligned}$$

Using a Riemann sum approximation argument we conclude that

$$\mathbf{E}\left[T_2^{'2}\right] \sim \frac{4d(H,q)^4(H'(2H'-1))^{2q} \;\times\; N^{2(2H'-2)}}{(4H'-3)(4H'-2)[((2H'-2)(q-1)+1)]^2[(H'-1)(q-1)+1]^2}.$$

Therefore, it follows that

$$\mathbf{E}\left[c_{1,H}^{-1}N^{2(2-2H')}T_2^2\right] \to_{N\to\infty} 1, \tag{3.6}$$

with $c_{1,H}$ as in (3.5).

Let us study now the term $T_4,\dots,T_{2q}$ given by (3.4). Here the function $\sum_{i=0}^{N-1} f_{i,N}\otimes_k f_{i,N}$ is no longer symmetric but we will show that the behavior of its L^2 norm is dominated by $\mathbf{E}\left[T_2^2\right]$. Since for any square integrable function g one has $\|\tilde{g}\|_{L^2}\leq\|g\|_{L^2}$, we have for $k=0,\dots,q-2$

$$\begin{aligned}
&\frac{1}{(2q-2k)!}\mathbf{E}\left[T_{2q-2k}^2\right] = N^{4H-2}\|\sum_{i=0}^{N-1} f_{i,N}\tilde{\otimes}_k f_{i,N}\|^2_{L^2([0,1]^{2q-2k})} \\
\leq \quad & N^{4H-2}\|\sum_{i=0}^{N-1} f_{i,N}\otimes_k f_{i,N}\|^2_{L^2([0,1]^{2q-2k})} \\
= \quad & N^{4H-2}\sum_{i,j=0}^{N-1}\langle f_{i,N}\otimes_k f_{i,N}, f_{j,N}\otimes_k f_{j,N}\rangle_{L^2([0,1]^{2q-2k})}
\end{aligned} \tag{3.7}$$

and proceeding as above, with $e_{H,q,k} := (2q-2k)!(H'(2H'-1))^{2q} d(H,q)^4$ we can write

$$\mathbf{E}\left[T_{2q-2k}^2\right] \leq e_{H,q,k} N^{4H-2} \sum_{i,j=0}^{N-1} \int_{I_i}\int_{I_i} dy_1 dz_1 \int_{I_j}\int_{I_j} dy_1' dz_1'$$
$$\times |y_1 - z_1|^{(2H'-2)k} |y_1' - z_1'|^{(2H'-2)k} |y_1 - y_1'|^{(2H'-2)(q-k)} |z_1 - z_1'|^{(2H'-2)(q-k)}$$

and using a change of variables as before,

$$\begin{aligned} & \mathbf{E}\left[T_{2q-2k}^2\right] \\ \leq\ & e_{H,q,k} N^{4H-2} N^{-4} N^{-(2H'-2)2q} \sum_{i,j=0}^{N-1} \int_{[0,1]^4} dy dz dy' dz' (|y-z||y'-z'|)^{(2H'-2)k} \\ & \times |y-y'+i-j|^{(2H'-2)(q-k)} |z-z'+i-j|^{(2H'-2)(q-k)} \\ =\ & \frac{(2q-2k)! d(H,q)^4}{a(H')^{-2q}} \frac{N^{(2H'-2)(2q-2k)}}{N^2} \sum_{\ell=2}^{N} \left(1 - \left(\frac{\ell-1}{N}\right)\right) \int_{[0,1]^4} dy dz dy' dz' \\ & \times (|y-z||y'-z'|)^{(2H'-2)k} \left(\left|\frac{y-y'}{N} + \frac{\ell-1}{N}\right| \left|\frac{z-z'}{N} + \frac{\ell-1}{N}\right|\right)^{(2H'-2)(q-k)} \end{aligned} \tag{3.8}$$

Since off a diagonal term (again of lower order), the terms $\frac{z-z'}{N}$ are dominated by $\frac{\ell}{N}$ for large l, N it follows that, for $1 \leq k \leq q-1$

$$\mathbf{E}\left[c_{q-k,H}^{-1} N^{(2-2H')(2q-2k)} T_{2q-2k}^2\right] = O(1) \tag{3.9}$$

when $N \to \infty$, with

$$c_{q-k,H} = 2\left(\int_0^1 (1-x) x^{(2H'-2)(2q-2k)} dx\right) a(H')^{-2} (2q-2k)! d(H,q)^2 a(H')^{2q}. \tag{3.10}$$

It is obvious that the dominant term in the decomposition of V_N is the term in the chaos of order 2. [The case $k=0$ is in the same situation for $H > \frac{3}{4}$ and for $H \in (\frac{1}{2}, \frac{3}{4})$ the term T_{2q} obtained for $k=0$ has to be renormalized by N, (as in proofs in the remainder of the paper); in any case it is dominated by the term T_2]. More specifically we have for any $k \leq q-2$,

$$\mathbf{E}\left[N^{2(2-2H')} T_{2q-2k}^2\right] = O\left(N^{-2\left(2-2H'\right)2(q-k-1)}\right). \tag{3.11}$$

Combining this with the orthogonality of chaos integrals, we immediately get that, up to terms that tend to 0, $N^{2-2H'} V_N$ and $N^{2-2H'} T_2$ have the same norm in $L^2(\Omega)$. This finishes the proof of the proposition. ■

Summarizing the spirit of the above proof, to understand the behavior of the renormalized sequence V_N it suffices to study the limit of the term

$$I_2\left(N^{2H-1} N^{(2-2H')} \sum_{i=0}^{N-1} f_{i,N} \otimes_{q-1} f_{i,N}\right) \tag{3.12}$$

with

$$
\begin{aligned}
& (f_{i,N} \otimes_{q-1} f_{i,N})(y,z) \qquad (3.13)\\
= \; & d(H,q)^2 a(H')^{q-1} \\
& \Big(1_{[0,\frac{i}{N}]}(y \vee z) \int_{I_i}\int_{I_i} dvdu \partial_1 K(u,y)\partial_1 K(v,z)|u-v|^{(2H'-2)(q-1)} \\
& + 1_{[0,\frac{i}{N}]}(y) 1_{I_i}(z) \int_{I_i}\int_z^{\frac{i+1}{N}} dvdu \partial_1 K(u,y)\partial_1 K(v,z)|u-v|^{(2H'-2)(q-1)} \\
& + 1_{I_i}(y) 1_{[0,\frac{i}{N}]}(z) \int_y^{\frac{i+1}{N}}\int_{I_i} dvdu \partial_1 K(u,y)\partial_1 K(v,z)|u-v|^{(2H'-2)(q-1)} \\
& + 1_{I_i}(y) 1_{I_i}(z) \int_y^{\frac{i+1}{N}}\int_z^{\frac{i+1}{N}} dvdu \partial_1 K(u,y)\partial_1 K(v,z)|u-v|^{(2H'-2)(q-1)} \Big).
\end{aligned}
$$

We will see in the proof of the next theorem that, of the contribution of the four terms on the right-hand side of (3.13), only the first one does not tend to 0 in $L^2(\Omega)$. Hence the following notation will be useful: f_2^N will denote the integrand of the contribution to (3.12) corresponding to that first term, and r_2 will be the remainder of the integrand in (3.12). In other words,

$$
f_2^N + r_2 = N^{2H-1} N^{(2-2H')} \sum_{i=0}^{N-1} f_{i,N} \otimes_{q-1} f_{i,N} \qquad (3.14)
$$

and

$$
\begin{aligned}
f_2^N(y,z) \; &: \; = N^{2H-1} N^{(2-2H')} d(H,q)^2 a(H')^{q-1} \\
\sum_{i=0}^{N-1} 1_{[0,\frac{i}{N}]}(y \vee z) & \int_{I_i}\int_{I_i} dvdu \partial_1 K(u,y)\partial_1 K(v,z)|u-v|^{(2H'-2)(q-1)}. \qquad (3.15)
\end{aligned}
$$

Theorem 3.2. *The sequence given by (3.12) converges in $L^2(\Omega)$ as $N \to \infty$ to the constant $c_{1,H}^{1/2}$ times a standard Rosenblatt random variable $Z_1^{(2,2H'-1)}$ with self-similarity parameter $2H'-1$ and H' is given by (2.5). Consequently, we also have that $c_{1,H}^{-1/2} N^{(2-2H')} c_2^{-1} V_N$ converges in $L^2(\Omega)$ as $N \to \infty$ to the same Rosenblatt random variable.*

Proof: The first statement of the theorem is that $N^{2-2H'} T_2$ converges to

$$
c_{1,H}^{1/2} Z_1^{(2,2H'-1)}
$$

in $L^2(\Omega)$. From (3.12) it follows that T_2 is a second-chaos random variable, with kernel

$$
N^{2H-1} \sum_{i=0}^{N-1} (f_{i,N} \otimes_{q-1} f_{i,N})
$$

(see expression in (3.13)), so we only need to prove this kernel converges in $L^2\left([0,1]^2\right)$. The first observation is that $r_2(y,z)$ defined in (3.14) converges to zero in $L^2([0,1]^2)$ as $N \to \infty$. The crucial fact is that the intervals I_i which are disjoints, appear in the expression of this term and this implies that the non-diagonal terms vanish when we take the square norm of the sum; in fact it can easily be seen that the norm in L^2 of r_2 corresponds to the diagonal part in the evaluation in ET_2^2 which is clearly dominated by the non-diagonal part, so this result comes as no surprise. The proof follows the lines of [20]. This shows $N^{(2-2H')}T_2$ is the sum of $I_2\left(f_2^N\right)$ and a term which goes to 0 in $L^2(\Omega)$. Our next step is thus simply to calculate the limit in $L^2(\Omega)$, if any, of $I_2(f_2^N)$ where f_2^N has been defined in (3.15). By the isometry property (2.1), limits of second-chaos r.v.'s in $L^2(\Omega)$ are equivalent to limits of their symmetric kernels in $L^2\left([0,1]^2\right)$. Note that f_2^N is symmetric. Therefore, it is sufficient to prove that f_2^N converges to the kernel of the Rosenblatt process at time 1. We have by definition

$$\begin{aligned} f_2^N(y,z) &= (H'(2H'-1))^{(q-1)}d(H,q)^2N^{2H-1}N^{2-2H'} \\ &\quad\times\sum_{i=0}^{N-1}\int_{I_i}\int_{I_i}|u-v|^{(2H'-2)(q-1)}\partial_1K^{H'}(u,y)\partial_1K^{H'}(v,z). \end{aligned}$$

Thus for every y,z,

$$\begin{aligned} & f_2^N(y,z) = d(H,q)^2(H'(2H'-1))^{(q-1)}N^{2H-1}N^{2-2H'} \\ & \times\sum_{i=0}^{N-1}\int_{I_i}\int_{I_i}|u-v|^{(2H'-2)(q-1)}\partial_1K^{H'}(u,y)\partial_1K^{H'}(v,z)dudv \\ = \; & d(H,q)^2(H'(2H'-1))^{(q-1)}N^{2H-1}N^{2-2H'}\sum_{i=0}^{N-1}\int_{I_i}\int_{I_i}|u-v|^{(2H'-2)(q-1)} \\ & \times\left(\partial_1K^{H'}(u,y)\partial_1K^{H'}(v,z)-\partial_1K^{H'}(i/N,z)\partial_1K^{H'}(i/N,z)\right)dudv \\ & +d(H,q)^2(H'(2H'-1))^{(q-1)}N^{2H-1}N^{2-2H'}\sum_{i=0}\int_{I_i}\int_{I_i}|u-v|^{(2H'-2)(q-1)} \\ & \times\partial_1K^{H'}(i/N,y)\partial_1K^{H'}(i/N,z)dudv \\ =: \; & A_1^N(y,z)+A_2^N(y,z). \end{aligned}$$

As in [20], one can show that $\mathbf{E}\left[\left\|A_1^N\right\|^2_{L^2([0,1]^2)}\right]\to 0$ as $N\to\infty$. Regarding the second term $A_2^N(y,z)$, the summand is zero if $i/N<y\vee z$, therefore we get that f_2^N is equivalent to

$$\begin{aligned} & N^{2H-1}N^{2-2H'}d(H,q)^2(H'(2H'-1))^{(q-1)}\sum_{i=0}^{N-1}\int_{I_i}\int_{I_i}|u-v|^{(2H'-2)(q-1)} \\ & \times\partial_1K^{H'}(i/N,y)\partial_1K^{H'}(i/N,z)dudv \end{aligned}$$

$$
\begin{aligned}
&= (H'(2H'-1))^{(q-1)} d(H,q)^2 N^{2H-1} N^{2-2H'} \\
&\quad \times \sum_{i=0}^{N-1} \partial_1 K^{H'}(i/N,y) \partial_1 K^{H'}(i/N,z) 1_{y\vee z<i/N} \int_{I_i}\int_{I_i} |u-v|^{(2H'-2)(q-1)} du dv \\
&= (H'(2H'-1))^{(q-1)} [((2H'-2)(q-1)+1)((H'-1)(q-1)+1)]^{-1} \\
&\quad \times \frac{N^{2H-1}N^{(2-2H')q}}{N^2} \sum_{i=0}^{N-1} \partial_1 K^{H'}(i/N,y) \partial_1 K^{H'}(i/N,z) 1_{y\vee z<i/N} \\
&= \frac{d(H,q)^2}{d(2H'-2,2)} \frac{(H'(2H'-1))^{(q-1)}}{((2H'-2)(q-1)+1)((H'-1)(q-1)+1)} \\
&\quad \times d(2H'-2,2) N^{-1} \sum_{i=0}^{N-1} \partial_1 K^{H'}(i/N,y) \partial_1 K^{H'}(i/N,z) 1_{y\vee z<i/N}.
\end{aligned}
$$

The sequence $d(2H'-2,2)N^{-1}\sum_{i=0}^{N-1} \partial_1 K^{H'}(i/N,y)\partial_1 K^{H'}(i/N,z)1_{y\vee z<i/N}$ is a Riemann sum that converges pointwise on $[0,1]^2$ to the kernel of the Rosenblatt process $Z^{2H'-1,2}$ at time 1. To obtain the convergence in $L^2\left([0,1]^2\right)$ we will apply the dominated convergence theorem. Indeed,

$$
\begin{aligned}
&\int_0^1\int_0^1 \left| \frac{1}{N}\sum_{i=0}^{N-1} \partial_1 K^{H'}(i/N,y)\partial_1 K^{H'}(i/N,z)1_{y\vee z<i/N}\right|^2 dydz \\
&= \frac{1}{N^2}\sum_{i,j=0}^{N-1} \left| \int_0^1 \partial_1 K^{H'}(i/N,y)\partial_1 K^{H'}(j/N,y) 1_{y<(i\wedge j)/N} dy \right|^2 \\
&\leq \frac{1}{N^2}\sum_{i,j=0}^{N-1} \left| \mathbf{E}\left[\Delta Z_{i/N}\Delta Z_{j/N}\right]\right|^2,
\end{aligned}
$$

where $\Delta Z_{i/N}$ is the difference $Z\left(\frac{i}{N}\right) - Z\left(\frac{i-1}{N}\right)$ for a Rosenblatt process Z. We now show that the above sum is always $\ll N^2$, which proves that the last expression, with the N^{-2} factor, is bounded. In fact for $H_1 = 2H'-1$

$$
\begin{aligned}
\sum_{i,j=0}^{N-1} \left|\mathbf{E}\left[\Delta Z_{i/N}\Delta Z_{j/N}\right]\right|^2 &= \sum_{i,j=0}^{N-1} \left| \left|\frac{i-j+1}{N}\right|^{2H_1} + \left|\frac{i-j-1}{N}\right|^{2H_1} - 2\left|\frac{i-j}{N}\right|^{2H_1} \right|^2 \\
&= \frac{N^{-4H_1}}{4} \sum_{i,j=0}^{N-1} \left| |i-j+1|^{2H_1} + |i-j-1|^{2H_1} - 2|i-j|^{2H_1} \right|^2 \\
&\leq \frac{N^{-4H_1}}{4} 2N \sum_{\ell=-N+1}^{N-1} \left| |\ell+1|^{2H_1} + |\ell-1|^{2H_1} - 2|\ell|^{2H_1} \right|^2.
\end{aligned}
$$

The function $g(\ell) = |\ell+1|^{2H_1} + |\ell-1|^{2H_1} - 2|\ell|^{2H_1}$ behaves like $H_1(2H_1-1)|\ell|^{2H_1-2}$ for large ℓ. We need to separate the cases of convergence and divergence of the series $\sum_{-\infty}^{\infty}|g(\ell)|^2$. It is divergent as soon as $H_1 \geq 3/4$, or equivalently $H' \geq 7/8$, in which case we get for some constant c not dependent on N,

$$\sum_{i,j=0}^{N-1} \left|\mathbf{E}\left[\Delta Z_{i/N}\Delta Z_{j/N}\right]\right|^2 \leq cN^{-4H_1+1+4H_1-3} = cN^{-2} \ll N^2.$$

The series $\sum_{-\infty}^{\infty} |g(\ell)|^2$ is convergent if $H' < 7/8$, in which case we get

$$\sum_{i,j=0}^{N-1} \left|\mathbf{E}\left[\Delta Z_{i/N}\Delta Z_{j/N}\right]\right|^2 \leq cN^{-4H_1+1}.$$

For this to be $\ll N^2$, we simply need $-4H_1+1<2$, i.e. $H' > 5/8$. However, since $q \geq 2$ and $H > 1/2$ we always have $H' > 3/4$. Therefore in all cases, the sequence $A_2^N(y,z)$ is bounded in $L^2\left([0,1]^2\right)$ and in this way we obtain the L^2 convergence to the kernel of a Rosenblatt process of order 1. The first statement of the theorem is proved. In order to show that $c_{1,H}^{-1/2}N^{(2-2H')}c_2^{-1}V_N$ converges in $L^2(\Omega)$ to the same Rosenblatt random variable as the normalized version of the quantity in (3.12), it is sufficient to show that, after normalization by $N^{2-2H'}$, each of the remaining terms of in the chaos expansion (3.2) of V_N, converge to zero in $L^2(\Omega)$, i.e. that $N^{(2-2H')}T_{2q-2k}$ converge to zero in $L^2(\Omega)$, for all $1 \leq k < q-1$. From (3.11) we have

$$\mathbf{E}\left[N^{2(2-2H')}T_{2q-2k}^2\right] = O\left(N^{-2\left(2-2H'\right)2(q-k-1)}\right)$$

which is all that is needed, concluding the proof of the theorem. ■

4. Reproduction property for the Hermite process

We now study the limits of the other terms in the chaos expansion (3.2) of V_N. We will consider first the convergence of the term of greatest order T_{2q} in this expansion. The behavior of T_{2q} is interesting because it differs from the behavior of the all other terms: its suitable normalization possesses a Gaussian limit if $H \in (1/2, 3/4]$. Therefore it inherits, in some sense, the properties of the quadratic variations for the fractional Brownian motion (results in [20]). We have

$$T_{2q} = N^{2H-1}I_{2q}\left(\sum_{i=0}^{N-1} f_{i,N}\otimes f_{i,N}\right)$$

and $\mathbf{E}\left[T_{2q}^2\right] = N^{4H-2}(2q)!\sum_{i,j=0}^{N-1}\langle f_{i,N}\tilde{\otimes}f_{i,N}, f_{j,N}\tilde{\otimes}f_{j,N}\rangle_{L^2[0,1]^{2q}}$. We will use the following combinatorial formula: if f,g are two symmetric functions in $L^2([0,1]^q)$,

$$\begin{aligned}&(2q)!\langle f\tilde{\otimes}f, g\tilde{\otimes}g\rangle_{L^2([0,1]^{2q})}\\ =\ &(q!)^2\langle f\otimes f, g\otimes g\rangle_{L^2([0,1]^{2q})} + \sum_{k=1}^{q-1}\binom{q}{k}^2(q!)^2\langle f\otimes_k g, g\otimes_k f\rangle_{L^2([0,1]^{2q-2k})}\end{aligned}$$

to obtain

$$\begin{aligned}\mathbf{E}\left[T_{2q}^2\right] &= N^{4H-2}(q!)^2\sum_{i,j=0}^{N-1}\langle f_{i,N}, f_{j,N}\rangle^2_{L^2([0,1]^q)}\\ &+N^{4H-2}(q!)^2\sum_{i,j=0}^{N-1}\sum_{k=1}^{q-1}\binom{q}{k}^2\langle f_{i,N}\otimes_k f_{j,N}, f_{j,N}\otimes_k f_{i,N}\rangle_{L^2([0,1]^{2q-2k})}.\end{aligned}$$

First note that $\mathbf{E}\left[\left(Z^{(q,H)}_{\frac{i+1}{N}}-Z^{(q,H)}_{\frac{i}{N}}\right)\left(Z^{(q,H)}_{\frac{j+1}{N}}-Z^{(q,H)}_{\frac{j}{N}}\right)\right]=\mathbf{E}\left[I_q(f_{i,N})I_q(f_{j,N})\right]=$ $q!\langle f_{i,N}, f_{j,N}\rangle_{L^2([0,1]^q)}$ and so the covariance structure of $Z^{(q,H)}$ implies

$$\begin{aligned}&(q!)^2\sum_{i,j=0}^{N-1}\langle f_{i,N}, f_{j,N}\rangle^2_{L^2([0,1]^q)}\\ &= \frac{1}{4}\sum_{i,j=0}^{N-1}\left[\left(\frac{i-j+1}{N}\right)^{2H}+\left(\frac{i-j-1}{N}\right)^{2H}-2\left(\frac{i-j}{N}\right)^{2H}\right]^2. \qquad (4.1)\end{aligned}$$

Secondly, the square norm of the contraction $f_{i,N}\otimes_k f_{j,N}$ has been computed before (actually its expression is obtained in the lines from formula (3.7) to formula (3.8)). By a simple polarization, we obtain

$$\begin{aligned}&N^{2H-1}\sum_{i,j=0}^{N-1}\langle f_{i,N}\otimes_k f_{j,N}, f_{j,N}\otimes_k f_{i,N}\rangle_{L^2([0,1]^{2q-2k})}\\ &= d(H,q)^4 a(H')^{2q}N^{4H-2}N^{(2H'-2)2q}\sum_{i,j=0}^{N-1}\int_0^1\int_0^1\int_0^1\int_0^1 dydzdy'dz'\\ &(|y-z+i-j||y'-z'+i-j|)^{(2H'-2)k}|(y-y'+i-j||z-z'+i-j|)^{(2H'-2)(q-k)}\end{aligned}$$

and as in the proof of Proposition 3.1 we can find that this term has to be renormalized by, if $H>\frac{3}{4}$

$$b_{H,k}N^{(2-2H')2q}=b_{H,k}N^{4-4H}$$

where $b_{1,H,k}=(q!)^2(C_k^q)^2 2d(H,q)^4a(H')^{2q}\int_0^1(1-x)x^{4H-4}dx$. If $H\in(\frac{1}{2},\frac{3}{4})$, then, the same quantity will be renormalized by $b_{2,H,k}N$ where

$$b_{2,H,k}=(q!)^2(C_k^q)^2d(H,q)^4a(H')^{2q}\sum_{k=1}^{\infty}\left(2k^{2H}-(k+1)^{2H}-(k-1)^{2H}\right)^2$$

while for $H=\frac{3}{4}$ the renormalization is of order $b_{3,H,k}N(\log N)^{-1}$ with $b_{3,H,k}=(q!)^2(C_k^q)^2 2d(H,q)^4a(H')^{2q}(1/2)$. As a consequence, we find a sum whose behavior is well-known (it is the same as the mean square of the quadratic variations of the fractional Brownian motion, see e.g. [20]) and we get, for large N

$$\begin{aligned}\mathbf{E}\left[T_{2q}^2\right] &\sim \frac{1}{N}x_{1,H}, \text{ if } H\in(\frac{1}{2},\frac{3}{4}); \mathbf{E}\left[T_{2q}^2\right]\sim N^{4H-4}x_{2,H}, \text{ if } H\in(\frac{3}{4},1)\\ \mathbf{E}\left[T_{2q}^2\right] &\sim \frac{\log N}{N}x_{3,H}, \text{ if } H=\frac{3}{4}\end{aligned}$$

where $x_{1,H} = \left(\sum_{l=1}^{q-1} b_{2,H,l} + 1 + (1/2)\sum_{k=1}^{\infty}\left(2k^{2H} - (k+1)^{2H} - (k-1)^{2H}\right)^2\right)$, $x_{2,H} = (\sum_{l=1}^{q-1} b_{1,H,l} + H^2(2H-1)/(4H-3))$ and $x_{3,H} = (\sum_{l=1}^{q-1} b_{3,H,l} + 9/16)$.

Remark 4.1. The fact that the normalizing factor for T_{2q} when $H < 3/4$ is $N^{-1/2}$ (in particular does not depend on H) is a tell-tale sign that normal convergence may occur. It is possible to show that the term in the chaos of order $2q$ converges, after its renormalization, to a Gaussian law. More precisely: a) suppose that $H \in (\frac{1}{2}, \frac{3}{4})$ and let $F_N := x_{1,H}^{-1/2}\sqrt{N}T_{2q}$; then, as $N \to \infty$, the sequence F_N convergence to the standard normal distribution $N(0,1)$; b) suppose $H = \frac{3}{4}$ and set $G_N := \sqrt{\frac{N}{\log N}} x_{3,H}^{-1/2} T_{2q}$; then, as $N \to \infty$, the sequence G_N convergence to the standard normal distribution $N(0,1)$. We refer to the extended version of our paper on arxiv for the basic ideas of the proof.

It is possible to give the limits of the terms T_{2q-2} to T_2 appearing in the decomposition of the statistics V_N. All these renormalized terms will converge to Hermite random variables of the same order as their indices (we have already proved this property in detail for T_2 in the previous section). This is a kind of "reproduction" of the Hermite processes through their variations.

Theorem 4.2.

- *For every* $H \in (\frac{1}{2}, 1)$ *and for every* $k = 1, \ldots, q-2$ *we have*

$$\lim_{N\to\infty} N^{(2-2H')(q-k)} T_{2q-2k} = z_{k,H} Z^{(2q-2k,(2q-2k)(H'-1)+1)}, \qquad \text{in } L^2(\Omega) \tag{4.2}$$

  where $Z^{(2q-2k,(2q-2k)(H'-1)+1)}$ *denotes a Hermite random variable with self-similarity parameter* $(2q-2k)(H'-1)+1$ *and*
  $z_{k,H} = d(H,q)^2 a(H')^k \left((H'-1)k+1\right)^{-1} \left(2(H'-1)+1\right)^{-1}$.
- *Moreover, if* $H \in (\frac{3}{4}, 1)$ *then*

$$\lim_{N\to\infty} N^{2-2H} x_{2,H}^{-1/2} T_{2q} = Z^{(2q,2H-1)}, \qquad \text{in } L^2(\Omega). \tag{4.3}$$

Proof: Recall that we have $T_{2k} = N^{2H-1} I_{2q-2k}\left(\sum_{i=0}^{N-1} f_{i,N} \otimes_k f_{i,N}\right)$, for $k = 1$ to $2q-2$, with $f_{i,N} \otimes_k f_{i,N}$ given by (3.1). The first step of the proof is to observe that the limit of $N^{(2H'-2)(q-k)} T_{2q-2k}$ is given by

$$N^{(2H'-2)(q-k)} N^{2H-1} I_{2q-2k}(f_{2q-2k}^N)$$

with $f_{2q-2k}^N(y_1, \ldots, y_{q-k}, z_1, \ldots, z_{q-k}) = d(H,q)^2 a(H')^k \sum_{i=0}^{N-1} 1_{[0,\frac{i}{N}]}(y_i) 1_{[0,\frac{i}{N}]}(z_i)$ $\int_{I_i}\int_{I_i} \partial_1 K(u, y_1) \ldots \partial_1 K(u, y_{q-k}) \partial_1 K(v, z_1) \ldots \partial_1 K(v, z_{q-k}) |u-v|^{(2H'-2)k} dvdu$.
The argument leading to the above fact is the same as in Theorem 3.2: a the remainder term r_{2q-2k}, which is defined by T_{2q-2k} minus $I_2(f_{2q-2k}^N)$ converges to zero similarly to the term r_2 in the proof of Theorem 3.2 because of the appearance of the indicator functions $1_{I_i}(y_i)$ or $1_{I_i}(z_i)$ in each of the terms that form this remainder. The second step of the proof is to replace $\partial_1 K(u, y_i)$ by $\partial_1 K(\frac{i}{N}, y_i)$ and $\partial_1 K(v, z_i)$ by $\partial_1 K(\frac{i}{N}, z_i)$ on the interval I_i inside the integrals du and dv.

This can be argued, as in the proof of Theorem 3.2, by a dominated convergence theorem. Therefore the term $N^{(2-2H')(q-k)}T_{2q-2k}$ will have the same limit as

$$
\begin{aligned}
& N^{(2-2H')(q-k)}N^{2H-1}d(H,q)^2a(H')^k\sum_{i=0}^{N-1}1_{[0,\frac{i}{N}]}(y_i)1_{[0,\frac{i}{N}]}(z_i) \\
& \prod_{j=1}^{q-k}\left[\partial_1K(\frac{i}{N},y_j)\partial_1K(\frac{i}{N},z_j)\right]\iint_{I_i^2}|u-v|^{(2-2H')k}dvdu \\
= \; & \frac{d(H,q)^2a(H')^k}{((H'-1)k+1)\,(2(H'-1)+1)}N^{(2-2H')(q-k)}N^{2H-1}N^{(2-2H')k+2} \\
& \sum_{i=0}^{N-1}1_{[0,\frac{i}{N}]^2}(y_i,z_i)\prod_{j=1}^{q-k}\left[\partial_1K(\frac{i}{N},y_j)\partial_1K(\frac{i}{N},z_j)\right] \\
= \; & \frac{z_{k,H}}{N}\sum_{i=0}^{N-1}1_{[0,\frac{i}{N}]^2}(y_i,z_i)\prod_{j=1}^{q-k}\left[\partial_1K(\frac{i}{N},y_j)\partial_1K(\frac{i}{N},z_j)\right].
\end{aligned}
$$

Now, the last sum is a Riemann sum that converges pointwise and in $L^2([0,1]^{2q-2k})$ to $z_{k,H}L_1$ where

$$L_1(y_1,\dots,y_{q-k},z_1,\dots,z_{q-k})=\int_{y_1\vee z_{q-k}}^1\prod_{j=1}^{q-k}\left[\partial_1K(u,y_j)\partial_1K(u,z_j)\right]du$$

which is the kernel of the Hermite random variable of order $2q-2k$ with self-similarity parameter $(2q-2k)(H'-1)+1$. The case $H\in(\frac{3}{4},1)$ and $k=0$ follows analogously. ■

5. Consistent estimation of H and its asymptotics

Theorem 3.2 can be immediately applied to the statistical estimation of H. We note that with V_N in (1.1) and S_N defined by

$$S_N=\frac{1}{N}\sum_{i=0}^{N-1}\left(Z^{(q)}_{\frac{i+1}{N}}-Z^{(q)}_{\frac{i}{N}}\right)^2, \tag{5.1}$$

we have

$$1+V_N=N^{2H}S_N \tag{5.2}$$

and $\mathbf{E}\left[S_N\right]=N^{-2H}$ so that $H=-\frac{\log\mathbf{E}[S_N]}{2\log N}$. To form an estimator of H, since Theorem 3.2 implies that S_N evidently concentrates around its mean, we will use S_N in the role of an empirical mean, instead of its true mean; in other words we let

$$\hat{H}=\hat{H}_N=-\frac{\log S_N}{2\log N}.$$

We immediately get from (5.2) that

$$\log\left(1+V_N\right)\quad=2H\log N+\log S_N=2\left(H-\hat{H}\right)\log N. \tag{5.3}$$

The first observation is that $\hat{H}_N$ is strongly consistent for the Hurst parameter.

Proposition 5.1. *We have that $\hat{H}_N$ converges to H almost surely as $N \to \infty$.*

Proof: Let us prove that V_N converges to zero almost surely as $N \to \infty$. We know that V_N converges to 0 in $L^2(\Omega)$ as $N \to \infty$ and an estimation for its variance is given by the formula (3.6). On the other hand this sequence is stationary because the increments of the Hermite process are stationary. We can therefore apply a standard argument to obtain the almost sure convergence for discrete stationary sequence under condition (3.6). This argument follows from Theorem 6.2, page 492 in [6] and it can be used exactly as in the proof of Proposition 1 in [4]. A direct elementary proof using the Borel-Cantelli lemma is almost as easy. The almost sure convergence of $\hat{H}$ to H is then obtained immediately via (5.3). ■

Owing to the fact that V_N is of small magnitude (it converges to zero almost surely by the last proposition's proof), $\log(1+V_N)$ can be confused with V_N. It then stands to reason that $\left(H-\hat{H}\right) N^{2-2H'} \log N$ is asymptotically Rosenblatt-distributed since the same holds for V_N by Theorem 3.2. Just as in that theorem, more is true: as we now show, the convergence to a Rosenblatt random variable occurs in $L^2(\Omega)$.

Proposition 5.2. *There is a standard Rosenblatt random variable R with self-similarity parameter $2H'-1$ such that*

$$\lim_{N\to\infty} \mathbf{E}\left[\left|2N^{2-2H'}\left(H-\hat{H}\right)\log N - c_2 c_{1,H,q}^{1/2} R\right|^2\right] = 0$$

Proof: To simplify the notation, we denote $c_2 c_{1,H,q}^{1/2}$ by c in this proof. Theorem 3.2 signifies that a standard Rosenblatt r.v. R with parameter $2H'-1$ exists such that

$$\lim_{N\to\infty} \mathbf{E}\left[\left|N^{2-2H'}V_N - cR\right|^2\right] = 0.$$

From (5.3) we immediately get

$$\begin{aligned} & \mathbf{E}\left[\left|2N^{2-2H'}\left(H-\hat{H}\right)\log N - cR\right|^2\right] = \mathbf{E}\left[\left|N^{2-2H'}\log(1+V_N) - cR\right|^2\right] \\ \leq \quad & 2\mathbf{E}\left[\left|N^{2-2H'}V_N - cR\right|^2\right] + 2N^{4-4H'}\mathbf{E}\left[\left|V_N - \log(1+V_N)\right|^2\right]. \end{aligned}$$

Therefore, we only need to show that $\mathbf{E}\left[\left|V_N - \log(1+V_N)\right|^2\right] = o\left(N^{4H'-4}\right)$. Using the inequality $x - \log(1+x) = |x - \log(1-x)| \leq x^2$ for $x \geq \frac{1}{2}$ one gets

$$\begin{aligned} & \mathbf{E}\left|V_N - \log(1+V_N)\right|^2 \\ & \leq 2\mathbf{E}\left|V_N - \log(1+V_N)\right|^2 1_{V_N \geq -\frac{1}{2}} + 2\mathbf{E}\left|V_N - \log(1+V_N)\right|^2 1_{V_N < -\frac{1}{2}} \\ & \leq 2\mathbf{E}V_N^4 + 4P\left(V_N < -\frac{1}{2}\right) + 4\mathbf{E}\left|\log(1+V_N)\right|^2 1_{V_N < -\frac{1}{2}}. \end{aligned}$$

The first term is bounded above and this is immediately dealt with using the following lemma.

Lemma 5.3. *For every $n \geq 2$, there is a constant c_n such that* $\mathbf{E}\left[|V_N|^{2n}\right] \leq c_n\ N^{(4H'-4)n}$.

Proof: In the proof of Proposition 3.1 we calculated the L^2 norm of the terms appearing in the decomposition of V_N, where V_N is a sum of multiple integrals. Therefore, from Proposition 5.1 of [10] we immediately get an estimate for any event moment of each term. Indeed, for $k = q-1$

$$\mathbf{E}\left[T_2^{2n}\right] \leq 1 \cdot 3 \cdot 5 \cdots (4n-1)\left(c_{1,H}^2\ N^{4H'-4}\right)^n = c_n N^{(4H'-4)n}$$

and for $1 \leq k < q-1$

$$\begin{aligned}\mathbf{E}\left[T_{2q-2k}^{2n}\right] &\leq 1 \cdot 3 \cdot 5 \cdots (2(2q-2k)n)\left(c_{1,q,H}^2\ N^{(2q-2k)(2H'-2)}\right)^n \\ &= c_{q,n} N^{(2q-2k)(2H'-2)n}\end{aligned}$$

The dominant term is again the T_2 term and the result follows immediately. ■

Then applying the above lemma for $n = 2$ and using a similar proof to prove that $\mathbf{E}\left|\log(1+V_N)\right|^2 1_{V_N < -\frac{1}{2}} = o(N^{4H'-4})$ and $P(V_N < -\frac{1}{2}) = O(N^{8H'-8}) = o(N^{4H'-4})$. This leads to Proposition 5.2 ■

A difficulty arises when applying the above proposition for model validation when checking the asymptotic distribution of the estimator $\hat{H}$: the normalization constant $N^{2-2H'}\log N$ depends on H. While it is not always obvious that one may replace this instance of H' by its estimator, in our situation, because of the $L^2(\Omega)$ convergences, this is legitimate, as the following theorem shows.

Theorem 5.4. *Let $\hat{H}' = 1 + (\hat{H}-1)/q$. There is a standard Rosenblatt random variable R with self-similarity parameter $2H'-1$ such that*

$$\lim_{N\to\infty} \mathbf{E}\left[\left|2N^{2-2\hat{H}'}\left(H-\hat{H}\right)\log N - c_2 c_{1,H,q}^{1/2} R\right|\right] = 0.$$

Proof: By the previous proposition, it is sufficient to prove that

$$\lim_{N\to\infty} \mathbf{E}\left[\left|\left(N^{2-2\hat{H}'} - N^{2-2H'}\right)\left(H-\hat{H}\right)\log N\right|\right] = 0.$$

We decompose the probability space depending on whether $\hat{H}$ is far or not from its mean. For a fixed value $\varepsilon > 0$ which will be chosen later, it is most convenient to define the event

$$D = \left\{\hat{H} > q\varepsilon/2 + 2H - 1\right\}.$$

We have

$$\begin{aligned}&\mathbf{E}\left[\left|\left(N^{2-2\hat{H}'} - N^{2-2H'}\right)\left(H-\hat{H}\right)\log N\right|\right] \\ =\ &\mathbf{E}\left[\mathbf{1}_D\left|\left(N^{2-2\hat{H}'} - N^{2-2H'}\right)\left(H-\hat{H}\right)\log N\right|\right] \\ &+\mathbf{E}\left[\mathbf{1}_{D^c}\left|\left(N^{2-2\hat{H}'} - N^{2-2H'}\right)\left(H-\hat{H}\right)\log N\right|\right] =: A + B.\end{aligned}$$

We study A first. Introducing the shorthand notation $x = \max\left(2-2H', 2-2\hat{H}'\right)$ and $y = \min\left(2-2H', 2-2\hat{H}'\right)$, we may write

$$\begin{aligned} & \left|N^{2-2\hat{H}'} - N^{2-2H'}\right| = e^{x\log N} - e^{y\log N} \\ = & \quad e^{y\log N}\left(e^{(x-y)\log N} - 1\right) \leq N^y (\log N)(x-y) N^{x-y} \\ = & \quad 2(\log N) N^x \left|H' - \hat{H}'\right| = 2q^{-1}(\log N) N^x \left|H - \hat{H}\right|. \end{aligned}$$

Thus

$$\begin{aligned} A & \leq 2q^{-1}\,\mathbf{E}\left[\mathbf{1}_D N^x \left|H-\hat{H}\right|^2 \log^2 N\right] \\ & = 2q^{-1}\,\mathbf{E}\left[N^{x-\left(4-4H'\right)}\mathbf{1}_D N^{4-4H'}\left|H-\hat{H}\right|^2 \log^2 N\right]. \end{aligned}$$

Now choose $\varepsilon \in (0, 2-2H')$. In this case, if $\omega \in D$, and $x = 2-2H'$, we get $x - (4-4H') = -x < -\varepsilon$. On the other hand, for $\omega \in D$ and $x = 2-2\hat{H}'$, we get $x - (4-4H') = 2-2\hat{H}' - (4-4H') < -\varepsilon$ as well. In conclusion, on D,

$$x - (4-4H') < -\varepsilon,$$

which implies immediately

$$A \leq N^{-\varepsilon} 2q^{-1}\mathbf{E}\left[N^{4-4H'}\left|H-\hat{H}\right|^2 \log^2 N\right];$$

this prove that A tends to 0 as $N \to \infty$, since the last expectation above is bounded (converges to a constant) by Proposition 5.2. Now we may study B. We are now operating with $\omega \in D^c$. In other words,

$$H - \hat{H} > 1 - H - q\varepsilon/2.$$

Still using $\varepsilon < 2-2H$', this implies that $H > \hat{H}$. Consequently, it is not inefficient to bound $\left|N^{2-2H'} - N^{2-2\hat{H}'}\right|$ above by $N^{2-2\hat{H}'}$. In the same fashion, we bound $\left|H-\hat{H}\right|$ above by H. Hence we have, using Hölder's inequality with the powers $2q$ and $p^{-1} + (2q)^{-1} = 1$.

$$\begin{aligned} B & \leq H\log N\ \mathbf{E}\left[\mathbf{1}_{D^c} N^{2-2\hat{H}'}\right] \\ & \leq H\log N\ \mathbf{P}^{1/p}\left[D^c\right]\ \mathbf{E}^{1/(2q)}\left[N^{\left(2-2\hat{H}'\right)2q}\right]. \end{aligned} \tag{5.4}$$

From Proposition 5.2, by Chebyshev's inequality, we have

$$\mathbf{P}^{1/p}\left[D^c\right] \leq \frac{\mathbf{E}^{1/p}\left[\left|H-\hat{H}\right|^2\right]}{\left(1-H-q\varepsilon/2\right)^{2/p}} \leq c_{q,H} N^{-\left(4-4H'\right)/p} \tag{5.5}$$

for some constant $c_{q,H}$ depending only on q and H. Dealing with the other term in the upper bound for B is a little less obvious. We must return to the definition of $\hat{H}$. By (5.3) we have

$$\begin{aligned} 1+V_N &= N^{2\left(H-\hat{H}\right)} = N^{2q\left(H'-\hat{H}'\right)} \\ &= N^{2q\left(2-2\hat{H}'\right)} N^{-2q\left(2-2H'\right)}. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbf{E}^{1/(2q)}\left[N^{\left(2-2\hat{H}'\right)2q}\right] &\leq N^{2-2H'}\mathbf{E}^{1/(2q)}\left[1+V_N\right] \\ &\leq 2N^{2-2H'}. \end{aligned} \tag{5.6}$$

Plugging (5.5) and (5.6) back into (5.4), we get

$$B \leq 2Hc_{q,H}\left(\log N\right) N^{-\left(4-4H'\right)(2/p-1)}.$$

Given that $2q \geq 4$ and p is conjugate to $2q$, we have $p \leq 4/3$ so that $2/p-1 \geq 1/2 > 0$, which proves that B goes to 0 as $N \to \infty$. This finishes the proof of the theorem. ■

Finally we state the extension of our results to $L^p(\Omega)$-convergence.

Theorem 5.5. *The convergence in Theorem 5.4 holds in any* $L^p(\Omega)$. *In fact, the* $L^2(\Omega)$*-convergences in all other results in this paper can be replaced by* $L^p(\Omega)$*-convergences.*

Sketch of proof. We only give the outline of the proof. Lemma 5.3 works because, in analogy to the Gaussian case, for random variables in a fixed Wiener chaos of order p, existence of second moments implies existence of all moments, and the relations between the various moments are given using a set of constants which depend only on p. This Lemma can be used immediately to prove the extension of Proposition 3.1 that

$$\mathbf{E}\left[V_N^{2p}\right] \simeq c_p N^{p(4H'-4)}.$$

Proving a new version of Theorem 3.2 with $L^p(\Omega)$-convergence can base itself on the above result, and requires a careful reevaluation of the various terms involved. Proposition 5.2 can then be extended to $L^p(\Omega)$-convergence thanks to the new $L^p(\Omega)$ versions of Theorem 3.2 and Proposition 3.1, and Theorem 5.4 follows easily from this new version of Proposition 3.1. ■

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