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Stein's lemma, Malliavin calculus, and tail bounds, with application to polymer fluctuation exponent

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Abstract

We consider a random variable X satisfying almost-sure conditions involving $G := \left\langle DX, -DL^{-1}X \right\rangle$ where DX is X's Malliavin derivative and L^{-1} is the pseudo-inverse of the generator of the Ornstein-Uhlenbeck semigroup. A lower- (resp. upper-) bound condition on G is proved to imply a Gaussian-type lower (resp. upper) bound on the tail $\mathbf{P}[X > z]$. Bounds of other natures are also given. A key ingredient is the use of Stein's lemma, including the explicit form of the solution of Stein's equation relative to the function $\mathbf{1}_{x>z}$, and its relation to G. Another set of comparable results is established, without the use of Stein's lemma, using instead a formula for the density of a random variable based on G, recently devised by the author and Ivan Nourdin. As an application, via a Mehler-type formula for G, we show that the Brownian polymer in a Gaussian environment, which is white-noise in time and positively correlated in space, has deviations of Gaussian type and a fluctuation exponent $\chi = 1/2$. We also show this exponent remains $1/2$ after a non-linear transformation of the polymer's Hamiltonian.

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1. Introduction

1.1. Background and context

Ivan Nourdin and Giovanni Peccati have recently made a long-awaited connection between Stein's lemma and the Malliavin calculus: see [9], and also [10]. Our article uses crucial basic elements from their work, to investigate the behavior of square-integrable random variables whose Wiener chaos expansions are not finite. Specifically, we devise conditions under which the tail of a random variable is bounded below by Gaussian tails, by using Stein's lemma and the Malliavin calculus. Our article also derives similar lower bounds by way of a new formula for the density of a random variable, established in [12], which uses Malliavin calculus, but not Stein's lemma. Tail upper bounds are also derived, using both methods.

Stein's lemma has been used in the past for Gaussian upper bounds, e.g. in [3] in the context of exchangeable pairs. Malliavin derivatives have been invoked for similar upper bounds in [22]. In the current paper, the combination of these two tools yields a novel criterion for a Gaussian tail lower bound. We borrow a main idea from Nourdin and Peccati [9], and also from [12]: to understand a random variable Z which is measurable with respect to a Gaussian field W, it is fruitful to consider the random variable

$$G := \langle DZ, -DL^{-1}Z \rangle_H,$$

where D is the Malliavin derivative relative to W, $\langle \cdot, \cdot \rangle_H$ is the inner product in the canonical Hilbert space H of W, and L is the generator of the Ornstein-Uhlenbeck semigroup. Details on D, H, L, and G, will be given below.

The function $g(z) = \mathbf{E}[G|Z = z]$ has already been used to good effect in the density formula discovered in [12]; this formula implied new lower bounds on the densities of some Gaussian processes' suprema. These results are made possible by fully using the Gaussian property and, in particular, by exploiting both upper and lower bounds on the process's covariance. The authors of [12] noted that, if Z has a density and an upper bound is assumed on G, in the absence of any other assumption on how Z is related to the underlying Gaussian process W, then Z's tail is sub-Gaussian. On the other hand, the authors of [12] tried to discard any upper bound assumption, and assume instead that G was bounded below, to see if they could derive a Gaussian lower bound on Z's tail; they succeeded in this task, but only partially, as they had to impose some additional conditions on Z's function g, which are of upper-bound type, and which may not be easy to verify in practice.

The techniques used in [12] are well adapted to studying densities of random variables under simultaneous lower and upper bound assumptions, but less so under single-sided assumptions. The point of the current paper is to show that, while the quantitative study of densities via the Malliavin calculus seems to require two-sided assumptions, as in [12], single-sided assumptions on G are, in essence, sufficient to obtain single sided bounds on tails of random variables, and there are two strategies to this end: Nourdin and Peccati's connection between Malliavin calculus and Stein's lemma, and exploiting the Malliavin-calculus-based density formula in [12].

A key new component in our work, relative to the first strategy, may be characterized by saying that, in addition to a systematic exploitation of the Stein-lemma–Malliavin-calculus connection (via Lemma 3.5 below), we carefully analyze the behavior of solutions of the so-called Stein equation, and use them profitably, rather than simply use the fact that there exist bounded solutions with bounded derivatives. We were inspired to work this way by the similar innovative use of the solution in the context of Berry-Esséen theorems in [10]. The novelty in our second strategy is simply to note that the difficulties inherent to using the density formula of [12] with only one-sided assumptions, tend to disappear when one passes to tail formulas.

Our work follows in the footsteps of Nourdin and Peccati's. One major difference in focus between our work and theirs, and indeed between ours and the main use of Stein's method since its inception in [19] to the most recent results (see [3,4,17], and references therein), is that Stein's method is typically concerned with convergence to the normal distribution while we are only interested in rough bounds of Gaussian or other types for single random variables (not sequences), without imposing conditions which would lead to normal or any other convergence. While the focus in [9] is on convergence theorems, its authors were already aware of the ability of Stein's lemma and the Malliavin calculus to yield bounds for fixed r.v.'s, not sequences: their work implies that a bound on the deviation of a single G from the value 1 has clear implications for the distance from Z's distribution to the normal law. In fact, the main technical tool therein ([9, Theorem 3.1]) is stated for fixed random variables, yielding bounds on various distances between the distributions of such r.v.'s and the normal law, based on expectation calculations using $G-1$ explicitly; also see [9, Remark 3.6]. Nourdin and Peccati's motivations only required them to make use of [9, Theorem 3.1] as applied to convergences of sequences.

One other difference between our motivations and theirs is that we do not consider the case of a single Wiener chaos. Their work does, in principle, apply to random variables with arbitrary infinite chaos expansions (see e.g. again [9, Theorem 3.1], and also [9, Remark 3.8]) but their motivation is largely to apply the general theorem to r.v.'s in a fixed Wiener chaos. That we systematically consider random variables with infinitely many non-zero Wiener chaos components, comes from the application which we also consider in this article, to the so-called *fluctuation exponent* χ of a polymer in a random environment. Details on this application, where we show that $\chi = 1/2$ for a certain class of environments, are in Section 5. There is a more fundamental obstacle to seeking upper or lower Gaussian tail bounds on an r.v. in a *single* Wiener chaos: unlike convergence results for *sequences* of r.v.'s, such as [15], a single qth chaos r.v. has a tail of order $\exp\left(-(x/c)^{2/q}\right)$ (see [2]), it never has a Gaussian behavior; our lower-bound results below (e.g. Theorem 1.3 Point 3) does apply to such an r.v., but the result cannot be sharp.

Since submitting the first version of this article, there have been rapid developments in the use of Malliavin calculus, Stein's lemma, and the random variable G, which apply to situations not restricted to single Wiener chaos: see, in particular [11], where the authors prove a second-order Poincaré inequality to again assess the distance between a single r.v.'s law and the Gaussian law, and [16] where the authors use the G-based density formula of [12] to find Gaussian upper and lower bounds for solutions of some stochastic heat equations.

1.2. Summary of results

We now describe our main theoretical results. All stochastic analytic concepts used in this introduction are described in Section 2. Let W be an isonormal Gaussian process relative to a Hilbert space $H = L^2(T, \mathcal{B}, \mu)$ (for instance if W is the Wiener process on [0, 1], then $T = [0, 1]$ and μ is the Lebesgue measure). The norm and inner products in H are denoted by $\|\cdot\|_H$ and $\langle \cdot; \cdot \rangle_H$. Let $L^2(\Omega)$ be the set of all random variables which are square-integrable and measurable with respect to W. Let D be the Malliavin derivative with respect to W (see Paul Malliavin's or David Nualart's texts [8,13]). Thus DX is a random element in $L^2(\Omega)$ with values in the Hilbert space H. The set of all $X \in L^2(\Omega)$ such that $\|DX\|_H \in L^2(\Omega)$ is called $\mathbf{D}^{1,2}$. Let $\bar{\Phi}$ be the tail of the standard normal distribution

$$\bar{\Phi}(u) := \int_u^\infty e^{-x^2/2} dx / \sqrt{2\pi}.$$

The following result, described in [22] as an elementary consequence of a classical stochastic analytic inequality found for instance in Üstünel's textbook [21, Theorem 9.1.1], makes use of a condition based solely on the Malliavin derivative of a given r.v. to guarantee that its tail is bounded above by a Gaussian tail.

Proposition 1.1. *For any $X \in \mathbf{D}^{1,2}$, if $\|DX\|_H$ is bounded almost surely by 1, then X is a standard sub-Gaussian random variable, in the sense that* $\mathbf{P}[|X - \mathbf{E}[X]| > u] \le 2e^{-u^2/2}$.

Remark 1.2. The value 1 in this proposition, and indeed in many places in this paper, has the role of a dispersion coefficient. Since the Malliavin derivative D is linear, the above proposition implies that for any $X \in \mathbf{D}^{1,2}$ such that $\|DX\|_H \le \sigma$ almost surely, then $\mathbf{P}[|X - \mathbf{E}[X]| > u] \le 2e^{-u^2/(2\sigma^2)}$. This trivial normalization argument can be used throughout this paper, because our hypotheses are always based on linear operators such as D. We use this argument in our application in Section 5.

The question of whether a lower bound on $\|DX\|_H^2$ gives rise to an inequality in the opposite direction, as in the above proposition, arises naturally. However, we were unable to find any proof of such a result. Instead, after reading Eulalia Nualart's article [14] where she finds a class of lower bounds by considering exponential moments on the divergence (Skorohod integral) of a covering vector field of X, we were inspired to look for other Malliavin calculus operations on X which would yield a Gaussian lower bound on X's tail. We turned to the quantity $G := \langle DX; -DL^{-1}X\rangle_H$, identified in [9], and used profitably in [12]. Here L^{-1}, the pseudo-inverse of the so-called generator of the Ornstein–Uhlenbeck semigroup, is defined in Section 3.2. This article's first theoretical result is that a lower (resp. upper) bound on G can yield a lower (resp. upper) bound similar to the upper bound in Proposition 1.1. For instance, summarizing the combination of some consequences of our results and Proposition 1.1, we have the following.

Theorem 1.3. *Let X be a random variable in $\mathbf{D}^{1,2}$. Let $G := \langle DX; -DL^{-1}X\rangle_H$.*

1. *If $G \ge 1$ almost surely, then*

$$\mathrm{Var}[X] = \mathbf{E}[G] \ge 1.$$

2. *If $G \ge 1$ almost surely, and if for some $c > 2$, $\mathbf{E}\left[X^c\right] < \infty$, then*

$$\limsup_{z\to\infty} \mathbf{P}[X - \mathbf{E}[X] > z] / \bar{\Phi}(z) \ge \frac{c-2}{c}. \tag{1}$$

3. *If $G \ge 1$ almost surely, and if there exist $c' \in (0, 1/2)$ and $z_0 > 0$, such that $G \le c'X^2$ almost surely when $X \ge z_0$, then for $z > z_0$,*

$$\mathbf{P}[X - \mathbf{E}[X] > z] \ge \left(1 - 2c'\right) \bar{\Phi}(z).$$

4. *If $G \le 1$ almost surely, and X has a density, then for every $z > 0$*

$$\mathbf{P}[X - \mathbf{E}[X] > z] \le \left(1 + \frac{1}{z^2}\right) \bar{\Phi}(z). \tag{2}$$

5. *If* $\|DX\|_H^2 \leq 1$ *almost surely, then* $\mathrm{Var}[X] \leq (\pi/2)^2$ *and for* $z > 0$,

$$\mathbf{P}[X - \mathbf{E}[X] > z] \leq \mathrm{e}^{-z^2/2}. \tag{3}$$

Remark 1.4. Item 1 in this theorem is Corollary 4.2 Point 1. Item 2 here comes from Corollary 4.2 Point 3. Item 3 here follows from Corollary 4.5 Point 1. Item 4 is from Theorem 4.1. Inequality (3) in Item 5 here is equivalent to Proposition 1.1. The variance upper bound in Item 5 here follows from [21, Theorem 9.2.3 part (iii)]. Other, non-Gaussian comparisons are also obtained in this article: see Corollary 4.5.

The results in Theorem 1.3 point to basic properties of the Malliavin derivative and generator of the Ornstein–Uhlenbeck semigroup when investigating tail behavior of random variables. The importance of the relation of G to the value 1 was already noticed in [9, Theorem 3.1] where its L^2-convergence to 1 for a sequence of r.v.'s was a basic building block for convergence to the standard normal distribution. Here we show what can still be asserted when the condition is significantly relaxed. An attempt was made to prove a version of the theorem above in [12, Section 4]; here we significantly improve that work by: (i) removing the unwieldy upper bound conditions made in [12, Theorem 4.2] to prove lower bound results therein; and (ii) improving the upper bound in [12, Theorem 4.1] while using a weaker hypothesis.

Our results should have applications in any area of pure or applied probability where Malliavin derivatives are readily expressed. In fact, Nourdin and Peccati [9, Remark 1.4, point 4] already hint that G is not always as intractable as one may fear. We present such an application in this article, in which the deviations of a random polymer in some random media are estimated, and its *fluctuation exponent* is calculated to be $\chi = 1/2$, a result which we prove to be robust to non-linear changes in the polymer's Hamiltonian.

The structure of this article is as follows. Section 2 presents all necessary background information from the theory of Wiener chaos and the Malliavin calculus needed to understand our statements and proofs. Section 3 recalls Stein's lemma and equation, presents the way it will be used in this article, and recalls the density representation results from [12]. Section 4 states and proves our main lower and upper bound results. Section 5 gives a construction of continuous random polymers in Gaussian environments, and states and proves the estimates on its deviations and its fluctuation exponent under Gaussian and non-Gaussian Hamiltonians, when the Gaussian environment has infinite-range correlations. Several interesting open questions are described in this section as well. The Appendix, contains the proofs of some lemmas.

2. Preliminaries: Wiener chaos and Malliavin calculus

For a complete treatment of this topic, we refer the reader to David Nualart's textbook [13].

We are in the framework of an *isonormal Gaussian process* W on a suitable probability space $(\Omega, \mathcal{F}, \mathbf{P})$: it is defined as a Gaussian field W on a Hilbert space $H = L^2(T, \mathcal{B}, \mu)$ where μ is a σ-finite measure that is either discrete or without atoms, and the covariance of W coincides with the inner product in H. This forces W to be linear on H; consequently, it can be interpreted as an abstract Wiener integral. For instance, if $T = [0, 1]$ and μ is the Lebesgue measure, then $W(f)$ represents the usual Wiener stochastic integral $\int_0^1 f(s)\,\mathrm{d}W(s)$ of a square-integrable non-random function f with respect to a Wiener process also denoted by W; i.e. we confuse the notation $W(t)$ and $W(\mathbf{1}_{[0,t]})$. In general for $\{f_i : i = 1, \dots, n\} \in H^n$, $(W(f_i) : i = 1, \dots, n)$ is a centered Gaussian vector, with covariance matrix given by $\sigma_{i,j}^2 = \langle f_i; f_j \rangle_H$. The set $\mathcal{H}_1$ of all Wiener integrals $W(f)$ when f ranges over all of H is called the first Wiener chaos of W. To construct higher-order chaoses, one may for example use iterated Itô integration in the case of standard Brownian motion, where $H = L^2[0,1]$. If we denote $I_0(f) = f$ for any non-random constant f, then for any integer $n \geq 1$ and any symmetric function $f \in H^n$, we let

$$I_n(f) := n! \int_0^1 \int_0^{s_1} \cdots \int_0^{s_{n-1}} f(s_1, s_2, \ldots, s_n)\, \mathrm{d}W(s_n) \cdots \mathrm{d}W(s_2)\, \mathrm{d}W(s_1).$$

This is the nth iterated Wiener integral of f w.r.t. W.

Definition 2.1. The set $\mathcal{H}_n := \{I_n(f) : f \in H^n\}$ is the nth Wiener chaos of W.

We refer to [13, Section 1.2] for the general definition of I_n and $\mathcal{H}_n$ when W is a more general isonormal Gaussian process.

Proposition 2.2. *$L^2(\Omega)$ is the direct sum – with respect to the inner product defined by expectations of products of r.v.'s – of all the Wiener chaoses. Specifically for any $X \in L^2(\Omega)$, there exists a sequence of non-random symmetric functions $f_n \in H^n$ with $\sum_{n=0}^{\infty} \|f_n\|_{H^n}^2 < \infty$ such that $X = \sum_{n=0}^{\infty} I_n(f_n)$. Moreover $\mathbf{E}[X] = f_0 = I_0(f_0)$ and $\mathbf{E}[I_n(f_n)] = 0$ for all $n \geq 1$, and $\mathbf{E}[I_n(f_n) I_m(g_m)] = \delta_{m,n} n! \langle f_n, g_n \rangle_{H^n}$ where $\delta_{m,n}$ equals 0 if $m \neq n$ and 1 if $m = n$. In particular $\mathbf{E}[X^2] = \sum_{n=0}^{\infty} n! \|f_n\|_{H^n}^2$.*

The Malliavin derivative operator is usually constructed via an extension starting from so-called simple random variables which are differentiable functions of finite-dimensional vectors from the Gaussian space $\mathcal{H}_1$. The reader can consult Nualart's textbook [13]. We recall the properties which are of use to us herein.

1. The Malliavin derivative operator D is defined from $\mathcal{H}_1$ into H by the formula: for all $r \in T$,

   $$D_r W(f) = f(r).$$

   The Malliavin derivative of a non-random constant is zero. For any m-dimensional Gaussian vector $G = (G_i)_{i=1}^m = (I_1(g_i))_{i=1}^m \in (\mathcal{H}_1)^m$, for any $F \in C^1(\mathbf{R}^m)$ such that $X = F(G) \in L^2(\Omega)$, we have $D_r X = \sum_{i=1}^m \frac{\partial F}{\partial x_i}(G)\, g_i(r)$.
2. The Malliavin derivative of an nth Wiener chaos r.v. is particularly simple. Let $X_n \in \mathcal{H}_n$, i.e. let f_n be a symmetric function in H^n and $X_n = I_n(f_n)$. Then

   $$D_r X = D_r I_n(f_n) = n I_{n-1}(f_n(r, \cdot)). \tag{4}$$

   The Malliavin derivative being linear, this extends immediately to any random variable X in $L^2(\Omega)$ by writing X as its Wiener chaos expansion $\sum_{n=0}^{\infty} I_n(f_n)$, which means that, using the covariance formulas in Proposition 2.2, $DX \in L^2(\Omega \times T)$ if and only if

   $$\mathbf{E}\left[\|DX\|_H^2\right] := \sum_{n=1}^{\infty} n\, n! \|f_n\|_{H^n}^2 < \infty. \tag{5}$$

   The set of all $X \in L^2(\Omega)$ such that $DX \in L^2(\Omega \times T)$ is denoted by $\mathbf{D}^{1,2}$.

Remark 2.3. The general chain rule of point 1 above generalizes to $D[h(X)] = h'(X)\, DX$ for any $X \in \mathbf{D}^{1,2}$ such that X has a density, and any function h which is continuous and piecewise differentiable with a bounded derivative. This is an immediate consequence of [13, Proposition 1.2.3].

3. Tools: Using Stein's lemma and Malliavin derivatives

3.1. Stein's lemma and equation

The version of Stein's lemma which we use can be found in [9]. Let Z be a standard normal random variable and $\bar{\Phi}(z) = \mathbf{P}[Z > z]$ its tail. Let h be a measurable function of one real variable. Stein's equation poses the following question: to find a continuous and piecewise differentiable function f with bounded derivative such that, for all $x \in \mathbf{R}$ where $f'(x)$ exists,

$$h(x) - \mathbf{E}[h(Z)] = f'(x) - xf(x). \tag{6}$$

The precise form of the solution to this differential equation for $h = 1_{(-\infty,z]}$, given in the next lemma, was derived in Stein's original work [19]; a recent usage is found in equalities (1.5), (2, 20), and (2.21) in [10].

Lemma 3.1. *Fix $z \in \mathbf{R}$. Let $h = 1_{(-\infty,z]}$. Then Stein's equation (6) has at a unique solution f satisfying $\|f'\|_\infty := \sup_{x\in\mathbf{R}} |f'(x)| \leq 1$. It is the following:*

- *for $x \leq z$, $f(x) = \sqrt{2\pi}e^{x^2/2}\left(1 - \bar{\Phi}(x)\right)\bar{\Phi}(z)$,*
- *for $x > z$, $f(x) = \sqrt{2\pi}e^{x^2/2}\left(1 - \bar{\Phi}(z)\right)\bar{\Phi}(x)$.*

Corollary 3.2. *Let $X \in L^2(\Omega)$. Setting $x = X$ in Stein's equation (6) and taking expectations we get*

$$\mathbf{P}[X > z] = \bar{\Phi}(z) - \mathbf{E}\left[f'(X)\right] + \mathbf{E}[Xf(X)].$$

The next section gives tools which will allow us to combine this corollary with estimates of the random variable $G := \langle DX; -DL^{-1}X\rangle_H$ in order to get tail bounds. It also shows how G can be used, as in [12], to express the density of X without using Stein's lemma.

3.2. Malliavin derivative tools

Definition 3.3. The generator of the Ornstein–Uhlenbeck semigroup L is defined as follows. Let $X = \sum_{n=1}^{\infty} I_n(f_n)$ be a centered r.v. in $L^2(\Omega)$. If $\sum_{n=1}^{\infty} n^2 n! |f_n|^2 < \infty$, then we define a new random variable LX in $L^2(\Omega)$ by $-LX = \sum_{n=1}^{\infty} nI_n(f_n)$. The pseudo-inverse of L operating on centered r.v.'s in $L^2(\Omega)$ is defined by the formula $-L^{-1}X = \sum_{n=1}^{\infty} \frac{1}{n} I_n(f_n)$. If X is not centered, we define its image by L and L^{-1} by applying them to $X - \mathbf{E}X$.

Definition 3.4. For $X \in \mathbf{D}^{1,2}$, we let $G := \langle DX; -DL^{-1}X\rangle_H$.

The following formula will play an important role in our proofs where we use Stein's lemma. It was originally noted in [9]. We provide a self-contained proof of this result in the Appendix, which does not use the concept of divergence operator (Skorohod integral).

Lemma 3.5. *For any centered $X \in \mathbf{D}^{1,2}$ with a density, with G from Definition 3.4, and for any deterministic continuous and piecewise differentiable function h such that h' is bounded,*

$$\mathbf{E}[Xh(X)] = \mathbf{E}\left[h'(X)G\right]. \tag{7}$$

On the other hand, the next result and its proof (see [12]), make no reference to Stein's lemma.

Definition 3.6. With $X \in \mathbf{D}^{1,2}$ and G as above in Definition 3.6, let the function g be defined almost everywhere on the support of X as the conditional expectation of G given X:

$$g(z) := \mathbf{E}[G|X = z]. \tag{8}$$

Proposition 3.7. *Let $X \in \mathbf{D}^{1,2}$ be centered with a density ρ which is supported on a set I. Then I is an interval $[a, b]$ and, with g as above, we have for almost all $z \in (a, b)$,*

$$\rho(z) = \frac{\mathbf{E}|X|}{2g(z)} \exp\left(-\int_0^z \frac{y\,\mathrm{d}y}{g(y)}\right).$$

Strictly speaking, the proof of this proposition is not contained in [12], since the authors there use the additional assumption that $g(x) \geq 1$ everywhere, which implies that ρ exists and that $I = \mathbf{R}$. However, the modification of their arguments to yield the proposition above is straightforward, and we omit it: for instance, that I is an interval follows from $X \in \mathbf{D}^{1,2}$ as seen in [13, Proposition 2.1.7].

As one can see from this proposition, and the statement of Theorem 1.3, it is important to have a technique to be able to calculate $DL^{-1}X$. We will use a device which can be found for instance in a different form in the proof of Lemma 1.5.1 in [13], and is at the core of the so-called Mehler formula, also found in [13]. It requires a special operator which introduces a coupling with an independent Wiener space. This operator R_θ replaces W by the linear combination $W\cos\theta + W'\sin\theta$ where W' is an independent copy of W. For instance, if W is Brownian motion and one writes the random variable X as $X = F(W)$ where F is a deterministic Borel-measurable functional on the space of continuous functions, then

$$R_\theta X := F\left(W\cos\theta + W'\sin\theta\right). \tag{9}$$

We have the following formula (akin to the Mehler formula, and proved in the Appendix), where $\mathrm{sgn}(\theta) = \theta/|\theta|$, $\mathrm{sgn}(0) = 1$ by convention, where $\mathbf{E}'$ represents the expectation w.r.t. the randomness in W' only, i.e. conditional on W, and where D' is the Malliavin derivative w.r.t. W' only.

Lemma 3.8. *For any $X \in \mathbf{D}^{1,2}$, for all $s \in T$,*

$$\begin{aligned}
-D_s\left(L^{-1}X\right) &= \frac{1}{2}\int_{-\pi/2}^{\pi/2} \mathbf{E}'\left[D'_s(R_\theta X)\right]\mathrm{sgn}(\theta)\,\mathrm{d}\theta \\
&= \frac{1}{2}\int_{-\pi/2}^{\pi/2} \mathbf{E}\mathbf{E}'\left[D'_s(R_\theta X)\,|W\right]\mathrm{sgn}(\theta)\,\mathrm{d}\theta.
\end{aligned}$$

To be specific, note that $D'_s(R_\theta X)$ can typically be expressed explicitly. For instance, for an elementary random variable $X = f\left((W(h_i))_{i=1,\ldots,n}\right)$ with $f \in C_b^\infty(\mathbf{R}^n)$ and $h_i \in H$ for all i,

$$\begin{aligned}
D'_s(R_\theta X) &= D'_s\left[f\left(\left(W(h_i)\cos\theta + W'(h_i)\sin\theta\right)_i\right)\right] \\
&= \sin\theta\sum_{i=1}^n h_i(s)\frac{\partial f}{\partial x_i}\left(\left(W(h_i)\cos\theta + W'(h_i)\sin\theta\right)_i\right).
\end{aligned}$$

We also note that if we rewrite this example abstractly by setting $DX = \Psi(W)$ where Ψ is a measurable function from Ω into H, then the above calculation shows that $D'(R_\theta X) =$

$(\sin\theta)\ \Psi(R_\theta W)$, and the result of the lemma above can be rewritten in a form which may be more familiar to users of Mehler's formula:

$$-D_s\left(L^{-1}X\right)=\frac{1}{2}\int_{-\pi/2}^{\pi/2}\mathbf{E}'\left[\Psi\left(R_\theta W\right)\right]\left|\sin\theta\right|\mathrm{d}\theta.$$

4. Main results

All results in this section are stated and discussed in the first two subsections, the first one dealing with consequences of Stein's lemma, the second with the function g. All proofs are in the third subsection.

4.1. Results using Stein's lemma

Our first result is tailored to Gaussian comparisons.

Theorem 4.1. *Let $X\in\mathbf{D}^{1,2}$ be centered. Assume that almost surely,*

$$G:=\left\langle DX;-DL^{-1}X\right\rangle_H\geq 1. \tag{10}$$

Then for every $z>0$,

$$\mathbf{P}[X>z]\geq\bar{\Phi}(z)-\frac{1}{1+z^2}\int_z^\infty(2x-z)\mathbf{P}[X>x]\,\mathrm{d}x.$$

Assume, instead, that X has a density and $G\leq 1$ almost surely; then for every $z>0$,

$$\mathbf{P}[X>z]\leq\left(1+\frac{1}{z^2}\right)\bar{\Phi}(z).$$

Before proving this theorem, we record some consequences of its lower bound result in the next Corollary. In order to obtain a more precise lower bound result on the tail $S(z):=\mathbf{P}[X>z]$, it appears to be necessary to make some regularity and integrability assumptions on S. This is the aim of the second point in the next corollary. The first and third points show what can be obtained by using only an integrability condition, with no regularity assumption: we may either find a universal lower bound on such quantities as X's variance, or an asymptotic statement on S itself.

Corollary 4.2. *Let $X\in\mathbf{D}^{1,2}$ be centered. Let $S(z):=\mathbf{P}[X>z]$. Assume that condition (10) holds.*

1. *We have*

$$\mathrm{Var}[X]=\mathbf{E}[G]\geq 1.$$

2. *Assume there exists a constant $c>2$ such that $\left|S'(z)\right|/S(z)\geq c/z$ holds for large z. Then for large z,*

$$\mathbf{P}[X>z]\geq\frac{(c-2)\left(1+z^2\right)}{c-2+cz^2}\bar{\Phi}(z)\simeq\frac{(c-2)}{c}\bar{\Phi}(z).$$

3. *Assume there exists a constant $c > 2$ such that $S(z) < z^{-c}$ holds for large z. Then, for large z,*

$$\sup_{x \geq z} x^c \mathbf{P}[X > x] \geq \frac{c-2}{c} z^c \bar{\Phi}(z).$$

Consequently,

$$\limsup_{z \to \infty} \frac{\mathbf{P}[X > z]}{\bar{\Phi}(z)} \geq \frac{c-2}{c}.$$

Let us discuss the assumptions and results in the corollary from a quantitative standpoint. The assumption of point 2, $|S'(z)|/S(z) \geq c/z$, when integrated, yields $S(z) \leq S(1) z^{-c}$, implies no more than existence of a moment of order larger than 2; it does, however, represent an additional monotonicity condition since it refers to S'. The assumption of point 3, which is weaker because it does not require any monotonicity, also implies the same moment condition. This moment condition is little more than the integrability required from X belonging to $\mathbf{D}^{1,2}$. If c can be made arbitrarily large (for instance in point 3, this occurs when X is assumed to have moments of all orders), asymptotically $(c-2)/c$ can be replaced by 1, yielding the sharpest possible comparison to the normal tail. If indeed S is close to the normal tail, it is morally not a restriction to assume that c can be taken arbitrarily large: it is typically easy to check this via a priori estimates.

4.2. Results using the function g

We now present results which do not use Stein's lemma, but refer only to the random variable $G := \langle DX; -DL^{-1}X \rangle_H$ and the resulting function $g(z) := \mathbf{E}[G|X = z]$ introduced in (8). We will prove the theorem below using the results in [12] on representation of densities. Its corollary shows how to obtain quantitatively explicit upper and lower bounds on the tail of a random variable, which are as sharp as the upper and lower bounds one might establish on g. A description of the advantages and disadvantages of using g over Stein's lemma follows the statements of the next theorem and its corollary.

Theorem 4.3. *Let $X \in \mathbf{D}^{1,2}$ be centered. Let $G := \langle DX; -DL^{-1}X \rangle_H$ and $g(z) := \mathbf{E}[G|X = z]$. Assume that X has a density which is positive on the interior of its support $(a, +\infty)$, where $a \in [-\infty, +\infty)$. For $x \geq 0$, let*

$$A(x) := \exp\left(-\int_0^x \frac{y\,\mathrm{d}y}{g(y)}\right).$$

Then for all $x > 0$,

$$\mathbf{P}[X > x] = \frac{\mathbf{E}|X|}{2} \left(\frac{A(x)}{x} - \int_x^\infty \frac{A(y)}{y^2} \mathrm{d}y\right). \tag{11}$$

Remark 4.4. The density formula in Proposition 3.7 shows that g must be non-negative [in fact, this was already known, and holds true almost surely for G itself, even if X is not known have a density, assuming only $X \in \mathbf{D}^{1,2}$: [9, Proposition 3.9]]. Assuming our centered $X \in \mathbf{D}^{1,2}$ has a density ρ, we have already noted that ρ must be positive on (a, b) and zero outside. To ensure that $b = +\infty$, as is needed in the above theorem, it is sufficient to assume that g is bounded below on

$[0, b)$ by a positive constant. If in addition we can assume, as in (10), that this lower-boundedness of g holds everywhere, then X has a density, and its support is $\mathbf{R}$.

Corollary 4.5. *Assume that for some $c' \in (0,1)$ and some $z_0 > 1$, we have for all $x > z_0$, $g(x) \le c' x^2$. Then, with $K := \frac{\mathbf{E}|X|}{2} \frac{(c')^{c'}}{(1+c')^{1+c'}}$, for $x > z_0$,*

$$\mathbf{P}[X > x] \ge K \frac{A(x)}{x}. \tag{12}$$

1. *Under the additional assumption (10), $g(x) \ge 1$ everywhere, and we have*
$$\mathbf{P}[X > z] \ge K \frac{1}{x} \exp\left(-\frac{x^2}{2}\right) \simeq \sqrt{2\pi} K \bar{\Phi}(z).$$
2. *If we have rather the stronger lower bound $g(x) \ge c'' x^2$ for some $c'' \in (0, c']$ and all $x > z_0$, then for $x > z_0$, and with some constant K' depending on g, c'' and z_0,*
$$\mathbf{P}[X > z] \ge K' x^{-1-1/c''}.$$
3. *If we have instead that $g(x) \ge c_1 x^p$ for some $c_1 > 0$, $p < 2$, and for all $x > z_0$, then for $x > z_0$, and with some constant K'' depending on g, c_1, p, and z_0,*
$$\mathbf{P}[X > z] \ge K'' \exp\left(-\frac{x^{2-p}}{(2-p)c_1}\right).$$
4. *In the last two points, if the inequalities on g in the hypotheses are reversed, the conclusions are also reversed, without changing any of the constants: i.e.*
   (a) *if $\exists c'' \le c', \exists z_0 > 0 : \forall x > z_0, g(x) \le c'' x^2$ then $x > z_0 \Rightarrow \mathbf{P}[X > z] \le K' x^{-1-1/c''}$;*
   (b) *if $\exists c_1 > 0, \exists p < 2, \exists z_0 > 0 : \forall x > z_0, g(x) \le c_1 x^p$ then $x > z_0 \Rightarrow \mathbf{P}[X > z] \le K'' \exp\left(-\frac{x^{2-p}}{(2-p)c_1}\right)$.*

The tail formula (11) in Theorem 4.3 readily implies asymptotic estimates on S of non-Gaussian type if one is able to compare g to a power function. Methods using Stein's lemma, at least in its form described in Section 3.1, only work efficiently for comparing S to the Gaussian tail. Arguments found in Nourdin and Peccati's articles (e.g. [9]) indicate that Stein's method may be of use in some specific non-Gaussian cases, which one could use to compare tails to the Gamma tail, and perhaps to other tails in the Pearson family, which would correspond to polynomial g with degree at most 2. The flexibility of our method of working directly with g rather than Stein's lemma, is that it seems to allow any type of tail. Stein's method has one important advantage, however: it is not restricted to having a good control on g; Theorem 4.1 establishes Gaussian lower bounds on tails by only assuming (10) and mild conditions on the tail itself. This is to be compared to the lower bound [12, Theorem 4.2] proved via the function g alone, where it required growth conditions on g which may not be that easy to check.

There is one intriguing, albeit perhaps technical, fact regarding the use of Stein's method: in Point 1 of the above Corollary 4.5, since the comparison is made with a Gaussian tail, one may wonder what the usage of Stein's lemma via Theorem 4.1 may produce when assuming, as in Point 1 of Corollary 4.5, that $g(x) \ge 1$ and g grows slower than x^2. As it turns out, Stein's method is not systematically superior to Corollary 4.5, as we now see.

Corollary 4.6 (*Consequence of* Theorem 4.1). *Assume that* $g(x) \geq 1$ *and, for some* $c' \in (0, 1/2)$ *and large* $x > z_0$, $g(x) \leq c'x^2$. *Then for* $z > z_0$,

$$\mathbf{P}[X > z] \geq \frac{1 + z^2}{1 + (1 - 2c')^{-1} z^2} \bar{\Phi}(z) \simeq \left(1 - 2c'\right) \bar{\Phi}(z).$$

When this corollary and Point 1 in Corollary 4.5 are used in an efficient situation, this means that X is presumably "sub-Gaussian" as well as being "super-Gaussian" as a consequence of assumption (10). For illustrative purposes, we can translate this roughly as meaning that for some $\alpha > 0$, $g(x)$ is in the interval $[1, 1 + \alpha]$ for all x. This implies that we can take $c' \to 0$ in both Corollaries 4.5 and 4.6; as a consequence, the first corollary yields $\mathbf{P}[X > z] \geq \bar{\Phi}(z)$, while the second gives $\mathbf{P}[X > z] \geq (\sqrt{2\pi}\mathbf{E}|X|/2)\ \bar{\Phi}(z)$. The superiority of one method over another then depends on how $\sqrt{2\pi}\mathbf{E}|X|/2$ compares to 1. It is elementary to check that, in "very sharp" situations, which means that α is quite small, $\sqrt{2\pi}\mathbf{E}|X|/2$ will be close to 1, from which one can only conclude that both methods appear to be equally efficient.

4.3. Proofs

We now turn to the proofs of the above results.

Proof of Theorem 4.1. *Step* 1: *exploiting the negativity of* f'. From Lemma 3.1, we are able to calculate the derivative of the solution f to Stein's equation:

- for $x \leq z$, $f'(x) = \bar{\Phi}(z)\left(1 + \sqrt{2\pi}\left(1 - \bar{\Phi}(x)\right) x e^{x^2/2}\right)$;
- for $x > z$, $f'(x) = \left(1 - \bar{\Phi}(z)\right)\left(-1 + \sqrt{2\pi}\,\bar{\Phi}(x)\, x e^{x^2/2}\right)$.

We now use the standard estimate, valid for all $x > 0$ (see [6, Problem 2.9.22, page 112]):

$$\frac{x}{(x^2 + 1)\sqrt{2\pi}} e^{-x^2/2} \leq \bar{\Phi}(x) \leq \frac{1}{x\sqrt{2\pi}} e^{-x^2/2}. \tag{13}$$

In the case $x > z$, since $z > 0$, the upper estimate yields $f'(x) \leq \left(1 - \bar{\Phi}(z)\right)(-1 + 1) = 0$. Now by the expression for $\mathbf{P}[X > z]$ in Corollary 3.2, the negativity of f' on $\{x > z\}$ implies for all $z > 0$,

$$\begin{aligned}\mathbf{P}[X > z] &= \bar{\Phi}(z) - \mathbf{E}\left[\mathbf{1}_{X \leq z} f'(X)\right] - \mathbf{E}\left[\mathbf{1}_{X > z} f'(X)\right] + \mathbf{E}[Xf(X)] \\ &\geq \bar{\Phi}(z) - \mathbf{E}\left[\mathbf{1}_{X \leq z} f'(X)\right] + \mathbf{E}[Xf(X)].\end{aligned}$$

Step 2: *Exploiting the positivities and the smallness of* f'. Using Step 1, we have

$$\mathbf{P}[X > z] \geq \bar{\Phi}(z) - \mathbf{E}\left[\mathbf{1}_{X \leq z} f'(X)\right] + \mathbf{E}\left[\mathbf{1}_{X \leq z} Xf(X)\right] + \mathbf{E}\left[\mathbf{1}_{X > z} Xf(X)\right].$$

We apply Lemma 3.5 to the function $h(x) = (f(x) - f(z))\mathbf{1}_{x \leq z}$; h is continuous everywhere; it is differentiable everywhere with a bounded derivative, equal to $f'(x)\mathbf{1}_{x \leq z}$, except at $x = z$. Applying this lemma is legitimized by the fact, proved in [12], that $G \geq 1$ a.s. implies X has a density (see the explanation given after the statement of Proposition 3.7). Thus we get

$$\begin{aligned}\mathbf{P}[X > z] &\geq \bar{\Phi}(z) - \mathbf{E}\left[\mathbf{1}_{X \leq z} f'(X)\right] + \mathbf{E}[Xh(X)] \\ &\quad + \mathbf{E}\left[\mathbf{1}_{X \leq z} X\right] f(z) + \mathbf{E}\left[\mathbf{1}_{X > z} Xf(X)\right] \\ &\geq \bar{\Phi}(z) + \mathbf{E}\left[\mathbf{1}_{X \leq z} f'(X)(-1 + G)\right] + \mathbf{E}\left[\mathbf{1}_{X \leq z} X\right] f(z) + \mathbf{E}\left[\mathbf{1}_{X > z} Xf(X)\right].\end{aligned} \tag{14}$$

When $x \leq z$, we can use the formula in Step 1 to prove that $f'(x) \geq 0$. Indeed this is trivial when $x \geq 0$, while when $x < 0$, it is proved as follows: for $x = -y < 0$, and using the upper bound in (13)

$$f'(x) = \bar{\Phi}(z)\left(1 + \sqrt{2\pi}\left(1 - \bar{\Phi}(x)\right) x e^{x^2/2}\right) = \bar{\Phi}(z)\left(1 - \sqrt{2\pi}\,\bar{\Phi}(y)\, y e^{y^2/2}\right) \geq 0.$$

By the lower bound hypothesis (10), we also have positivity of $-1 + G$. Thus the second term on the right-hand side of (14) is non-negative. In other words we have

$$\mathbf{P}[X > z] \geq \bar{\Phi}(z) + \mathbf{E}\left[\mathbf{1}_{X \leq z} X\right] f(z) + \mathbf{E}\left[\mathbf{1}_{X > z} X f(X)\right] \tag{15}$$

$$=: \bar{\Phi}(z) + A. \tag{16}$$

The sum of the last two terms on the right-hand side of (15), which we call A, can be rewritten as follows, using the fact that $\mathbf{E}[X] = 0$:

$$\begin{aligned} A &:= \mathbf{E}\left[\mathbf{1}_{X \leq z} X\right] f(z) + \mathbf{E}\left[\mathbf{1}_{X > z} X f(X)\right] \\ &= \mathbf{E}\left[\mathbf{1}_{X \leq z} X\right] f(z) + \mathbf{E}\left[\mathbf{1}_{X > z} X\left(f(X) - f(z)\right)\right] + f(z)\,\mathbf{E}\left[\mathbf{1}_{X > z} X\right] \\ &= \mathbf{E}\left[\mathbf{1}_{X > z} X\left(f(X) - f(z)\right)\right]. \end{aligned}$$

This quantity A is slightly problematic since, f being decreasing on $[z, +\infty)$, we have $A < 0$. However, we can write $f(X) - f(z) = f'(\xi)(X - z)$ for some $\xi > z$. Note that this ξ is random and depends on z; in fact on the event $X > z$, we have $\xi \in [z, X]$, but we will only need to use the lower bound on ξ: we use the lower bound in (13) to get that for all $\xi > z$,

$$\begin{aligned} \left|f'(\xi)\right| &= -f'(\xi) = \left(1 - \bar{\Phi}(z)\right)\left(1 - \sqrt{2\pi}\,\bar{\Phi}(\xi)\,\xi e^{\xi^2/2}\right) \leq 1 \cdot \left(1 - \frac{\xi^2}{1 + \xi^2}\right) \\ &= \frac{1}{1 + \xi^2}. \end{aligned} \tag{17}$$

This upper bound can obviously be further bounded above uniformly by $\left(1 + z^2\right)^{-1}$, which means that

$$|A| \leq \mathbf{E}\left[\mathbf{1}_{X > z} X (X - z)\right] \frac{1}{1 + z^2}.$$

By using this estimate in (16) we finally get

$$\mathbf{P}[X > z] \geq \bar{\Phi}(z) - \mathbf{E}\left[\mathbf{1}_{X > z} X (X - z)\right] \frac{1}{1 + z^2}. \tag{18}$$

Step 3: *Integrating by parts*. For notational compactness, let $S(z) := \mathbf{P}[X > z]$. We integrate the last term in (18) by parts with respect to the positive measure $-\mathrm{d}S(x)$. We have, for any $z > 0$,

$$\begin{aligned} \mathbf{E}\left[\mathbf{1}_{X > z} X (X - z)\right] &= -\int_z^\infty x (x - z)\,\mathrm{d}S(x) \\ &= z(z - z) S(z) - \lim_{x \to +\infty} x (x - z) S(x) + \int_z^\infty (2x - z) S(x)\,\mathrm{d}x \\ &\leq \int_z^\infty (2x - z) S(x)\,\mathrm{d}x. \end{aligned}$$

The conclusion (18) from the previous step now implies

$$S(z) \geq \bar{\Phi}(z) - \frac{1}{1+z^2}\int_z^\infty (2x-z)S(x)\,\mathrm{d}x,$$

which finishes the proof of the theorem's lower bound.

Step 4: *Upper bound*. The proof of the upper bound is similar to, not symmetric with, and less delicate than, the proof of the lower bound. Indeed, we can take advantage of a projective positivity result on the inner product of DX and $-DL^{-1}X$, namely [9, Proposition 3.9] which says that $\mathbf{E}[G|X] \geq 0$. This allows us to avoid the need for any additional moment assumptions. Since X is assumed to have a density, we may use Lemma 3.5 directly with the function $h = f$, which is continuous, and differentiable everywhere except at $x = z$: we have

$$\begin{aligned}\mathbf{P}[X > z] &= \bar{\Phi}(z) - \mathbf{E}\left[f'(X)\right] + \mathbf{E}\left[f'(X)\,G\right]\\ &= \bar{\Phi}(z) + \mathbf{E}\left[\mathbf{1}_{X\leq z} f'(X)(-1+G)\right] + \mathbf{E}\left[\mathbf{1}_{X>z} f'(X)(-1+G)\right]\\ &\leq \bar{\Phi}(z) + \mathbf{E}\left[\mathbf{1}_{X>z} f'(X)(-1+G)\right] \end{aligned}\tag{19}$$

where the last inequality simply comes from the facts that by hypothesis $-1+G$ is negative, and when $x \leq z$, $f'(x) \geq 0$ (see previous step for proof of this positivity). It remains to control the term in (19): since $\mathbf{E}[G|X] \geq 0$, and using the negativity of f' on $x > z$,

$$\begin{aligned}\mathbf{E}\left[\mathbf{1}_{X>z} f'(X)(-1+G)\right] &= \mathbf{E}\left[\mathbf{1}_{X>z} f'(X)\,\mathbf{E}[(-1+G)\,|X]\right]\\ &\leq -\mathbf{E}\left[\mathbf{1}_{X>z} f'(X)\right] = \mathbf{E}\left[\mathbf{1}_{X>z}\left|f'(X)\right|\right].\end{aligned}$$

This last inequality together with the bound on f' obtained in (17) imply

$$\mathbf{E}\left[\mathbf{1}_{X>z} f'(X)(-1+G)\right] \leq \mathbf{P}[X > z]\,\frac{1}{1+z^2}.$$

Thus we have proved that

$$\mathbf{P}[X > z] \leq \bar{\Phi}(z) + \mathbf{P}[X > z]\,\frac{1}{1+z^2}$$

which implies the upper bound of the theorem, finishing its proof. ■

Proof of Corollary 4.2. To simplify the expressions of the constants in this corollary, we have ignored the term z in the factor $(2x - z)$ in the lower bound of Theorem 4.1, which of course yields a stronger lower bound statement. One also notes that, by a result in [12], condition (10) implies that X has a density.

Proof of Point 1. Since X has a density, we can apply Lemma 3.5, from which it trivially follows that $\mathrm{Var}[X] = \mathbf{E}[XX] = \mathbf{E}[G] \geq 1$.

Proof of Point 2. Since X has a density, S' is defined, and we get

$$\begin{aligned}F(z) := \int_z^\infty x\mathbf{P}[X > x]\,\mathrm{d}x &= \int_z^\infty xS(x)\,\mathrm{d}x \leq \frac{1}{c}\int_z^\infty x^2\left|S'(x)\right|\mathrm{d}x\\ &= \frac{1}{c}\left(z^2 S(z) - \lim_{x\to\infty} x^2 S(x) + \int_z^\infty 2xS(x)\,\mathrm{d}x\right)\\ &\leq \frac{1}{c}\left(z^2 S(z) + 2F(z)\right)\end{aligned}$$

which implies

$$F(z) \leq \frac{1}{c-2} z^2 S(z).$$

With the lower bound conclusion of Theorem 4.1, we obtain

$$S(z) \geq \bar{\Phi}(z) - \frac{2z^2}{1+z^2}\frac{1}{c-2} S(z)$$

which is equivalent to the statement of Point 2.

Proof of Point 3. From Theorem 4.1, we have for large z,

$$\begin{aligned} S(z) &\geq \bar{\Phi}(z) - \frac{1}{1+z^2}\int_z^\infty 2x^{1-c} x^c S(x)\,\mathrm{d}x \\ &\geq \bar{\Phi}(z) - \frac{1}{1+z^2}\sup_{x>z}\left[x^c S(x)\right]\int_z^\infty 2x^{1-c}\mathrm{d}x = \bar{\Phi}(z) \\ &\quad - \frac{z^{2-c}2/(c-2)}{1+z^2}\sup_{x>z}\left[x^c S(x)\right] \end{aligned}$$

which implies

$$\sup_{x>z}\left[x^c S(x)\right]\left(\frac{2}{c-2}+1\right) \geq z^c \bar{\Phi}(z)$$

which is equivalent to the first part of the statement of Point 3, the second part following from the fact that $z^c \bar{\Phi}(z)$ is decreasing for large z. ■

Proof of Theorem 4.3. By Proposition 3.7, with $L = \mathbf{E}|X|/2$, for $x \in (a, +\infty)$,

$$\rho(x) = LA(x)/g(x).$$

By definition we also get $A'(x) = -xA(x)/g(x) = -xL^{-1}\rho(x)$, and thus

$$\mathbf{P}[X > x] =: S(x) = L\int_x^{+\infty}\frac{-A'}{y}\mathrm{d}y = L\left(\frac{A(x)}{x} - \lim_{y\to\infty}\frac{A(y)}{y} - \int_x^{+\infty}\frac{A(y)}{y^2}\mathrm{d}y\right).$$

Since g is non-negative, A is bounded, and the term $\lim_{y\to\infty} A(y)/y$ is thus zero. Equality (11) follows immediately, proving the theorem. ■

Proof of Corollary 4.5. *Proof of inequality* (12). From Theorem 4.3, with $L = \mathbf{E}|X|/2$, and $k > 1$, and using the fact that A is decreasing, we can write

$$\begin{aligned} S(x) =: \mathbf{P}[X > x] &= L\left(\frac{A(x)}{x} - \int_x^{kx}\frac{A(y)}{y^2}\mathrm{d}y - \int_{kx}^{+\infty}\frac{A(y)}{y^2}\mathrm{d}y\right) \\ &\geq L\left(\frac{A(x)}{x} - \frac{A(x)}{x}\left(1-\frac{1}{k}\right) - \frac{A(kx)}{kx}\right) \\ &= L\frac{A(x)}{x}\frac{1}{k}\left(1 - \frac{A(kx)}{A(x)}\right). \end{aligned}$$

It is now just a matter of using the assumption $g(x) \leq c'x^2$ to control $A(kx)/A(x)$. We have for large x,

$$\frac{A(kx)}{A(x)} = \exp\left(-\int_x^{kx}\frac{y\mathrm{d}y}{g(y)}\right) \leq \exp\left(-\frac{1}{c'}\log k\right) = k^{-1/c'}.$$

This proves

$$S(x) \geq L\frac{A(x)}{x}\frac{1}{k}\left(1-k^{-1/c'}\right).$$

The proof is completed simply by optimizing this over the values of $k > 1$: the function $k \mapsto \left(1-k^{-1/c'}\right)/k$ reaches its maximum of $(c')^{c'}(1+c')^{-c'-1}$ at $(1+1/c')^{c'}$.

Proof of Points 1, 2, 3, *and* 4. Point 1 is immediate since $g(x) \geq 1$ implies $A(x) \geq \exp\left(-x^2/2\right)$. Similarly, for Point 2, we have

$$A(x) \geq \exp\left(-\int_0^{y_0}\frac{y\mathrm{d}y}{g(y)}\right)\exp\left(-\frac{1}{c''}\int_{y_0}^{x}\frac{\mathrm{d}y}{y}\right) = cst\, x^{-1/c''},$$

and Point 3 follows in the same fashion. Point 4 is shown identically by reversing all inequalities, concluding the proof of the Corollary. ■

Proof of Corollary 4.6. This is in fact a corollary of the proof of Theorem 4.1. At the end of Step 2 therein, in (18), we prove that (10), the lower bound assumption $G \geq 1$, implies

$$S(z) \geq \bar{\Phi}(z) - \mathbf{E}\left[\mathbf{1}_{X>z}X(X-z)\right]\frac{1}{1+z^2}. \tag{20}$$

Let us investigate the term $B := \mathbf{E}\left[\mathbf{1}_{X>z}X(X-z)\right]$. Using Lemma 3.5 with the function $h(x) = (x-z)\mathbf{1}_{x>z}$, we have

$$B = \mathbf{E}\left[\mathbf{1}_{X>z}G\right] = \mathbf{E}\left[\mathbf{1}_{X>z}g(X)\right].$$

Now use the upper bound assumption on g: we get, for all $z \geq z_0$,

$$\begin{aligned} B &\leq c'\mathbf{E}\left[\mathbf{1}_{X>z}X^2\right] = c'\mathbf{E}\left[\mathbf{1}_{X>z}X(X-z)\right] + c'z\mathbf{E}\left[\mathbf{1}_{X>z}X\right] \\ &= c'B + c'z\mathbf{E}\left[\mathbf{1}_{X>z}X\right] = c'B + c'z\left(zS(z) + \int_z^\infty S(x)\,\mathrm{d}x\right), \end{aligned} \tag{21}$$

where we used integration by parts for the last inequality. Again using integration by parts, but directly on the definition of $B := \int_z^\infty \left(x^2 - zx\right)\rho(x)\,\mathrm{d}x$, yields

$$B = 2\int_z^\infty xS(x)\,\mathrm{d}x - z\int_z^\infty S(x)\,\mathrm{d}x.$$

Introducing the following additional notation: $D := z\int_z^\infty S(x)\,\mathrm{d}x$ and $E := 2\int_z^\infty xS(s)\,\mathrm{d}x$, we see that $B = E - D$ and also that $E \geq 2D$. Moreover, in (21), we also recognize the appearance of D. Therefore we have

$$(E-D)\left(1-c'\right) \leq c'D + c'z^2S(z) \leq \left(c'/2\right)E + c'z^2S(z).$$

With $E - D \geq E/2$, we now get $E\left(1-c'\right) \leq c'E + 2c'z^2S(z)$, i.e.

$$B \leq E \leq \frac{2c'}{1-2c'}z^2S(z).$$

From (20), we now get

$$S(z) \geq \bar{\Phi}(z) - \frac{2c'\left(1-2c'\right)^{-1}z^2}{1+z^2}S(z)$$

from which we obtain, for $z \geq z_0$

$$S(z) \geq \frac{1+z^2}{1+\left(2c'(1-2c')^{-1}+1\right)z^2}\bar{\Phi}(z),$$

finishing the proof of the corollary. ■

5. Fluctuation exponent and deviations for polymers in Gaussian environments

Lemma 3.8 provides a way to calculate $G := \langle DX; -DL^{-1}X\rangle_H$ in order to check, for instance, whether it is bounded below by a positive constant c^2. If $c^2 \neq 1$, because of the bilinearity of Condition (10), one only needs to consider X/c instead of X in order to apply Theorem 4.1, say. To show that such a tool can be applied with ease in a non-trivial situation, we have chosen the issue of fluctuation exponents for polymers in random environments.

We can consider various polymer models in random environments constructed by analogy with the so-called stochastic Anderson models (see [18,5]). A polymer's state space R can be either $\mathbf{R}^d$ or the d-dimensional torus $\mathbf{S}^d$, or also $\mathbf{Z}^d$ or $\mathbf{Z}/p\mathbf{Z}$; we could also use any Lie group for R. We can equip R with a Markov process b on $[0,\infty)$ whose infinitesimal generator, under the probability measure P_b, is the Laplace(-Beltrami) operator or the discrete Laplacian. Thus for instance, b is Brownian motion when $R = \mathbf{R}^d$, or is the simple symmetric random walk when $R = \mathbf{Z}^d$; it is the image of Brownian motion by the imaginary exponential map when $R = \mathbf{S}^1$. To simplify our exposition, we can and will typically assume, unless explicitly stated otherwise, that $R = \mathbf{R}$, but our constructions and proofs can be adapted to any of the above choices.

5.1. *The random environment*

Let W be a Gaussian field on $\mathbf{R}_+ \times \mathbf{R}$ which is homogeneous in space and is Brownian in time for fixed space parameter: the covariance of W is thus

$$\mathbf{E}[W(t,x)\,W(s,y)] = \min(s,t)\,Q(x-y),$$

for some homogeneous covariance function Q on $\mathbf{R}$. We assume that Q is continuous and that its Fourier transform is a measure with a density denoted by $\hat{Q}$. Note that $\hat{Q}$ is a positive function, and $|Q|$ is bounded by $Q(0)$. The field W can be represented using a very specific isonormal Gaussian process: there exists a white noise measure M on $\mathbf{R}_+ \times \mathbf{R}$ such that

$$W(t,x) = \int_0^t \int_{\mathbf{R}} M(\mathrm{d}s,\mathrm{d}\lambda)\,\sqrt{\hat{Q}(\lambda)}\,\mathrm{e}^{\mathrm{i}\lambda\cdot x},$$

where the above integral is the Wiener integral of $(s,\lambda) \mapsto \mathbf{1}_{[0,t]}(s)\sqrt{\hat{Q}(\lambda)}\,\mathrm{e}^{\mathrm{i}\lambda\cdot x}$ with respect to M. This M is the Gaussian white noise measure generated by an isonormal Gaussian process whose Hilbert space is $H = L^2(\mathbf{R}_+ \times \mathbf{R}, \mathrm{d}r\mathrm{d}\lambda)$, i.e. the control measure of M is the Lebesgue measure on $\mathbf{R}_+ \times \mathbf{R}$. Malliavin derivatives relative to M will take their parameters (s,λ) in $\mathbf{R}_+ \times \mathbf{R}$, and inner products and norms are understood in H. There is a slight possibility of notational confusion since now the underlying isonormal Gaussian process is called M, with the letter W – the traditional name of the polymer potential field – being a linear transformation of M.

The relation between D and W is thus that $D_{s,\lambda}W(t,x) = \mathrm{e}^{\mathrm{i}\lambda\cdot x}\sqrt{\hat{Q}(\lambda)}\mathbf{1}_{[0,t]}(s)$. We will make use of the following similarly important formulas: for any measurable function f:

$$D_{s,\lambda}\int_{\mathbf{R}}\int_0^t M(\mathrm{d}s,\mathrm{d}\lambda)\sqrt{\hat{Q}(\lambda)}\mathrm{e}^{\mathrm{i}\lambda\cdot f(s)} = \sqrt{\hat{Q}(\lambda)}\mathrm{e}^{\mathrm{i}\lambda\cdot f(s)}\mathbf{1}_{[0,t]}(s); \tag{22}$$

$$\int_0^t\int_{\mathbf{R}}\mathrm{d}s\,\hat{Q}(\lambda)\,\mathrm{d}\lambda\,\mathrm{e}^{\mathrm{i}\lambda\cdot f(s)} = \int_0^t Q(f(s))\,\mathrm{d}s. \tag{23}$$

The last equality is obtained by Fubini's theorem and the definition of the function $\hat{Q}$ as the Fourier transform of the univariate function Q. Quantitatively, formula (23) will be particularly useful as a key to easy upper bounds by noting the fact that $\max_{x\in\mathbf{R}} Q(x) = Q(0)$ is positive and finite. On the other hand, if Q is positive and non-degenerate, lower bounds will easily follow.

In order to use the full strength of our estimates in Section 4, we will also allow Q to be inhomogeneous, and in particular, unbounded. This is easily modeled by specifying that

$$W(t,x) = \int_0^t\int_{\mathbf{R}} M(\mathrm{d}s,\mathrm{d}\lambda)\,q(\lambda,x)$$

where $\int_{\mathbf{R}} q(\lambda,x)\,q(\lambda,y)\,\mathrm{d}\lambda = Q(x,y)$. Calculations similar to (22) and (23) then ensue.

We may also devise polymer models in non-Gaussian environments by considering W as a mixture of Gaussian fields. This means that we consider Q to be random itself, with respect to some separate probability space. We will place only weak restrictions on this randomness: under a probability measure $\mathcal{P}$, we assume $\hat{Q}$ is a non-negative random field on $\mathbf{R}$, integrable on $\mathbf{R}$, with $Q(0) = \int_{\mathbf{R}}\hat{Q}(\lambda)\,\mathrm{d}\lambda$ integrable with respect to $\mathcal{P}$.

5.2. The polymer and its fluctuation exponent

Let the Hamiltonian of a path b in $\mathbf{R}$ under the random environment W be defined, up to time t, as

$$H_t^W(b) = \int_0^t W(\mathrm{d}s,b_s) = \int_{\mathbf{R}}\int_0^t M(\mathrm{d}s,\mathrm{d}\lambda)\sqrt{\hat{Q}(\lambda)}\mathrm{e}^{\mathrm{i}\lambda\cdot b_s}.$$

Since W is a symmetric field, we have omitted the traditional negative sign in front of the definition of H_t^W. For fixed path b, this Hamiltonian $H_t^W(b)$ is a Gaussian random variable w.r.t W.

The polymer $\tilde{P}_b$ based on b in the random Hamiltonian H^W is defined as the law whose Radon-Nykodym derivative with respect to P_b is $Z_t(b)/E_b[Z_t(b)]$ where

$$Z_t(b) := \exp H_t^W(b).$$

We use the notation u for the partition function (normalizing constant) for this measure:

$$u(t) := E_b[Z_t(b)].$$

The process $u(t)$ is of special importance: its behavior helps one understand the behavior of the whole measure $\tilde{P}_b$. When $b_0 = x$ instead of 0, the resulting $u(t,x)$ is the solution of a stochastic heat equation with multiplicative noise potential W, and the logarithm of this solution solves a so-called stochastic Burgers equation.

It is known that $t^{-1}\log u(t)$ typically converges almost surely to a non-random constant λ called the almost sure Lyapunov exponent of u (see [18] and references therein for instance; the case of random Q is treated in [7]; the case of inhomogeneous Q on compact space is discussed in [5]). The speed of concentration of $\log u(t)$ around its mean has been the subject of some debate recently. One may consult [1] for a discussion of the issue and its relation to the so-called wandering exponent in non-compact space. The question is to evaluate the asymptotics of $\log u(t) - \mathbf{E}[\log u(t)]$ for large t, or to show that it is roughly equivalent to t^χ, where χ is called the *fluctuation exponent*. The most widely used measure of this behavior is the asymptotics of $\mathrm{Var}[\log u(t)]$. Here, we show that if the space is compact with positive correlations, or if W has infinite spatial correlation range, then $\mathrm{Var}[\log u(t)]$ behaves as t, i.e. the fluctuation exponent χ is $1/2$. This result is highly robust to the actual distribution of W, since it does not depend on the law of Q under $\mathcal{P}$ beyond its first moment. We also provide a class of examples in which H^W is replaced by a non-linear functional of W, and yet the fluctuation exponent, as measured by the power behavior of $\sqrt{\mathrm{Var}}[\log u(t)]$, is still $1/2$.

We hope that our method will stimulate the study of this problem for other correlation structures not covered by the theorem below, in particular in infinite space when the correlation range of W is finite or decaying at a certain speed at infinity, or in the case of space-time white-noise in discrete space, i.e. when the Brownian motions $\{W(\cdot, x) : x \in \mathbf{Z}^d\}$ form an IID family. We conjecture that χ will depend on the decorrelation speed of W. It is at least believed by some that in the case of space-time white noise, $\chi < 1/2$.

The starting point for studying $\mathrm{Var}[\log u(t)]$ is the estimation of the function g relative to the random variable $\log u(t) = \log E_b[\exp H_t^W(b)]$. Here because the integral $H_t^W(b) = \int_0^t W(\mathrm{d}s, b_s)$ has to be understood as $\int_0^t \int_{\mathbf{R}} M(\mathrm{d}s, \mathrm{d}\lambda) \sqrt{\hat{Q}(\lambda)} \mathrm{e}^{\mathrm{i}\lambda \cdot b_s}$, we must calculate the Malliavin derivative with parameters r and λ. We will use the consequence of Mehler's formula described in Lemma 3.8 of Section 3.2. More specifically, we have the following.

Lemma 5.1. *Assume Q is homogeneous. Let*

$$X := \frac{\log u(t) - \mathbf{E} \log u(t)}{\sqrt{t}}.$$

Then

$$D_{s,\lambda} X = \frac{1}{\sqrt{t}} \frac{1}{u(t)} E_b \left[\sqrt{\hat{Q}(\lambda)} \mathrm{e}^{\mathrm{i}\lambda \cdot b_s} \mathrm{e}^{H_t^W(b)} \right] \mathbf{1}_{[0,t]}(s),$$

and

$$G := \left\langle DX, -DL^{-1}X \right\rangle_H = \frac{1}{2t} \int_{-\pi/2}^{\pi/2} |\sin\theta| \,\mathrm{d}\theta \, \mathbf{E}' E_{b,\bar{b}} \times \left[\int_0^t \mathrm{d}s \, Q\left(b_s - \bar{b}_s\right) \frac{\exp H_t^W(b)}{u(t)} \frac{\exp H_t^{R_\theta W}(\bar{b})}{R_\theta u(t)} \right], \tag{24}$$

where $E_{b,\bar{b}}$ is the expectation w.r.t. two independent copies b and $\bar{b}$ of Brownian motion, and $R_\theta W$ was defined in (9). When Q is inhomogeneous, the above formula still holds, with $Q(b_s - \bar{b}_s)$ replaced by $Q(b_s, \bar{b}_s)$.

Proof. By formula (22) and the chain rule for Malliavin derivatives, we have for fixed b,

$$D_{s,\lambda}\left(\mathrm{e}^{H_t^W(b)}\right) = \sqrt{\hat{Q}(\lambda)} \mathrm{e}^{\mathrm{i}\lambda \cdot b_s} \mathrm{e}^{H_t^W(b)} \mathbf{1}_{[0,t]}(s)$$

and therefore by linearity of the expectation E_b, and the chain rule again, the first statement of the lemma follows immediately.

Now we investigate $DL^{-1}X$. To use Lemma 3.8 relative to W, we note that the expression for $R_\theta X$ is straightforward, since X is defined as a non-random non-linear functional of an expression involving b and W with the latter appearing linearly via $H_t^W(b)$; in other words, $R_\theta X$ is obtained by replacing $H_t^W(b)$ by $H_t^{R_\theta W}(b)$, so we simply have

$$R_\theta X = \frac{\log E_b\left[\exp\left(H_t^W(b)\cos\theta + H_t^{W'}(b)\sin\theta\right)\right] - \mathbf{E}\log u(t)}{\sqrt{t}}.$$

Thus by Lemma 3.8,

$$-D_{s,\lambda}L^{-1}X = \int_{-\pi/2}^{\pi/2} \mathrm{d}\theta\, \frac{\operatorname{sgn}(\theta)}{2\sqrt{t}} E_b\mathbf{E}'\left[\sqrt{\hat{Q}(\lambda)}\mathrm{e}^{\mathrm{i}\lambda\cdot b_s}\sin(\theta)\,\frac{\exp H_t^{R_\theta W}(b)}{R_\theta u(t)}\right].$$

We may thus calculate explicitly the inner product $G := \langle DX, -DL^{-1}X\rangle_H$, using Eq. (23), obtaining the second announced result (24). The proof of the first statement is identical in structure to the above arguments. The last statement is obtained again using identical arguments. ■

It is worth noting that a similar expression as for $G := \langle DX, -DL^{-1}X\rangle_H$ can be obtained for $\|DX\|_H^2$. Using the same calculation technique as in the above proof, we have

$$\begin{aligned}\|DX\|_H^2 &= \|DX\|^2_{L^2([0,t]\times\mathbf{R})} = \frac{1}{t}E_{b,\bar{b}}\left[\frac{\mathrm{e}^{H_t^W(b)}\mathrm{e}^{H_t^W(\bar{b})}}{u^2(t)}\int_0^t \mathrm{d}s\, Q\left(b_s,\bar{b}_s\right)\right]\\ &= \frac{1}{t}\tilde{E}_{b,\bar{b}}\left[\int_0^t \mathrm{d}s\, Q\left(b_s,\bar{b}_s\right)\right],\end{aligned} \tag{25}$$

where the last expression involves the expectation w.r.t. the polymer measure $\tilde{P}$ itself, or rather w.r.t. the product measure $\mathrm{d}\tilde{P}_{b,\bar{b}} = \mathrm{e}^{H_t^W(b)}\mathrm{e}^{H_t^W(\bar{b})}u^{-2}(t)\,\mathrm{d}P_b\times\mathrm{d}P_{\bar{b}}$ of two independent polymers $(b,\bar{b})$ in the same random environment W. This measure is called the two-replica polymer measure, and the quantity $\tilde{E}_{b,\bar{b}}\left[\int_0^t \mathrm{d}s\, Q\left(b_s,\bar{b}_s\right)\right]$ is the so-called *replica overlap* for this polymer. This notion should be familiar to those studying spin glasses such as the Sherrington–Kirkpatrick model (see [20]). The strategy developed in this article suggests that the expression $G := \langle DX, -DL^{-1}X\rangle_H$ may be better suited than the rescaled overlap $\|DX\|_H^2$ in seeking lower bounds on $\log u$'s concentration.

Notation 5.2. *In order to simplify the notation in the next theorem, when Q is not homogeneous, we denote $Q(0) = \max_{x\in\mathbf{R}} Q(x,x)$. We then have, in all cases, $Q(0) \geq |Q(x,y)|$ for all $x,y\in\mathbf{R}$. Similarly we denote $Q_m = \min_{x,y\in\mathbf{R}} Q(x,y)$. In the homogeneous case Q_m thus coincides with $\min_{x\in\mathbf{R}} Q(x)$. When Q is random, assumptions about Q below are to be understood as being required $\mathcal{P}$-almost surely.*

Definition 5.3. To make precise statements about the fluctuation exponent, it is convenient to use the following definition:

$$\chi := \lim_{t\to\infty}\frac{\log \operatorname{Var}\left[\log u(t)\right]}{2\log t}.$$

Theorem 5.4. 1. *Assume $Q(0)$ is finite. We have for all $a, t > 0$,*

$$\mathbf{P}\left[\left|\log u(t) - \mathbf{E}\left[\log u(t)\right]\right| > a\sqrt{t}\right] \leq 1 \wedge \frac{2Q(0)^{1/2}}{a\sqrt{2\pi}} \exp\left(-\frac{a^2}{2Q(0)}\right). \tag{26}$$

If Q is random, one only needs to take an expectation $\mathbf{E}_{\mathcal{P}}$ of the above right-hand side.

2. *Assume $Q(0)$ is finite. Then for all t,*

$$\operatorname{Var}\left[\log u(t)\right] \leq \left(\frac{\pi}{2}\right)^2 \mathbf{E}_{\mathcal{P}}\left[Q(0)\right] t. \tag{27}$$

3. *Assume Q_m is positive. Then for all t,*

$$\operatorname{Var}\left[\log u(t)\right] \geq \mathbf{E}_{\mathcal{P}}\left[Q_m\right] t. \tag{28}$$

4. *Assume Q_m is positive and $Q(0)$ is finite. Then, in addition to (26), we have for any $K \in (0, 1)$ and all a large,*

$$\mathbf{P}\left[\left|\log u(t) - \mathbf{E}\left[\log u(t)\right]\right| > a\sqrt{t}\right] \geq K\frac{Q_m^{1/2}}{a} \exp\left(-\frac{a^2}{2Q_m}\right) \tag{29}$$

Moreover, in this case, the conclusions (27) and (28) hold simultaneously, so that the fluctuation exponent is $\chi = 1/2$ as soon as $Q(0) \in L^1[\mathcal{P}]$.

The hypotheses in Points 3 and 4 of this theorem are satisfied if the state space $\mathbf{R}$ is replaced by a compact set such as $\mathbf{S}^1$, or a finite set, and Q is positive everywhere: then indeed $Q_m > 0$. Although the hypothesis of uniform positivity of Q can be considered as restrictive for non-compact state space, one notes that there is no restriction on how small Q_m can be compared to $Q(0)$; in this sense, the slightest persistent correlation of the random environment at distinct sites results in a fluctuation exponent $\chi = 1/2$. In sharp contrast is the case of space-time white noise in discrete space, which is not covered by our theorem, since then $Q(x) = 0$ except if $x = 0$; the main open problem in discrete space is to prove that $\chi < 1/2$ in this white noise case.

In relation to the overlap $\|DX\|_H^2$, we see that under the assumptions of Point 4 above, $\|DX\|_H$ is also bounded above and below by non-random multiples of $t^{1/2}$. Hence, while our proofs cannot use $\|DX\|_H^2$ directly to prove $\chi = 1/2$, the situation in which we can prove $\chi = 1/2$ coincides with a case where the overlap has the same rough large-time behavior as $\operatorname{Var}\left[\log u(t)\right]$. We believe this is in accordance with common intuition about related spin glass models.

More generally, we consider it an important open problem to understand the precise deviations of $\log u(t)$. The combination of the sub-Gaussian and super-Gaussian estimates (26) and (29) are close to a central limit theorem statement, except for the fact that the rate is not sharply pinpointed. Finding a sharper rate is an arduous task which will require a finer analysis of the expression (24), and should depend heavily and non-trivially on the correlations of the covariance function, just as the obtaining of a $\chi < 1/2$ should depend on having correlations that decay at infinity sufficiently fast. There, we believe that a fine analysis will reveal differences between G and the overlap $\|DX\|_H^2$, so that precise quantitative asymptotics of $\log u(t)$ can only be understood by analyzing G, not merely $\|DX\|_H^2$. For instance, it is trivial to prove that $\mathbf{E}[G] \leq \mathbf{E}\left[\|DX\|_H^2\right]$, and we conjecture that this inequality is asymptotically strict for large t, while the deviations of G and $\|DX\|_H^2$ themselves from their respective means are quite small, so that their means' behavior is determinant.

Answering these questions is beyond this article's scope; we plan to pursue them actively in the future.

Proof of Theorem 5.4. *Proof of Point* 1. Since $Q(x,y) \le Q(0)$ for all x, y, from Lemma 5.1, we have

$$G \le \frac{Q(0)}{2t} \int_{-\pi/2}^{\pi/2} |\sin\theta| \, \mathrm{d}\theta \, t \, \mathbf{E}' E_{b,\bar{b}} \left[\frac{\exp H_t^W(b)}{u(t)} \frac{\exp H_t^{R_\theta W}(\bar{b})}{R_\theta u(t)} \right] = Q(0),$$

where we used the trivial facts that $E_b\left[\exp H_t^W(b)\right] = u(t)$ and $E_b\left[\exp H_t^{R_\theta W}(b)\right] = R_\theta u(t)$. The upper bound result in Theorem 4.1, applied to the random variable $\tilde{X} = X/\sqrt{Q(0)}$, now yields

$$\mathbf{P}[X > z] = \mathbf{P}\left[\tilde{X} > zQ(0)^{-1/2}\right] \le \left(1 + \frac{Q(0)}{z^2}\right) \bar{\Phi}\left(\frac{z}{Q(0)^{1/2}}\right)$$

and the upper bound statement (26).

Proof of Points 2 *and* 3. Now we note that, since all terms in the integrals in Lemma 5.1 are positive, our hypothesis that $Q(x,y) \ge Q_m > 0$ for all x, y implies

$$\begin{aligned} G &\ge \frac{Q_m}{2t} \int_{-\pi/2}^{\pi/2} |\sin\theta| \, \mathrm{d}\theta \, t \, \mathbf{E}' E_{b,\bar{b}} \left[\frac{\exp H_t^W(b)}{u(t)} \frac{\exp H_t^{R_\theta W}(\bar{b})}{R_\theta u(t)} \right] \\ &= \frac{Q_m}{2} \int_{-\pi/2}^{\pi/2} |\sin\theta| \, \mathrm{d}\theta \, \mathbf{E}' \left[\frac{E_b\left[\exp H_t^W(b)\right]}{u(t)} \frac{E_b\left[\exp H_t^{R_\theta W}(\bar{b})\right]}{R_\theta u(t)} \right] = Q_m. \end{aligned}$$

Applying Point 1 in Corollary 4.2 to the random variable $\tilde{X} = X/\sqrt{Q_m}$, the lower bound of (28) in Point 3 follows. The upper bound (27) of Point 2 can be proved using the result (26) of Point 1, although one obtains a slightly larger constant than the one announced. The constant $(\pi/2)^2$ is obtained by using the bound $\|DX\|_H^2 \le Q(0)$ which follows trivially from (25), and then applying the classical result $\mathrm{Var}[X] \le (\pi/2)^2 \mathbf{E}\left[\|DX\|_H^2\right]$, found for instance in [21, Theorem 9.2.3].

Proof of Point 4. Since $Q(0)$ is finite and Q_m is positive, using $\tilde{X} = X/\sqrt{Q_m}$ in Corollary 4.6, we have that $g(x) \ge 1$ and $g(x) \le Q(0)/Q_m$, so that we may use any value $c' > 0$ in the assumption of that corollary, with thus $K = 1 - 2c'$ arbitrarily close to 1; the corollary's conclusion is the statement of Point 4. This finishes the proof of the theorem. ■

5.3. Robustness of the fluctuation exponent: A non-Gaussian Hamiltonian

The statements of Point 4 of Theorem 5.4 show that if the random environment's spatial covariance is bounded above and below by positive constants, then the partition function's logarithm $\log u(t)$ is both sub-Gaussian and super-Gaussian, in terms of its tail behavior (tail bounded respectively above and below by Gaussian tails). We now provide an example of a polymer subject to a non-Gaussian Hamiltonian, based still on the same random environment, whose logarithmic partition function may not be sub-Gaussian, yet still has a fluctuation exponent equal to $1/2$. It is legitimate to qualify the persistence of this value $1/2$ in a non-Gaussian example as a type of robustness.

Let

$$X_t^W(b) := \int_0^t W(\mathrm{d}s, b_s).$$

With $F(t,x) = x + x|x|/(2t)$, we define our new Hamiltonian as

$$H_t^W(b) := F\left(t, X_t^W(b)\right). \tag{30}$$

Similarly to Lemma 5.1, the Chain Rule for Malliavin derivatives proves that

$$D_{s,\lambda} X = \frac{1}{\sqrt{t}} \frac{1}{u(t)} E_b\left[\sqrt{\hat{Q}(\lambda)} e^{i\lambda \cdot b_s} e^{H_t^W(b)} \left(1 + \frac{\left|X_t^W(b)\right|}{t}\right)\right] \mathbf{1}_{[0,t]}(s), \tag{31}$$

and

$$\begin{aligned} G := \left\langle DX, -DL^{-1}X\right\rangle_H = &\frac{1}{2t} \int_{-\pi/2}^{\pi/2} |\sin\theta|\, d\theta\, \mathbf{E}' E_{b,\bar{b}} \\ &\times \left[\int_0^t ds\, Q\left(b_s, \bar{b}_s\right) \frac{\exp H_t^W(b)}{u(t)} \frac{\exp H_t^{R_\theta W}\left(\bar{b}\right)}{R_\theta u(t)}\right. \\ &\times \left.\left(1 + \frac{\left|X_t^W(b)\right|}{t}\right)\left(1 + \frac{\left|X_t^{R_\theta W}\left(\bar{b}\right)\right|}{t}\right)\right]. \end{aligned} \tag{32}$$

Theorem 5.5. *Consider $u(t) = E_b\left[\exp H_t^W(b)\right]$ where the new Hamiltonian H_t^W is given in (30). The random environment W is as it was defined in Section 5.1, and $Q(0)$ and Q_m are given in Notation 5.2, and are non-random.*

1. *Assume $Q(0) < 1/4$. Then* $\mathrm{Var}\left[\log u(t)\right] \leq 64(\pi/2)^2 Q^3(0)\, t + O\left(t^{-1}\right)$.
2. *Assume Q_m is positive. Then* $\mathrm{Var}\left[\log u(t)\right] \geq Q_m t$.

If both assumptions of Points 1 and 2 hold, the fluctuation exponent of Definition 5.3 is $\chi = 1/2$, and the conclusion of Point 4 in Theorem 5.4 holds.

The theorem above also works when $Q(0)$ and Q_m are random. We leave it to the reader to check that the conclusions of Points 1 and 2 above hold with expectations $\mathbf{E}_P$ on the right-hand sides, and with $O\left(t^{-1}\right) = t^{-1}\, \mathbf{E}_P\left[Q(0)\left(1 + \log^2(1 - 4Q(0))\right)\right]$.

We suspect that the logarithmic partition function $\log u(t)$ given by the non-Gaussian Hamiltonian in (30) is eminently non-Gaussian itself; in fact, the form of its derivative in (31), with the additional factors of the form $(1 + X(b))/t$, can presumably be compared with X. We conjecture, although we are unable to prove it, that the corresponding $g(y)$ grows linearly in y. This would show, via Corollary 4.5 Point 3, that $\log u(t)$ has exponential tails. Other examples of non-Gaussian Hamiltonians can be given, using the formulation (30) with other functions F, such as $F(t,x) = x + x|x|^p / t^{(1+p)/2}$ for $p > 0$. It should be noted, however, that in our Gaussian environment, any value $p > 1$ results in a partition function $u(t)$ with infinite first moment, in which case the arguments we have given above for proving that $\chi = 1/2$ will not work. This does not mean that the logarithmic partition function cannot be analyzed using finer arguments; it can presumably be proved to be non-Gaussian with heavier-than-exponential tails when $p > 1$.

Proof of Theorem 5.5. Since the additional terms in (32), compared to Lemma 5.1, are factors greater than 1, the conclusion of Point 2 follows immediately using the proof of Points 2 and 3 of Theorem 5.4.

To prove that Point 1 holds, we will use again the classical fact $\mathrm{Var}[X] \leq (\pi/2)^2 \mathbf{E}\left[\|DX\|_H^2\right]$. Here from (31) note first that

$$\begin{aligned}
&\int_{\mathbf{R}} \left|D_{s,\lambda}X\right|^2 \mathrm{d}\lambda \\
&= \frac{1}{t}\frac{1}{u^2(t)} E_b E_{b'}\left[\int_{\mathbf{R}} \mathrm{d}\lambda \sqrt{\hat{Q}(\lambda)} \mathrm{e}^{\mathrm{i}\lambda\cdot(b_s - b'_s)} \mathrm{e}^{H_t^W(b)} \mathrm{e}^{H_t^W(b')} \left(1 + \frac{\left|X_t^W(b)\right|}{t}\right)\right. \\
&\quad \times \left.\left(1 + \frac{\left|X_t^W(b')\right|}{t}\right)\right] \\
&= \frac{1}{t}\frac{1}{u^2(t)} E_b E_{b'}\left[Q(b_s - b'_s) \mathrm{e}^{H_t^W(b)} \mathrm{e}^{H_t^W(b')} \left(1 + \frac{\left|X_t^W(b)\right|}{t}\right)\left(1 + \frac{\left|X_t^W(b')\right|}{t}\right)\right] \\
&\leq \frac{Q(0)}{t}\frac{1}{u^2(t)} E_b E_{b'}\left[\mathrm{e}^{H_t^W(b)} \mathrm{e}^{H_t^W(b')} \left(1 + \frac{\left|X_t^W(b)\right|}{t}\right)\left(1 + \frac{\left|X_t^W(b')\right|}{t}\right)\right] \\
&= \frac{Q(0)}{t}\left(1 + E_b\left[\frac{\mathrm{e}^{H_t^W(b)}}{u(t)}\frac{\left|X_t^W(b)\right|}{t}\right]\right)^2.
\end{aligned}$$

Therefore we have immediately

$$\|DX\|_H^2 \leq Q(0)\left(1 + E_b\left[\frac{\mathrm{e}^{H_t^W(b)}}{u(t)}\frac{\left|X_t^W(b)\right|}{t}\right]\right)^2.$$

Therefore, to get an upper bound on the variance of X uniformly in t we only need to show that the quantity

$$B := \mathbf{E}\left[\left(E_b\left[\frac{\mathrm{e}^{H_t^W(b)}}{u(t)}\frac{\left|X_t^W(b)\right|}{t}\right]\right)^2\right]$$

is bounded in t. We see that, using Jensen's inequality w.r.t. the polymer measure $\mathrm{d}\tilde{P}_b = \mathrm{e}^{H_t^W(b)} u(t)^{-1}\, \mathrm{d}P_b$, and then w.r.t. the random medium's expectation,

$$\begin{aligned}
B &= \frac{1}{t^2}\mathbf{E}\left[\left(E_b\left[\frac{\mathrm{e}^{H_t^W(b)}}{u(t)} \log \mathrm{e}^{\left|X_t^W(b)\right|}\right]\right)^2\right] \\
&\leq \frac{1}{t^2}\mathbf{E}\left[\left(\log E_b\left[\frac{\mathrm{e}^{H_t^W(b)+\left|X_t^W(b)\right|}}{u(t)}\right]\right)^2\right] \\
&\leq \frac{1}{t^2}\log^2\left(e - 1 + \mathbf{E}\left[E_b\left[\frac{\mathrm{e}^{H_t^W(b)+\left|X_t^W(b)\right|}}{u(t)}\right]\right]\right).
\end{aligned}$$

Here we used the fact that $E_b\left[e^{H_t^W(b)+\left|X_t^W(b)\right|}\right]/u(t) \geq E_b\left[H_t^W(b)\right]/u(t) = 1$ and that $x \mapsto \log^2(x+e-1)$ is convex for $x \geq 1$. Now we evaluate

$$\mathbf{E}\left[E_b\left[\frac{e^{H_t^W(b)+\left|X_t^W(b)\right|}}{u(t)}\right]\right] = E_b\mathbf{E}\left[\frac{e^{X_t^W(b)+\left|X_t^W(b)\right|^2/(2t)+\left|X_t^W(b)\right|}}{u(t)}\right]$$
$$\leq \mathbf{E}^{1/2}\left[u(t)^{-2}\right] E_b\mathbf{E}^{1/2}\left[e^{4\left|X_t^W(b)\right|+\left|X_t^W(b)\right|^2/t}\right]$$
$$\leq \mathbf{E}^{1/2}\left[E_b\left[e^{-2H_t^W(b)}\right]\right] E_b\mathbf{E}^{1/2}\left[e^{4\left|X_t^W(b)\right|+\left|X_t^W(b)\right|^2/t}\right]. \quad (33)$$

The first term in the above product is actually less than the second. For the second, we note that for any fixed b, the random variable $X_t^W(b)$ is Gaussian centered, with a variance bounded above by $Q(0)t$. Therefore we have that $\mathbf{E}\left[e^{2\left|X_t^W(b)\right|^2/t}\right]$ is bounded by the constant $\mathbf{E}\left[e^{2Q(0)Z^2}\right] = (1-4Q(0))^{-1/2}$ (here Z denotes a standard normal); that expectation is finite because $Q(0) < 1/4$ by assumption. Similarly for fixed b, $\mathbf{E}\left[e^{8\left|X_t^W(b)\right|}\right] = \mathbf{E}\left[e^{8\sqrt{Q(0)t}|Z|}\right] \leq 2e^{32Q(0)t}$. We now apply these two estimates and Schwartz's inequality to the last factor in (33), to get:

$$\mathbf{E}^{1/2}\left[e^{4\left|X_t^W(b)\right|+\left|X_t^W(b)\right|^2/t}\right] \leq 2^{1/4}\mathbf{E}^{1/4}\left[e^{8\left|X_t^W(b)\right|}\right]\mathbf{E}^{1/4}\left[e^{2\left|X_t^W(b)\right|^2/t}\right]$$
$$\leq 2^{1/4}e^{8Q(0)t}(1-4Q(0))^{-1/8}.$$

Combining this with the inequality in (33) now yields

$$B \leq \frac{1}{t^2}\log^2\left(e-1+\frac{2^{1/4}e^{8Q(0)t}}{(1-4Q(0))^{1/8}}\right)$$
$$\leq \frac{1}{t^2}\left(8Q(0)t+1+4^{-1}\log 2-8^{-1}\log(1-4Q(0))\right)^2$$
$$= 64Q^2(0)+O\left(t^{-2}\right),$$

where $O\left(t^{-2}\right)$ is non-random (depends only on $Q(0)$), proving Point 1, and the theorem. ■

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Appendix

To prove Lemma 3.5, we begin with an intermediate result in the nth Wiener chaos.

Lemma A.1. *Let $n \in \mathbf{N}$ and $f_n \in H^n$ be a symmetric function. Let $Y \in \mathbf{D}^{1,2}$. Then*

$$\begin{aligned}\mathbf{E}\left[I_n\left(f_n\right) Y\right] &= \frac{1}{n}\mathbf{E}\left[\left\langle D_\cdot\left(I_n\left(f_n\right)\right); D_\cdot Y\right\rangle_H\right] \\ &= \mathbf{E}\left[\left\langle I_{n-1}\left(f_n\left(\star,\cdot\right)\right); D_\cdot Y\right\rangle_H\right],\end{aligned}$$

where we used the notation $I_{n-1}\left(f_n\left(\star,\cdot\right)\right)$ to denote the function $r \mapsto I_{n-1}\left(f_n\left(\star,r\right)\right)$ where I_{n-1} operates on the $n-1$ variables "$\star$" of $f_n\left(\star,r\right)$.

Proof. This is an immediate consequence of formula (4) and the famous relation $\delta D = -L$ (where δ is the divergence operator (Skorohod integral), adjoint of D, see [13, Proposition 1.4.3]).

Here, however, we present a direct proof. Note that, because of the Wiener chaos expansion of Y in Proposition 2.2, and the fact that all chaos terms of different orders are orthogonal, without loss of generality, we can assume $Y = I_n\left(g_n\right)$ for some symmetric $g_n \in H^n$; then, using the formula for the covariance of two nth-chaos r.v.'s in Proposition 2.2, we have

$$\begin{aligned}\mathbf{E}\left[\left\langle I_{n-1}\left(f_n\left(\star,\cdot\right)\right); D_\cdot Y\right\rangle_H\right] &= \mathbf{E}\left[\left\langle I_{n-1}\left(f_n\left(\star,\cdot\right)\right); nI_{n-1}\left(g_n\left(\star,\cdot\right)\right)\right\rangle_H\right] \\ &= n\int_T \mathbf{E}\left[I_{n-1}\left(f_n\left(\star,r\right)\right) I_{n-1}\left(g_n\left(\star,r\right)\right)\right]\mu\left(\mathrm{d}r\right) \\ &= n\int_T \left(n-1\right)!\left\langle f_n\left(\star,r\right), g_n\left(\star,r\right)\right\rangle_{L^2\left(T^{n-1},\mu^{\otimes n-1}\right)}\mu\left(\mathrm{d}r\right) \\ &= n!\left\langle f_n; g_n\right\rangle_{L^2\left(T^n,\mu^{\otimes n}\right)} = \mathbf{E}\left[I_n\left(f_n\right) Y\right]\end{aligned}$$

which, together with formula (4), proves the lemma. ∎

Proof of Lemma 3.5. Since $X \in \mathbf{D}^{1,2}$ and is centered, it has a Wiener chaos expansion $X = \sum_{n=1}^{\infty} I_n\left(f_n\right)$. We calculate $\mathbf{E}\left[Xh\left(X\right)\right]$ via this expansion and the Malliavin calculus, invoking Remark 2.3 and using Lemma A.1:

$$\begin{aligned}\mathbf{E}\left[Xh\left(X\right)\right] &= \sum_{n=1}^{\infty}\mathbf{E}\left[I_n\left(f_n\right)h\left(X\right)\right] \\ &= \sum_{n=1}^{\infty}\frac{1}{n}\mathbf{E}\left[\int_T D_r I_n\left(f_n\right)\ D_r h\left(X\right)\ \mu\left(\mathrm{d}r\right)\right] \\ &= \mathbf{E}\left[h'\left(X\right)\int_T D_r\left(\sum_{n=1}^{\infty}\frac{1}{n}I_n\left(f_n\right)\right)D_r X\,\mu\left(\mathrm{d}r\right)\right]\end{aligned}$$

which by the definition of $-L$ is precisely the statement (7). ∎

Proof of Lemma 3.8. The proof goes exactly as that of Lemma 1.5.1 in [13], with only computational changes. We give it here for completeness. It is sufficient to assume that $X = p\left(\left(W\left(h_i\right)\right)_{i=1}^n\right)$ where p is, for instance, a polynomial in n variables. Thus $R_\theta X = p\left(\left(W\left(h_i\right)\cos\theta + W'\left(h_i\right)\sin\theta\right)_{i=1}^n\right)$, so that $D'_s\left(R_\theta X\right) = \left(\sin\theta\right)R_\theta\left(D_s X\right)$. Using the Mehler formula (formula (1.54) in [13]) with $t > 0$ such that $\cos\theta = \mathrm{e}^{-t}$, and $T_t = \mathrm{e}^{tL}$, we get $\mathbf{E}'\left[D'_s\left(R_\theta X\right)\right] = \left(\sin\theta\right)T_t\left(D_s X\right)$, which we can rewrite as

$$\mathbf{E}'\left[D'_s\left(R_\theta X\right)\right] = \sum_{n=0}^{\infty}\sin\theta\cos^n\theta\, J_n D_s X.$$

Integrating this expression over $\theta \in [-\pi/2, \pi/2]$ yields

$$\frac{1}{2}\int_{-\pi/2}^{\pi/2} \operatorname{sgn}(\theta)\, \mathbf{E}'\left[D'_s\left(R_\theta X\right)\right] \mathrm{d}\theta = \sum_{n=0}^{\infty} \left(\frac{1}{2}\int_{-\pi/2}^{\pi/2} |\sin\theta| \cos^n\theta \mathrm{d}\theta\right) J_n D_s X$$
$$= \sum_{n=0}^{\infty}\left(\int_0^{\pi/2} \sin\theta\cos^n\theta \mathrm{d}\theta\right) J_n D_s X = \sum_{n=0}^{\infty} \frac{1}{n+1} J_n D_s X.$$

It is now an elementary property of multiplication operators to check that the last expression above equals $-D_s L^{-1} X$ (see the commutativity relationship (1.63) in [13]), finishing the proof of the lemma. ■

For completeness, we finish with a short proof of the upper bound in Theorem 1.3, which is equivalent to Proposition 1.1.

Proof of Proposition 1.1. Assume $X \in \mathbf{D}^{1,2}$ is centered and W is the standard Wiener space. By the Clark–Ocone representation formula

$$X = \int_0^1 \mathbf{E}\left[D_s X | \mathcal{F}_s\right] \mathrm{d}W(s)$$

(see [13, Proposition 1.3.5]), we can define a continuous square-integrable martingale M with $M(1) = X$, via the formula $M(t) := \int_0^t \mathbf{E}\left[D_s X|\mathcal{F}_s\right] \mathrm{d}W(s)$. The quadratic variation of M is equal to $[M]_t = \int_0^t \left|\mathbf{E}\left[D_s X|\mathcal{F}_s\right]\right|^2 \mathrm{d}s$; therefore, by hypothesis, $[M]_t \leq t$. Using the Doleans–Dadec exponential martingale $\mathcal{E}(\lambda M)$ based on λM, defined by $\mathcal{E}(\lambda M)_t = \exp\left(\lambda M_t - \frac{\lambda^2}{2}[M]_t\right)$ we now have

$$\mathbf{E}\left[\exp \lambda X\right] = \mathbf{E}\left[\mathcal{E}(\lambda M)_1 \exp\left(\frac{\lambda^2}{2}[M]_1\right)\right] \leq \mathbf{E}\left[\mathcal{E}(\lambda M)_1\right] \mathrm{e}^{\lambda^2/2} = \mathrm{e}^{\lambda^2/2}.$$

The proposition follows using a standard optimization calculation and Chebyshev's inequality. [21, Theorem 9.1.1] can be invoked to prove the same estimate for a general isonormal Gaussian process W, as done in [22]. ■

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