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PORTFOLIO OPTIMIZATION WITH CONSUMPTION IN A FRACTIONAL BLACK-SCHOLES MARKET

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Abstract. We consider the classical Merton problem of finding the optimal consumption rate and the optimal portfolio in a Black-Scholes market driven by fractional Brownian motion B^H with Hurst parameter $H > 1/2$. The integrals with respect to B^H are in the Skorohod sense, not pathwise which is known to lead to arbitrage. We explicitly find the optimal consumption rate and the optimal portfolio in such a market for an agent with logarithmic utility functions. A true self-financing portfolio is found to lead to a consumption term that is always favorable to the investor. We also present a numerical implementation by Monte Carlo simulations.

1. Introduction

Fractional Brownian motion (fBm) with Hurst parameter $H \in (0,1)$ is the centered Gaussian process $\left\{B^H(t,\omega) : t \geq 0, \omega \in \Omega\right\}$ on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with the covariance structure

$$\mathbf{E}\left[B_s^H B_t^H\right] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right) \tag{1.1}$$

for s, $t \geq 0$. Alternatively, we can specify the fractional Brownian motion by setting $B_0^H = 0$ and

$$\mathbf{E}\left[\left(B_s^H - B_t^H\right)^2\right] = |t-s|^{2H}. \tag{1.2}$$

When $H = 1/2$ we obtain the standard Brownian motion (BM).

Originally, fBm was defined and studied by Kolmogorov within a Hilbert space framework influenced by his interest in modeling turbulence. Kolmogorov used the name "Wiener spiral" for this process. The name "fractional Brownian motion" comes from the paper by Mandelbrot and Van Ness [13], where they introduced fBm as a centered Gaussian process and gave the first representation of it as an integral with respect to standard BM. The Hurst parameter H is named after the hydrologist H. E. Hurst who noticed in the 1950's that the levels of water in the Aswan dam in Egypt followed a random motion with a self-similarity parameter.

The value of Hurst parameter H characterizes fBm in such a way that it accounts not only for the sign of the increments' correlation and their rate of long-range decay, but also for the regularity of the sample paths. Indeed, for $H > 1/2$ the increments are positively correlated, and for $H < 1/2$ they are negatively

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correlated. Furthermore, for every $\beta \in (0, H)$, the sample paths of fBm are almost surely Hölder continuous with index β. This result follows from (1.2) and Kolmogorov's lemma (see [16, Theorem I.2.1]).

fBm holds a significant property known as self-similarity, *i.e.*, the processes

$$\left\{B_{ct}^H\right\}_{t\geq 0} \quad \text{and} \quad \left\{c^H B_t^H\right\}_{t\geq 0}$$

are identical in distribution for any fixed $c > 0$. When $H > 1/2$, it implies the so-called long-range dependence, which says specifically that the correlation between $B_{t+1}^H - B_t^H$ and $B_{t+n+1}^H - B_{t+n}^H$ is of order n^{2H-2} when n is large. This behavior also holds for $H < 1/2$, but since the function n^{2H-2} is non-summable iff $H > 1/2$, consistent with the econometric nomenclature, only the case $H > 1/2$ merits the appellation "long memory". This is the only case we treat in this article.

For $H \neq 1/2$, fBm is not a semimartingale (see [12, Example 2 of Section 4.9.13]) and we cannot apply the stochastic calculus developed by Itô in order to define stochastic integrals with respect to fBm. We refer the reader to [1], [2], [3], [7], [15] and references therein for a survey of numerous articles contributing to the development of the theory of integration with respect to fBm.

Self-similarity and long-range dependence of fBm with $H > 1/2$ make it a natural candidate as a model of noise in mathematical modeling of financial markets (see, for example, [5], [9], [17] and references therein). One proposal that has been made, which we take up here, is to model stock returns as increments of fBm.

It was discovered (see [17]) that if pathwise integration theory (see [6], [11]) is used, the corresponding markets may have arbitrage opportunities. Recently, it was established in [4] that such arbitrages are perhaps not truly achievable since they would require arbitrarily fast trading. On the other hand, the use of Skorohod integration theory (see [1], [2], [3], [7]) in connection to finance was proposed by Hu and Øksendal [9] as another way to have an arbitrage-free model. Using this integration theory the markets appear to be arbitrage-free; however, the definition of a self-financing portfolio in [9] is criticized for the clarity of its economic interpretation. While this criticism remains a problem, in the situation of portfolio optimization with consumption, it typically becomes a moot point since the consumption can be adjusted to account for any deviation of the "Skorohod-sense" notion of self-financement from an actual self-financing portfolio. In Section 5 of this part, in the context of logarithmic utility, we show precisely how such an adjustment pans out, and in particular we prove that any discrepancy will always be in favor of the investor.

Section 2 summarizes the basic results of the Skorohod integration theory used in this article. Section 3 gives the details of the financial model we consider. It is that which is used by Hu and Øksendal in [9], and is simply the fractional generalization of the geometric Brownian motion, as one can see immediately in formula (3.3), where the model parameters r, a, σ still have the standard interpretation of risk-free rate, mean rate of return of the stock, and volatility of the stock.

Hu, Øksendal and Sulem [10] solved a portfolio optimization problem with consumption based on this model using power utility functions. They proved that the martingale method for classical BM can be adapted to work for fBm as well. In Section 4, we solve a portfolio optimization problem as in [10], using a logarithmic utility function instead, and derive the optimal consumption and portfolio via the "martingale" method for fBm. Most significantly, we use Itô's formula for fBm to simplify our results further than had been previously thought possible, by eliminating the need to refer to expressions involving Malliavin derivatives. Our work, which also applies in the case studied in [10], is thus a significant improvement on [10] from the computational viewpoint.

Specifically, to follow our optimal trading strategy, the practitioner will only need to use our formulas for the optimal holdings α^* and β^* of risk-free account and stock, and the optimal consumption c^*, as given in Theorem 4.2. With the help of expressions (4.26) and (4.27) which are obtained by Itô's formula, the formulas for α^*, β^*, and c^* involve only universal non-random functions (such as φ in (2.1), K in (3.7), and ζ in (4.14)), the model parameters r, a, σ, other functions based on the above (such as g_1 in (4.18) and g_2 in (4.20)), and stochastic integrals of these functions with respect to B_t^H or $\widehat{B}_t^H = B_t^H + \frac{a-r}{\sigma}\, t$. Because the stochastic integrals are with non-random integrands only, they can be calculated as Stieltjes integrals, where the increments of B^H, and thus of $\widehat{B}_t^H$, are directly observable from the fact that the stock price is explicitly given by the geometric fractional Brownian motion model (3.3).

As a consequence of the explicitness of our expressions, we show that a numerical implementation is straightforward. Section 7 presents the results of simulations for such an implementation in the case of no consumption, including an explanation of how to approximate the stochastic integrals needed in the numerical scheme. Our method does better than one which would use Merton's classical formulas for the case $H = 1/2$; but as an added bonus, Section 5 shows that the investor recuperates a positive consumption when using a truly self-financing portfolio. This result also means that the optimal portfolio for truly self-financing conditions is not equal to the one we express herein. To find the former, one may reinvest the positive consumption obtained in Section 5 into stock and bond optimally. However, this would not lead to a strategy that can be calculated explicitly as we do here.

Our technique for deriving explicit formulas also works in the power utility case: in Section 6 we present the result of using Itô's formula to simplify the formulas given by Hu, Øksendal, and Sulem [10]; again, our formulas would make it simple to devise a numerical implementation.

2. Preliminaries

In order to present a self-contained account for the sake of readability, in this section, we present the terminology and the results that we will use from other references. Let $\Omega = C_0([0,T], \mathbf{R})$ be the space of real-valued continuous functions on $[0,T]$ with the initial value zero and the topology of local uniform convergence. There is a probability measure μ_H on $(\Omega, \mathcal{F}_T^{(H)})$, where $\mathcal{F}_T^{(H)}$ is the Borel σ-algebra, such that on the probability space $(\Omega, \mathcal{F}_T^{(H)}, \mu_H)$ the coordinate process $B^H : \Omega \to \mathbf{R}$, defined by $B_t^H(\omega) = \omega(t)$, for all $\omega \in \Omega$ and $t \in [0,T]$, is an fBm. B^H constructed in this way is referred to as the *canonical* fBm. We will use this canonical fBm and its associated probability space in our study.

Duncan et al. [7] define the Skorohod integral with respect to fBm, $\int_0^T f(t)\, dB_t^H$, for certain class of functions f, using Wick products. Alòs and Nualart [3] give an equivalent definition using techniques of Malliavin calculus (see also [14]). Since both of the constructions are quite lengthy, we will not say any further about this matter and refer the reader to the references mentioned. Note that this integral has zero mean.

Now, consider the filtration $\{\mathcal{F}_t^{(H)}\}_{t\in[0,T]}$ of B^H, *i.e.*, $\mathcal{F}_t^{(H)}$ is the σ-algebra generated by the random variables $B_s^H,\ s \le t$. Define

$$\varphi(s,t) = H(2H-1)|s-t|^{2H-2} \tag{2.1}$$

and define, for g measurable on $[0,T]$,

$$|g|_\varphi^2 = \int_0^T \int_0^T g(s)\, g(t)\, \varphi(s,t)\, ds dt, \tag{2.2}$$

as a Riemann integral when it exists.

Define the space $\hat{L}_\varphi^2([0,T]^n)$ to be the set of symmetric functions $f(x_1,\cdots,x_n)$ on $[0,T]^n$ such that

$$\|f\|_{\hat{L}_\varphi^2([0,T]^n)} := \int_{[0,T]^n\times[0,T]^n} |f(u_1,\cdots,u_n)\, f(v_1,\cdots,v_n)|\, \varphi(u_1,v_1) \cdots \varphi(u_n,v_n)\, du_1\cdots du_n dv_1\cdots dv_n < \infty.$$

For each $\mathcal{F}_T^{(H)}$-measurable random variable F in $L^2(\mu_H)$, there exists (see [7]) $f_n \in \hat{L}_\varphi^2([0,T]^n)$, $n = 0,1,2,\ldots$ such that

$$F = \sum_{n=0}^\infty \int_{[0,T]^n} f_n d\left(B^H\right)^{\otimes n} \quad \left(\text{convergence in } L^2(\mu_H)\right), \tag{2.3}$$

where

$$\int_{[0,T]^n} f_n d\left(B^H\right)^{\otimes n} = n! \int_{0\le s_1<\cdots<s_n\le T} f_n(s_1,\cdots,s_n)\, dB_{s_1}^H \cdots dB_{s_n}^H$$

is the iterated Skorohod integral.

If there exists $q \in \mathbf{N}$ such that the formal expansion F of the form (2.3) satisfies

$$\sum_{n=0}^\infty n!\, \|f_n\|_{\hat{L}_\varphi^2([0,T]^n)}\, e^{-2qn} < \infty, \tag{2.4}$$

Hu and Øksendal [9, Definition 4.9] defined the *quasi-conditional expectation* of F by

$$\widetilde{\mathbf{E}}_{\mu_H}\left[F \,\middle|\, \mathcal{F}_t^{(H)}\right] = \sum_{n=0}^\infty \int_{[0,t]^n} f_n d\left(B^H\right)^{\otimes n}.$$

They show that

$$\widetilde{\mathbf{E}}_{\mu_H}\left[F \,\middle|\, \mathcal{F}_t^{(H)}\right] = F \quad \text{a.s.} \quad \Longleftrightarrow \quad F \text{ is } \mathcal{F}_t^{(H)}\text{-measurable},$$

but in general $\widetilde{\mathbf{E}}_{\mu_H}\left[F \,\middle|\, \mathcal{F}_t^{(H)}\right] \ne \mathbf{E}_{\mu_H}\left[F \,\middle|\, \mathcal{F}_t^{(H)}\right]$.

Definition 2.1. A (t,ω)-measurable, $\mathcal{F}_t^{(H)}$-adapted process $M = \{M(t,\omega) : t \in [0,T],\ \omega \in \Omega\}$ is said to be a quasi-martingale if $M(t)$ has an expansion of the form (2.3) which satisfies (2.4) for all t and furthermore, for all $t \geq s$, $\widetilde{\mathbf{E}}_{\mu_H}\left[M(t)\middle|\mathcal{F}_s^{(H)}\right] = M(s)$ a.s.

Lemma 2.2 (Lemma 1.1 in [10]). *Let f be a Skorohod integrable function. Then*

$$M(t) := \int_0^t f(s)\,dB_s^H, \quad t \geq 0$$

is a quasi-martingale. In particular, $\mathbf{E}_{\mu_H}[M(t)] = \mathbf{E}_{\mu_H}[M(0)] = 0$ *for all* $t \geq 0$.

This result enables us to employ many of the useful martingale methods valid for Brownian motion when we replace conditional expectation by quasi-conditional expectation. Since we will use it in our calculations, let us mention the following example (see [10, Example 1.1]): let $f \in \hat{L}_\varphi^2([0,T])$, then

$$M(t) := \exp\left\{\int_0^t f(s)\,dB_s^H - \frac{1}{2}\left|f \cdot \mathbf{1}_{[0,t]}\right|_\varphi^2\right\}$$

is a quasi-martingale. We will use the following fractional version of Girsanov theorem.

Theorem 2.3 (Theorem 3.18 in [9]). *Fix $T > 0$ and let $u : [0,T] \to \mathbf{R}$ be a continuous deterministic function. Suppose $K : [0,T] \to \mathbf{R}$ is a deterministic function satisfying the equation*

$$\int_0^T K(s)\,\varphi(s,t)\,ds = u(t), \quad 0 \leq t \leq T$$

and extend K to $\mathbf{R}$ by defining $K(s) = 0$ outside $[0,T]$. Define the probability measure $\hat{\mu}_H$ on $\mathcal{F}_T^{(H)}$ by

$$\frac{d\hat{\mu}_H}{d\mu_H} = \exp\left(-\int_0^T K(s)\,dB_s^H - \frac{1}{2}|K|_\varphi^2\right).$$

Then $\widehat{B}_t^H := B_t^H + \int_0^t u(s)\,ds$ is an fBm with the same Hurst parameter H with respect to the measure $\hat{\mu}_H$.

3. Standard Framework of Black-Scholes Market Driven by fBm

We consider in our model that there are two investment vehicles described as following:

(i) A bank or risk-free account, where the price $A(t)$ at time t, $0 \leq t \leq T$, is given by,

$$\begin{aligned} dA(t) &= rA(t)\,dt, \\ A(0) &= 1 \end{aligned} \tag{3.1}$$

for a constant $r > 0$; since r is a nonrandom constant, $A(t) = e^{rt}$ can also be called *the bond*.

(ii) A stock, where the price $S(t)$ at time t, $0 \le t \le T$, is given by,

$$\begin{aligned} dS(t) &= aS(t)\,dt + \sigma S(t)\,dB_t^H, \\ S(0) &= s_0 > 0, \end{aligned} \tag{3.2}$$

where $a > r > 0$ and $\sigma \neq 0$ are constants. Here dB^H is understood in the Skorohod sense.

It is proved in [9] that the solution of (3.2) is

$$S(t) = s_0 \exp\left\{at - \frac{1}{2}\sigma^2 t^{2H} + \sigma B_t^H\right\}. \tag{3.3}$$

Suppose that an investor's portfolio is given by $\theta(t) = (\alpha(t), \beta(t))$, where $\alpha(t)$ and $\beta(t)$ are the number of bonds and stocks held at time t, respectively. We also allow the investor to choose a consumption process $c(t) \ge 0$. We assume that α, β and c are $\{\mathcal{F}_t^{(H)}\}$-adapted processes, and that $(t,\omega) \to \alpha(t,\omega), \beta(t,\omega), c(t,\omega)$ are measurable with respect to $\mathcal{B}[0,T] \times \mathcal{F}_T^{(H)}$, where $\mathcal{B}[0,T]$ is the Borel σ-algebra on $[0,T]$.

The wealth process is given by

$$Z(t) = \alpha(t) A(t) + \beta(t) S(t). \tag{3.4}$$

We say that θ is *(Skorohod) self-financing* with respect to c, if

$$dZ(t) = \alpha(t)\,dA(t) + \beta(t)\,dS(t) - c(t)\,dt. \tag{3.5}$$

See Section 5 for the relation with the natural notion of self-financing. From (3.4) we get

$$\alpha(t) = A^{-1}(t)\left[Z(t) - \beta(t) S(t)\right].$$

Substituting this into (3.5) and using (3.1), we obtain

$$d\left(e^{-rt} Z(t)\right) + e^{-rt} c(t)\,dt = \sigma e^{-rt} \beta(t) S(t) \left(\frac{a-r}{\sigma} dt + dB_t^H\right). \tag{3.6}$$

Let

$$K(s) = \frac{(a-r)\left(Ts - s^2\right)^{\frac{1}{2}-H} \mathbf{1}_{[0,T]}(s)}{2\sigma H \cdot \Gamma(2H) \cdot \Gamma(2-2H) \cdot \cos\left(\pi\left(H - \frac{1}{2}\right)\right)}, \tag{3.7}$$

and define a new measure $\hat{\mu}_H$ on $\mathcal{F}_T^{(H)}$ by

$$\frac{d\hat{\mu}_H}{d\mu_H} = \exp\left(-\int_0^T K(s)\,dB_s^H - \frac{1}{2}|K|_\varphi^2\right) =: \eta(T). \tag{3.8}$$

Then by the fractional Girsanov formula (Theorem 2.3), the process

$$\widehat{B}_t^H := B_t^H + \frac{a-r}{\sigma}\, t \tag{3.9}$$

is a fractional Brownian motion with the same Hurst parameter H with respect to $\hat{\mu}_H$. In terms of $\widehat{B}^H$, we can write (3.6) as

$$e^{-rt} Z(t) + \int_0^t e^{-ru} c(u)\,du = Z(0) + \int_0^t \sigma e^{-ru} \beta(u) S(u)\,d\widehat{B}_u^H. \tag{3.10}$$

We also have

$$|K|_{\varphi}^{2} = \int_0^T \int_0^T K(s) K(t) \varphi(s,t)\, ds dt = \frac{a-r}{\sigma} \int_0^T K(s)\, ds. \tag{3.11}$$

If $Z(0) = z > 0$, we denote the corresponding wealth process $Z(t)$ in (3.10) by $Z_z^{c,\theta}(t)$.

We say that (c,θ) is *admissible* with respect to z and write $(c,\theta) \in \mathcal{A}(z)$ if βS is Skorohod integrable, α satisfies (3.4), θ is self-financing with respect to c and $Z_z^{c,\theta}(T) \geq 0$. In this case, it follows from Lemma 2.2 that

$$M(t) := \int_0^t \sigma e^{-ru} \beta(u) S(u)\, d\widehat{B}_u^H$$

is a quasi-martingale with respect to $\hat{\mu}_H$. In particular, $\mathbf{E}_{\hat{\mu}_H}[M(T)] = 0$. Therefore, from (3.10) we obtain the budget constraint

$$\mathbf{E}_{\hat{\mu}_H}\left[e^{-rT} Z_z^{c,\theta}(T) + \int_0^T e^{-ru} c(u)\, du\right] = z, \tag{3.12}$$

which holds for all admissible (c,θ).

We finish this section with a result from [10] that will be used in Section 4.

Lemma 3.1 (Lemma 2.1 in [10]). *Let $c(t) \geq 0$ be a given consumption rate and let F be a given $\mathcal{F}_T^{(H)}$-measurable random variable such that*

$$G := e^{-rT} F + \int_0^T e^{-ru} c(u)\, du$$

satisfies $\mathbf{E}_{\hat{\mu}_H}\left[G^2\right] < \infty$. Then the following two statements are equivalent:

(i) *There exists a portfolio θ such that $(c,\theta) \in \mathcal{A}(z)$ and $Z_z^{c,\theta}(T) = F$ a.s.*
(ii) $\mathbf{E}_{\hat{\mu}_H}[G] = z$.

4. Optimal Consumption and Portfolio

Let $D_1 > 0$, $\delta \geq 0$ and $T > 0$ be given constants. Consider the following total expected logarithmic utility obtained from the consumption rate $c(t) \geq 0$ and the terminal wealth $F := Z_z^{c,\theta}(T)$, where $Z(0) = z > 0$,

$$J^{c,\theta}(z) = \mathbf{E}_{\mu_H}\left[\int_0^T e^{-\delta t} \log c(t)\, dt + D_1 \log F\right]. \tag{4.1}$$

We want to find $(c^*, \theta^*) \in \mathcal{A}(z)$ and $V(z)$ such that

$$V(z) = \sup_{(c,\theta)\in\mathcal{A}(z)} J^{c,\theta}(z) = J^{c^*,\theta^*}(z). \tag{4.2}$$

By Lemma 3.1, this problem is equivalent to the following constrained optimization problem

$$V(z)=\sup_{c,F\geq 0}\left\{\mathbf{E}_{\mu_H}\left[\int_0^T e^{-\delta t}\log c(t)\,dt+D_1\log F\right];\ \text{given that}\right.\\ \left.\mathbf{E}_{\widehat{\mu}_H}\left[\int_0^T e^{-ru}c(u)\,du+e^{-rT}F\right]=z\right\},\tag{4.3}$$

where the supremum is taken over all $c(t)\geq 0$ and $\mathcal{F}_T^{(H)}$-measurable F such that

$$\int_0^T e^{-ru}c(u)\,du+e^{-rT}F\in L^2(\hat{\mu}_H).$$

Optimization problem (4.3) can be solved by applying Lagrange multiplier method. Consider for each $\lambda>0$ the following unconstrained optimization problem (with $\mathbf{E}=\mathbf{E}_{\mu_H}$)

$$V_\lambda(z)=\sup_{c,F\geq 0}\left\{\mathbf{E}\left[\int_0^T e^{-\delta t}\log c(t)\,dt+D_1\log F\right]\right.\\ \left.-\lambda\left(\mathbf{E}_{\widehat{\mu}_H}\left[\int_0^T e^{-ru}c(u)\,du+e^{-rT}F\right]-z\right)\right\}.\tag{4.4}$$

We can rewrite this as

$$\begin{aligned}V_\lambda(z)&=\sup_{c,F\geq 0}\mathbf{E}\Big[\int_0^T\big(e^{-\delta t}\log c(t)-\lambda\eta(T)\,e^{-rt}c(t)\big)\,dt\\&\qquad+D_1\log F-\lambda\eta(T)\,e^{-rT}F\Big]+\lambda z\\&=\sup_{c,F\geq 0}\mathbf{E}\Big[\int_0^T\big(e^{-\delta t}\log c(t)-\lambda\rho(t)\,e^{-rt}c(t)\big)\,dt\\&\qquad+D_1\log F-\lambda\eta(T)\,e^{-rT}F\Big]+\lambda z,\end{aligned}\tag{4.5}$$

where $\eta(T)$ is given by (3.8) and

$$\rho(t)=\mathbf{E}\left[\eta(T)\,\Big|\mathcal{F}_t^{(H)}\right].\tag{4.6}$$

To get (4.5) we use the fact that

$$\mathbf{E}[\eta(T)\,c(t)]=\mathbf{E}\left[\mathbf{E}[\eta(T)\,c(t)\,\Big|\mathcal{F}_t^{(H)}]\right]=\mathbf{E}\left[c(t)\,\mathbf{E}[\eta(T)\,\Big|\mathcal{F}_t^{(H)}]\right]=\mathbf{E}\left[c(t)\,\rho(t)\right].$$

The unconstrained problem (4.5) can be solved simply by maximizing the following functions for each $t\in[0,T]$ and $\omega\in\Omega$:

$$g(c)=e^{-\delta t}\log c-\lambda\rho(t)\,e^{-rt}c,$$
$$h(F)=D_1\log F-\lambda\eta(T)\,e^{-rT}F.$$

We have $g'(c)=0$ for

$$c=\frac{e^{-\delta t}e^{rt}}{\lambda\rho(t)},\tag{4.7}$$

and by concavity this is the maximum point of g.

Similarly, we get the maximum point of h

$$F = \frac{D_1 e^{rT}}{\lambda \eta (T)}. \tag{4.8}$$

We now look for λ^* such that the constraint in (4.3) holds, *i.e.*,

$$\mathbf{E}\left[\left(\int_0^T e^{-ru} c(u)\, du + e^{-rT} F\right) \eta(T)\right] = z.$$

Substituting (4.7) and (4.8) into the above and solving for λ, we obtain

$$\lambda^* = \frac{1}{Mz}, \qquad \text{where} \qquad M = \left(\frac{1 - e^{-\delta t}}{\delta} + D_1\right)^{-1}. \tag{4.9}$$

Now substitute λ^* into (4.7) and (4.8) to get

$$c^*(t) := c_{\lambda^*}(t) = Mz e^{-\delta t} e^{rt} \frac{1}{\rho(t)} \tag{4.10}$$

and

$$F^* := F_{\lambda^*} = D_1 Mz e^{rT} \frac{1}{\eta(T)}. \tag{4.11}$$

This is the optimal (c, F) for the problem (4.2) and we conclude that the optimal utility is given by

$$\begin{aligned} V(z) &= \mathbf{E}\left[\int_0^T e^{-\delta t} \log c^*(t)\, dt + D_1 \log F^*\right] \\ &= \int_0^T \{\log(Mz) + (r - \delta) t\} e^{-\delta t} dt + D_1 \{\log(D_1 Mz) + rT\} \\ &\quad + \int_0^T e^{-\delta t} \mathbf{E}\left[\log \frac{1}{\rho(t)}\right] dt + D_1 \mathbf{E}\left[\log \frac{1}{\eta(T)}\right]. \end{aligned}$$

By the definition of $\eta(T)$ given in (3.8),

$$\begin{aligned} \mathbf{E}\left[\log \frac{1}{\eta(T)}\right] &= \mathbf{E}\left[\int_0^T K(s)\, dB_s^H + \frac{1}{2} |K|_\varphi^2\right] \\ &= \frac{1}{2} |K|_\varphi^2 = \frac{a - r}{2\sigma} \int_0^T K(t)\, dt = \frac{(a - r)^2}{2\sigma^2} \cdot \Lambda_H \cdot T^{2-2H}, \end{aligned}$$

where

$$\Lambda_H = \frac{\Gamma^2\left(\frac{3}{2} - H\right)}{2H \cdot (2 - 2H) \cdot \Gamma(2H) \cdot \Gamma(2 - 2H) \cdot \cos\left(\pi\left(H - \frac{1}{2}\right)\right)}. \tag{4.12}$$

It was proved by Hu [8] that

$$\rho(t) = \mathbf{E}\left[\eta(T) \,\middle|\, \mathcal{F}_t^{(H)}\right] = \exp\left(-\int_0^t \zeta_t(s)\, dB_s^H - \frac{1}{2} |\zeta_t|_\varphi^2\right), \tag{4.13}$$

where ζ_t is determined by the equation

$$(-\Delta)^{-\left(H-\frac{1}{2}\right)} \zeta_t(s) = -(-\Delta)^{-\left(H-\frac{1}{2}\right)} K(s), \quad 0 \le s \le t,$$
$$\zeta_t(s) = 0, \quad s < 0 \text{ or } s > t.$$

The following solution for ζ_t is also given in [8]:

$$\zeta_t(s) = -\kappa_H s^{\frac{1}{2}-H} \frac{d}{ds} \int_s^t w^{2H-1} (w-s)^{\frac{1}{2}-H} \times \left(\frac{d}{dw} \int_0^w z^{\frac{1}{2}-H} (w-z)^{\frac{1}{2}-H} g(z)\, dz \right) dw, \tag{4.14}$$

where $g(z) = -(-\Delta)^{-\left(H-\frac{1}{2}\right)} K(z)$ and

$$\kappa_H = \frac{2^{2H-2}\sqrt{\pi}\,\Gamma\left(H-\frac{1}{2}\right)}{\Gamma(1-H)\,\Gamma^2\left(\frac{3}{2}-H\right)\cos\left(\pi\left(H-\frac{1}{2}\right)\right)}.$$

Hence,

$$\mathbf{E}\left[\log\frac{1}{\rho(t)}\right] = \mathbf{E}\left[\int_0^t \zeta_t(s)\, dB_s^H + \frac{1}{2}|\zeta_t|_\varphi^2\right] = \frac{1}{2}|\zeta_t|_\varphi^2.$$

Thus we obtain

$$\begin{aligned} V(z) &= \delta^{-2}(r-\delta)\left[1-e^{-\delta T}(1+\delta T)\right] + \delta^{-1}\left(1-e^{-\delta T}\right)\log(Mz) \\ &\quad + D_1\left(\log(D_1 M z) + rT\right) \\ &\quad + \frac{1}{2}\int_0^T e^{-\delta t}|\zeta_t|_\varphi^2\, dt + \frac{D_1(a-r)^2}{2\sigma^2}\Lambda_H T^{2-2H}, \end{aligned} \tag{4.15}$$

where the constants M and Λ_H are given by (4.9) and (4.12), respectively. This proves the following theorem.

Theorem 4.1. *The value function of the optimal consumption and portfolio problem (4.1) is given by (4.15). The corresponding optimal consumption c^* and the optimal terminal wealth $Z_z^{c_{\lambda^*},\theta^*}(T) = F^*$ are given by (4.10) and (4.11), respectively.*

It remains to find the optimal portfolio $\theta^* = (\alpha^*, \beta^*)$ for problem (4.1). Let $G = e^{-rT}F^* + \int_0^T e^{-rt}c^*(t)\, dt$. In the proof of Lemma 3.1, it was shown that $G = z + \int_0^T \widetilde{\mathbf{E}}_{\widehat{\mu}}\left[\widehat{D}_t G \,\middle|\, \mathcal{F}_t^{(H)}\right] d\widehat{B}_t^H$, where $\widehat{D}$ denotes the Malliavin derivative with respect to $\hat{\mu}_H$ ($\widehat{D}$ is not to be confused with the utility legacy scale constant D_1), and

$$\begin{aligned} \beta^*(t) &= \frac{e^{rt}}{\sigma S(t)} \widetilde{\mathbf{E}}_{\widehat{\mu}}\left[\widehat{D}_t G \,\middle|\, \mathcal{F}_t^{(H)}\right] \\ &= \frac{e^{rt}}{\sigma S(t)} \left(\widetilde{\mathbf{E}}_{\widehat{\mu}}\left[\widehat{D}_t\left(e^{-rT}F^*\right) \,\middle|\, \mathcal{F}_t^{(H)}\right] + \widetilde{\mathbf{E}}_{\widehat{\mu}}\left[\widehat{D}_t\left(\int_0^T e^{-ru}c^*(u)\, du\right) \,\middle|\, \mathcal{F}_t^{(H)}\right] \right) \\ &= \frac{e^{rt}}{\sigma S(t)} \Big(Y_1(t) + Y_2(t) \Big), \end{aligned} \tag{4.16}$$

when we set

$$Y_1(t) := \widetilde{\mathbf{E}}_{\widehat{\mu}}\left[\widehat{D}_t\left(e^{-rT}F^*\right)\middle|\mathcal{F}_t^{(H)}\right],$$
$$Y_2(t) := \widetilde{\mathbf{E}}_{\widehat{\mu}}\left[\widehat{D}_t\left(\int_0^T e^{-ru}c^*(u)\,du\right)\middle|\mathcal{F}_t^{(H)}\right].$$

To compute $Y_1(t)$ and $Y_2(t)$, we first compute the following.

$$\begin{aligned}\frac{1}{\eta(T)} &= \exp\left\{\int_0^T K(s)\,dB_s^H + \frac{1}{2}|K|_\varphi^2\right\}\\ &= \exp\left\{\int_0^T K(s)\,d\widehat{B}_s^H + \frac{1}{2}|K|_\varphi^2 - \frac{a-r}{\sigma}\int_0^T K(s)\,ds\right\},\\ &= \exp\left\{\int_0^T K(s)\,d\widehat{B}_s^H - \frac{1}{2}|K|_\varphi^2\right\},\end{aligned}$$

$$\begin{aligned}\widehat{D}_t\left(\frac{1}{\eta(T)}\right) &= \widehat{D}_t\left(\exp\left\{\int_0^T K(s)\,d\widehat{B}_s^H\right\}\right)\exp\left\{-\frac{1}{2}|K|_\varphi^2\right\}\\ &= K(t)\exp\left\{\int_0^T K(s)\,d\widehat{B}_s^H\right\}\exp\left\{-\frac{1}{2}|K|_\varphi^2\right\},\end{aligned}$$

$$\begin{aligned}\frac{1}{\rho(u)} &= \exp\left\{\int_0^u \zeta_u(s)\,dB_s^H + \frac{1}{2}|\zeta_u|_\varphi^2\right\}\\ &= \exp\left\{\int_0^u \zeta_u(s)\,d\widehat{B}_s^H - \frac{a-r}{\sigma}\int_0^u \zeta_u(s)\,ds + \frac{1}{2}|\zeta_u|_\varphi^2\right\},\end{aligned}$$

When $t \leq u$,

$$\begin{aligned}\widehat{D}_t\left(\frac{1}{\rho(u)}\right) &= \widehat{D}_t\left(\exp\left\{\int_0^u \zeta_u(s)\,d\widehat{B}_s^H\right\}\right)\exp\left\{-\frac{a-r}{\sigma}\int_0^u \zeta_u(s)\,ds + \frac{1}{2}|\zeta_u|_\varphi^2\right\}\\ &= \zeta_u(t)\exp\left\{\int_0^u \zeta_u(s)\,d\widehat{B}_s^H\right\}\exp\left\{-\frac{a-r}{\sigma}\int_0^u \zeta_u(s)\,ds + \frac{1}{2}|\zeta_u|_\varphi^2\right\},\end{aligned}$$

and

$$\widehat{D}_t\left(\frac{1}{\rho(u)}\right) = 0,$$

if $t > u$.

Therefore,

$$
\begin{aligned}
Y_1(t) &= \widetilde{\mathbf{E}}_{\widehat{\mu}}\left[\widehat{D}_t\left(e^{-rT}F^*\right)\Big|\mathcal{F}_t^{(H)}\right] \\
&= \widetilde{\mathbf{E}}_{\widehat{\mu}}\left[\widehat{D}_t\left(D_1Mz\frac{1}{\eta(T)}\right)\Big|\mathcal{F}_t^{(H)}\right] \\
&= D_1MzK(t)\exp\left\{-\frac{1}{2}|K|_\varphi^2\right\}\widetilde{\mathbf{E}}_{\widehat{\mu}}\left[\exp\left\{\int_0^T K(s)\,d\widehat{B}_s^H\right\}\Big|\mathcal{F}_t^{(H)}\right] \\
&= D_1MzK(t)\,\widetilde{\mathbf{E}}_{\widehat{\mu}}\left[\exp\left\{\int_0^T K(s)\,d\widehat{B}_s^H-\frac{1}{2}|K|_\varphi^2\right\}\Big|\mathcal{F}_t^{(H)}\right] \\
&= D_1MzK(t)\exp\left\{\int_0^t K(s)\,d\widehat{B}_s^H-\frac{1}{2}\left|K\cdot\mathbf{1}_{[0,t]}\right|_\varphi^2\right\} \\
&= g_1(t)\,K(t)\exp\left\{\int_0^t K(s)\,d\widehat{B}_s^H\right\},
\end{aligned}
\tag{4.17}
$$

where

$$
g_1(t) := D_1Mz\exp\left\{-\frac{1}{2}\left|K\cdot\mathbf{1}_{[0,t]}\right|_\varphi^2\right\}. \tag{4.18}
$$

Similarly,

$$
\begin{aligned}
Y_2(t) &= \widetilde{\mathbf{E}}_{\widehat{\mu}}\left[\widehat{D}_t\left(\int_0^T e^{-ru}c^*(u)\,du\right)\Big|\mathcal{F}_t^{(H)}\right] \\
&= \widetilde{\mathbf{E}}_{\widehat{\mu}}\left[\widehat{D}_t\left(\int_0^T Mze^{-\delta u}\frac{1}{\rho(u)}du\right)\Big|\mathcal{F}_t^{(H)}\right] \\
&= \widetilde{\mathbf{E}}_{\widehat{\mu}}\left[\int_0^T Mze^{-\delta u}\widehat{D}_t\left(\frac{1}{\rho(u)}\right)du\Big|\mathcal{F}_t^{(H)}\right] \\
&= \widetilde{\mathbf{E}}_{\widehat{\mu}}\Bigg[\int_t^T Mze^{-\delta u}\zeta_u(t)\exp\Bigg\{\int_0^u \zeta_u(s)\,d\widehat{B}_s^H \\
&\qquad\qquad -\frac{a-r}{\sigma}\int_0^u \zeta_u(s)\,ds+\frac{1}{2}|\zeta_u|_\varphi^2\Bigg\}du\Big|\mathcal{F}_t^{(H)}\Bigg] \\
&= \int_t^T Mze^{-\delta u}\zeta_u(t)\,\widetilde{\mathbf{E}}_{\widehat{\mu}}\Bigg[\exp\Bigg\{\int_0^u \zeta_u(s)\,d\widehat{B}_s^H \\
&\qquad\qquad -\frac{a-r}{\sigma}\int_0^u \zeta_u(s)\,ds+\frac{1}{2}|\zeta_u|_\varphi^2\Bigg\}\Big|\mathcal{F}_t^{(H)}\Bigg]du \\
&= \int_t^T Mze^{-\delta u}\zeta_u(t)\exp\left\{|\zeta_u|_\varphi^2-\frac{a-r}{\sigma}\int_0^u \zeta_u(s)\,ds\right\} \\
&\qquad\qquad \times\widetilde{\mathbf{E}}_{\widehat{\mu}}\left[\exp\left\{\int_0^u \zeta_u(s)\,d\widehat{B}_s^H-\frac{1}{2}|\zeta_u|_\varphi^2\right\}\Big|\mathcal{F}_t^{(H)}\right]du
\end{aligned}
$$

$$= \int_t^T Mze^{-\delta u}\zeta_u(t)\exp\left\{|\zeta_u|_\varphi^2 - \frac{a-r}{\sigma}\int_0^u \zeta_u(s)\,ds\right\} \\ \times \exp\left\{\int_0^t \zeta_u(s)\,d\widehat{B}_s^H - \frac{1}{2}\left|\zeta_u \cdot \mathbf{1}_{[0,t]}\right|_\varphi^2\right\}du$$

$$= \int_t^T g_2(u,t)\,\zeta_u(t)\exp\left\{\int_0^t \zeta_u(s)\,d\widehat{B}_s^H\right\}du, \tag{4.19}$$

where

$$g_2(u,t) := Mze^{-\delta u}\exp\left\{|\zeta_u|_\varphi^2 - \frac{a-r}{\sigma}\int_0^u \zeta_u(s)\,ds - \frac{1}{2}\left|\zeta_u \cdot \mathbf{1}_{[0,t]}\right|_\varphi^2\right\}. \tag{4.20}$$

We summarize our calculations in the following theorem.

Theorem 4.2. *The optimal portfolio $\theta^*(t) = (\alpha^*(t), \beta^*(t))$ for problem (4.1) is given by*

$$\beta^*(t) = \frac{e^{rt}}{\sigma S(t)}\{Y_1(t) + Y_2(t)\}, \tag{4.21}$$

where $Y_1(t)$ and $Y_2(t)$ are given by (4.17) and (4.19), respectively; and

$$\alpha^*(t) = e^{-rt}\{Z^*(t) - \beta^*(t)\,S(t)\} = e^{-rt}Z^*(t) - \frac{1}{\sigma}\{Y_1(t) + Y_2(t)\} \tag{4.22}$$

where the optimal wealth process, $Z^*(t)$, can be obtained from

$$e^{-rt}Z^*(t) + \int_0^t e^{-rs}c^*(s)\,ds = z + \int_0^t \sigma e^{-rs}\beta^*(s)\,S(s)\,d\widehat{B}_s^H \tag{4.23}$$

and $c^*(s)$ is given by (4.10).

In order to determine $\alpha^*(t)$ explicitly, our next goal is to calculate

$$\int_0^t \sigma e^{-rs}\beta^*(s)\,S(s)\,d\widehat{B}_s^H = \int_0^t \{Y_1(s) + Y_2(s)\}\,d\widehat{B}_s^H \tag{4.24}$$

which simplifies (4.23) in Theorem 4.2. This is contained in formulas (4.26) and (4.27) below. We summarize the strategy for calculating the optimal consumption and portfolio explicitly:

Compute Y_1 and Y_2. The quantities Y_1 and Y_2 are given in (4.17) and (4.19). These formulas are evaluated using the non-random quantities g_1 and g_2 given in (4.18) and (4.20). The Wiener stochastic integrals $\int_0^t K(s)\,d\widehat{B}_s^H$ and $\int_0^t \zeta_u(s)\,d\widehat{B}_s^H$ can be estimated simply using Riemann-sum approximations, based on the observed increments of $\widehat{B}_t^H := B_t^H + \left(\frac{a-r}{\sigma}\right)t$, since the integrands K and ζ are non-random. More information on computing such integrals is in Section 7.

Compute β^*. Since $S(t)$ is also observable, the optimal number of stocks β^* follows directly from (4.21)

Compute c^*. With formula (4.10), we see that the optimal consumption c^* can be calculated using non-random quantities, and the Wiener stochastic integral $\int_0^t \zeta_u(s)\,d\widehat{B}_s^H$, which is approximated from the osbservations using Riemann sums.

Compute the stochastic integrals of Y_1 and Y_2. The stochastic integral $\int_0^t Y_1(s)\, d\widehat{B}_s^H$ is given in formula (4.26) using again Riemann integrals, the function g_1 in (4.18), and the stochastic integral $\int_0^s K(s)\, d\widehat{B}_s^H$, discussed above. Similarly, the stochastic integral $\int_0^t Y_1(s)\, d\widehat{B}_s^H$ in (4.27) requires only Riemann integrals, g_2 from (4.20), and $\int_0^t \zeta_u(s)\, d\widehat{B}_s^H$ as above.

Compute Z^*. From (4.23), (4.24), we have

$$e^{-rt} Z^*(t) = z - \int_0^t e^{-rs} c^*(s)\, ds + \int_0^t \{Y_1(s) + Y_2(s)\}\, d\widehat{B}_s^H,$$

where c^* was found above, and the stochastic integral is the sum of the two integrals computed in the last step above.

Compute α^*. Finally, the optimal number of risk-free units (bonds) α^* is obtained immediately from Z^*, Y_1, and Y_2 thanks to (4.22).

We now calculate the stochastic integrals of Z_1 and Z_2. We will use Itô's formula for fBm (see [7, Corollary 4.4] or [3, Theorem 8]) to calculate $\int_0^t Y_1(v)\, d\widehat{B}_v^H$. Let $b_t = \int_0^t a_s d\widehat{B}_s^H$, where a is deterministic and Skorohod integrable. Then, for a $C^{1,2}$ function $f : [0,T] \times \mathbf{R} \to \mathbf{R}$ we have

$$\begin{aligned} f(t, b_t) =& f(0,0) + \int_0^t \frac{\partial f}{\partial s}(s, b_s)\, ds + \int_0^t \frac{\partial f}{\partial x}(s, b_s)\, a_s d\widehat{B}_s^H \\ &+ \int_0^t \frac{\partial^2 f}{\partial x^2}(s, b_s) \int_0^s a_v \varphi(s,v)\, dvds. \end{aligned} \tag{4.25}$$

Letting $b_t = \int_0^t K(s)\, d\widehat{B}_s^H$ and $f(t,x) = g_1(t)\, e^x$ in (4.25) yields

$$\begin{aligned} g_1(t)\, e^{b_t} =& g_1(0) + \int_0^t g_1'(s)\, e^{b_s} ds + \int_0^t g_1(s)\, e^{b_s} K(s)\, d\widehat{B}_s^H \\ &+ \int_0^t g_1(s)\, e^{b_s} \int_0^s K(v)\, \varphi(s,v)\, dvds \\ =& g_1(0) + \int_0^t g_1'(s)\, e^{b_s} ds + \int_0^t Y_1(s)\, d\widehat{B}_s^H \\ &+ \int_0^t g_1(s)\, e^{b_s} \int_0^s K(v)\, \varphi(s,v)\, dvds, \end{aligned}$$

and from that we obtain

$$\begin{aligned} \int_0^t Y_1(s)\, d\widehat{B}_s^H =& -g_1(0) + g_1(t) \exp\left\{\int_0^t K(s)\, d\widehat{B}_s^H\right\} \\ &- \int_0^t g_1'(s) \exp\left\{\int_0^s K(u)\, d\widehat{B}_u^H\right\} ds \\ &- \int_0^t g_1(s) \exp\left\{\int_0^s K(u)\, d\widehat{B}_u^H\right\} \int_0^s K(v)\, \varphi(s,v)\, dvds. \end{aligned} \tag{4.26}$$

Using Fubini's theorem and the same argument as above, we calculate

$$\begin{aligned}
\int_0^t Y_2(s)\, d\widehat{B}_s^H &= \int_0^t \int_s^T g_2(u,s)\, \zeta_u(s) \exp\left\{\int_0^s \zeta_u(v)\, d\widehat{B}_v^H\right\} du\, d\widehat{B}_s^H \\
&= \int_0^t \int_0^u g_2(u,s)\, \zeta_u(s) \exp\left\{\int_0^s \zeta_u(v)\, d\widehat{B}_v^H\right\} d\widehat{B}_s^H du \\
&\quad + \int_t^T \int_0^t g_2(u,s)\, \zeta_u(s) \exp\left\{\int_0^s \zeta_u(v)\, d\widehat{B}_v^H\right\} d\widehat{B}_s^H du \\
= \int_0^t \Bigg(& -g_2(u,0) + g_2(u,u) \exp\left\{\int_0^u \zeta_u(v)\, d\widehat{B}_v^H\right\} \\
& - \int_0^u \frac{\partial g_2}{\partial s}(u,s) \exp\left\{\int_0^s \zeta_u(v)\, d\widehat{B}_v^H\right\} ds \\
& - \int_0^u g_2(u,s) \exp\left\{\int_0^s \zeta_u(v)\, d\widehat{B}_v^H\right\} \int_0^s \zeta_u(\tau)\, \varphi(s,\tau)\, d\tau ds \Bigg) du \\
+ \int_t^T \Bigg(& -g_2(u,0) + g_2(u,t) \exp\left\{\int_0^t \zeta_u(v)\, d\widehat{B}_v^H\right\} \\
& - \int_0^t \frac{\partial g_2}{\partial s}(u,s) \exp\left\{\int_0^s \zeta_u(v)\, d\widehat{B}_v^H\right\} ds \\
& - \int_0^t g_2(u,s) \exp\left\{\int_0^s \zeta_u(v)\, d\widehat{B}_v^H\right\} \int_0^s \zeta_u(\tau)\, \varphi(s,\tau)\, d\tau ds \Bigg) du.
\end{aligned} \tag{4.27}$$

It is clear that the only randomness in the formula for β^* in Theorem 4.2 is given in terms of Wiener integrals with respect to fBm. However, that was not the case for α^*. With these last two calculations based on the fractional Itô formula, we are now able to express the randomness in α^* in terms of only Wiener integrals as well. This represents a practical advance over previous works where solutions are presented in terms of general Skorohod integrals and/or using Malliavin derivatives, (quasi-)conditional expectations, and the like, since there are no numerical methods available for these general objects. The study presented here simplifies the solution as much as possible for numerical implementation purposes. In Section 7 we present the results of such an implementation.

5. Truly Self-Financing Portfolio; Positive Consumption

A common criticism of the framework used in Section 3, and used in our sources [9], [10], is that the definition of self-financing using a Skorohod stochastic integral does not correspond to the true notion of a self-financing portfolio. We discuss this issue here. If our purpose was to provide a framework for pricing derivatives, we would need indeed to construct a portfolio with the true self-financing property. However, because we are only trying to find a strategy maximizing an expected future utility using a certain class of admissible strategies, it is up to us to decide what class of strategies we wish to use, and our Skorohod-self-financing ones are certainly an option. One may then argue in disfavor of it by asking whether there is any guarantee that it is indeed financially possible to follow such a strategy. The purpose of this Section is to prove that it is, and that one always gets more than what one bargained for, because following it results in additional positive consumption.

In particular, we now calculate the discrepancy between the two notions of self-financing portfolios in the framework of our portfolio optimization, and we conclude that this discrepancy is always in favor of the investor, in the sense of the proposition below. To compare the two notions, recall first that the wealth Z in (3.4) in a consumption-free "Skorohod-self-financing" portfolio defined by the strategy $(\alpha(t), \beta(t))_{t\geq 0}$ satisfies:

$$Z(t) = Z(0) + \int_0^t \alpha(u)\, dA(u) + \int_0^t \beta(u)\, dS(u), \tag{5.1}$$

where, as for the second differential in the Skorohod-self-financing condition (3.5), the second integral in (5.1) is in the Skorohod sense. However, since $H > \frac{1}{2}$, the *pathwise* integral of β with respect to S can also be defined, and it is the one which yields the true notion of self-financing, because it can be approximated by Riemann-Stieltjes sums in a natural way. We omit the details. We simply say that a portfolio trading strategy defined by $(\alpha(t), \beta(t))_{t\geq 0}$ is "truly self-financing with consumption process $C(t)$" if its wealth Z, still given by (3.4), satisfies

$$Z(t) = Z(0) + \int_0^t \alpha(u)\, dA(u) + \int_0^t \beta(u)\, d^P S(u) - C(t), \tag{5.2}$$

where the integral $\int_0^t \beta(u)\, d^P S(u)$ is in the pathwise sense. Note that here we use the notation C for the cummulative consumption, and that C is related to the usual notation c via $dC(t) = c(t)dt$. A number of articles on fractional Brownian motion can be consulted for the definition of the pathwise integral; for instance, we refer to [15], which also contains the following formula relating this integral to the Skorohod integral:

$$\int_0^t \beta(s)\, d^P S(s) - \int_0^t \beta(s)\, dS(s) = \alpha_H \int_0^t \int_0^t D_s\left[\beta(\tau) S(\tau)\right] |\tau - s|^{2H-2}\, d\tau ds, \tag{5.3}$$

where $\alpha_H = H(2H-1)$. As a consequence, we prove the following result.

Proposition 5.1. *Assume that the trading strategy $(\alpha(t), \beta(t))_{t\geq 0}$ is the optimal portfolio θ^* identified in Theorem 4.2, assuming no consumption ($\delta = +\infty$). Then the wealth process Z given by $Z(t) = \alpha^*(t) A(t) + \beta^*(t) S(t)$ corresponds to a truly self-financing portfolio, with initial wealth z, satisfying (5.2), with consumption process $C(t)$ given by*

$$C(t) = \frac{\alpha_H e^{rt}}{\sigma} \int_0^t d\tau \int_0^\tau g_1(\tau) K(\tau) \exp\left\{\int_0^\tau K(u)\, d\hat{B}_u^H\right\} K(s) (\tau - s)^{2H-2}\, ds$$

where $\hat{B}$ and K are given in (3.9) and (3.7), while, in accordance with (4.18) below,

$$g_1(\tau) = z \exp\left\{-\frac{\alpha_H}{2} \int_0^\tau \int_0^\tau K(u) K(v) |u - v|^{2H-2}\, du dv\right\}.$$

Most notably, $C(t)$ is positive almost surely for all t.

The precise formula for the term $C(t)$ above is not as important as the fact that it is always positive. In this sense, the optimal portfolio of Theorem 4.2 is a truly self-financing strategy which both maximizes the expected future utility for Skorohod-self-fnancing strategies, and provides the investor with free additional consumption. The formula $Z(t) = \alpha^*(t) A(t) + \beta^*(t) S(t)$ is in fact sufficient to allow the investor to keep track of her consumption. Indeed, $Z(t)$ is obviously directly calculable from the observed values of $A(t)$ and $S(t)$, and the computed optimal values $\alpha^*(t)$ and $\beta^*(t)$, both also based only on A and S (see Theorem 4.2, and relation (3.3)); then the formula

$$dC(t) = -dZ(t) + \alpha^*(t)\, dA(t) + \beta^*(t)\, d^P S(t),$$

where the latter differential is in the pathwise sense, can be calculated in an adapted way for any fixed realization of the process S.

Proof. **[of Proposition 5.1]**

Applying (5.3) to (5.1) with $(\alpha, \beta) = (\alpha^*, \beta^*) = \theta^*$ as in Theorem 4.2, since in the notation of the statement and proof of that theorem (Section 4), $\beta^*(t) = \frac{e^{rt}}{\sigma S(t)} [Y_1(t) + Y_2(t)]$, we find that (5.2) holds with

$$\begin{aligned} C(t) &= \alpha_H \int_0^t \int_0^t D_s [\beta^*(\tau) S(\tau)] |\tau - s|^{2H-2} d\tau ds \\ &= \frac{\alpha_H e^{rt}}{\sigma} \int_0^t \int_0^t D_s [Y_1(\tau) + Y_2(\tau)] |\tau - s|^{2H-2} d\tau ds. \end{aligned}$$

However, since we are in the case of no consumption for the Skorohod-self-financing portfolio ($\delta = +\infty$), one sees that $Y_2 \equiv 0$. Since

$$Y_1(t) = g_1(t) K(t) \exp\left\{ \int_0^t K(s)\, d\hat{B}_s^H \right\},$$

we obtain

$$D_s Y_1(\tau) = g_1(\tau) K(\tau) \exp\left\{ \int_0^\tau K(u)\, d\hat{B}_u^H \right\} K(s)\, \mathbf{1}_{\{s \le \tau\}}.$$

It then follows that

$$\begin{aligned} C(t) &= \frac{\alpha_H e^{rt}}{\sigma} \int_0^t \int_0^t D_s Y_1(\tau) |\tau - s|^{2H-2} d\tau ds \\ &= \frac{\alpha_H e^{rt}}{\sigma} \int_0^t d\tau \int_0^\tau g_1(\tau) K(\tau) \exp\left\{ \int_0^\tau K(u)\, d\hat{B}_u^H \right\} K(s) (\tau - s)^{2H-2} ds. \end{aligned}$$

Our expression for g_1 follows immediately from (2.2) and (4.18) when one notices that in the case $\delta = +\infty$, we obtain $MD_1 = 1$. The positivity of $C(t)$ is also immediate, since the formula for $C(t)$ contains the factor K twice, and K is proportional to a positive function (with proportionality constant equal to $a - r$, whose constant sign, which is typically positive, is nonetheless irrelevant). $\square$

6. The Case of Power Utility Functions

Hu, Øksendal and Sulem [10] solve the optimization problem in the framework of Section 3 using power utility functions. In this section we improve their results using our techniques from Section 4.

Let $D_1, D_2 > 0$, $T > 0$and $\gamma \in (-\infty, 1) \setminus \{0\}$ be constants. The quantity

$$J^{c,\theta}(z) = \mathbf{E}_{\mu_H}\left[\int_0^T \frac{D_1}{\gamma} c^{\gamma}(t)\, dt + \frac{D_2}{\gamma}\left(Z_z^{c,\theta}(T)\right)^{\gamma}\right],$$

where $(c, \theta) \in \mathcal{A}(z)$, can be regarded as the total expected (power) utility obtained from the consumption rate $c(t) \geq 0$ and the terminal wealth $Z_z^{c,\theta}(T)$. As before, the problem is to find $(c^*, \theta^*) \in \mathcal{A}(z)$ and $V(z)$ such that

$$V(z) = \sup_{(c,\theta)\in\mathcal{A}(z)} J^{c,\theta}(z) = J^{c^*,\theta^*}(z), \quad z > 0.$$

For the rest of this section, we present the solution to this optimization problem by listing the formulas without proof, since the calculations are very similar to what we have done in Section 4. Letting

$$\begin{aligned} N = {} & \frac{1}{D_1}\int_0^T \exp\left\{\frac{r\gamma t}{1-\gamma} + \frac{\gamma}{2(1-\gamma)^2}\left|\zeta_t\right|_\varphi^2\right\} dt \\ & + \frac{1}{D_2}\exp\left\{\frac{r\gamma T}{1-\gamma} + \frac{\gamma(a-r)^2\Lambda_H T^{2-2H}}{2(1-\gamma)^2\sigma^2}\right\}, \end{aligned}$$

the optimal consumption rate, optimal terminal wealth, value function of the optimal consumption and portfolio problem, and the optimal portfolio $\theta^*(t) = (\alpha^*(t), \beta^*(t))$ are given (respectively) by

$$c^*(t) = \frac{z}{D_1 N}\exp\left\{\frac{rt}{1-\gamma}\right\}\rho(t)^{\frac{1}{\gamma-1}},$$

$$F^* = \frac{z}{D_2 N}\exp\left\{\frac{rT}{1-\gamma}\right\}\eta(T)^{\frac{1}{\gamma-1}},$$

$$\begin{aligned} V(z) = \frac{z^\gamma}{\gamma}\Bigg\{ & D_1^{1-\gamma}N^{-\gamma}\int_0^T \exp\left\{\frac{r\gamma t}{1-\gamma} + \frac{2\gamma^2-\gamma}{2(1-\gamma)^2}\left|\zeta_t\right|_\varphi^2\right\} dt \\ & + D_2^{1-\gamma}N^{-\gamma}\exp\left\{\frac{r\gamma T}{1-\gamma} + \frac{\gamma(a-r)^2\Lambda_H T^{2-2H}}{2(1-\gamma)^2\sigma^2}\right\}\Bigg\}, \end{aligned}$$

$$\beta^*(t) = \frac{e^{rt}}{\sigma S(t)}\left(Y_1(t) + Y_2(t)\right),$$

$$\alpha^*(t) = e^{-rt}Z^*(t) - \frac{1}{\sigma}\left(Y_1(t) + Y_2(t)\right),$$

where

$$Z^*(t) = z - \int_0^t e^{r(t-s)}c^*(s)\, ds + e^{rt}\int_0^t \left\{Y_1(s) + Y_2(s)\right\} d\widehat{B}_s^H,$$

$$
\begin{aligned}
h_1(t) :=& \frac{z}{D_2 N} \exp\left\{ \frac{r\gamma T}{1-\gamma} - \frac{1}{2(1-\gamma)^2} \left| K \cdot \mathbf{1}_{[0,t]} \right|_{\varphi}^2 \right. \\
& \left. + \frac{2-\gamma}{2(1-\gamma)^2} |K|_{\varphi}^2 - \frac{a-r}{\sigma(1-\gamma)} \int_0^t K(s)\, ds \right\},
\end{aligned}
$$

$$
\begin{aligned}
h_2(u,t) :=& \frac{z}{D_1 N} \exp\left\{ \frac{r\gamma u}{1-\gamma} - \frac{1}{2(1-\gamma)^2} \left| \zeta_u \cdot \mathbf{1}_{[0,t]} \right|_{\varphi}^2 \right. \\
& \left. + \frac{2-\gamma}{2(1-\gamma)^2} |\zeta_u|_{\varphi}^2 - \frac{a-r}{\sigma(1-\gamma)} \int_0^u \zeta_u(s)\, ds \right\},
\end{aligned}
$$

$$
\begin{aligned}
Y_1(t) =& \widetilde{\mathbf{E}}_{\widehat{\mu}} \left[\widehat{D}_t \left(e^{-rT} F^* \right) \middle| \mathcal{F}_t^{(H)} \right] \\
=& h_1(t) \frac{K(t)}{1-\gamma} \exp\left\{ \frac{1}{1-\gamma} \int_0^t K(s)\, d\widehat{B}_s^H \right\},
\end{aligned}
$$

$$
\begin{aligned}
Y_2(t) =& \widetilde{\mathbf{E}}_{\widehat{\mu}} \left[\widehat{D}_t \left(\int_0^T e^{-ru} c^*(u)\, du \right) \middle| \mathcal{F}_t^{(H)} \right] \\
=& \int_t^T h_2(u,t) \frac{\zeta_u(t)}{1-\gamma} \exp\left\{ \frac{1}{1-\gamma} \int_0^t \zeta_u(s)\, d\widehat{B}_s^H \right\} du,
\end{aligned}
$$

$$
\begin{aligned}
\int_0^t Y_1(s)\, d\widehat{B}_s^H =& -h_1(0) + h_1(t) \exp\left\{ \frac{1}{1-\gamma} \int_0^t K(s)\, d\widehat{B}_s^H \right\} \\
& - \int_0^t h_1'(s) \exp\left\{ \frac{1}{1-\gamma} \int_0^s K(u)\, d\widehat{B}_u^H \right\} ds \\
& - \frac{1}{1-\gamma} \int_0^t h_1(s) \exp\left\{ \frac{1}{1-\gamma} \int_0^s K(u) d\widehat{B}_u^H \right\} \int_0^s K(v)\varphi(s,v) dv ds,
\end{aligned}
$$

$$
\int_0^t Y_2(s)\, d\widehat{B}_s^H = \int_0^t \int_s^T h_2(u,s) \frac{\zeta_u(s)}{1-\gamma} \exp\left\{ \frac{1}{1-\gamma} \int_0^s \zeta_u(v)\, d\widehat{B}_v^H \right\} du d\widehat{B}_s^H
$$

$$
\begin{aligned}
=& \int_0^t \left(-h_2(u,0) + h_2(u,u) \exp\left\{ \frac{1}{1-\gamma} \int_0^u \zeta_u(s)\, d\widehat{B}_s^H \right\} \right. \\
& - \int_0^u \frac{\partial h_2}{\partial s}(u,s) \exp\left\{ \frac{1}{1-\gamma} \int_0^s \zeta_u(v)\, d\widehat{B}_v^H \right\} ds \\
& \left. - \frac{1}{1-\gamma} \int_0^u h_2(u,s) \exp\left\{ \frac{1}{1-\gamma} \int_0^s \zeta_u(v)\, d\widehat{B}_v^H \right\} \int_0^s \zeta_u(\tau) \varphi(s,\tau)\, d\tau ds \right) du \\
& + \int_t^T \left(-h_2(u,0) + h_2(u,t) \exp\left\{ \frac{1}{1-\gamma} \int_0^t \zeta_u(s)\, d\widehat{B}_s^H \right\} \right. \\
& - \int_0^t \frac{\partial h_2}{\partial s}(u,s) \exp\left\{ \frac{1}{1-\gamma} \int_0^s \zeta_u(v)\, d\widehat{B}_v^H \right\} ds \\
& \left. - \frac{1}{1-\gamma} \int_0^t h_2(u,s) \exp\left\{ \frac{1}{1-\gamma} \int_0^s \zeta_u(v)\, d\widehat{B}_v^H \right\} \int_0^v \zeta_u(\tau) \varphi(s,\tau)\, d\tau dv \right) du.
\end{aligned}
$$

7. Numerical Results

In this Section we implement our portfolio optimization problem. For the sake of simplicity, we present the case of no consumption. We have the following simplifications:

$$V(z) = \log z + rT + \frac{1}{2}\left(\frac{a-r}{\sigma}\right)^2 \Lambda_H T^{2-2H}, \tag{7.1}$$

where Λ_H is given by (4.12), and $Y_2(t) = 0$.

We ran 2000 scenarios with the parameters $T = 1$, $\Delta t = 0.001$, $H = 0.65$, $s_0 = 100$, $a = 0.0375$, $r = 0.0350$, $\sigma = 0.25$, $\alpha(0) = 1$, $\beta(0) = 1$, $D_1 = 1$. We simulated fBm's using the method of Wood and Chan [18], and calculated α^*, β^* and the corresponding optimal wealth process Z^*. Figure 1 shows a sample path of the stock price process, S, and Figure 2 shows the corresponding Z^*.

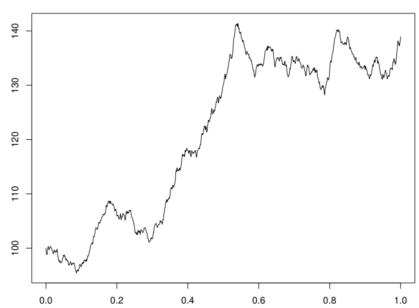

Figure 1. A sample path of the stock price process given by a geometric fBm (see (3.3)) with the parameters given in the text.

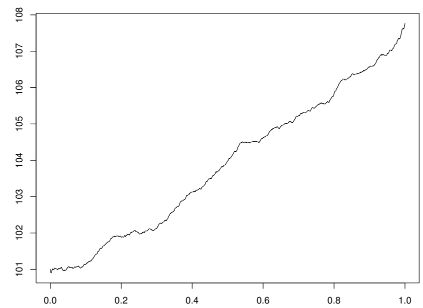

Figure 2. Optimal wealth process (Z^*) corresponding to the geometric fBm of Figure 1 with the parameters given in the text.

It is instructive to compare the explicit formula (7.1) for the value function to the corresponding classical Black-Scholes-Merton situation with standard Brownian motion, $\{W_t\}_{t\in[0,T]}$. In the latter case, the optimal wealth process is given by

$$Z^*_{BM}(t) = z \exp\left\{\left(r + \frac{1}{2}\left(\frac{a-r}{\sigma}\right)^2\right) t + \frac{a-r}{\sigma} W_t\right\} \tag{7.2}$$

and the value function is given by

$$V_{BM}(z) = \log z + rT + \frac{1}{2}\left(\frac{a-r}{\sigma}\right)^2 T. \tag{7.3}$$

An immediate comparison of (7.3) with (7.1) shows that the value function V for the fBm model exceeds that of the standard Black-Scholes-Merton value function V_{BM} for all initial wealth if and only if

$$T \leq \Lambda_H^{\frac{1}{2H-1}}. \tag{7.4}$$

This can be rephrased as saying that for short enough maturity, one is better off in a fractional market, while we expect a standard Black-Scholes market to be more profitable in the long run. On the other hand, this interpretation depends highly on the value of H. It is elementary to check, using properties of the Gamma function, that for H close to 1, the threshold in (7.4) is extremely large (tends to infinity as H tends to 1), which means that for all practical purposes, a fBm-driven market has a higher expected utility. When H is very close to $\frac{1}{2}$, where the two value functions V and V_{BM} tend to each other as they should, nevertheless the fBm value function is still the largest one for "small and moderate" T, since the right-hand side of (7.4) can be expanded as follows:

$$\Lambda_H^{\frac{1}{2H-1}} = \exp\left\{ \left|\Gamma'(1)\right| + \left(2 + 2\left|\Gamma'(1)\right|^2 + \frac{\pi^2}{4}\right)\left(H - \frac{1}{2}\right) + O\left(\left(H - \frac{1}{2}\right)^2\right) \right\}.$$

We now discuss a more difficult question with regards to comparing (7.3) and (7.1), which is beyond the scope of this article, but for which we give some indication of what might occur nonetheless. It is the issue of robustness of fBm models with respect to H. What happens if a statistical misspecification of H occurs? Of particular importance is the case where one wrongly assumes that $H = \frac{1}{2}$ and one follows the classical Merton portfolio selection scheme, in a market where the true H exceeds $\frac{1}{2}$. We conjecture that the resulting portfolio, which will necessarily be suboptimal, will in fact always lead to a significantly smaller expected future utility than the one leading to V, for any maturity. The comparison in the previous paragraph is a strong indication that this difference should be exacerbated when H is closer to 1. A more general question, still of the same nature, is to find the inefficiency due to a small misspecification of H around any fixed true $H > \frac{1}{2}$. If the convexity of the function Λ_H, as studied in the previous paragraph, is any indication, robustness of the optimization scheme should be higher for H closer to $\frac{1}{2}$.

Our numerical work can be used to investigate empirically the order of magnitude of the utility's variance, but also gives a tool to predict the average future wealth itself, without any utility function. The following output gives Monte Carlo averages of $Z^*(T)$ and $\log(Z^*(T))$ in the case of fBm and BM, as well as the value functions of the optimal portfolio problem evaluated at the initial wealth for 2000 scenarios:

```
>>>>>>>> fBm case:
Monte Carlo average of terminal optimal wealth  = 106.426
                                 Standard error = 1.04225 (~0.98%)
Monte Carlo average of log-term. optimal wealth = 4.66741
                                 Standard error = 0.00979 (~0.21%)
                    Value function at z0: V(101) = 4.65018602895
                 |(log-terminal wealth) - V(z0)| = 0.0172252

>>>>>>>> BM case:
Monte Carlo average of terminal optimal wealth  = 104.610
                                 Standard error = 1.05260 (~1.01%)
```

```
Monte Carlo average of log-term. optimal wealth = 4.65017
                                 Standard error = 0.01006 (~0.22%)
                 Value function at z0: V_BM(101) = 4.65017051684
              |(log-terminal wealth) - V_BM(z0)| = 0.0002646
```

The last line in each case shows the error in the Monte Carlo simulation, which should be proportional to the utility's standard deviation s_V, since we have $\mathbf{E}[\log(Z^*(T))] = V(z_0)$, theoretically. Firstly, the agreement between our Monte Carlo average and the theoretical value function indicates that our code runs correctly. More importantly, we see a significant increase in variance from the BM case to the fBm case. Yet the empirical result in the fBm case indicates that s_V is of the order of 0.4%, which is certainly an acceptable level. The average terminal wealth is not of any theoretical mathematical significance for logarithmic utility maximization, but we have included these numerical values to indicate that, with our choice of parameters, an fBm market can be expected to provide 2% more than a standard BM market.

We finish with a note regarding the actual numerical evaluation of Wiener stochastic integrals (*i.e.*, with deterministic integrands) with respect to fBm. The first observation is that, when integrands are deterministic, the various versions (forward, Stratonovich, Skorohod, etc...) of stochastic integrals with respect to fBm coincide. For our simulations, we only need to simulate the stochastic integral

$$\int_0^t K(s)\, d\widehat{B}_s^H, \tag{7.5}$$

where K is the function given in (3.7). A standard reflex for stochastic integrals is to use an Itô-type Riemann sum approximation, *i.e.*, $\sum_i K(t_i)(\widehat{B}^H_{t_{i+1}} - \widehat{B}^H_{t_i})$. However, since $K(0) = +\infty$, this would force us to drop the first term. It may thus be more efficient to use a formula in which this singularity is not an issue. The generalized Stratonovich integral of Russo and Vallois, also known as the symmetric regularized stochastic integral, as presented for instance in Alòs and Nualart's paper [3], claims that for ε tending to 0,

$$\frac{1}{2\varepsilon}\int_0^t K(s)\left(\widehat{B}^H_{s+\varepsilon} - \widehat{B}^H_{s-\varepsilon}\right) ds \tag{7.6}$$

tends to the stochastic integral (7.5) in $L^2(\Omega)$. Using $\varepsilon = \frac{t_{i+1}-t_i}{2}$, and using a further Riemann approximation for the Riemann integral in (7.6), we approximate (7.5) by

$$\sum_i K\Big(\frac{t_{i+1}+t_i}{2}\Big)(\widehat{B}^H_{t_{i+1}} - \widehat{B}^H_{t_i}) = \sum_i K\Big(\frac{t_{i+1}+t_i}{2}\Big)\left(B^H_{t_{i+1}} - B^H_{t_i} + \frac{a-r}{\sigma}(2\varepsilon)\right).$$

A theorem justifying that this approximation actually works can also be found in the paper [3], Proposition 3.

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