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Skorohod integration and stochastic calculus beyond the fractional Brownian scale

Oana Mocioalca[a], Frederi Viens[b,*,1]

[a]*Department of Mathematics, Purdue University, 150 N. University St., West Lafayette, IN 47907-2067, USA*

[b]*Department of Statistics, Purdue University, 150 N. University St., West Lafayette, IN 47907-2067, USA*

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Abstract

We extend the Skorohod integral, allowing integration with respect to Gaussian processes that can be more irregular than any fractional Brownian motion. This is done by restricting the class of test random variables used to define Skorohod integrability. A detailed analysis of the size of this class is given; it is proved to be non-empty even for Gaussian processes which are not continuous on any closed interval. Despite the extreme irregularity of these stochastic integrators, the Skorohod integral is shown to be uniquely defined, and to be useful: an Ito formula is established; it is employed to derive a Tanaka formula for a corresponding local time; linear additive and multiplicative stochastic differential equations are solved; an analysis of existence for the stochastic heat equation is given.

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* Corresponding author. Department of Mathematics and Statistics, Purdue University, 150 N. University St., West Lafayette, IN 47907-2067, USA.

E-mail addresses: oana@math.purdue.edu (O. Mocioalca), viens@stat.purdue.edu (F. Viens).

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1. Introduction

The purpose of this paper is to establish a stochastic calculus for processes that may have longer-range negative interactions than even *fractional Brownian motion* (fBm) with small *Hurst parameter H* (see next section for definitions). This general class of stochastic processes will encompass fBm with $H < \frac{1}{2}$, and more generally any class of Gaussian stochastic processes defined by any given scale of almost-sure uniform continuity that is bounded below by the modulus of continuity of Brownian motion. For processes that are not less regular than Brownian motion, a different construction could be used, which we do not discuss here.

The topic of stochastic calculus, which originated more than 60 years ago, with legendary associated names such as Levy, Ito, Stratonovich, saw a renewed interest in the late 1980s, when, for example, a study of stochastic integration of non-adapted processes with respect to Brownian motion first appeared in [15] in the context of two-sided integration, and subsequently in [14] a general theory of anticipating stochastic integration was developed using the connection to Skorohod integrals. Our work inscribes itself in this context, which uses as its main tool the *Malliavin calculus* (see for example Nualart's book [13] on the topic).

The most recent trend in Malliavin calculus has been in the study of fBm, anticipating stochastic integration being particularly well-suited for the study of this process whose increments are not independent, but can still be represented using standard Brownian motion. The theory of stochastic calculus for fBm is becoming relatively solid, whose main results include Ito formulas (the chain rule for non-random functions of fBm) which can be found for example in [1,2]. Other approaches to stochastic integration w.r.t. fBm include the so-called Russo–Vallois integral, with recent stochastic calculus results in [12,11]. However, both approaches have had difficulties in establishing the Ito formula, which is the cornerstone of the stochastic calculus, particularly when the fBm's regularity, as measured by its Hurst parameter H, is in the range $H \in (0; \frac{1}{4}]$. The Russo–Vallois integral has a limit of $H > \frac{1}{6}$, according to the recent results of [11], despite some intriguing results for a special version of the Russo–Vallois integral in [11] in which no restriction on H is needed. On the other hand, Cheridito and Nualart propose in the preprint [6] a new, relaxed way of defining Skorohod integration which results in an Ito formula with no restriction on $H > 0$. The idea is to restrict the space of test random variable needed to define the notion of Skorohod integrability. It is this idea which we adopt here. Our techniques and tools differ significantly from the preprint [6] by their increased simplicity (freeing ourselves from the use of fractional integrals and derivatives), and by their scope (going beyond the Hölder regularity scale, including even unbounded integrators). Lastly we mention the only other work which proposes an Ito formula for fBm with $H \leqslant \frac{1}{4}$: that of [3], which is in the context of white noise calculus, and also uses fractional integrals and derivatives crucially.

We begin our study in Section 2 by proposing a basic and wide class of Gaussian processes which contains processes very close to fBm, as well as processes which may be much more irregular than fBm, including processes that are neither uniformly continuous nor bounded. Section 3 shows briefly how to define Wiener integration w.r.t.

our processes, and gives a detailed account on how to approximate the filtrations generated by our processes. Before establishing the Malliavin calculus, Skorohod integration, and the Ito formula for our processes in Section 5, we take some time and effort in Section 4 to evaluate the precise size of the Hilbert spaces on which our test random variables will be constructed; this is particularly important since there is no a priori guarantee that the spaces will not be empty, which would result in a failed definition of the Skorohod integral. As applications of the Ito formula (Theorem 31), we establish a Tanaka formula and discuss related issues for the local time of our processes in Section 6. Then, we solve some simple finite and infinite-dimensional stochastic differential equations in Section 7. These last two sections of this article are meant as an illustration of our theory of Skorohod integration: we do not seek the most general results that may be readily available at this stage, as we hope to encourage further research on the topic.

We are grateful to a wise question of Michael Röckner which lead us to include Section 3.2 and the uniqueness result for the Skorohod integral.

2. Gaussian noise with arbitrary correlation

2.1. Definition

We begin by considering a class of Gaussian processes that may have arbitrary correlation between increments. First, recall the scale of fBm is defined as the class of centered Gaussian processes B^H on $\mathbf{R}_+$ such that with H fixed in [0,1], $B^H(0) = 0$ and $E|B^H(t) - B^H(s)|^2 = |t-s|^{2H}$. It is natural to generalize this class as follows. Let γ be a continuous increasing function on $\mathbf{R}_+$ or possibly only on a neighborhood of 0 in $\mathbf{R}_+$, such that $\lim_0 \gamma = 0$. The most naive idea is to attempt to define B^γ to be the Gaussian process such that

$$E|B^\gamma(t) - B^\gamma(s)|^2 = \gamma^2(|t-s|),$$
$$B^\gamma(0) = 0.$$

However, the corresponding process may not exist because, depending on the exact form of γ, its covariance function may not be of positive type (symmetric and non-negative definite). Therefore, for any fixed γ as above, we will be satisfied with finding a Gaussian process B such that the following hold:

(i) defining the canonical metric δ of B on $(\mathbf{R}_+)^2$ by

$$\delta^2(s,t) := E|B(t) - B(s)|^2$$

and denoting that two functions f and g are *commensurable* ($f \asymp g$) if there exist positive constants c, C such that $cg(x) \leqslant f(x) \leqslant Cg(x)$ for all values of a common variable x, we have

$$\delta(s,t) \asymp \gamma(|t-s|);$$

(ii) $B(0)=0$.

It is most comforting to assume that B can also be taken to satisfy the following:

(iii) B is adapted to a Brownian filtration.

If $\gamma(r) \gg r^H$ for all $H > 0$ then neighboring increments of B are negatively correlated, and the range of correlation is longer than for any fBm. Consider for example the following choice of γ: $\gamma^2(r) \asymp (\log \frac{1}{r})^{-1}$. It is worth noting that by Gaussian regularity theory (see [18] for example), the corresponding process B is not almost-surely uniformly continuous. We will analyze this and other examples in more detail below.

Proposition 1. *Let W be a standard Brownian motion on $\mathbf{R}_+$ with respect to the probability space $(\Omega, \mathcal{F}, P)$ and the filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$. Assume γ^2 is of class C^2 everywhere in $\mathbf{R}_+$ except at 0 and that $d\gamma^2/dr$ is non-increasing. The following centered Gaussian process satisfies conditions* (i), (ii) *and* (iii) *with respect to $\{\mathcal{F}_t\}_{t\geqslant 0}$: for any $t \geqslant 0$,*

$$B(t) = B^\gamma(t) := \int_0^t \varepsilon(t-s)\, dW(s), \tag{1}$$

where

$$\varepsilon(r) := \left(\frac{d(\gamma^2)}{dr}\right)^{1/2}.$$

In fact the constants c and C in (i) *can be taken as* 1 *and* $\sqrt{2}$ *respectively*.

Remark 2. The C^2 assumption on γ is not restrictive in terms of the magnitude of the almost-sure modulus of continuity of B, and can be achieved without loss of generality given the possibility of multiplying γ by a factor that is bounded above and away from zero.

Remark 3. The modulus of continuity γ for a non-Lipshitz function $\gamma(r)$ satisfies $\lim_{r\to 0}(d\gamma/dr) = +\infty$. Thus, we can assume, again without loss of generality, that $d\gamma/dr$ is decreasing. Moreover, since we are aiming to study processes that are less regular than Brownian motion (for which $\gamma^2(r) = r$), we can assume without loss of generality that $d\gamma^2/dr$ is non-increasing.

Proof of Proposition 1. Assume $t > s$. Then

$$E(B^\gamma(t) - B^\gamma(s))^2 = E\left(\int_0^s [\varepsilon(t-r) - \varepsilon(s-r)]\, dW(r) + \int_s^t \varepsilon(t-r)\, dW(r)\right)^2$$

$$= \int_0^s [\varepsilon(t-r) - \varepsilon(s-r)]^2 \, dr + \int_s^t \varepsilon^2(t-r) \, dr$$

$$= \int_0^s [\varepsilon(t-r) - \varepsilon(s-r)]^2 \, dr + \gamma^2(t-s) \tag{2}$$

because $\int \varepsilon^2 = \gamma^2$. Thus it is sufficient to show that

$$\int_0^s [\varepsilon(t-r) - \varepsilon(s-r)]^2 \, dr \leqslant \gamma^2(t-s).$$

We calculate

$$\int_0^s [\varepsilon(t-r) - \varepsilon(s-r)]^2 \, dr$$
$$= \gamma^2(t) - \gamma^2(t-s) + \gamma^2(s) - 2\int_0^s \sqrt{\varepsilon^2(t-r)\varepsilon^2(s-r)} \, dr.$$

We assumed that ε^2 is non-increasing, so that

$$\begin{aligned}
&\int_0^s [\varepsilon(t-r) - \varepsilon(s-r)]^2 \, dr \\
&\leqslant \gamma^2(t) - \gamma^2(t-s) + \gamma^2(s) - 2\int_0^s \varepsilon^2(t-r) \, dr \\
&= \gamma^2(t) - \gamma^2(t-s) + \gamma^2(s) - 2(\gamma^2(t) - \gamma^2(t-s)) \\
&= \gamma^2(t-s) - (\gamma^2(t) - \gamma^2(s)) \\
&\leqslant \gamma^2(t-s),
\end{aligned}$$

which finishes the proof. □

Stochastic integration with respect to the increments of a process B defined by the previous proposition can be achieved by means of the Malliavin calculus, as we will see below. The first step, however, is to understand the Wiener integral with respect to B, and indeed much work can be achieved without stochastic calculus, using only the Wiener integral, including linear additive stochastic evolution equations (see Section 7.2); the range of the Wiener integral (see Section 3.2) is even crucial in the development of the stochastic calculus (see proof of Proposition 28). Before covering these integrals, we conclude this section with some remarks on fBm.

2.2. Relation to fractional Brownian motion

The fractional Brownian motion B^H, defined in the previous section, is a process that also satisfies properties (i)–(iii). However, it does not quite satisfy a representation formula (1). What we have shown above is that by letting $\gamma(r) = \gamma_H(r) = r^H$, we have constructed a process B^{γ_H} whose covariance structure is commensurate with that of fBm, which is to say that it shares the same regularity properties as fBm, and several other crucial properties ((i)–(iii)); however, B^{γ_H} is arguably easier to work with in terms of stochastic calculus than fBm. This is an indication that for any other fixed γ, B^{γ} is a good choice for a process with covariance structure satisfying (i). A reader who is familiar with the technicalities inherent in the use of fractional integrals and derivatives needed to establish stochastic calculus for standard fBm, may appreciate the ease with which we establish the Ito formula and other results in this article.

Our process B^{γ_H} shares another important property with standard fBm: that of *self-similarity*, a.k.a. the power scaling property. We recall the relevant concept before stating the result.

Definition 4. A stochastic process X defined on $\mathbf{R}_+$ is said to be self-similar with parameter H if for any $a > 0$, the law of $\{X(at) : t \in \mathbf{R}_+\}$ and the law of $\{a^H X(t) : t \in \mathbf{R}_+\}$ are identical.

Proposition 5. *B^{γ_H} is self-similar with parameter H.*

Proof. By construction, B^{γ_H} is a separable Gaussian process. Therefore, we only need to check the self-similarity property on the first two moments of the finite-dimensional distributions of B^{γ_H}. In other words, we need only to check that if f is a polynomial of degree 2 on $(\mathbf{R}_+)^m$ and $t_1, \ldots, t_m$ are fixed times, we have

$$E[f(B^{\gamma_H}(at_1), \ldots, B^{\gamma_H}(at_m))] = E[f(a^H B^{\gamma_H}(t_1), \ldots, a^H B^{\gamma_H}(t_m))].$$

Since B^{γ_H} is centered, it is thus sufficient to check this equality for monomials of degree 2. Equivalently, we can calculate, for $0 \leqslant s < t$ fixed, the following two second moments:

$$\sigma_t^2 := E[B^{\gamma_H}(t)^2] \quad \text{and} \quad \delta(s,t)^2 := E[(B^{\gamma_H}(t) - B^{\gamma_H}(s))^2].$$

We get by definition of ε that $\sigma_{at}^2 = \int_0^{at} \varepsilon^2(at-s)\, ds = a^{2H} t^{2H} = a^{2H} \sigma_t^2$, which is the self-similarity property for this moment, while for the other term, the calculation in the proof of the previous proposition yields immediately from (2):

$$\begin{aligned}
\delta(as, at)^2 &= \int_0^{as} [\varepsilon(at-r) - \varepsilon(as-r)]^2\, dr + \gamma^2(at-as) \\
&= \int_0^{s} 2H[(a(t-r'))^{2H-1} - (a(s-r'))^{2H-1}]^2 a\, dr' + (a(t-s))^{2H} \\
&= a^{2H} \left[\int_0^{s} 2H[(t-r')^{2H-1} - (s-r')^{2H-1}]^2\, dr' + (t-s)^{2H} \right] \\
&= a^{2H} \delta(s,t)^2.
\end{aligned}$$

This finishes the proof of the proposition. □

With so many shared properties between standard fBm and our process B^{γ_H}, one may ask what difference there is between the two processes. It is well-known (see for example [16]) that fBm is the only stochastic process with finite variance that is self-similar with parameter H *and* has stationary increments. Therefore B^{γ_H} cannot have stationary increments. Of course, we can easily calculate by how much the increments of B^{γ_H} fail to be stationary. We record this in the following:

Remark 6. Standard fBm B^H has stationary increments in the sense that for all $s, t, h \in \mathbf{R}_+$,

$$E[(B^H(t+h) - B^H(t))^2] = E[(B^H(s+h) - B^H(s))^2].$$

The proof of Proposition 1 (line (2)) shows that

$$\begin{aligned}\frac{1}{2}\, E[(B^{\gamma_H}(s+h) - B^{\gamma_H}(s))^2] &\leqslant E[(B^{\gamma_H}(t+h) - B^{\gamma_H}(t))^2]\\ &\leqslant 2E[(B^{\gamma_H}(s+h) - B^{\gamma_H}(s))^2].\end{aligned}$$

In other words B^{γ_H} only fails to have stationary increments by factors no greater than 2.

3. Wiener integral with respect to B^γ

3.1. Definition

Let $(B^\gamma(t))_{t\in[0,T]}$ be the centered Gaussian process defined by its Wiener integral representation as in (1). We can formally take the differential of (1), to get

$$dB^\gamma(t) = dt \int_0^t \varepsilon'(t-s)\, dW(s) + \varepsilon(t-t)\, dW(t).$$

However, we are well aware of the fact that $\varepsilon(0) = +\infty$. Thus, we formally perform the following transformation to properly define the Wiener integral: for a deterministic function f

$$\begin{aligned}\int_0^t f(t)\, dB^\gamma(t) &= \int_0^t ds\, f(s) \int_0^s \varepsilon'(s-r)\, dW(r) + \int_0^t f(s)\varepsilon(s-s)\, dW(s)\\ &= \int_0^t ds \int_0^s (f(s) - f(r))\varepsilon'(s-r)\, dW(r)\\ &\quad + \int_0^t ds \int_0^s f(r)\varepsilon'(s-r)\, dW(r) + \int_0^t f(s)\varepsilon(s-s)\, dW(s)\\ &= \int_0^t ds \int_0^s (f(s) - f(r))\varepsilon'(s-r)\, dW(r)\end{aligned}$$

$$+\int_0^t f(r)\,dW(r)(\varepsilon(t-r)-\varepsilon(r-r))+\int_0^t f(s)\varepsilon(s-s)\,dW(s)$$
$$=\int_0^t ds\int_0^s (f(s)-f(r))\varepsilon'(s-r)\,dW(r)+\int_0^t f(r)\,dW(r)\varepsilon(t-r).$$

This justifies the following definition of the Wiener integral with respect to B^γ.

Definition 7. Let B^γ be defined as in (1). Let f be a deterministic measurable function on $\mathbf{R}_+$. We define the operator $K^*=K^*_\gamma$ on f by

$$K^*_\gamma f(r):=\left[f(r)\varepsilon(T-r)+\int_r^T (f(s)-f(r))\varepsilon'(s-r)\,ds\right]$$

if it exists. If $K^*_\gamma f(\cdot)$ is in $L^2(dr)$ then we say that f belongs to the space $L^2_\gamma([0,T])$, and we denote

$$\|f\|^2_\gamma=\|K^*_\gamma f\|^2_{L^2([0,T])}=\int_0^T\left|f(r)\varepsilon(T-r)+\int_r^t (f(s)-f(r))\varepsilon'(s-r)\,ds\right|^2 dr.$$

This L^2_γ is the so-called *canonical* Hilbert space of B^γ on $[0,T]$. We will also denote it by $\mathcal{H}$. For any f in $\mathcal{H}$ we define the stochastic integral of f with respect to B^γ on $[0,T]$ as the Gaussian random variable given by

$$\int_0^T f(t)\,dB^\gamma(t)=\int_0^T dW(r)\,K^*_\gamma f(r).$$

3.2. Sigma-fields

Let $\mathcal{F}_T$ be the sigma field generated by our Brownian motion W up to time T. Let $\mathcal{G}_T$ be the sigma field generated by all the Gaussian random variables $\int_0^T f(t)\,dB^\gamma(t)$ for $f\in\mathcal{H}$. By the previous definition, we immediately have $\mathcal{G}_T\subset\mathcal{F}_T$. The converse inclusion does not seem to hold. To give an idea why, and in order to introduce the result we can actually prove in this subsection, consider the following. If the opposite inclusion did hold, we would have for each $t\in[0,T]$ the existence of a function $f\in\mathcal{H}$ such that $K^*f=1_{[0,t]}$. Assume t is fixed and strictly positive. Let g be a function in $L^2([0,T])$. Consider the function f in $L^2([0,T])$ defined by $f(r)=g(r)1_{[0,t]}(r)$. For $r>t$ we clearly have $K^*f(r)=0$. For $r\leqslant t$, we have

$$K^*f(r)=g(r)\varepsilon(T-r)+\int_r^t [g(s)-g(r)]\varepsilon'(s-r)\,ds-\int_t^T g(r)\varepsilon'(s-r)\,ds$$
$$=\varepsilon(t-r)g(r)+\int_r^t [g(s)-g(r)]\varepsilon'(s-r)\,ds.$$

This shows that K^*f does not depend on T. Note as a consequence that $(\mathcal{G}_t)_{t\leqslant T}$ is a filtration, the natural filtration of B^γ.

It would remain to show that g can be chosen so that the last expression above is equal to 1 for all $r \leqslant t$. We can rewrite this equation as $Lg = g$ where L is a linear operator defined by

$$Lg = f_0 + L_0 g,$$

where for all $r \leqslant t$,

$$f_0(r) = \frac{1}{\varepsilon(t-r)}, \quad L_0 g(r) = \frac{1}{\varepsilon(t-r)} \int_r^t [g(r) - g(s)]\varepsilon'(s-r)\, ds.$$

Thus, we need to solve a fixed point equation. Unfortunately it does not seem possible to show that the operator L_0 is stable over any Banach space of functions. Alternately, one easily checks that for general γ the Picard iteration based on the above fixed point equation yields a diverging term after three iteration, even if the initial function f_0 is in C_b^∞, as is the case for the function f_0 above.

Abandoning this negative situation, we now establish positive results in the direction of approximating the field $\mathcal{G}_T$ generated by B^γ. These results will be crucial in the sequel. For any function $g \in L^2([0,T])$ we define $\hat{g}(k)$ as its kth Fourier coefficient, so that with $e_k(x) = \exp(2\pi T^{-1} ik)$ the following equality holds in $L^2([0,T])$:

$$g(x) = \sum_{k\in\mathbf{Z}} \hat{g}(k) e_k(x).$$

Lemma 8. *Let ε be as in Proposition 1. For all x in a neighborhood of 0, let $E(x) = x^{-1}\int_0^x \varepsilon(s)\, ds$. Let F^E be the Banach space of functions defined by*

$$F^E = \left\{ g : g \in L^2([0,T]); \sum_{k\in\mathbf{Z}} |\hat{g}(k)| E(1/k) < \infty \right\}.$$

For every $g \in F^E$, there exists a sequence $(g_n)_n$ of functions in C_b^∞ such that $\lim_n g_n = g$ and $\lim_n K^* g_n = K^* g$ where the limits hold in $L^2([0,T])$.

Proof. Without loss of generality we assume $T = 2\pi$ to simplify the notation. We use $g_n(x) = \sum_{|k|\leqslant n} \hat{g}(k) e_k(x)$. The L^2-convergence $\lim_n g_n = g$ follows by definition. Moreover, we have

$$K^*(g - g_n)(r) = G_1(r) + G_2(r),$$

where $G_1(r) = \varepsilon(T-r)(g-g_n)(r)$ and

$$G_2(r) = \sum_{|k|\geqslant n+1} e^{ikr}\hat{g}(k)\int_r^T (e^{ik(s-r)}-1)\varepsilon'(s-r)\,ds.$$

For the first term we have

$$\int_0^T |G_1(r)|^2\,dr \leqslant \int_0^T |\varepsilon(T-r)|^2 \sum_{|k|\geqslant n+1} |\hat{g}(k)e^{ikr}|\,dr = \gamma^2(T)\sum_{|k|\geqslant n+1}|\hat{g}(k)|,$$

which converges to 0 by the definition of F^E since for small x, $E(x)\geqslant\varepsilon(x)\geqslant 1$. For the second term, we bound $|e^{ikx}-1|$ above by 2 for $x>1/k$ and by kx for $x\leqslant 1/k$, yielding

$$\begin{aligned}\int_r^T (e^{ik(s-r)}-1)\varepsilon'(s-r)\,ds &\leqslant \int_0^{1/k} ks|\varepsilon'(s)|\,ds + 2\int_{1/k}^T |\varepsilon'(s)|\,ds\\ &= k\left[-\frac{1}{k}\varepsilon\left(\frac{1}{k}\right)+\int_0^{1/k}\varepsilon(s)\,ds\right] - 2\varepsilon(T) + 2\varepsilon\left(\frac{1}{k}\right)\\ &\leqslant E(1/k)+\varepsilon(1/k)\leqslant 2E(1/k).\end{aligned}$$

Therefore

$$\int_0^T |G_2(r)|^2\,dr \leqslant T\left(\sum_{|k|\geqslant n+1}|\hat{g}(k)|2E(1/k)\right)^2$$

from which the lemma follows, again by definition of F^E. □

The proof of the next proposition requires the use of a sharp summability lemma introduced in [18] in the context of Gaussian regularity theory.

Proposition 9. *Let $\mathcal{G}^\infty$ be the sigma field generated by the random variables $\{\int_0^T g(r)\,dB^\gamma(r) : g \in C_b^\infty\}$. Then $\mathcal{G}^\infty = \mathcal{G}_T$.*

Proof. It is sufficient to show that for any fixed $g\in\mathcal{H}$ the Gaussian variable $\int_0^T g(r)\,dB^\gamma(r) = \int_0^T K^*g(r)\,dW(r)$ is the limit in $L^2(\Omega,\mathcal{F}^T,P)$ of Gaussian variables of the form $\int_0^T g_n(r)\,dB^\gamma(r)$ for $g_n\in C_b^\infty$. The Gaussian property implies that it is sufficient to show K^*g_n converges in $L^2([0,T])$ to K^*g. By the previous lemma, it is sufficient to show that $g\in F^E$.

Since $g\in L^2([0,T])$ we can decompose g into a Fourier series; we will do so by extending g into an even function on $[-T,T]$, so that its Fourier series contains only cosine terms: $g(r) = \sum_{k=0}^\infty a_k\cos(2\pi kr/T)$. Since F^E is a vector space, by decomposing g into the sum of its Fourier terms for $a_k \geqslant 0$ and the sum for $a_k \leqslant 0$, we can restrict ourselves to $a_k \geqslant 0$. Moreover note that since $g \in \mathcal{H}$, the function

$$r \mapsto \int_r^T (g(s) - g(r))\varepsilon'(s-r)\, ds \tag{3}$$

is in $L^2([0,T])$. This means that for almost all r, the function g has to be (Hölder-)continuous at r in order for the above integral to converge. We assume without loss of generality that $r = 0$ is such a point: otherwise the Fourier analysis in this proof just needs to be shifted around an $r \neq 0$. Hence, the formula

$$\delta^2(r) := g(0) - g(r) = \sum_{k=1}^{\infty} a_k(1 - \cos(kr)) \tag{4}$$

defines a *canonical metric function* in the sense of [18]. Theorem 2 therein implies that for any continuous decreasing function h in a neighborhood of 0 such that $\int_0 h < \infty$,

$$\sum_{k=1}^{\infty} a_k h(\delta^2(1/k)) < \infty.$$

It now suffices to show that there exists a function h as above such that $h(\delta^2(x)) \geqslant E(x)$ near 0. We rewrite the integral in (3) for $r = 0$

$$\begin{aligned}\infty > & \left| \int_0^T (g(s) - g(0))|\varepsilon'(s)|\, ds \right| = \left| \int_0^T \delta^2(r)|\varepsilon'(s)|\, ds \right| \\ = & \left| \lim_0 \delta^2 \varepsilon - \delta^2(T)\varepsilon(T) + \int_0^T d(\delta^2)(s)\varepsilon(s) \right|,\end{aligned}$$

where the last integral can be understood in the generalized Stieltjes sense. We leave it to the reader to check that the last limit above is zero. Formula (4) defines δ^2 as a continuous increasing function in a neighborhood $[0, \alpha]$ of 0. In particular, we get

$$\infty > \int_0^{\alpha} d(\delta^2)(s)\varepsilon(s) = \int_0^{\delta^2(\alpha)} \varepsilon((\delta^2)^{-1}(u))\, du.$$

Therefore, we have found a decreasing continuous integrable function h on $(0, \alpha] : h(u) = \varepsilon((\delta^2)^{-1}(u))$, i.e. for all x near 0

$$h(\delta^2(x)) = \varepsilon(x).$$

The last step in the proof is to show that $E(x) \leqslant 2\varepsilon(x)$. Since ε^2 is integrable at 0, we can assume without loss of generality that $\varepsilon^2(x)x$ is increasing near 0. This means $2\varepsilon(x)\varepsilon'(x)x + \varepsilon^2(x) \geqslant 0$, which implies for all x near 0

$$\frac{|\varepsilon'(x)|}{\varepsilon(x)} \leqslant \frac{1}{2x}.$$

Thus, we have

$$\int_0^x \varepsilon(r)\,dr \geqslant 2\int_0^x |\varepsilon'(r)| r\,dr = -2\varepsilon(x)x + 2\int_0^x \varepsilon(r)\,dr$$

which implies

$$\int_0^x \varepsilon(r)\,dr \leqslant 2\varepsilon(x)x.$$

This is exactly the statement that $E(x) \leqslant 2\varepsilon(x)$, which finishes the proof. □

4. The canonical spaces $\mathcal{H}$ and $\mathcal{H}_2$

The above notion of Wiener integral, apart from being defined only for non-random integrands, has the additional uninviting property that it is only defined for members of $\mathcal{H}$; we will see shortly that this canonical space is uncomfortably small, since it may not even contain $C^{1/2}$ when γ is very irregular. This implies that there should be no hope of defining stochastic integrals of standard processes such as semimartingales, or even standard Brownian motion, with respect to B^γ. However, when integrating a random function, the notion of Skorohod integral can be extended to include such processes, and many more, including B^γ itself. In this section, we prepare the field by defining a new space of test functions.

Definition 10. Let us fix the time interval $[0, T]$, with $T < 1$. We define the operator $K_\gamma^{*,a}$ to be the adjoint of the operator K_γ^* in $L^2([0, T])$. We denote by $\mathcal{H}$ the set

$$\mathcal{H} = (K_\gamma^*)^{-1}(L^2([0, T])$$

and by $\mathcal{H}_2$ the set

$$\mathcal{H}_2 = (K_\gamma^*)^{-1}((K_\gamma^{*,a})^{-1}(L^2([0, T])).$$

Note that this new definition of $\mathcal{H}$ coincides with $\mathcal{H} = L^2_\gamma$, introduced previously as the domain of the Wiener integral.

Remark 11. The space $\mathcal{H}$ endowed with the inner product

$$\langle f, g\rangle_{\mathcal{H}} = \langle K_\gamma^* f, K_\gamma^* g\rangle_{L^2([0,T])}$$

is a Hilbert space.

Remark 12. Observe that if we denote by $\mathcal{H}' = \{f \in L^2([0,T]), K_\gamma^{*,a} f \in L^2([0,T])\}$ then by definition

$$\langle K_\gamma^{a,*} f, g\rangle_{L^2([0,T])} = \langle f, K_\gamma^* g\rangle_{L^2([0,T])}$$

for every $f \in \mathcal{H}'$ and $g \in \mathcal{H}$.

Next, we study the richness of the spaces $\mathcal{H}$, $\mathcal{H}'$, and $\mathcal{H}_2$ by showing that these spaces contain sets of functions with specific moduli of continuity.

Definition 13. Let η be a continuous increasing function on a neighborhood of 0 in $\mathbf{R}_+$, with $\lim_{0_+} \eta = 0$. The space C^η is defined as the space of all functions defined on $[0,T]$ that admit η as a uniform modulus of continuity

$$C^\eta = \{f \in L^2[0,T] : \sup_{0 \leqslant r < s \leqslant T} |f(s) - f(r)|/\eta(s-r) < \infty\}.$$

Proposition 14. *If η satisfies*

$$\int_0 \eta(s)|\varepsilon'(s)|\, ds < \infty \tag{C_η}$$

then $\mathcal{H}$ contains C^η. Moreover condition (C_η) is equivalent to the following:

(D_h) *There exists a positive function h defined and decreasing on a neighborhood of 0 in $\mathbf{R}_+ - \{0\}$ such that $\int_0 h < \infty$ and for small $r > 0$*

$$\eta(r) = \int_0^r \frac{h(s)}{\varepsilon(s)}\, ds. \tag{5}$$

Proof. For the first statement, for fixed η, we only need to show that if $f \in C^\eta$, then $K_\gamma^* f \in L^2([0,T])$. We treat the two terms in the sum defining $K_\gamma^* f$ separately. First, observe that if $f \in C^\eta$ then f is bounded, so that by definition of γ,

$$\int_0^T f^2(r)\varepsilon^2(T-r)\, dr \leqslant (\|f\|_\infty)^2 \int_0^T \varepsilon^2(T-r)\, dr = (\|f\|_\infty)^2 \gamma^2(T) < \infty.$$

For the $L^2[0,T]$-norm of the second term in $K_\gamma^* f$, since there exists a constant C_f such that $|f(s)-f(r)| \leqslant C_f \eta(s-r)$, we have

$$\begin{aligned}\int_0^T \left[\int_r^T (f(s)-f(r))\varepsilon'(s-r)\,ds\right]^2 dr \\ \leqslant C_f^2 \int_0^T \left[\int_r^T \eta(s-r)|\varepsilon'(s-r)|\,ds\right]^2 dr \\ = C_f^2 \int_0^T \left[\int_0^{T-r} \eta(s)|\varepsilon'(s)|\,ds\right]^2 dr.\end{aligned}$$

Note that if η is not defined up to T, we can simply extend η as an arbitrary constant by adjusting the constant C_f since f is bounded. Now $\eta|\varepsilon'|$ is integrable on all of $[0,T]$, including at 0, because of hypothesis (C_η) and the assumption that ε is differentiable except at 0 and η is continuous everywhere. This proves that $K_\gamma^* f \in L^2([0,T])$.

To prove the second statement, first note that we can assume that ε' is non-positive (see Remark 3). Now consider the following calculation, assuming (D_h):

$$\begin{aligned}\int_0^T \eta(r)|\varepsilon'(r)|\,dr &= \int_0^T \left(\int_0^r \frac{h(s)}{\varepsilon(s)}\,ds\right)(-\varepsilon'(r))\,dr \\ &= \int_0^T \left(\int_s^T -\varepsilon'(r)\,dr\right)\frac{h(s)}{\varepsilon(s)}\,ds \\ &= \int_0^T (\varepsilon(s)-\varepsilon(T))\,\frac{h(s)}{\varepsilon(s)}\,ds \\ &\leqslant \int_0^T h(s)\,ds < \infty.\end{aligned}$$

This proves (D_h) implies (C_η). The proof of the converse implication is more technical. However, the result is less important since, with the first implication, we can already guarantee that C^η is contained in $\mathcal{H}$ as soon as η is of form (5). Thus we leave the details of "(C_η) implies (D_h)" to the reader. □

Proposition 15. *Let η and C^η be as in the previous definition and proposition. Then C^η is contained in $\mathcal{H}'$ and the adjoint operator $K_\gamma^{*,a}$ can be explicitly calculated for any $f \in C^\eta$ according to*

$$K_\gamma^{*,a} f(x) = f(x)\varepsilon(x) + \int_0^x [f(y)-f(x)]\varepsilon'(x-y)\,dy =: Gf(x)$$

Proof. We will show that for $f, g \in C^\eta$ we have

$$\langle Gf, g\rangle_{L^2([0,T])} = \langle f, K_\gamma^* g\rangle_{L^2([0,T])}. \tag{6}$$

First observe that the left-hand side of the equality is well defined, i.e. for $f \in C^\eta$ we have $Gf \in L^2([0,T])$. Indeed, if $f \in C^\eta$ then $f\varepsilon$ is bounded hence in $L^2([0,T])$. Moreover $|[f(y) - f(x)]\varepsilon'(y-x)|$ is bounded above by $\eta(|y-x|)|\varepsilon'(|y-x|)|$ which implies as in the proof of the previous proposition that the second term in the definition of Gf is in $L^2([0,T])$. Thus, we have $C^\eta \subset \mathcal{H}'$. Now we only need to show equality (6). We denote by $P(T) = \langle Gf, g\rangle_{L^2([0,T])}$ and by $Q(T) = \langle f, K_\gamma^* g\rangle_{L^2([0,T])}$, then we have

$$P(T) = \int_0^T \left[f(x)g(x)\varepsilon(x) + g(x)\int_0^x (f(y) - f(x))\varepsilon'(x-y)\,dy \right] dx$$

and

$$Q(T) = \int_0^T \left[f(x)g(x)\varepsilon(T-x) + f(x)\int_x^T (g(y) - g(x))\varepsilon'(y-x)\,dy \right] dx$$

and we observe that $P(0) = Q(0)$. Hence, it is enough to show that $P'(T) = Q'(T)$ in order to conclude the proposition. But

$$\begin{aligned} P'(T) &= \frac{\delta}{\delta T}\int_0^T \left[f(x)g(x)\varepsilon(x) + g(x)\int_0^x (f(y) - f(x))\varepsilon'(x-y)\,dy \right] dx \\ &= f(T)g(T)\varepsilon(T) + g(T)\int_0^T f(y) - f(T))\varepsilon'(T-y)\,dy \end{aligned}$$

while

$$\begin{aligned} Q'(T) &= \frac{\delta}{\delta T}\int_0^T f(x)g(x)\varepsilon(T-x)\,dx \\ &\quad + \frac{\delta}{\delta T}\int_0^T f(x)\int_x^T (g(y) - g(x))\varepsilon'(y-x)\,dy]\,dx \\ &= \frac{\delta}{\delta T}\int_0^T f(T-x)g(T-x)\varepsilon(x)\,dx + f(T)\cdot 0 \\ &\quad + \int_0^T f(x)[g(T) - g(x)]\varepsilon'(T-x)\,dx \\ &= f(0)g(0)\varepsilon(T) + \int_0^T (fg)'(T-x)\varepsilon(x)\,dx \end{aligned}$$

$$
\begin{aligned}
&+\int_0^T [f(x)g(T) - f(x)g(x)]\varepsilon'(T-x)\,dx \\
&= f(0)g(0)\varepsilon(T) + \int_0^T (fg)'(x)\varepsilon(T-x)\,dx \\
&+\int_0^T f(x)g(T) - f(x)g(x)]\varepsilon'(T-x)\,dx,
\end{aligned}
$$

where the last equality was obtained by the change of variable $T - x \to x$. Therefore, by denoting $k(x) = f(x)g(x)$, it is clear that $k \in C^\eta$, so we may write

$$
\begin{aligned}
P'(T) - Q'(T) &= k(T)\varepsilon(T) + \int_0^T [g(T)f(x) - g(T)f(T)]\varepsilon'(T-x)\,dx - k(0)\varepsilon(T) \\
&- \int_0^T f(x)g(T) - f(x)g(x)]\varepsilon'(T-x)\,dx \\
&- \int_0^T (fg)'(x)\varepsilon(T-x)\,dx \\
&= (k(T) - k(0))\varepsilon(T) + \int_0^T [k(x) - k(T)]\varepsilon'(T-x)\,dx \\
&- \int_0^T k'(x)\varepsilon(T-x)\,dx \\
&= (k(T) - k(0))\varepsilon(T) + \int_0^T [(k(T) - k(x))\varepsilon(T-x)]'\,dx \\
&= (k(T) - k(0))\varepsilon(T) + [k(T) - k(x)]\varepsilon(T-x)|_0^T \\
&= (k(T) - k(0))\varepsilon(T) - (k(T) - k(0))\varepsilon(T) = 0.
\end{aligned}
$$

The last equality follows from the fact that $\lim_0 \eta\varepsilon = 0$. To prove this last statement, note that by statement (D_h) in the previous proposition, we have the existence of a positive decreasing integrable function h defined on a neighborhood of 0 in $\mathbf{R}_+ - \{0\}$ such that

$$\eta(r) = \int_0^r h(s)/\varepsilon(s)\,ds.$$

Therefore, since ε is also decreasing, we get

$$\eta(r)\varepsilon(r) = \int_0^r \varepsilon(r)h(s)/\varepsilon(s)\,ds \leqslant \int_0^r h(s)\,ds.$$

Since h is integrable at 0, the function $r \mapsto \int_0^r h(s)\,ds$ tends to 0 when r tends to 0, which proves the claim that $\lim_0 \eta\varepsilon = 0$, and finishes the proof of the proposition. □

For the next proposition on the size of $\mathcal{H}_2$, we need the following additional assumption on γ.

(A) Assume that near 0, ε is thrice continuously differentiable, ε' is non-decreasing, and ε'' is non-increasing. This can be assumed without loss of generality. Also recall that $\varepsilon^{(k)}$ has the sign of $(-1)^k$. Additionally, assume that $\varepsilon'''\varepsilon'(\varepsilon'')^{-2}$ is bounded near 0.

(A′) Assume that there exist an $\alpha > 0$ such that near 0, $r^{-\alpha} = o(\varepsilon(r))$.

Condition (A) will be satisfied in all the examples we will encounter below; it does not reduce the generality of the scale of processes that we may consider. Condition (A′) reduces the scale only very slightly: it is satisfied for the full irregular fBm scale for which $\varepsilon(r) \asymp r^{H-1/2}$ with $H < \frac{1}{2}$, but it is not satisfied for $\varepsilon(r) \asymp r^{-1/2} f(r)$ where f is negligible in front of any power. Given the fact that standard Skorohod integration is known to allow a stochastic calculus as soon as $H > \frac{1}{4}$, we are only really concerned with the case $H \leqslant \frac{1}{4}$, and in this sense Condition (A′) is certainly not a restriction.

Proposition 16. *Assume Conditions* (A) *and* (A′). *Let* ζ *and* C^ζ *be as the* η *and* C^η *in Definition* 13. *Assume moreover that* ζ *satisfies the following condition*:

(E_h) *There exists a positive function* h *defined, decreasing, and differentiable on a neighborhood of* 0 *in* $\mathbf{R}_+ - \{0\}$ *such that* $\int_0 h < \infty$ *and for small* $r > 0$,

$$\zeta(r) = -\frac{1}{\varepsilon''(r)} \frac{d(h/\varepsilon)}{dr}(r).$$

Then $\mathcal{H}_2$ *contains* C^ζ.

Proof. *Step* 0 (Setup): Observe that for any $f \in \mathcal{H}_2$, we have

$$K_\gamma^{*,a} K_\gamma^* f(x) = K_\gamma^* f(x)\varepsilon(x) + \int_0^x (K_\gamma^* f(x) - K_\gamma^* f(y))\varepsilon'(x-y)\,dy.$$

In order to show that $K_\gamma^{*,a} K_\gamma^* f(x) \in L^2([0,T])$ it is enough to show that

$$\int_0^x (K_\gamma^* f(x) - K_\gamma^* f(y))\varepsilon'(x-y)\,dy \in L^2([0,T]). \tag{7}$$

Indeed,

$$\|K_\gamma^* f\varepsilon\|_{L^2([0,T])} \leqslant \|K_\gamma^* f\|_{L^2([0,T])} \|\varepsilon\|_{L^2([0,T])} = \|K_\gamma^* f\|_{L^2([0,T])}\gamma(T) < \infty.$$

From the proof of the previous proposition, we know that (7) holds as soon as $K_\gamma^* f$ is included in C^η where η satisfies Condition (C_η) or (D_η). Again, this clearly reduces to requiring that the function J defined by

$$J(r) := \int_r^T (f(s) - f(r))\varepsilon'(s-r)\,ds$$

belongs to C^η. We must consider, for a fixed pair (x, y), with say $x < y$, the quantity $L(x, y) = J(x) - J(y)$. We rewrite $L = L_1 + L_2 + L_3$ where

$$L_1(x, y) = \int_x^y \varepsilon'(s-x)(f(s) - f(x))\,ds,$$

$$L_2(x, y) = \int_y^T (\varepsilon'(s-x) - \varepsilon'(s-y))(f(s) - f(y))\,ds,$$

$$L_3(x, y) = (f(y) - f(x))(\varepsilon(T-x) - \varepsilon(y-x)).$$

Our assumption is that for some constant c, which we can take to be $c = 1$ to simplify the notation, we have for all r, $r' > 0$,

$$|f(r) - f(r')| \leqslant c\zeta(|r-r'|) = c\,\frac{1}{\varepsilon''(|r-r'|)}\,\frac{d(h/\varepsilon)}{dr}\,(|r-r'|). \tag{8}$$

We only need to show the three L_i's are bounded in absolute value by $\eta(y-x)$.

Step 1 (ζ is acceptable): It would be well-advised to first check that the function ζ defined in Condition (E_h) is a bona-fide modulus of continuity function. Since h is integrable at 0, we can assume that $(-h')(r) \ll (r^2\log(r^{-1}))^{-1}$ near 0. Then since $(-h/\varepsilon)' = (-h')/\varepsilon + h\varepsilon'/\varepsilon^2 \leqslant (-h')/\varepsilon$ we get $(-h/\varepsilon)'(r) \ll (\varepsilon(r)r^2\log(r^{-1}))^{-1}$. By Condition (A′), we can assume without loss of generality that $\varepsilon(x)x^\alpha$ is decreasing near 0, so that $\varepsilon(x)\alpha x^{\alpha-1} + \varepsilon'(x)x^\alpha < 0$, so that we obtain $|\varepsilon'(r)| > \alpha\varepsilon(r)/r$. We can apply this argument to ε' instead of ε, since Condition (A′) certainly holds for ε'. This yields $\varepsilon''(r) \geqslant \alpha^2\varepsilon(r)r^{-2}$. This proves that $\lim_0 \zeta = 0$. The continuity of ζ is trivial given our hypotheses. The positivity of ζ near 0 can be obtained as follows. Note that we can choose h so that $h(r) \gg r^{-\alpha}$ for any $\alpha < 1$, while by the definition of ε (since ε^2 has to be integrable at 0), we must have $\varepsilon(r) \ll r^{-1/2}$; therefore as r tends to 0, h/ε tends to infinity faster than $r^{-\alpha+1/2}$. Because of the flexibility of choice for h, this limit can be attained in an increasing fashion, hence $(h/\varepsilon)' < 0$; the positivity of ε'', which is part of our hypothesis, now guarantees that ζ is positive. To guarantee that ζ is increasing, we can again invoke the flexibility on the choice of h, combined with the fact that $\zeta = o(1)$.

Step 2 (Estimate for L_2): Using the notation $\delta = y - x$, the fact that ε' is non-decreasing, and the hypothesis (8) on *f*, we have

$$|L_2(x, y)| \leqslant \int_0^{T-y} (\varepsilon'(r+\delta) - \varepsilon'(r))\zeta(r)\,dr.$$

We can split up this estimate for L_2 into two pieces

$$\begin{aligned}|L_2(x, y)| &\leqslant \int_0^{\delta} (\varepsilon'(r+\delta) - \varepsilon'(r))\zeta(r)\,dr + \int_{\delta}^{T} (\varepsilon'(r+\delta) - \varepsilon'(r))\zeta(r)\,dr \\ &:= L_{21}(\delta) + L_{22}(\delta).\end{aligned}$$

For the second piece we obtain, using the mean-value theorem and the hypothesis that ε'' is positive and decreasing,

$$\begin{aligned}L_{22}(\delta) &\leqslant \int_{\delta}^{T} \delta\varepsilon''(r)\zeta(r)\,dr = -\delta \int_{\delta}^{T} (h/\varepsilon)'(r)\,dr \\ &= \delta[(h/\varepsilon)(\delta) - (h/\varepsilon)(T)] \leqslant \delta(h/\varepsilon)(\delta) \leqslant \eta(\delta),\end{aligned}$$

where the last inequality follows from the definition of $\eta(\delta) = \int_0^{\delta} h/\varepsilon$ and the fact that h/ε is decreasing. To estimate the first piece we need an integration by parts. We write $L_{21}(\delta) = \int_0^{\delta} u\,dv$ where

$$\begin{aligned}u &= (\varepsilon'(r+\delta) - \varepsilon'(r))\left(-\frac{1}{\varepsilon''(r)}\right) \\ dv &= (h/\varepsilon)'(r)\,dr\end{aligned}$$

so that

$$-\frac{du}{dr} = \frac{1}{\varepsilon''(r)}(\varepsilon''(r+\delta) - \varepsilon''(r)) - \frac{\varepsilon'''(r)}{\varepsilon''(r)^2}(\varepsilon'(r+\delta) - \varepsilon'(r)).$$

The first term in $-du/dr$ is negative. The second term is positive and bounded above by $2\varepsilon'\varepsilon'''(\varepsilon'')^{-2}$, which is bounded by Assumption (A), say by a constant c. Therefore

$$-\int_0^{\delta} v\,du \leqslant c\int_0^{\delta} \frac{h}{\varepsilon} = c\eta(\delta)$$

and since u is negative,

$$[uv]_0^{\delta} \leqslant \lim_{r\to 0} \frac{h}{\varepsilon}(r)(\varepsilon'(r+\delta) - \varepsilon'(r))\frac{1}{\varepsilon''(r)} \leqslant \lim_{r\to 0} \frac{h}{\varepsilon}(r)\frac{2|\varepsilon'(r)|}{\varepsilon''(r)}.$$

We can show that this limit is 0. Indeed, we know $h \ll 1/r$. Moreover, using the argument with ϕ in Step 1, we have $|\varepsilon'|/\varepsilon'' < r$. The required limit is then obtained since by assumption, $1/\varepsilon$ tends to 0. Therefore, we have proved

$$L_{21}(\delta) \leqslant c\eta(\delta),$$

which implies

$$|L_2(x, y)| \leqslant (c+1)\eta(y-x).$$

Step 3 (Estimate for L_3): By assumption (8), using again $\delta = y - x$, and also using the estimate $(\varepsilon/\varepsilon'')(\delta) < \delta^2$ which was proved in Step 1, we have

$$|L_3(x, y)| \leqslant -\left(\frac{h}{\varepsilon}\right)'(\delta)\,\frac{\varepsilon(\delta)}{\varepsilon''(\delta)} \leqslant -\delta^2\left(\frac{h}{\varepsilon}\right)'(\delta).$$

By definition $\eta(\delta) = \int_0^\delta h/\varepsilon$. Therefore, $h/\varepsilon = \eta'$ and $(h/\varepsilon)' = \eta''$. First, we can see that $\eta'(\delta) < \eta(\delta)/\delta$ because η is increasing and concave and $\eta(0) = 0$, the concavity coming from the fact that $\eta' = h/\varepsilon$ can be chosen to be decreasing by an appropriate choice of h since $\varepsilon \ll r^{-1/2}$ while $h \gg r^{-1/2}$. Next, since η' decreases from $+\infty$, we can also assume that $-\eta''$ is decreasing; then by the mean value theorem, we have $\eta'(\delta/2) - \eta'(\delta) \geqslant -\eta''(\delta)\delta/2$ which yields $-\eta''(\delta) \leqslant \eta'(\delta/2)2/r$. Putting this together with the estimate on η' we get $-\eta''(\delta) \leqslant 4\delta^{-2}\eta(\delta/2)$. Since η is increasing, we finally get

$$|L_3(x, y)| \leqslant -\delta^2\eta''(\delta) \leqslant 4\eta(\delta/2) \leqslant 4\eta(\delta).$$

Step 4 (Estimate for L_1): By assumption (8), still using $\delta = y - x$,

$$\begin{aligned} |L_1(x, y)| &= \left| \int_x^y \varepsilon'(s-x)(f(s) - f(x))\, ds \right| \\ &\leqslant \int_0^\delta \varepsilon'(s)\left(\frac{h}{\varepsilon}\right)'(s)\,\frac{1}{\varepsilon''(s)}\, ds \\ &\leqslant c\eta(\delta), \end{aligned}$$

where the last inequality is established by repeating the method of estimation of $\int_0^\delta u\, dv$ in Step 2.

This finishes the proof of the proposition. □

The previous proposition is crucial in showing that $\mathcal{H}_2$ is non-empty (modulo constant functions), which is a *sine qua non* condition for the validity of our stochastic calculus below. It will also be of crucial importance when we investigate the uniqueness of the

Skorohod integral below. The next corollary aids in showing how sharp the previous proposition is, and the examples following show precisely how large we can expect $\mathcal{H}_2$ to be in specific cases of interest.

Corollary 17. *Let*

$$\tilde{\zeta}(r) = -\frac{1}{\varepsilon'(r)} \frac{d(\tilde{h}/\varepsilon')}{dr}(r),$$

where $\tilde{h}$ satisfies the same hypotheses as h except that $\int_0 \tilde{h} = +\infty$. This $\tilde{h}$ can be chosen so that $\tilde{\zeta}$ is a bona-fide modulus of continuity, and $\mathcal{H}_2$ does not contain $C^{\tilde{\zeta}}$.

Proof. The proof of the corollary uses the estimates in the proof of the previous propositions. We give only the main parts of the argument, leaving some of the details to the reader, since the corollary is not used in the remainder of the paper. First, we can invoke the same argument as in Step 1 of the proof of Proposition 16 to justify that with $\tilde{h}(r) = r \log^{-1}(r^{-1})$ we do have $\tilde{\zeta}$ non-negative, increasing and continuous at 0 with $\tilde{\zeta}(0) = 0$. Recall then that the dominant term in the calculation of the $\mathcal{H}_2$-norm of a function f is

$$\int_0^y (K_\gamma^* f(y) - K_\gamma^* f(x))\varepsilon'(y-x)\,dx. \tag{9}$$

We will show that f can be chosen in $C^{\tilde{\zeta}}$ so as to make the above term infinite for all x close to a fixed y. We treat the case $y = T$; smaller values of y are treated similarly, although the calculations are slightly more involved. The dominant term in $K_\gamma^* f(y) - K_\gamma^* f(x)$ is

$$\int_x^T (f(s) - f(x))\varepsilon'(s-x)\,ds - \int_y^T (f(s) - f(y))\varepsilon'(s-y)\,ds$$
$$= \int_x^y (f(s) - f(x))\varepsilon'(s-x)\,ds.$$

Since, the integral of the above expression, as a function of x in the space $L^1([0,y], \varepsilon'(y-x)\,dx)$, is required, by definition of $\mathcal{H}_2$, to be a member of $L^2([0,T], dy)$ after x-integration, we deduce that the expression must be absolutely integrable except possibly for a null set of values of (x, y). There exists a function f in $C^{\tilde{\zeta}}$ such that for all $0 \leqslant x \leqslant s \leqslant y$, $|f(x) - f(s)| \geqslant \tilde{\zeta}(s-x)$. Thus

$$\int_x^y |f(s) - f(x)|\varepsilon'(s-x)\,ds \geqslant \int_x^y \left(\frac{\tilde{h}}{\varepsilon'}\right)'(s-x)\,ds = \int_0^{y-x} \left(\frac{\tilde{h}}{\varepsilon'}\right)' = \frac{\tilde{h}(y-x)}{\varepsilon'(y-x)},$$

where the last step is because of the fact that in all cases which we study, $\varepsilon'(r) \geqslant 1/r$, since we study only processes B^γ that are more irregular than Brownian motion. Integrating this last expression against $\varepsilon'(y-x)\,dx$ in $[0,y]$ yields infinity by definition of $\tilde{h}$. Since this holds for all x and y the expression in (9) cannot be in $L^2([0,T])$. $\square$

Definition 18. Let $\beta > 0$ be fixed. Let γ be defined by

$$\gamma^2(r) = [\log(1/r)]^{-\beta}$$

so that

$$\varepsilon^2(r) = \beta r^{-1}[\log(1/r)]^{-\beta-1}.$$

We call the corresponding process B^γ, as defined in Proposition 1, the *logarithmic Brownian motion* (*logBm*) with parameter β.

Note that since γ has a singularity at $r = 1$, it is safe to define logBm only on closed intervals in $[0, 1)$. For larger intervals, simple scaling can be used; for infinite intervals, it is best to modify the behavior of γ for large r.

From Proposition 1, we have that the canonical metric δ of B^γ is commensurate with γ. It is then well-known that if $\xi(r) := \gamma(r)\log^{1/2}(r^{-1})$ is continuous, it is almost-surely a uniform modulus of continuity for B^γ, i.e. $B^\gamma \in C^\xi$ a.s. In fact this property is sharp: if $B^\gamma \in C^\xi$ then γ is bounded below by a constant multiple of $\xi(r)\log^{-1/2}(r^{-1})$; this property was established for homogeneous Gaussian processes in [18]; here a slightly modified argument, using the estimates in the proof of Proposition 1, can be invoked; we leave this to the reader, since the result is only tangential to our main results. What we can see immediately is that B^γ is a.s. uniformly continuous if and only if $\beta > 1$.

It can also be established that if $\beta \leqslant 1$ then B^γ is unbounded. It is often quoted in the literature that a homogeneous Gaussian process is either a.s. uniformly continuous or is a.s. unbounded, from which it is sometimes inferred that in the unbounded case, the process is discontinuous. However, even though we know of no proof of this fact, we believe that even if $\beta \leqslant 1$, the logBm is still pointwise continuous a.s., even if only at countably many points; this certainly does not contradict its unboundedness. But more importantly it would explain, heuristically, why we are able to define a stochastic calculus and a non-trivial local time with respect to it. We now summarize the above discussion, and give an indication of the sizes of $\mathcal{H}$ and $\mathcal{H}_2$, by applying Propositions 14 and 16, with $h(r) = r^{-1}\log^{-\alpha}(1/r)$ for some $\alpha > 1$.

- *Fractional Brownian scale*: The process B^γ has a canonical metric that is commensurate with that of H-fBm if $\gamma(r) = r^H$, or more generally if $\gamma(r) \asymp r^H$. In this case, our Skorohod integral defined in Section 5.2 has the same properties as that defined in [6]. It is interesting to note that our results above prove that $\mathcal{H}_2$ is indeed non-empty in this case, although this question did not seem concerning in [6]. According to our results, $\mathcal{H}_2$ contains $C^{1-2H'}$ for any $H' < H < \frac{1}{2}$. The larger H is, the bigger $\mathcal{H}_2$ is. However, Corollary 17 shows that $\mathcal{H}_2$ does not contain the space

$C^{\tilde{\zeta}}$ where $\tilde{\zeta}(r) = r^{1-2H}\log^{-1}(1/r)$. A slight historical digression on the Skorohod integral for fBm for $H \in (\frac{1}{4}; \frac{1}{2})$ might be relevant at this stage. However, we refer to the last paragraph in Section 5.2 below for such a development.

- *Regular logBm*: case $\beta > 1$. The logBm process B^γ is a.s. uniformly continuous with modulus of continuity $\xi(r) := \log^{(1-\beta)/2}(r)$. $\mathcal{H}$ contains the space C^η for

$$\eta(r) = r^{1/2}\log^{\beta-\alpha}(1/r)$$

for any $\alpha > 1$, so in particular it contains a space bigger than $C^{1/2}$. $\mathcal{H}_2$ contains the space C^ζ for

$$\zeta(r) = r\log^{\beta+1-\alpha}(1/r)$$

for any $\alpha > 1$, which is non-empty if $\alpha \leqslant \beta + 1$.
- *Irregular logBm*: case $\beta \in (0; 1]$. The logBm process B^γ is a.s. unbounded. $\mathcal{H}$ contains the space C^η for η as defined in the previous case for any $\alpha > 1$, which is never empty, but does not contains a space bigger than $C^{1/2}$. $\mathcal{H}_2$ contains the space C^ζ for ζ as defined in the previous case for any $\alpha > 1$, which is non-empty if $1 < \alpha \leqslant \beta + 1$; therefore $\mathcal{H}_2$ is non-empty for any $\beta > 0$, and a stochastic calculus w.r.t. B^γ will be defined below.
- *Highly irregular processes*: One could study examples such as $\gamma(r) \asymp \log^{-1}(\log(1/r))$, or even using multiple iterations of the logarithm. One can check that $\mathcal{H}_2$ is non-empty in these cases, although the size of $\mathcal{H}_2$ decreases "dangerously". However, since the transition between the continuous and discontinuous processes occurs within the logBm scale, we have not yet found any compelling reasons to expand on these other examples.

5. Stochastic calculus

5.1. The derivative operator

We denote by $\mathcal{S}$ the set of smooth cylindrical random variables of the form

$$F = f(B^\gamma(\phi_1), \ldots, B^\gamma(\phi_n)), \quad n \geqslant 1, \quad f \in C^\infty(R^n), \quad \phi_i \in \mathcal{H}. \tag{10}$$

We define the differential operator D on $\mathcal{S}$ by

$$DF = \sum_{i=1}^{n} \frac{\delta f}{dx_i}(B^\gamma(\phi_1), \ldots, B^\gamma(\phi_n))\phi_i.$$

Remark 19. DF is an element of $L^2(\Omega, \mathcal{G}_T, P; \mathcal{H})$.

Remark 20. For all $p \geqslant 1$, $F \to DF$ is closable from $L^p(\Omega, \mathcal{G}_T, P)$ into $L^p(\Omega, \mathcal{G}_T, P; \mathcal{H})$. The domain of D in $L^p(\Omega)$ is denoted by $D^{1,p}$, meaning that $D^{1,p}$ is the closure of the smooth random variables $\mathcal{S}$ with respect to the norm

$$\|F\|_{1,p} = [E(|F|^p) + E(\|DF\|^p_{L^2(T)})]^{\frac{1}{p}}.$$

Remark 21. For $p = 2$ the space $D^{1,2}$ is the Hilbert space with the scalar product

$$\langle F, G\rangle_{1,2} = E(FG) + E(\langle DF, DG\rangle_{\mathcal{H}}).$$

The Hermite polynomials are given by $H_0(x) = 1$ and

$$H_m(x) = \frac{(-1)^m}{m!} e^{\frac{x^2}{2}} \frac{d^m}{dx^m} (e^{-\frac{x^2}{2}}), \quad m \geqslant 1.$$

Theorem 22. *Let $\phi \in \mathcal{H}$ be an element of norm 1. Then it holds that*

$$m! H_m(B^\gamma(\phi)) = \int_{[0,T]^m} \phi(t_1)\phi(t_2) \cdots \phi(t_m) B^\gamma(dt_1) \cdots B^\gamma(dt_m).$$

This theorem is a direct application of the well-known result on multiple Wiener integrals, which can be found in Nualart's book [13], established for all isonormal Gaussian processes, and hence for our particular class of processes and their associated Hilbert spaces $\mathcal{H}$. Moreover, the proof of the following chain rule can also be found in [13]:

Proposition 23. *Let $\phi : \mathbf{R}^m \to \mathbf{R}$ be a continuous differentiable function with bounded partial derivatives, and fix $p \geqslant 1$. Suppose that $F = (F^1, \ldots, F^m)$ is a random vector whose components belong to the space $D^{1,p}$. Then*

$$D(\phi(F)) = \sum_{i=1}^m \frac{\delta\phi}{\delta x_i} (F) DF^i.$$

5.2. The divergence operator and its extension

Definition 24. The divergence operator δ is defined to be the adjoint of the derivative operator D viewed as an operator from $L^2(\Omega, \mathcal{G}_T, P) \to L^2(\Omega, \mathcal{G}_T, P; \mathcal{H})$. We will use the standard notation $\delta(u) = \int_{[0,T]} u_t \delta B^\gamma_t$ and we will refer to this random variable as the *Skorohod integral* of u with respect to B^γ.

Remark 25. δ is an operator from $L^2(\Omega, \mathcal{G}_T, P; \mathcal{H})$ into $L^2(\Omega, \mathcal{G}_T, P)$, and its domain denoted by $Dom\,\delta$ is the space of processes $u \in L^2(\Omega, \mathcal{G}_T, P; \mathcal{H})$ such that $F \to E(\langle DF, u\rangle_{\mathcal{H}})$ is a bounded linear functional on $(\mathcal{S}, \|\cdot\|_2)$. Since for such u the functional $F \to E(\langle DF, u\rangle_{\mathcal{H}})$ is linear and bounded, we can write it as an inner product, hence there is a unique element $\delta(u)$ in $L^2(\Omega, \mathcal{G}_T, P)$ such that

$$E(\langle DF, u\rangle_{\mathcal{H}}) = E(\delta(u)F)$$

for all $F \in \mathcal{S}_{\mathcal{H}}$.

It is now understood that for fBm B^H with parameter $H \leqslant \frac{1}{4}$, B^H is not in the domain $Dom\,\delta$ of its own Skorohod integral; see [6]. The same argument used therein can be applied to our process B^γ with $\gamma(r) \geqslant r^{1/4}$. An extension of the Skorohod integral was developed in [6] for which B^H is integrable. We are about to see how to extend the Skorohod integral and the Ito formula in general to allow for all our processes B^γ to be integrable. Our proof of the Ito formula uses the same algebraic ideas based on Gaussian chaos and Hermite polynomials as in [6]. However all our definitions and proofs are simpler, since they do not require the use of fractional calculus, and wider-ranging, since they are not restricted to the Hölder scale for fBm.

Definition 26. We denote by $\mathcal{S}_{\mathcal{H}_2}$ the set of smooth cylindrical random variables of the form

$$F = f(B^\gamma(\phi_1), \ldots, B^\gamma(\phi_n)), \quad n \geqslant 1, \;\; f \in C^\infty(R^n), \;\; \phi_n \in \mathcal{H}_2.$$

Definition 27. Let $\{u_t, t \in [0, T]\}$ be such that $E\int_0^T u_t^2\, dt < \infty$. We say that $u \in Dom^*\,\delta$ if there exists $\delta(u) \in L^2(\Omega, \mathcal{G}_T, P)$ such that for all $F \in \mathcal{S}_{\mathcal{H}_2}$ we have

$$\int_0^T E[u_t[K_\gamma^{*,a} K_\gamma^*](D, F)(t)]\, dt = E[\delta(u)F]. \tag{11}$$

The divergence operator defined for $u \in Dom^*\,\delta$ will also be called the Skorohod integral of u with respect to B^γ.

This definition does not require that $\delta(u)$ be uniquely defined by it. Propositions 9 and 16 will now be used to settle the question of uniqueness.

Proposition 28. *For each u in $Dom^*\,\delta$, the Skorohod integral $\delta(u)$ is uniquely defined in $L^2(\Omega, \mathcal{G}_T, P)$.*

Proof. Assume $u \in Dom^*\,\delta$ and assume there exists a random variable V in $L^2(\Omega, \mathcal{G}_T, P)$ such that for all $F \in \mathcal{S}_{\mathcal{H}_2}$, $E(\delta(u)F)$ and $E(VF)$ are both equal to the left-hand side of (11). Thus, in particular, for all $\varphi \in \mathcal{H}_2$, and for all $n \in \mathbf{N}$,

$$E[(\delta(u) - V)H_n(B^\gamma(\varphi))] = 0.$$

The result of the proposition follows from the next lemma. □

Lemma 29. *If $G \in L^2(\Omega, \mathcal{G}_T, P)$ is such that for all $H_n(B^\gamma(\varphi))$ where $\varphi \in \mathcal{H}_2$ and $n \in \mathbf{N}$, we have $E(GH_n(B^\gamma(\varphi))) = 0$, then G is 0 in $L^2(\Omega, \mathcal{G}_T, P)$.*

Proof. The first half of this proof is essentially borrowed from [13]. Let $m \in \mathbf{N}$. Since the monomial $x \mapsto x^m$ can be written as $x^m = \sum_{k=0}^m a_k H_k(x)$ for some coefficients a_k, we obtain $E[(B^\gamma(\varphi))^m G] = 0$, and since for any $t \in \mathbf{R}$, $\exp(tB^\gamma(\varphi))$ is in $L^2(\Omega, \mathcal{G}_T, P)$, we also have $E[G \exp(tB^\gamma(\varphi))]$. Since $\mathcal{H}_2$ is a vector space, we can translate this as

$$E\left[G \exp\left(\sum_{i=1}^n t_i B^\gamma(\varphi_i)\right)\right] = 0$$

for any $\varphi_i \in \mathcal{H}_2$, $t_i \in \mathbf{R}$, $i = 1, \ldots, n$, $n \in \mathbf{N}$. This means the Laplace transform of the signed measure on the Borel sets of $\mathbf{R}^n$

$$\nu(B) := E[G\mathbf{1}_B((B^\gamma(\varphi_i)_{i=1}^n))]$$

is zero. Therefore, the measure ν is 0. Now let $\mathcal{G}'$ be the sigma field generated by $\{B^\gamma(\varphi) : \varphi \in \mathcal{H}_2\}$, so that $\mathcal{G}' \subset \mathcal{G}_T$. We have proved that for all $A \in \mathcal{G}'$,

$$E[G\mathbf{1}_A] = 0,$$

which means G is zero in $L^2(\Omega, \mathcal{G}', P)$. Now recall from Proposition 9 that the sigma field generated by the random variables $\{B^\gamma(g) : g \in C_b^\infty\}$ actually equals $\mathcal{G}_T$. But by Proposition 16, $\mathcal{H}_2$ contains the space C^ζ, and therefore $\mathcal{H}_2$ contains C_b^∞. Combining these two results proves that $\mathcal{G}' = \mathcal{G}_T$, which finishes the proof of the lemma. □

Remark 30. The following properties now are easily seen.

(1) $Dom\,\delta \subset Dom^*\,\delta$.
(2) If $u \in Dom^*\,\delta$ then $E(u) \in \mathcal{H}$.
(3) If u is deterministic then $u \in Dom^*\,\delta$ iff $u \in \mathcal{H}$ iff $u \in Dom\,\delta$.
(4) δ is a closed operator from $Dom^*\,\delta \subset L^2(\Omega, \mathcal{G}_T, P; \mathcal{H})$ to $L^2(\Omega, \mathcal{G}_T, P)$. In other words, if $u_k \to u$ in $L^2(\Omega, \mathcal{G}_T, P; \mathcal{H})$ and $u_k \in Dom^*\,\delta$ and $\delta(u_k)$ converges in $L^2(\Omega, \mathcal{G}_T, P)$ to some random variable V, then $u \in Dom^*\,\delta$ and $V = \delta(u)$.

Before presenting the Ito formula, we finish this subsection with some remarks and questions on the relation with standard Skorohod integration. Before preprint [6] was circulated, it was commonly thought that Skorohod integration for fBm had a lower limit, and the threshold $H > \frac{1}{4}$ was often quoted as the most irregular level for which a Skorohod integration could be defined. Skorohod integration in [6] is pushed beyond this level by modifying the size of the space of test functions needed to assert integrability. Our results here show that in the range $H \in (\frac{1}{4}; \frac{1}{2})$, the size of the space of test functions in [6] or in this article is significantly smaller than the original test space for Skorohod integration; indeed (compare line (10) and Definition (24)), the latter is based on $\mathcal{H}$ while the former is based on $\mathcal{H}_2$. In view of this, one may ask to what extent our Skorohod integral, or that of [6], generalizes the standard Skorohod integral for $H \in (\frac{1}{4}, \frac{1}{2})$. Proposition 28 implies that the Skorohod integrals actually coincide as members of $L^2(\Omega, \mathcal{G}_T, P)$. The coincidence of the Ito formulas in the standard case and in the case of [6] (Lemma 9 therein) is another aspect of the same phenomenon, signifying that the Ito formula contains much information about the process. It is not surprising, for example, that a study of local time is possible. On the other hand, our Ito formula (Theorem 31 below) has exactly the same form again, even though our process B^{γ_H} is not identical to fBm (see Remark 6: it does not even have the same covariance structure). Thus the Ito formula for deterministic functions of B^γ is not a thorough test for comparing Skorohod integrals and/or processes.

5.3. The Itô formula

Following the arguments of Cheridito and Nualart in [6], in this section we will prove the basic result of stochastic calculus. Our proof does not require the use of fractional derivatives—in fact we had to find a way to do without them, since we do not work in the power scale. Some other aspects of the proof have presumably well-known structures, and are similar to some arguments in [6], such as the proof of the algebraic identities using Hermite polynomials (18)–(20). We have included brief proofs of all such claims, for the sake of readability.

Theorem 31. *Let $f \in C^\infty(\mathbf{R})$ be a function such that for all $n \geqslant 0$ there exist constants C_n and D_n with $D_n < \frac{1}{2}\log\frac{1}{T}$ such that*

$$|f^{(n)}(y) \leqslant C_n e^{D_n y^2}, \quad y \in \mathbf{R}.$$

Then for all $t \leqslant T$ the process $f'(B_s^\gamma)1_{(0,t]}(s) \in Dom^*\,\delta$ and we have

$$\delta(f'(B_s^\gamma)1_{(0,t]}) = f(B_t^\gamma) - f(0) - \int_0^t f''(B_s^\gamma)\gamma(s)\gamma'(s)\,ds.$$

Proof. The process $f'(B^\gamma_s)1_{(0,t]}(s) \in Dom^*\,\delta$ and the formula in the above theorem is true iff for all $F \in \mathcal{S}_{\mathcal{H}_2}$ we have

$$\int_0^T E(f'(B^\gamma_s)1_{(0,t]}(s)K^{*,a}_\gamma K^*_\gamma D_s F)\,ds$$
$$= E\left(\left(f(B^\gamma_t) - f(0) - \int_0^t f''(B^\gamma_s)\gamma(s)\gamma'(s)\,ds\right)F\right). \tag{12}$$

Since $H_n(B^\gamma(\phi))$, $n \geqslant 1$, with H_n being the nth Hermitian polynomial, are dense in $\mathcal{S}_{\mathcal{H}_2}$ it is enough to show (12) for F of this type.

However, $D_s H_n(B^\gamma(\phi)) = H_{n-1}(B^\gamma(\phi))\phi(s)$; hence (12) is equivalent to

$$\int_0^T E(f'(B^\gamma_s)1_{[0,t]}(s)K^{*,a}_\gamma K^*_\gamma H_{n-1}(B^\gamma(\phi))\phi(s)))\,ds$$
$$= E\left[\left(f(B^\gamma_t) - f(0) - \int_0^t f''(B^\gamma_s)\gamma(s)\gamma'(s)\,ds\right)H_n(B^\gamma(\phi))\right]. \tag{13}$$

Since $H_{n-1}(B^\gamma(\phi))$ does not depend on s we can rewrite (13) as

$$\int_0^T E(f'(B^\gamma_s)H_{n-1}(B^\gamma(\phi))(K^{*,a}_\gamma K^*_\gamma H_{n-1}\phi)(s))\,ds$$
$$= E\left[\left(f(B^\gamma_t) - f(0) - \int_0^t f''(B^\gamma_s)\gamma(s)\gamma'(s)\,ds\right)H_n(B^\gamma(\phi))\right]. \tag{14}$$

Let us compute $E(f^{(n)}(B^\gamma_t))$.

The heat kernel $p(\sigma, y) := (2\pi\sigma)^{-1/2}\exp(-\frac{1}{2}\frac{y^2}{\sigma})$, $\sigma > 0$, $y \in \mathbf{R}$, satisfies $\frac{\partial p}{\partial \sigma} = \frac{1}{2}\frac{\partial^2 p}{\partial y^2}$. Then

$$\begin{aligned}
\frac{d}{dt}E(f^{(n)}(B^\gamma_t)) &= \frac{d}{dt}\int_{\mathbf{R}} p(\gamma^2(t), y)f^{(n)}(y)\,dy \\
&= \int_{\mathbf{R}} \frac{\partial p}{\partial \sigma}(\gamma^2(t), y)2\gamma(t)\gamma'(t)f^{(n)}(y)\,dy \\
&= \int_{\mathbf{R}} \frac{\partial^2 p}{\partial y^2}(\gamma^2(t), y)f^{(n)}(y)\gamma(t)\gamma'(t)\,dy \\
&= \int_{\mathbf{R}} \gamma(t)\gamma'(t)p(\gamma^2(t), y)f^{(n+2)}(y)\,dy \\
&= \gamma(t)\gamma'(t)E(f^{(n+2)}(B^\gamma_t)).
\end{aligned} \tag{15}$$

The fourth equality is obtained using the properties of f and integration by parts applied twice. Indeed,

$$\begin{aligned}\int_{\mathbf{R}} \frac{\partial^2 p}{\partial y^2}(\gamma^2(t), y) f^{(n)}(y)\, dy = & \int_{\mathbf{R}} p(\gamma^2(t), y) f^{(n+2)}(y)\, dy \\ & + \frac{\partial p}{\partial y}(\gamma^2(t), y) f^{(n)}(y)\Big|_{-\infty}^{\infty} \\ & - p(\gamma^2(t), y) f^{(n+1)}(y)|_{-\infty}^{\infty}. \end{aligned} \tag{16}$$

But $\frac{\partial p}{\partial y}(\gamma^2(t), y) f^{(n)}(y) \leqslant -\frac{C_n}{2\gamma(t)} e^{y^2(-\frac{1}{\gamma(t)}+D_n)}$. Since $D_n < -\frac{1}{\gamma(t)}$ for all $t \leqslant T$ we conclude that the term $\frac{\partial p}{\partial y}(\gamma^2(t), y) f^{(n)}(y)|_{-\infty}^{\infty} = 0$. Similarly $p(\gamma^2(t), y) f^{(n+1)}(y)|_{-\infty}^{\infty} = 0$. Hence the 4th equality.

Now, we proceed to verify equality (14). For $n = 0$, the left-hand side of equality (14) is 0, and the right-hand side is

$$E((f(B_t^{\gamma}) - f(0) - \int_0^t f''(B_s^{\gamma})\gamma(s)\gamma'(s)\, ds) \cdot 1) = 0$$

by equality (15), so the equality is verified. For $n \geqslant 1$, for all $s \in (0, t]$ we have

$$\begin{aligned}\langle 1_{(0,s]}, \phi\rangle_{\mathcal{H}} & = \langle K_\gamma^* 1_{(0,s]}, K_\gamma^* \phi\rangle_{L^2([0,T])} \\ & = \langle 1_{(0,s]}, K_\gamma^{*,a} K_\gamma^* \phi\rangle_{L^2([0,T])} \\ & = \int_0^s K_\gamma^{*,a} K_\gamma^* \phi(\mu)\, d\mu \end{aligned}$$

and

$$\begin{aligned}\frac{d}{ds}(E[f^{(n)}(B_s^{\gamma})]\langle 1_{(0,s]}, \phi\rangle_{\mathcal{H}}^n) = & \gamma(s)\gamma'(s) E[f^{(n+2)}(B_s^{\gamma})]\langle 1_{(0,s]}, \phi\rangle_{\mathcal{H}}^n \\ & + n E[f^{(n)}(B_s^{\gamma})]\langle 1_{(0,s]}, \phi\rangle^{n-1} K_\gamma^{*,a} K_\gamma^* \phi(s). \end{aligned}$$

Hence,

$$\begin{aligned} & E[f^{(n)}(B_t^{\gamma})]\langle 1_{(0,t]}, \phi\rangle_{\mathcal{H}}^n \\ & \quad = \int_0^t \gamma(s)\gamma'(s) E[f^{(n+2)}(B_s^{\gamma})]\langle 1_{(0,s]}, \phi\rangle_{\mathcal{H}}^n\, ds \\ & \quad + n \int_0^t E[f^{(n)}(B_s^{\gamma})]\langle 1_{(0,s]}, \phi\rangle^{n-1} K_\gamma^{*,a} K_\gamma^* \phi(s)\, ds. \end{aligned} \tag{17}$$

Now, let us show that

$$E[f^{(n)}(B_t^\gamma)]\langle 1_{(0,t]}, \phi\rangle_{\mathcal{H}}^n = n!E[f(B_t^\gamma)H_n(B^\gamma(\phi))] \tag{18}$$

and

$$E[f^{(n)}(B_t^\gamma)]\langle 1_{(0,t]}, \phi\rangle_{\mathcal{H}}^{n-1} = (n-1)!E[f'(B_t^\gamma)H_{n-1}(B^\gamma(\phi))] \tag{19}$$

and also

$$E[f^{(n+2)}(B_t^\gamma)]\langle 1_{(0,t]}, \phi\rangle_{\mathcal{H}}^n = n!E[f''(B_t^\gamma)H_n(B^\gamma(\phi))]. \tag{20}$$

We know $E\langle u, DF\rangle_{\mathcal{H}} = E[\delta(u)F]$. Also, by Theorem 1.1.2 in [13],

$$u = H_{k-1}(B^\gamma(\phi))\phi(t) = \frac{1}{(k-1)!}\int_{[0,T]^{k-1}} \phi(t_1)\cdots\phi(t_{k-1})\phi(t)B^\gamma(dt_1)\cdots B^\gamma(dt_{k-1}))$$

and

$$\begin{aligned}\delta(u) &= \frac{1}{(k-1)!}\int_{[0,T]^k} \phi(t_1)\cdots\phi(t_{k-1})\phi(t)B^\gamma(dt_1)\cdots B^\gamma(dt_{k-1}B^\gamma(dt))) \\ &= \frac{k!}{(k-1)!}H_k(B^\gamma(\phi)) = kH_k(B^\gamma(\phi))\end{aligned}$$

and $u \in Dom(\delta)$. Also observe that

$$\begin{aligned}E(f(B_t^\gamma))\langle 1_{[0,T]}, \phi\rangle_{\mathcal{H}} &= E(f(B_t^\gamma))\langle K_\gamma^* 1_{[0,T]}, K_\gamma^*\phi\rangle_{L^2(),T)} \\ &= \langle E(f(B_t^\gamma))1_{[0,T]}, K_\gamma^{*,a}K_\gamma^*\phi\rangle_{L^2([0,T])} \\ &= \int_0^t E(f(B_t^\gamma))1_{[0,T]}K_\gamma^{*,a}K_\gamma^*\phi = E(\delta(\phi)f(B_t^\gamma)).\end{aligned}$$

We prove the first equality by induction. The other two have a similar proof. For $n = 1$ we have

$$E(f'(B_t^\gamma))\langle 1_{[0,T]}, \phi\rangle_{\mathcal{H}} = E(\delta(\phi)f(B_t^\gamma)) = E[H_1(B^\gamma(\phi))f(B_t^\gamma)).$$

Now assume the equality is true for $n = k$ and prove it is true for $n = k + 1$.

$$(k+1)!E[f(B_t^\gamma)H_{k+1}(B^\gamma(\phi))] = k!E[f(B_t^\gamma)\delta(H_k(B^\gamma(\phi))\phi(t))]$$

$$
\begin{aligned}
&k!\int_0^t E(f(B_s^\gamma))H_k(B^\gamma(\phi))K_\gamma^{*,a}K_\gamma^*\phi(s)\,ds\\
&\quad=\int_0^t E(f^{(k)}(B_s^\gamma))\langle 1_{[0,t]},\phi\rangle_{\mathcal{H}}^k K_\gamma^{*,a}K_\gamma^*\phi(s)\,ds\\
&\quad=\int_0^t E(f^{(k)}(B_s^\gamma))K_\gamma^{*,a}K_\gamma^*\phi(s)\,ds\langle 1_{[0,t]},\phi\rangle_{\mathcal{H}}^k\\
&\quad=E(f^{(k+1)}(B_s^\gamma))\langle 1_{[0,t]},\phi\rangle_{\mathcal{H}}^{k+1},
\end{aligned}
$$

where the last equality is deduce by the induction step.

Using (18)–(20) into (17) we obtain

$$
\begin{aligned}
n!E[f(B_t^\gamma)H_n(B^\gamma(\phi))]=&\int_0^t \gamma(s)\gamma'(s)n!E[f''(B_s^\gamma)]H_n(B^\gamma(\phi))\,ds\\
&+n\int_0^t (n-1)!E[f'(B_t^\gamma)H_{n-1}(B^\gamma(\phi))]K_\gamma^{*,a}K_\gamma^*\phi(s)\,ds,
\end{aligned}
$$

which is equivalent to (14). □

6. Local time

6.1. Introduction

There are two distinct "natural" ways of defining the local time of a Gaussian process. If one attempts to keep the highest possible analogy with the standard Brownian case, one can define λ_t as the occupation measure $\lambda_t(A) = \int_0^t 1_A(B_s^H)\,ds$ and use the same notation abusively to define its density with respect to Lebesgue measure. This was done for example originally in Berman's paper [4]. On the other hand, and more recently, several stochastic analysts working on fractional Brownian motion have chosen to consider a different occupation measure because it yields a connection to stochastic calculus via the Itô-Tanaka formula: see for example [8]; also see the summary on local time for fBm-based processes in [19].

We use herein the same type of definition, since our motivations are of the same nature. Specifically, we let L_t^a be the density at point a of the occupation measure

$$
A \mapsto \int_0^t \mathbf{1}_A(B_s^\gamma)d(\gamma^2)(s) = \int_0^t \mathbf{1}_A(B_s^\gamma)2\gamma(s)\gamma'(s)\,ds. \tag{21}
$$

This is the same definition as for fBm in articles such as [8], since there $\gamma^2(s) = s^{2H}$. In this section, we establish the existence of this occupation density. We will prove a

Tanaka formula and the following result

$$L_t^a = \int_0^t 2\gamma(s)\gamma'(s)\lambda^a(ds).$$

While L_t can be interpreted as the density of an "occupation time measure", it is important to note that the word "time" cannot have the same interpretation as for λ_t; indeed, for L_t, time is heavily weighted at the origin. In a forthcoming publication, we will show that L has a version that is Hölder-continuous in t uniformly in a on any set bounded away from the line $t = 0$, but at $t = 0$, a singularity occurs; we will show that the regularity of L^a at 0 is on the order of that of B^γ, which means that if B^γ is not uniformly continuous, L^a cannot be continuous on any interval containing 0. Because of this difficulty, the existence and the above formula for L^a are non-trivial to prove. We found no easier path than to give first the chaos decomposition for L.

6.2. Chaos decomposition

The main tool for proving L exists is a chaos decomposition calculation. The main arguments we follow can be considered classical, and are found for example in [8]. Let us denote by $p_\varepsilon(x) = (2\pi\varepsilon)^{-1/2}\exp(-x^2/(2\varepsilon))$ the heat kernel. Note that we are using the letter ε for a small parameter and for the kernel $\varepsilon^2 = (\gamma^2)'$. Which meaning is being used should be clear from the context.

Proposition 32. *For each $a \in \mathbf{R}$ and $t \in [0, T]$ the following convergence of random variables*:

$$\lim_{\varepsilon\to 0}\int_0^t p_\varepsilon(B_s - a)\,d\gamma^2(s) = \sum_{n=0}^{\infty}\int_0^t p_{\gamma^2(s)}(a)H_n\left(\frac{a}{\gamma(s)}\right)I_n(\varepsilon(s-.)^{\otimes n})\,d\gamma^2(s)$$

occurs in $L^2(\Omega)$, with H_n the nth Hermite polynomial, and $I_n(f^{\otimes n})$ denotes the iterated Skorohod integral with respect to B^γ of the tensor product of n copies of the deterministic function f.

Proof. Since $p_\varepsilon(B_s - a) \in \mathbf{D}^{\infty,2} = \cap_N \mathbf{D}^{N,2}$ by Theorem 1.1.2 in [13] we have

$$p_\varepsilon(B_s - a) = \sum_{m=0}^{\infty} I_m(f_m)$$

and by the Stroock formula ([13, Exercise 1.2.6]) we have

$$f_m = \frac{1}{m!}E(D^m(p_\varepsilon(B_s - a))),$$

where D^m is the mth iteration of the derivation operator D. But

$$D^n[p_\varepsilon(B_s - a)] = p_\varepsilon^{(n)}(B_s - a)\varepsilon(s - \cdot)^{\otimes_n}$$

and

$$E(p_\varepsilon(B_s - a)) = p_{\gamma^2(s)+\varepsilon}(a).$$

Hence

$$\begin{aligned} E(p_\varepsilon^{(n)}(B_s - a)) &= (-1)^n \frac{\partial^n}{\partial a^n} E(p_\varepsilon(B_s - a)) \\ &= (-1)^n \frac{\partial^n}{\partial a^n} p_{\gamma^2(s)+\varepsilon}(a) \\ &= n!(\gamma^2(s) + \varepsilon)^{-\frac{n}{2}} p_{\gamma^2(s)+\varepsilon}(a) H_n\left(\frac{a}{\sqrt{\gamma^2(s) + \varepsilon}}\right). \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} p_\varepsilon(B_s - a) &= \sum_{m=0}^{\infty} \frac{1}{m!} m!(\gamma^2(s) + \varepsilon)^{-\frac{m}{2}} p_{\gamma^2(s)+\varepsilon}(a) H_m\left(\frac{a}{\sqrt{\gamma^2(s) + \varepsilon}}\right) I_m(\varepsilon(s - \cdot)^{\otimes_n}) \\ &= \sum_{m=0}^{\infty} (\gamma^2(s) + \varepsilon)^{-\frac{m}{2}} p_{\gamma^2(s)+\varepsilon}(a) H_m\left(\frac{a}{\sqrt{\gamma^2(s) + \varepsilon}}\right) I_m(\varepsilon(s - \cdot)^{\otimes_n}) \end{aligned}$$

or

$$\int_0^t p_\varepsilon(B_s - a)\, d\gamma^2(s) = \sum_{m=0}^{\infty} \int_0^t \beta_{m,\varepsilon} I_m(\varepsilon(s - \cdot)^{\otimes_n})\, d\gamma^2(s),$$

where

$$\beta_{m,\varepsilon} = (\gamma^2(s) + \varepsilon)^{-\frac{m}{2}} p_{\gamma^2(s)\varepsilon}(a) H_m\left(\frac{a}{\sqrt{\gamma^2(s) + \varepsilon}}\right).$$

By some algebra manipulation it can be shown that $|H_n(y)e^{-\frac{y^2}{2}}| \leqslant C/(2^{\frac{n}{2}}[\frac{n}{2}]!)$. Therefore, we obtain

$$|\beta_{m,\varepsilon}(s)| \leqslant (\gamma^2(s) + \varepsilon)^{-\frac{m+1}{2}} \frac{C}{2^{\frac{m}{2}}[\frac{m}{2}]!} \leqslant \gamma(s)^{-(m+1)} \frac{C}{2^{\frac{m}{2}}[\frac{m}{2}]!}.$$

Let

$$\alpha_{m,\varepsilon} = E\left[\left(\int_0^t \beta_{m,\varepsilon} I_m(\varepsilon(s-\cdot)^{\otimes_n})\,d\gamma^2(s)\right)^2\right].$$

Next, we proceed to estimate $\alpha_{m,\varepsilon}$.

$$\begin{aligned}\alpha_{m,\varepsilon} &= m!\int_0^t\int_0^t E[\beta_{m,\varepsilon}(u)I_m(\varepsilon(u-\cdot)^{\otimes_n})\beta_{m,\varepsilon}(v)I_m(\varepsilon(v-\cdot)^{\otimes_n})]\,d\gamma^2(u)\,d\gamma^2(v)\\ &= m!\int_0^t\int_0^t E[I_m(\varepsilon(u-\cdot)^{\otimes_n})I_m(\varepsilon(v-\cdot)^{\otimes_n})]\beta_{m,\varepsilon}(u)\beta_{m,\varepsilon}(v)\,d\gamma^2(u)\,d\gamma^2(v)\\ &= m!\int_0^t\int_0^u \left[\frac{1}{2}\left(\gamma^2(u)+\gamma^2(v)-\gamma^2(u-v)\right)\right]^n\Big)\,\beta_{m,\varepsilon}(u)\beta_{m,\varepsilon}(v)\,d\gamma^2(u)\,d\gamma^2(v).\end{aligned}$$

Combining everything we get

$$\begin{aligned}\alpha_{m,\varepsilon} &\leqslant \frac{Cm!}{2^m([\frac{m}{2}]!)^2}\int_0^t\int_0^u\left[\frac{1}{2}(\gamma^2(u)+\gamma^2(v)-\gamma^2(u-v))\right]^m\\ &\quad\times\gamma(u)^{-(m+1)}\gamma(v)^{-(m+1)}\,d\gamma^2(u)d\gamma^2(v)\\ &= \frac{Cm!}{2^m([\frac{m}{2}]!)^2}\int_0^t\int_0^u\left[\frac{1}{2}(\gamma^2(u)+\gamma^2(v)-\gamma^2(u-v))\right]^n\\ &\quad\times\gamma(u)^{-(m)}\gamma(v)^{-m}\gamma'(u)\gamma'(v)\,dv\,du\\ &= \frac{Cm!}{2^m([\frac{m}{2}]!)^2}\int_0^t\int_0^u\left[\frac{\gamma^2(u)+\gamma^2(v)-\gamma^2(u-v)}{2\gamma(u)\gamma(v)}\right]^m\gamma'(u)\gamma'(v)\,dv\,du.\end{aligned}$$

Now, first observe that

$$\frac{(\gamma^2(u)+\gamma^2(v)-\gamma^2(u-v))}{2\gamma(u)\gamma(v)}\leqslant 1.$$

Indeed, since γ is an increasing, concave function it verifies $\gamma(u-v)\geqslant\gamma(u)-\gamma(v)$. Then observe that

$$\gamma^2(u-v)\geqslant\gamma^2(u)+\gamma^2(v)-2\gamma(u)\gamma(v).$$

By Stirling formula $\frac{C_m!}{2^m([\frac{m}{2}]!)^2}$ behaves as $\frac{1}{\sqrt{n}}$ therefore if we notice that for any positive numbers $a < 1$ and $1 < p < 2$ one can show that there is a constant c_p such that

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} a^n \leqslant c_p \frac{1}{(1-a^p)^{\frac{1}{p}}}$$

we conclude that

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \int_0^t \int_0^u \left[\frac{\gamma^2(u)+\gamma^2(v)-\gamma^2(u-v)}{2\gamma(u)\gamma(v)}\right]^n \gamma'(u)\gamma'(v)\,dv\,du$$
$$\leqslant c_p \int_0^t \int_0^u \left[1-\left[\frac{\gamma^2(u)+\gamma^2(v)-\gamma^2(u-v)}{2\gamma(u)\gamma(v)}\right]^p\right]^{-\frac{1}{p}} \gamma'(u)\gamma'(v)\,dv\,du.$$

Because $\gamma^2(s)$ is an increasing, concave function we have $\gamma^2(u-v) \geqslant \gamma^2(u) - \gamma^2(v)$ hence

$$\int_0^t \int_0^u \left[1-\left[\frac{\gamma^2(u)+\gamma^2(v)-\gamma^2(u-v)}{2\gamma(u)\gamma(v)}\right]^p\right]^{-\frac{1}{p}} \gamma'(u)\gamma'(v)\,dv\,du$$
$$\leqslant \int_0^t \int_0^u \left[1-\left[\frac{2\gamma^2(v)}{2\gamma(u)\gamma(v)}\right]^p\right]^{-\frac{1}{p}} \gamma'(u)\gamma'(v)\,dv\,du$$
$$\times \int_0^t \int_0^u \left[1-\left[\frac{\gamma(v)}{\gamma(u)}\right]^p\right]^{-\frac{1}{p}} \gamma'(u)\gamma'(v)\,dv\,du.$$

With the change of variable $z = \frac{\gamma(v)}{\gamma(u)}$ we obtain

$$\int_0^t \int_0^u \left[1-\left[\frac{\gamma(v)}{\gamma(u)}\right]^p\right]^{-\frac{1}{p}} \gamma'(u)\gamma'(v)\,dv\,du = \frac{\gamma^2(t)}{2} \int_0^1 \frac{1}{(1-z^p)^{\frac{1}{p}}}\,dz = C < \infty$$

and the desired convergence holds. □

6.3. Existence

Proposition 33. *The local time L_t^a exists as the density of the measure defined in* (21). *It satisfies the following equality*:

$$L_t^a = \lim_{\varepsilon \to 0} \int_0^t p_\varepsilon(B_s - a)\,d\gamma^2(s). \tag{22}$$

In particular, it has the following Wiener chaos expansion:

$$L_t^a = \sum_{n=0}^{\infty} \int_0^t p_{\gamma^2(s)}(a) H_n \left(\frac{a}{\gamma(s)} \right) I_n(\varepsilon(s-\cdot)^{\otimes n})\, d\gamma^2(s).$$

Proof. From the chaos decomposition proposition and its proof we deduce that $\int_0^t p_\varepsilon(B_s - y)\, d\gamma^2(s)$ converges uniformly in y as ε goes to 0. We can write now, for each continuous function with compact support

$$\begin{aligned}
&\int_{\mathbf{R}} \left(\lim_{\varepsilon\to 0} \int_0^t p_\varepsilon(B_s - y) d\gamma^2(s) \right) g(y)\, dy \\
&= \lim_{\varepsilon\to 0} \int_{\mathbf{R}} \left(\int_0^t p_\varepsilon(B_s - y) d\gamma^2(s) \right) g(y)\, dy \\
&= \lim_{\varepsilon\to 0} \int_0^t \left(\int_{\mathbf{R}} p_\varepsilon(B_s - y) g(y) dy \right) d\gamma^2(s).
\end{aligned}$$

Since g is continuous with compact support it will have a maximum on $[0,t]$ and applying dominated convergence we obtain

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \int_0^t \left(\int_{\mathbf{R}} p_\varepsilon(B_s - y) g(y)\, dy \right) d\gamma^2(s) \\
&= \int_0^t \left(\lim_{\varepsilon\to 0} \int_{\mathbf{R}} p_\varepsilon(B_s - y) g(y)\, dy \right) d\gamma^2(s) \\
&= \int_0^t g(B_s)\, d\gamma^2(s).
\end{aligned}$$

Therefore for each continuous function g with compact support we have

$$\int_{\mathbf{R}} \left(\lim_{\varepsilon\to 0} \int_0^t p_\varepsilon(B_s - y)\, d\gamma^2(s) \right) g(y)\, dy = \int_{\mathbf{R}} g(y) L_t^y\, dy,$$

which implies (22). □

Lemma 34. *Almost surely, for almost all a, we have*

$$\lim_{t\to 0} \varepsilon^2(t) \lambda_t^a = 0. \tag{23}$$

Additionally, for any measurable set A, almost surely we have

$$\lim_{t\to 0} \int_A \varepsilon^2(t) \lambda_t^a\, da = 0. \tag{24}$$

Proof. First, we show $\lim_{t\to 0} L_t^a = 0$ for almost all a. Indeed, for any set A, since L_t^a is an increasing function in t we have

$$\int_A \lim_{t\to 0} L_t^a da = \lim_{t\to 0} \int_A L_t^a\, da = \lim_{t\to 0} \int_0^t 1_A(B_s)\, d\gamma^2(s) \leqslant \lim_{t\to 0} \gamma^2(t) = 0.$$

Now, observe that

$$\lambda_t^a = \lim_{\alpha\to 0} \frac{1}{2\alpha} \int_0^t 1_{[a-\alpha,a+\alpha]}(B_s)\, ds \leqslant \lim_{\alpha\to 0} \int_0^t p_\alpha(B_s - a)\, ds.$$

Then, we have

$$\begin{aligned}
\lim_{t\to 0} \varepsilon^2(t)\lambda_t^a &\leqslant \lim_{t\to 0} \varepsilon^2(t) \lim_{\alpha\to 0} \int_0^t p_\alpha(B_s - a)\, ds \\
&= \lim_{t\to 0} \lim_{\alpha\to 0} \int_0^t \varepsilon^2(t) p_\alpha(B_s - a)\, ds \\
&\leqslant \lim_{t\to 0} \lim_{\alpha\to 0} \int_0^t \varepsilon^2(s) p_\alpha(B_s - a)\, ds \\
&= \lim_{t\to 0} \lim_{\alpha\to 0} \int_0^t p_\alpha(B_s - a)\, d\gamma^2(s) \\
&= \lim_{t\to 0} L_t^a = 0.
\end{aligned}$$

In a similar fashion we can show that $\lim_{t\to 0} \int_A \varepsilon^2(t)\lambda_t^a\, da \leqslant \lim_{t\to 0} \int_A L_t^a\, da = 0.$ □

The following proposition gives the relationship between L_t^a and λ_t^a.

Proposition 35. *The following equality holds almost surely for almost every a*

$$L_t^a = \int_0^t \varepsilon^2(s)\lambda^a(ds). \tag{25}$$

Proof. Let A be a measurable set. We have

$$\begin{aligned}
\int_A L_t^a\, da &= \int_0^t \varepsilon^2(s) 1_A(B_s)\, ds = \left[\varepsilon^2(s) \int_0^s 1_A(B_r)\, dr\right]_0^t \\
&\quad - \int_0^t \frac{d}{ds}(\varepsilon^2(s)) \int_0^s 1_A(B_r)\, dr\, ds \\
&= \left[\varepsilon^2(s) \int_A \lambda_s^y\, dy\right]_0^t - \int_0^t \frac{d}{ds}(\varepsilon^2(s)) \int_A \lambda_s^y\, dy\, ds
\end{aligned}$$

$$=\int_A \varepsilon^2(t)\lambda_t^y\,dy - \lim_{s\to 0}\int_A \varepsilon^2(s)\lambda_s^y\,dy - \int_0^t \frac{d}{ds}(\varepsilon^2(s))\int_A \lambda_s^y\,dy\,ds$$

$$=\int_A \varepsilon^2(t)\lambda_t^y\,dy - \int_A\int_0^t \frac{d}{ds}(\varepsilon^2(s))\lambda_s^y\,ds\,dy \tag{26}$$

$$=\int_{\tilde{A}}\left[\varepsilon^2(t)\lambda_t^y - \int_0^t \frac{d}{ds}(\varepsilon^2(s))\lambda_s^y\,ds\right]dy$$

$$=\int_A\left[[\varepsilon^2(s)\lambda_s^y]_0^t - \int_0^t \frac{d}{ds}(\varepsilon^2(s))\lambda_s^y\,ds\right]dy$$

$$=\int_A\int_0^t \varepsilon^2(s)\,d\lambda^y(s). \tag{27}$$

Here the second and last equalities are obtained by integration by parts using the fact that ε^2 is integrable at 0. Lines (26) and (27) are found by applying (24) and (23), respectively, with $\tilde{A}$ denoting $A\cap\tilde{\Omega}$ where $\tilde{\Omega}^c$ is a set of Lebesgue measure 0 off of which (23) holds. We proved that for any measurable set A we have

$$\int_A L_t^a da = \int_A\int_0^t \varepsilon^2(s)\,d\lambda^y(s).$$

From this we can deduce (25). □

6.4. Tanaka formula

Theorem 36. *Let $x\leqslant T$ and $y\in\mathbf{R}$. Then $1_{(y,\infty)}B^\gamma(\cdot)1_{(0,x]}(\cdot)\in Dom^*\,\delta$ and*

$$\int_0^x 1_{(y,\infty)}(B^\gamma(s))\,dB^\gamma(s) := \delta(1_{(y,\infty)}(B^\gamma)1_{(0,x]}(\cdot)) = (B_x^\gamma - y)_+ - \frac{1}{2}L_x^y.$$

Proof. For $\varepsilon>0$, denote by $p_\varepsilon(x)=(2\pi\varepsilon)^{-1/2}\exp(-x^2/2\varepsilon)$ and by

$$f_\varepsilon(\alpha) = \int_{-\infty}^\alpha\int_{-\infty}^v p_\varepsilon(z-y)\,dz\,dv,\quad \alpha\in\mathbf{R}.$$

Observe now that $f_\varepsilon(\alpha)\to(\alpha-y)^+$ and $f_\varepsilon'(\alpha)=\int_{-\infty}^\alpha p_\varepsilon(z-y)\,dz\to\frac{1}{2}1_{\{0\}}(\alpha)+1_{(y,\infty)}(\alpha)$. Hence $f_\varepsilon(B_x^\gamma)\to(B_x^\gamma-y)_+$ in $L^2(\Omega)$ and $f_\varepsilon'(B_t^\gamma)1_{(0,x]}(t)\to 1_{(y,\infty)}(B_t^\gamma)1_{(0,x]}(t)$ in $L^2(\Omega\times\mathbf{R})$.

Moreover, since the functions f_ε satisfy the conditions of Ito formula we deduce that $f_\varepsilon'(B_t^\gamma)1_{(0,x]}(t)\in Dom^*\,\delta$ and

$$\delta[f_\varepsilon'(B^\gamma)1_{(0,x]}(\cdot)] = f_\varepsilon(B_x^\gamma) - f_\varepsilon(B_0^\gamma) - \int_0^x f_\varepsilon''(B_s^\gamma)\gamma(s)\gamma'(s)\,ds. \tag{28}$$

Therefore if we show that

$$\lim_{\varepsilon\to 0} \delta[f'_\varepsilon(B^\gamma_\cdot)1_{(0,x]}(\cdot)] = \delta(1_{(y,\infty)}(B^\gamma_\cdot)1_{(0,x]}(\cdot)) \tag{29}$$

and that

$$\lim_{\varepsilon\to 0} \int_0^x f_\varepsilon''(B_s^\gamma)2\gamma(s)\gamma'(s)\,ds = L_x^y \tag{30}$$

the theorem will be proved.

Convergence (29) follows from the fact that δ is a closed operator on $Dom^*\,\delta$, i.e. if $u_n,\ u \in Dom^*\,\delta \cap L^2(\Omega, L^2(\mathbf{R}_+))$ are such that $\lim_{n\to\infty} u_n = u$ in $L^2(\Omega, L^2(\mathbf{R}_+))$ and if there is $U \in L^2(\Omega)$ such that $\lim_{n\to\infty} \delta(u_n) = U$ in $L^2(\Omega)$ then $u \in Dom^*\,\delta$ and $\delta(u) = U$. In our case $u_\varepsilon = f'_\varepsilon(B^\gamma_\cdot)1_{(0,x]}(\cdot)$ and using Cauchy convergence in (28) we obtain the convergence of $\delta(u_\varepsilon)$ hence (29). Convergence (30) follows from (22). □

7. Finite and infinite-dimensional stochastic differential equations

A well-known difficulty with Skorohod stochastic integration w.r.t. fBm is that solving even the simplest non-linear differential equation is yet an open problem. There are two notable exceptions, however: the linear additive and the linear multiplicative equations, yielding the so-called fractional Ornstein–Uhlenbeck and Geometric fractional Brownian motion processes, respectively. In this section, we show that this can be done for integration with respect to our processes B^γ. We keep our formulations to a minimal level of complexity. Additional non-linear terms in the drift parts can be considered using variants of the arguments given in some of the references cited in the introduction; we will not investigate these details here.

7.1. Finite-dimensional equations

Proposition 37. *Let γ and B^γ be fixed as in Proposition* 1. *Consider the stochastic differential equation*

$$X(t) = X_0 + \int_0^t bX(s)\,ds + \int_0^t \sigma X(s)\delta B^\gamma(s), \quad t \geqslant 0, \tag{31}$$

where X_0, b, σ are fixed non-random constants, and where $\int_0^t \sigma X(s)\delta B^\gamma(s)$ represents the Skorohod integral $\delta(\mathbf{1}_{[0,t]}(\cdot)\sigma X)$ as in Definition 27.

This linear multiplicative stochastic differential equation (31) *has a solution given by the following geometric* γ*-Brownian motion* $(G_\gamma Bm)$:

$$X(t) = X_0 \exp\left(\sigma B^\gamma(t) + bt - \frac{1}{2}\,\sigma^2\gamma^2(t)\right). \tag{32}$$

This solution is unique in the class of processes Z such that $Z(t) = g(t, B^\gamma(t))$ *where g is a deterministic function in* $C^{1,2}$ *satisfying the conditions of Theorem* 31 *uniformly in t.*

Proof. Ito's formula (Theorem 31) can be extended to include functions that depend also on time. This can be proven by approximation of such functions with respect to the time parameter. We omit the details. We thus have for any function f of class $C^{1,2}$ on $\mathbf{R}_+ \times \mathbf{R}$ satisfying the hypotheses of Theorem 31 with respect to the second parameter uniformly in the first parameter, that $\frac{\partial f}{\partial x}(\cdot, B^\gamma(\cdot))\mathbf{1}_{[0,t]}$ is in $Dom^*\,\delta$ and for all $t \geqslant 0$

$$\begin{aligned} f(t, B^\gamma(t)) &= f(0,0) + \int_0^t \frac{\partial f}{\partial s}\,(s, B^\gamma(s))\,ds \\ &\quad + \int_0^t \frac{\partial f}{\partial x}\,(s, B^\gamma(s))\delta B^\gamma(s) + \int_0^t \frac{\partial^2 f}{\partial x^2}\,(s, B^\gamma(s))\gamma\gamma'(s)\,ds. \end{aligned} \tag{33}$$

We apply this with $f(t,x) = X_0 \exp(\sigma x + bt - (1/2)\sigma^2\gamma^2(t))$ which immediately yields Eq. (31).

For the uniqueness, let Y be another solution to (31). Since $Y(t) = g(t, B^\gamma(t))$ for some g, we can use Ito's formula to show that for any function h such that $h \circ g$ is of class $C^{1,2}$ and satisfies the conditions of Theorem 31 uniformly in t, the following version of Ito holds for Y:

$$h(Y(t)) = h(Y(0)) + \int_0^t h'(Y(s))\delta Y(s) + \frac{1}{2}\int_0^t h''(Y(s))d[Y,Y]_\gamma(s),$$

where the notations $\delta Y(s)$ and $d[Y,Y]_\gamma(s)$ are defined as follows:

$$\begin{aligned} \delta Y(s) &:= \frac{\partial g}{\partial x}\,(s, B^\gamma(s))\delta B^\gamma(s) + \frac{\partial^2 g}{\partial x^2}\,(s, B^\gamma(s))\gamma\gamma'(s)\,ds, \\ d[Y,Y]_\gamma(s) &:= \left|\frac{\partial g}{\partial x}\,(s, B^\gamma(s))\right|^2 2\gamma\gamma'(s)\,ds. \end{aligned}$$

With this Ito formula in hand, we can now consider the processes $U = \log X$ and $V = \log Y$. A trivial calculation then yields that both U and V are solutions of the following equation in Z:

$$Z(t) = \log X_0 + \sigma B^\gamma(t) + bt - \sigma^2 \int_0^t \gamma\gamma'(s)\, ds.$$

Obviously, this is a trivial equation since the right-hand side is explicit, which proves uniqueness. □

Remark 38. We are not able to find a simple proof of uniqueness for the above equation in a wider space, because of the restrictive range of our Ito formula, valid only for deterministic functions of B^γ. Extending the validity of the Ito formula will be the subject of another article.

Proposition 39. *Let γ and B^γ be fixed as in Proposition* 1. *Consider the stochastic differential equation*

$$X(t) = X_0 + \int_0^t aX(s)ds + B^\gamma(t), \quad t \geqslant 0, \tag{34}$$

where X_0 and a are fixed non-random constants.

This linear additive stochastic differential equation (31) *has a solution given by the following γ-Ornstein–Uhlenbeck process (γOU):*

$$X(t) = X_0 e^{at} + \int_0^t e^{a(t-s)} \delta B^\gamma(s), \tag{35}$$

where the last integral is in the Wiener sense (Section 3). *The solution is unique, up to indistinguishability, in the class of all separable processes in $L^2(\Omega)$.*

Proof. Although this proposition can be considered as a consequence of the results presented in the next section, we include a quick proof for completeness. First note that the Wiener integral in (35) is well-defined since the smooth function $\exp(-as)$ is obviously in $\mathcal{H}$ which contains any C^α, $\alpha > \frac{1}{2}$. Assume that X exists satisfying (34). Then we define Y by $Y(t) = \exp(-at)X(t)$. Then Y is the sum of a differentiable process $ae^{-at} \int_0^t X(s)\, ds$ and of $e^{-at} B^\gamma(t)$ which is of the form $g(t, B^\gamma(t))$ where g is deterministic. By Ito's formula (33) and Eq. (34) we see that

$$\begin{aligned} Y(t) =& X_0 - \int_0^t ae^{-as} a \left[\int_0^s X(r)\, dr \right] ds - a \int_0^t e^{-as} B^\gamma(s)\, ds \\ &+ \int_0^t e^{-as} (aX(s)\, ds + \delta B^\gamma(s)). \end{aligned}$$

Now from (34) again, we may replace $a[\int_0^s X(r)\,dr]$ by $X(s) - B^{\gamma}(s)$, yielding simply

$$Y(t) = X_0 + \int_0^t e^{-as}\delta B^{\gamma}(s),$$

which proves uniqueness. This same calculation also shows that X given by (35) solves (34). □

7.2. Stochastic heat equations

An equation is commonly called the stochastic heat equation on $\mathbf{R}$ if it is of the form

$$u(t,x) = u_0(x) + \int_0^t \Delta_x u(s,x)\,ds + \int_0^t \sigma(u(s,x))\,dW(s) \tag{36}$$

for some Gaussian noise term W and some possibly non-linear function σ. As announced above, because of the difficulties inherent in Skorohod integration, we restrict ourselves to $\sigma = Id$ or $\sigma = 1$. For the case $\sigma = Id$, we present our results as conjectures.

The additive stochastic heat equation

$$u(t,x) = u_0(x) + \int_0^t \Delta_x u(s,x)\,ds + B^{\gamma}(t,x) \tag{37}$$

can be interpreted in its *evolution* form, as is often done, in the manner of Da Prato and Zabczyk [9], as

$$u(t,x) = P_t u_0(x) + \int_0^t P_{t-s} B^{\gamma}(\delta_s,\cdot)(x), \tag{38}$$

where $B^{\gamma}(t,\cdot)$ is an infinite-dimensional version of our $B^{\gamma}(t)$. Obviously here, since the right-hand side of the equation does not contain u, this u given by (38) *is* the unique (evolution) solution to (37), when it exists. It is well-known however that u may exist even if a strong-sense solution of (37) fails to exist, hence the use of the terminology *evolution solution*.

To be specific, let us assume B^{γ} is a centered Gaussian random field on $\mathbf{R}_+ \times S^1$ where S^1 is the circle (parameterized by $[0, 2\pi)$) with a given covariance structure Q in space and the same behavior as our one-dimensional B^{γ} defined in Proposition 1. In other words it can be written as

$$B^{\gamma}(t,x) = \int_0^t \varepsilon(t-s)W(ds,x),$$

where W has covariance $E[W(t,x)W(s,y)] = Q(x,y)\min(s,t)$. The operator P is the semigroup generated by Δ. In other words, for any test function f in $L^2(\mathbf{R})$

$$P_t f(x) = \int_{\mathbf{R}} (2\pi t)^{-1/2} \exp\left(-(x-y)^2/(2t)\right) f(y)\, dy.$$

The notation $\int_0^t P_{t-s} B^\gamma(\delta_s, \cdot)(x)$ is best understood if W (and consequently B^γ) can be expanded in a basis of a convenient space of functions. We use the trigonometric basis for $L^2(\mathbf{R})$, which are also the set of eigenfunctions of the Laplacian Δ. To make matters as simple as possible, we assume that $u_0 = 0$ and that W is spatially homogeneous. In this case, we know W can be expanded along the trigonometric basis, with identical coefficients for like sine and cosine terms. Consequently, we have

$$B^\gamma(t,x) = \sqrt{q_0} B_0^\gamma(t) + \sum_{n=1}^\infty \sqrt{q_n} \bar{B}_n^\gamma(t) \sin(nx) + \sum_{n=1}^\infty \sqrt{q_n} B_n^\gamma(t) \cos(nx),$$

where $(B_n^\gamma)_n$ and $(\bar{B}_n^\gamma)_n$ are independent families of independent copies of the B^γ in Proposition 1, and $(q_n)_n$ is a sequence of non-negative numbers. Since $\sin nx$ and $\cos nx$ share the eigenvalue $-n^2$ with respect to Δ, they have the eigenvalue $\exp(-n^2 t)$ for P_t. Consequently, we can rewrite (38) as

$$\begin{aligned} u(t,x) = & \sqrt{q_0} \int_0^t e^{-(t-s)n^2} B_0^\gamma(\delta_s) \\ & + \sum_{n=1}^\infty \sqrt{q_n} \cos(nx) \int_0^t B_n^\gamma(\delta s) e^{-(t-s)n^2} \\ & + \sum_{n=1}^\infty \sqrt{q_n} \sin(nx) \int_0^t \bar{B}_n^\gamma(\delta s) e^{-(t-s)n^2}. \end{aligned}$$

This shows, in particular, that for fixed t, $u(t,\cdot)$ is a homogeneous Gaussian process. It is worth noting that the above solution u may exist as a bona-fide Gaussian process even if $B^\gamma(t,\cdot)$ is not a bona-fide process in the space variable. The random element $B^\gamma(t,\cdot)$ may be generalized-function-valued (Schwartz-distribution-valued). For a precise description of such an object, the reader is referred to Section 3, and in particular, Section 3.1 in [17]. However, it is enough to notice that if $\sum q_n = \infty$ then $B^\gamma(t,\cdot)$ is a random generalized function. The next theorem gives a precise result in this direction.

Theorem 40. *Let γ and ε be as in Proposition* 1. *Assume moreover that ε' satisfies*

$$-\varepsilon'(r) = |\varepsilon'(r)| \asymp r^{-3/2} f(r),$$

where f is an increasing differentiable function. Also define

$$F(x) = \int_0^x \left(\frac{1}{s} \int_0^s \varepsilon(r)\, dr \right)^2 ds.$$

Assume also that f satisfies the following technical assumptions:

1. *f'/f is bounded by λ on the interval* $[1/(4\lambda), 1]$;
2. *for some $a > 0$, for all $r \leqslant a$, $f'(r) \leqslant f(r)/(2r)$*;
3. *with $g(x) = (xf'(x)f(x))^{1/2}$, $g'(r) \leqslant g(r)/(2r)$.*

Then the evolution solution $u(t,x)$ to Eq. (37) *exists and is unique as a random field in $L^2(\Omega \times [0,t] \times S^1)$ as soon as*

$$\sum_{n=1}^{\infty} q_n (F(n^{-2}) + f^2(n^{-2})) < \infty. \tag{39}$$

The second statement in the following corollary shows that the above theorem is sharp, since it reproduces the sufficient condition of [17] which was also shown therein to be necessary in the case of fBm itself. It also shows that in the two basic scales of regularity, the functions F and f^2 are commensurate.

Corollary 41. *The functions f and F can be estimated in the cases of logBm and fBm scales. Specifically, we have that in the following two cases, the technical conditions on γ and ε all hold, and the theorem translates as follows*:

- logarithmic Brownian scale: *If $\varepsilon'(r) \asymp r^{-3/2} \log^{-(\beta+1)/2}(r)$ for some $\beta > 0$, so that $\gamma(r) \asymp \log^{-\beta/2}(r)$, then* (39) *can be replaced by*

$$\sum_{n=1}^{\infty} q_n \log^{-(\beta+1)}(n) < \infty.$$

- fractional Brownian scale: *If $\varepsilon'(r) \asymp r^{H-3/2}$ for $H \in (0, \frac{1}{2}]$, so that $\gamma(r) \asymp r^H$, then* (39) *can be replaced by*

$$\sum_{n=1}^{\infty} q_n n^{-4H} < \infty.$$

Proof of Theorem 40. The proof's structure is identical to the general theorem relative to infinite-dimensional fBm in [17], proved in Section 3.3 therein. We give only the main difference in the calculation. It is regarding, in the notation of [17], the bounding of the term $I_2(\lambda, t)$. In our context, one can check that the only relevant values of λ are $\lambda = n^2$, $n \in \mathbf{N}$, and that we have

$$I_2(\lambda, t) = \int_0^t e^{-2\lambda s} \left(\int_0^s (e^{\lambda r} - 1)\varepsilon'(r)\, dr \right)^2 ds.$$

We will assume in this proof that $t \leqslant 1$, and we will indeed replace t by this value for all upper bounds below. More generally, to be able to consider arbitrary bounded intervals $[0, T]$, cases such as logBm must be modified in consequence to ensure that γ is defined and bounded on such intervals, e.g. in the case of logBm by replacing $\gamma(r)$ by $\gamma(r/2T)$ say. The results in the theorem hold for the existence of u for all $t \in [0, \infty)$ as long as one begins with a locally bounded γ. We omit the details.

We rewrite

$$\begin{aligned} I_2(\lambda, t) \leqslant I_2(\lambda, 1) = & \int_0^{1/\lambda} e^{-2\lambda s} \left(\int_0^s (e^{\lambda r} - 1)\varepsilon'(r)\, dr \right)^2 ds \\ & + \int_{1/\lambda}^1 e^{-2\lambda s} \left(\int_0^{1/\lambda} (e^{\lambda r} - 1)\varepsilon'(r)\, dr \right)^2 ds \\ & + \int_{1/\lambda}^1 e^{-2\lambda s} \left(\int_{1/\lambda s} (e^{\lambda r} - 1)\varepsilon'(r)\, dr \right)^2 ds \\ := & I_{2,0}(\lambda) + I_{2,1}(\lambda) + I_{2,2}(\lambda). \end{aligned}$$

The first term $I_{2,0}(\lambda)$ is controlled as follows. Up to universal constants, we bound $e^{\lambda r} - 1$ above by λr, and $e^{-2\lambda s}$ by 1, yielding

$$I_{2,0}(\lambda) \leqslant \int_0^{1/\lambda} \lambda^2 \left(\int_0^s r|\varepsilon'(r)|\, dr \right)^2 ds.$$

By integration by parts we get that

$$\int_0^s r|\varepsilon'(r)| dr = \int_0^s \varepsilon(r)\, dr - s\varepsilon(s) + \lim_{h \to 0} h\varepsilon(h).$$

The limit above is 0 since $\varepsilon(r) \ll r^{-1/2}$. Since ε is decreasing, $\int_0^s \varepsilon(r)\, dr$ exceeds $s\varepsilon(s)$, and thus we decide to ignore the smaller of the two, yielding an upper bound.

It follows that:

$$I_{2,0}(\lambda) \leqslant \int_0^{1/\lambda} \lambda^2 \left(\int_0^s \varepsilon(r)\, dr \right)^2 ds$$
$$\leqslant \int_0^{1/\lambda} \left(\frac{1}{s} \int_0^s \varepsilon(r)\, dr \right)^2 ds,$$

which is the required estimate.

For the second term, using the integration by parts calculation above,

$$I_{2,1}(\lambda) \leqslant \int_{1/\lambda}^1 e^{-2s\lambda} \left(\int_0^{1/\lambda} \lambda r |\varepsilon'(r)|\, dr \right)^2 ds$$
$$\leqslant \int_{1/\lambda}^1 e^{-2s\lambda} \left(\lambda \int_0^{1/\lambda} \varepsilon(r)\, dr \right)^2 ds$$
$$= \frac{1}{2\lambda} (e^{-2} - e^{-2\lambda}) \left(\lambda \int_0^{1/\lambda} \varepsilon(r)\, dr \right)^2$$
$$\leqslant \int_0^{1/\lambda} \left(\frac{1}{s} \int_0^s \varepsilon(r)\, dr \right)^2 ds,$$

where the last inequality comes from the fact that the function $h(s) = s^{-1} \int_0^s \varepsilon(r)\, dr$ is decreasing on $[0, \frac{1}{\lambda}]$. This fact can be seen as follows: $h'(s) = s^{-2}(s\varepsilon(s) - \int_0^s \varepsilon(r)\, dr) < 0$ since ε itself is decreasing.

The last term can be rewritten using a scalar change of variables, and then integration by parts, as follows:

$$I_{2,2}(\lambda) \leqslant \lambda^{-3} \int_1^{\lambda} e^{-2s} \left(\int_1^s e^r \left| \varepsilon'\left(\frac{r}{\lambda}\right) \right| dr \right)^2 ds.$$

Now, we use the representation $|\varepsilon'(r)| \asymp r^{-3/2} f(r)$ with f differentiable and increasing, and $|\varepsilon'|$ decreasing:

$$I_{2,2}(\lambda) \leqslant \int_1^{\lambda} e^{-2s} \left(\int_1^s e^r r^{-3/2} f\left(\frac{r}{\lambda}\right) dr \right)^2 ds.$$

We decompose the inside integral into three parts, and exploit the monotonicity of the integrands in each corresponding interval:

$$\int_1^s e^r r^{-3/2} f\left(\frac{r}{\lambda}\right) dr$$

$$= \int_1^{s/4} e^r r^{-3/2} f\left(\frac{r}{\lambda}\right) dr + \int_{s/4}^{s/2} e^r r^{-3/2} f\left(\frac{r}{\lambda}\right) dr + \int_{s/2}^{s} e^r r^{-3/2} f\left(\frac{r}{\lambda}\right) dr$$

$$\leqslant e^{s/2} f\left(\frac{s}{4\lambda}\right) + e^{s/2}\left(\frac{s}{4}\right)^{-3/2} f\left(\frac{s}{2\lambda}\right) + e^2\left(\frac{s}{2}\right)^{-3/2} f\left(\frac{s}{2\lambda}\right)$$

$$= I_{2,2,0}(s,\lambda) + I_{2,2,1}(s,\lambda) + I_{2,2,2}(s,\lambda).$$

To deal with $I_{2,2,0}(s,\lambda)$, we first integrating by parts, using the condition that f'/f is bounded by λ on the interval $[1/(4\lambda), 1]$:

$$\begin{aligned} I_{2,2,0}(\lambda) &:= \int_1^{\lambda} I_{2,2,0}(s,\lambda)^2 e^{-2s}\, ds = \int_1^{\lambda} e^{-s} f^2\left(\frac{s}{4\lambda}\right) ds \\ &= e^{-1} f^2\left(\frac{1}{4\lambda}\right) - e^{-\lambda} f^2(1/4) + \frac{1}{4\lambda}\int_1^{\lambda} e^{-s} 2 f\left(\frac{s}{4\lambda}\right) f'\left(\frac{s}{4\lambda}\right) ds \\ &\leqslant e^{-1} f^2\left(\frac{1}{4\lambda}\right) + \frac{1}{2}\int_1^{\lambda} e^{-2} f^2\left(\frac{s}{4\lambda}\right) ds \\ &= e^{-1} f^2\left(\frac{1}{4\lambda}\right) + \frac{1}{2} I_{2,2,0}(\lambda). \end{aligned}$$

This implies that

$$I_{2,2,0}(\lambda) \leqslant f^2\left(\frac{1}{4\lambda}\right).$$

For the next term we can see that can be dealt with exactly as $I_{2,2,0}(\lambda)$; in fact, it is of smaller order because of the factor s^{-3}.

The last term is more delicate. We begin by noting that

$$\begin{aligned} I_{2,2,2}(\lambda) &:= \int_1^{\lambda} I_{2,2,0}(s,\lambda)^2 e^{-2s}\, ds = 8\int_1^{\lambda} s^{-3} f^2\left(\frac{s}{2\lambda}\right) ds \\ &= 8\lambda^{-2}\int_{1/\lambda}^{1} r^{-3} f^2(r/2)\, dr. \end{aligned}$$

To bound this quantity, we introduce a modified version of it: for fixed $a > 0$, we let

$$I_g^a(x) := x^2 \int_x^1 r^{-3} g^2(r)\, dr,$$

where the function g will be chosen to be equal to $g(x) = (x f'(x) f(x))^{1/2}$. With this choice we do see that according to the hypotheses of the theorem, for some $a > 0$, assuming $\lambda > a^{-1}$, for all $r \leqslant a$, $g'(r) \leqslant g(r)/(2r)$. We calculate I_g^a by integration by parts, using the parts $v = r^{-2}$ and $du = r^{-1}g^2(r)$ so that $u(r) = \int_0^r y^{-1}g^2(y)$ and $dv = -2r^{-3}\,dr$:

$$I_g^a(x) = x^2u(1) - u(x) + x^2 \int_x^1 r^{-3}2u(r)\,dr. \tag{40}$$

Now let $J_g^a(x) := x^2 \int_x^1 r^{-3}2u(r)\,dr$. By hypothesis, $2g(r)g'(r) \leqslant g^2(r)/r$ for all $r \in [0, a]$, which implies for all such r that

$$g^2(r) - g(0) = \int_0^r 2g(y)g'(y)\,dy \leqslant u(r).$$

We have that $g(0) = \lim_0 g = 0$. Indeed, by hypothesis, $g^2(x) = 2xf'(x)f(x) \leqslant f^2(x)$ which tends to 0 at 0. This implies

$$I_g^a(x) \leqslant x^2 \int_x^1 r^{-3}u(r)\,dr = \frac{1}{2} J_g^a(x). \tag{41}$$

Combining (40) and (41) we obtain

$$\begin{aligned} J_g^a(x) &= I_g^a(x) - x^2u(1) + u(x) \\ &\leqslant \frac{1}{2} J_g^a(x) - x^2u(1) + u(x), \end{aligned}$$

which implies

$$J_g^a(x) \leqslant 2u(x). \tag{42}$$

Returning now to the definition of J_g^a and u we have

$$\begin{aligned} u(x) &= \int_0^x r^{-1}g^2(r)\,dr \\ &= \int_0^x r^{-1}rf'(r)f(r)\,dr \\ &= \frac{1}{2} f^2(x) \end{aligned} \tag{43}$$

and

$$J_g^a(x) = x^2 \int_x^1 r^{-3}f^2(r)\,dr.$$

With $x = \lambda^{-1}$, we recognize a piece of the integral defining $I_{2,2,2}(\lambda)$. In fact we have by (42) and (43) that

$$\begin{aligned} I_{2,2,2}(\lambda) &= 8\lambda^{-2} \int_{1/\lambda}^{1} r^{-3} f^2(r/2)\, dr \\ &\leqslant 8\lambda^{-2} \int_{1/\lambda}^{1} r^{-3} f^2(r)\, dr \\ &= 8 J_g^a(\lambda^{-1}) + 8\lambda^{-2} \int_a^1 r^{-3} f^2(r)\, dr \\ &\leqslant 4 f^2\left(\frac{1}{\lambda}\right) + \frac{1}{\lambda^2} K_f, \end{aligned}$$

where K_f is a constant depending only on f. The second term above is negligible compared to the first, since we know that $f(r) \geqslant r^{-1/2}$.

In conclusion, we have for large λ and for all $t \leqslant 1$, with $F(x) = \int_0^x (\frac{1}{s} \int_0^s \varepsilon(r)\, dr)^2\, ds$,

$$I_2(\lambda, t) \leqslant F\left(\frac{1}{\lambda}\right) + 48 f^2\left(\frac{1}{\lambda}\right),$$

which is the result required to obtain the first statement of the theorem. □

Proof of the Corollary 41. The statements in the corollary regarding the fBm and logBm scales are readily verified by trivial estimations of f and F in these cases. □

We finish this article by mentioning a conjecture on the multiplicative stochastic heat equation. This is the case $\sigma(u) = u$.

Conjecture 42. *The evolution form of Eq.* (36) *with* $\sigma(u) = u$, *namely*

$$u(t, x) = P_t u_0(x) + \int_0^t P_{t-s}[B^\gamma(\delta_s, \cdot) u(s, \cdot)](x)$$

has a unique solution in $L^2(\Omega \times [0, t] \times S^1)$ *as soon as* $\sum q_n < \infty$, *and it is given by the following Feynman–Kac formula*:

$$u(t, x) = E^b \left[u_0(x + b_t) \exp\left(\int_0^t B^\gamma(\delta r, x + b_t - b_r) - Q(0)\gamma^2(t)/2 \right) \right],$$

where b *is a standard Brownian motion independent of* B^γ, *and where* E^b *is the expectation with respect to* b.

A joint paper in preparation by one of the two authors of this paper establishes this Feynman–Kac formula for fBm in the case of $H > \frac{1}{2}$. It uses a Wiener chaos decomposition and some associated estimates. We do not believe that these estimates are yet available for $H < \frac{1}{2}$, making proving the above conjecture non-trivial, although the result is readily believable.

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