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Sharp Gaussian regularity on the circle, and applications to the fractional stochastic heat equation

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Abstract

A sharp regularity theory is established for homogeneous Gaussian fields on the unit circle. Two types of characterizations for such a field to have a given almost-sure uniform modulus of continuity are established in a general setting. The first characterization relates the modulus to the field's canonical metric; the full force of Fernique's zero-one laws and Talagrand's theory of majorizing measures is required. The second characterization ties the modulus to the field's random Fourier series representation. As an application, it is shown that the fractional stochastic heat equation has, up to a non-random constant, a given spatial modulus of continuity if and only if the same property holds for a fractional antiderivative of the equation's additive noise; a random Fourier series characterization is also given.
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1. Introduction

This article has two purposes: to present a sharp *regularity theory* for Gaussian fields on the unit circle S^1, and to apply this theory to formulate a spatial regularity theory for the fractional stochastic heat equation (SHE). In this introduction, we describe the results that we aim at, and the road that we take to achieve them.

We have in mind the equation

$$\frac{\partial X}{\partial t}(t,x) = \Delta_x X(t,x) + \frac{\partial B^H}{\partial t}(t,x): \quad x \in S^1; \quad t \in [0,1]; \quad u(0,x) = 0, \tag{1}$$

where Δ is the standard Laplacian (Laplace–Beltrami) operator on S^1, and where B^H is a Gaussian field on $[0,1] \times S^1$ whose behavior in time is that of fractional Brownian motion (fBm) with any parameter $H \in (0,1)$, and whose behavior in space is homogeneous, and can be completely arbitrary within that restriction. In the sequel, we will omit the superscript H and simply call the fBm B.

By "regularity theory" for a Gaussian field Y we mean a *characterization* of almost-sure modulus of continuity for Y that can be written using information about Y's covariance. We seek necessary *and* sufficient conditions whenever possible, hence the use of the word "characterization". By "spatial regularity theory" for the stochastic heat equation, we mean a characterization of the almost-sure modulus of continuity for the equation's solution in its space parameter $x \in S^1$, that can be written using information about the spatial covariance of the equation's data (additive Gaussian fractional noise $\partial B/\partial t$), or that can be formulated in exact relation to the data's almost sure modulus of continuity in x.

Let us be more specific about the distinction between the various characterizations. Let $Y(x) := (I - \Delta_x)^{-H} B(1,x)$. This defined a homogeneous Gaussian field on S^1. We can also abusively use the notation Y for the random field $Y := (I - \Delta)^{-H} B$ on $[0,1] \times S^1$, which can be called the "$2H$-fractional spatial antiderivative" of B. It is well understood (for the Brownian case, see [15], or more recently [10,13,14]) that in our one-dimensional situation, B does not need to be a bonafide function in x for the SHE (1) to have a solution. In fact only Y needs to be a bonafide function; in [12] it is shown that this is a necessary and sufficient condition even in the fractional Brownian case.

Once a condition for existence is given, it is natural to seek conditions for regularity. We consider two types of conditions for guaranteeing/characterizing the fact that the solution X of the SHE (1) admits a given fixed function f as an almost-sure uniform modulus of continuity:

- Type I (an *intrinsic* or *pathwise* condition): the fact that the same almost-sure continuity holds for $Y := (I - \Delta)^{-H} B$;
- Type II (a condition *on the distribution*): a condition that can be written using the covariance function of Y.

From the applied physical point of view, the necessary Type I condition may be quite useful. Indeed, the solution of a stochastic PDE can be a model for a turbulent physical observation, such as the velocity of a fluid flow; the necessary Type I condition asserts that the level of regularity (turbulence) that is observed is precisely related to the regularity of the equation's forcing term; but in many physical situations, this forcing term is unobserved; our result allows to draw almost-sure conclusions about the turbulence—and even the distribution—of the unobserved forcing term from one sample observation, without any statistical (Type II) analysis. This is of course an ideal situation, which has a lot to do with the Gaussian framework we have chosen.

This article goes fairly far in proving both Type I and Type II necessary and sufficient conditions for X's almost sure spatial continuity. This is achieved in Section 3. The results are entirely new, even in the case of standard white noise ($H = 1/2$). Among the small number of recent works on stochastic PDEs driven by fBm [4,6,8,9], those which study regularity questions only look at $H > 1/2$, and only consider the time parameter, with the exception of [6] where standard techniques are used to study spatial regularity. Our study of spatial regularity uses Gaussian tools instead. These tools are inspired by work done by two of the three authors of this article in [13,14]. However, these articles had led the two authors to formulate conjectures on the nature of a necessary and sufficient Type II condition; we prove herein that this conjecture was erroneous (see Remark 4 below). Moreover, still in [13], a necessary and sufficient Type I condition had been proved, but only the Hölder scale $f(r) = r^{\alpha}$ had been considered, and the way the Type I condition was stated left gaps within the Hölder scale, which could not be filled using the techniques in that article. In particular, it was believed that Type II conditions were a first step in proving Type I conditions. We prove herein that such a strategy is unnecessary (see Remark 5), that the two types of conditions can be proved essentially disjointedly, and we believe that this more efficient method of proof is the reason we were able to fill the gaps left in the Hölder scale in [13,14].

The proofs of our Type I and Type II theorems rely heavily on a more basic set of underlying results for a single Gaussian field $Y = Y(x)$, proved in Section 2, which constitute our general Gaussian regularity theory on the circle. In particular we prove

- *regularity* theorem (the basis for our Type I theorem): Y admits a fixed function f as an almost-sure modulus of continuity if and only if f exceeds, up to a non-random constant, a canonical (entropy-based) modulus of continuity;
- *summability* theorem (the basis for our Type II theorem): any bound on the canonical metric function $\delta(x) = [\mathbf{E}(Y(x) - Y(0))^2]^{1/2}$ is equivalent to a summability condition on the Fourier coefficients of Y's covariance.

The "if" part in our regularity theorem is well-known as part of the Fernique–Talagrand theory (see [1]), while the "only if" part seems entirely new. Our proof relies on revisiting the original work of Talagrand in [11], sharpening the proofs by relying on the particularly simple structure of the circle. Our summability theorem seems entirely new; it can be considered as an elementary—albeit non-trivial—result in the harmonic analysis of the circle.

The theorems of Section 3 also depend in a crucial way on sharp estimates developed in [12] in proving that the solution X to (1) exists as a member of $L^2(\Omega \times [0,1] \times S^1)$ if and only if the same holds for Y. These estimates, and the theory of the SHE (1), are recalled in Section 3.1. The basic result is of the following form.

- Let $t \in [0,1]$ be fixed and let r_n and s_n be the Fourier coefficients of the covariance functions of $X(t,\cdot)$ and $Y(t,\cdot)$, respectively. Then up to a constant K that may depend on t, we have $K^{-1}r_n \leqslant s_n \leqslant Kr_n$.

In our effort to give results that are as sharp as possible, we have specialized to the case of the Laplacian on the one-dimensional circle S^1. It is easy to extend all our sufficient conditions for continuity of X to higher-dimensional spaces, and/or much more general operators; the difficulty is in extending the necessary conditions. We will tackle the issue of sharp necessary conditions ("lower bounds") for more general operators and spaces in a subsequent publication. The extension to a uniformly hypoelliptic operator on a smooth compact one-dimensional manifold should be trivial, and we will not comment on this further. The reader may notice that our "lower bounds" proofs (necessary conditions) below make extensive use of a property of spatial isotropy for B, meaning that the spatial part of the homogeneous covariance function depends only on the parameter's distance to the origin. Note that this can also be characterized as saying that Δ commutes with the spatial covariance operator of B. Since we always assume that B is spatially homogeneous, in the case of the circle S^1, isotropy is always satisfied. In higher-dimensional problems, we believe there is hope of extending our one-dimensional lower bound results only in the isotropic case.

To establish the "necessary", or "lower bound" portion of the Type I condition, we will need a technical assumption (Condition B, see below) which amounts to requiring that X is not Hölder-continuous. Without this assumption, in the Hölder scale, we will show a slightly weaker result. The "necessary" or "lower bound" Type II condition, and the "sufficient", or "upper bound" Type II condition require separate mild technical assumptions which do not limit the regularity scales one may wish to consider.

Extending the range of validity of the necessary condition of Type I to include Holder-continuous moduli will be the subject of future work. However, in Corollary 2 below (also see Corollary 3), it can be seen that a Type I condition is necessary without any restriction on the regularity, provided one is satisfied with knowing that a certain canonical function f_Y is the common modulus of continuity for Y and X. This function f_Y is known only if one has precise statistical (Type II) information on B. In other words, the result is useful only if one knows the distribution of B with some accuracy. In this sense, one is dealing with a hybrid Type I–Type II condition. An arbitrary measurement of the regularity of X cannot alone be guaranteed to transfer sharply to Y without the "non-Hölder" regularity condition B. This is why we have spent the extra effort to exploit the full sharpness of Gaussian regularity theory, despite the mathematical price we have to pay in assuming Condition B.

Summarizing our results on the stochastic heat equation,

- the Type I condition is always sufficient;
- Type II conditions are necessary and sufficient, modulo mild technical assumptions;
- the Type I condition is necessary if the modulus is not too regular, i.e. not Hölder;
- in the Hölder scale, the Type I condition is nearly necessary;
- a Type I condition is necessary and sufficient with no regularity restriction if one is satisfied with having an a.s. modulus of continuity that is defined in terms of a formula of Type II.

2. Sharp Gaussian regularity theory on the circle

This section presents the basic regularity theory for homogeneous Gaussian fields on the circle. It is not difficult to modify the arguments to fit many one-dimensional situations; as alluded to above, many isotropic higher-dimensional settings as well.

2.1. Definitions and statements of the regularity results

We start with some definitions.

Definition 1. Let f be a continuous increasing function on $\mathbb{R}_+$ such that $\lim_{0^+} f = 0$. Let $\{Y(x) \colon x \in S^1\}$ be a bonafide random field on S^1 (meaning $Y(\cdot)$ is almost-surely a bonafide function on S^1).

- We say that f is an almost-sure spatial uniform modulus of continuity for Y if there exists an almost-surely positive (non-zero) random variable α_0 such that

$$\alpha < \alpha_0 \Rightarrow \sup_{x,y \in S_1; |x-y| < \alpha} \{|Y(x) - Y(y)|\} \leqslant f(\alpha).$$

- The canonical metric δ of Y is defined for all $x, y \in S^1$ by

$$\delta(x,y) = (E[(Y(x) - Y(y))^2])^{1/2}.$$

- Y is said to be homogeneous if for any x in S^1, Y and $Y(\cdot + x)$ have the same distribution.

Remark 1. If Y is a homogeneous Gaussian field on S^1 then $\delta(x,y) = \delta(|x-y|)$ where $\mathbb{R}_+ \ni r \mapsto \delta(r)$ is some function on a neighborhood of 0. Indeed, by homogeneity there exists some function δ such that $\delta(x,y) = \delta(x-y)$, and by symmetry this also equals $\delta(y-x)$, i.e. $\delta(r) = \delta(|r|)$. If Y is a separable random field, then δ will be continuous on $\mathbb{R}_+$ and increasing in a neighborhood of 0. We leave the proof of this last sentence to the reader.

Definition 2. We call the function $\delta(\cdot)$ on $\mathbb{R}_+$ the canonical metric function of Y.

Remark 2. There are precious few processes for which an exact modulus of continuity is known; we cite the standard Brownian motion W for which

$$\limsup_{h\to 0} \sup_{s,t\leqslant h} (W(t) - W(s))/((t-s)\log(t-s)^{-1})^{1/2} = +1$$

while for the lim inf of the inf we get the value -1, so that $f(r) = (r\log r^{-1})^{1/2}$ is an exact uniform mod. of cont. In this article, we are concerned only with *sharp* moduli of continuity, in the sense that any (preferably non-random) multiple of an exact modulus of continuity is still a sharp modulus of continuity.

Notation. We use the notation $\asymp$ for *commensurate* quantities: for positive functions A and B of any variable ξ, $A(\xi)\asymp B(\xi)$ means that the ratio $A(\xi)/B(\xi)$ is bounded away from 0 and ∞.

Notation. We denote by $\check{\delta}$ the inverse function of δ.

Note for example that for scalar fBm $\{B^H(t)\colon t\in[0,1]\}$ we have $\delta(r) = r^H$. One can construct a random field on S^1 that is homogeneous and that has the same regularity properties than fBm; in particular we can construct it so that its canonical metric function satisfies $\delta(r)\asymp r^H$; we leave it to the reader to check this can be done by taking $r_n = n^{-(1+2H)}$ in the random Fourier representation (3) below. Alternately, one may also try to define a stationary fractional Ornstein–Uhlenbeck bridge.

One should note however that there exists a major difference between this process and the fBm on $[0,1]$: the latter is adapted to a Brownian filtration, while even the concept of being adapted is unclear for a random field on S^1.

Assumption A. We will assume throughout that δ is differentiable except perhaps at 0, and that $\lim_{0+}\delta' = +\infty$. Without loss of generality this implies that δ is concave and strictly increasing in a neighborhood of 0.

In terms of regularity properties of Y, differentiability of δ except at 0 introduces no loss of generality. The condition that δ' at zero is infinite introduces no loss of generality outside of the very narrow class of processes Y that are a.s. β-Hölder-continuous for all $\beta<1$ but that are not a.s. of class C^1. For this class of processes, similar results to those we prove herein hold, but the methods of proof are substantially different. We do not comment on these processes further.

Remark 3 (Random Fourier series representation). For any spatially homogeneous Gaussian field Y on S^1 with canonical metric function δ, there exists a sequence

$\{r_n\}_{n=0}^{\infty}$ of non-negative terms such that

$$\delta(r)^2 = \sum_{n=1}^{\infty} r_n(1 - \cos(nr)). \tag{2}$$

Indeed, any such Y can be written as a random Fourier series

$$Y(x) = Y_0\sqrt{r_0} + 2\sum_{n=1}^{\infty} \sqrt{r_n}\{Y_n \cos(nx) + Z_n \sin(nx)\}, \tag{3}$$

where all Y_n's and Z_n's are IID $N(0,1)$ r.v.'s. Then just calculating δ^2 yields the former statement.

We now introduce a condition needed for the lower bound proof of the next theorem which characterizes the regularity of homogeneous Gaussian processes. This condition is satisfied for the class of functions δ defined by $\delta(r) = (\log(1/r))^{-p}$ for any $p > 0$ no matter how large, and for all functions that are more irregular than this class, but is not satisfied in the power scale defined by $\delta(r) = r^{\alpha}$ for any $\alpha \in (0,1)$. The case of the power scale will be treated separately without this condition.

Condition B. There exists a constant $c > 0$ such that in a neighborhood of 0 we have

$$\int_0^{\alpha} \delta(r) \frac{dr}{r\sqrt{\log(r^{-1})}} > c\delta(\alpha)\sqrt{\log(\alpha^{-1})}$$

Our first result is the main elementary *regularity* theorem, which is key to our Type I characterization.

Theorem 1. *Let Y be a Gaussian random field on S^1 with canonical metric function δ. Let*

$$f_{\delta}(\alpha) = \int_0^{\delta(\alpha)} \sqrt{\log \frac{1}{\check{\delta}(\varepsilon)}}\, d\varepsilon \tag{4}$$

$$= \int_0^{\alpha} \delta(r) \frac{dr}{2r\sqrt{\log(r^{-1})}} + \delta(\alpha)\sqrt{\log(\alpha^{-1})} \tag{5}$$

$$= \int_0^1 \frac{\delta(\min(r,\alpha))dr}{2r\sqrt{\log(r^{-1})}} \tag{6}$$

$$= \int_0^{\infty} \delta(\min(e^{-x^2}, \alpha))dx. \tag{7}$$

There is a non-random constant K depending only on the law of Y such that the following hold.

(a) *Lower bound/necessary condition. If Assumption* A *and Condition* B *hold, if f is an almost-sure uniform modulus of continuity for Y on S^1, then for all α small enough,*

$$Kf(\alpha) \geqslant f_\delta(\alpha). \tag{8}$$

(b) *Upper bound/sufficient condition. If* $\lim_{0^+} f_\delta = 0$ *then Kf_δ is an almost-sure modulus of continuity for Y on S^1.*

This theorem shows that the function f_δ is, up to a constant, an exact uniform modulus of continuity for Y, as long as Y is more irregular than Hölder. The next corollary shows that one can do nearly as well in the Hölder scale.

Corollary 1. *Assume $\delta(r) \asymp r^\alpha$ for some $\alpha > 0$ (the "fractional Brownian" scale). Then the Lower bound* (a) *in Theorem* 1 *holds even though Condition* B *is not satisfied, if one replaces $f(\alpha)$ by $f(\alpha)\log(1/\alpha)$ in line* (8).

Part (b) of the theorem, which is valid in all cases, is the well-known sufficient condition for knowing that f_δ is an almost sure uniform modulus of continuity. It seems to be essentially due to Dudley and Fernique, as can be seen from Theorem 4.4 and Corollary 4.7 in [1]. Our contribution here is the lower bound of statement (a). To our knowledge, no one has ever stated or proved any such result. The result that is usually quoted when speaking about necessary and sufficient conditions for regularity of homogeneous Gaussian processes is that the so-called *entropy integral*, which here equals $f_\delta(1)$, is finite if and only if the process is continuous, and that this quantity is commensurable with the expected value $E\sup_I Y$ of the supremum of the process Y over its index set I. In our first attempt to prove part (a) of the theorem, we had thought that the lower bound of the Dudley–Fernique theorem, which here reads

$$f_\delta(r) \leqslant KE \sup_{[0,r]} Y \tag{9}$$

for any fixed r near 0, with K a universal constant, could be used to derive (a). Indeed, if we could invoke a result based on (9) but with $Y = \{Y(x) \colon x \in [0,r]\}$ replaced by $Y = \{Y(x,y) \coloneqq Y(x) - Y(y) : (x,y) \in S^1 \times S^1; |x-y| < r\}$, then a Fernique zero-one law could be invoked much as in Step 5 of the proof of Theorem 1 below to conclude (a). Unfortunately, it is well-known that results such as (9) only hold for homogeneous Gaussian fields; when a Gaussian field is inhomogeneous, the entropy $f_\delta(r)$ must be replaced by Talagrand's *majorizing measure integral* (see [11, Theorem 17, part (b)]). This is especially crucial when the Gaussian field is blatantly inhomogeneous; that is the case in our application here even if $Y(x)$ is homogeneous, since when x and y are close together, $Y(x,y)$ is very small, while when x and y are further apart, $Y(x,y)$ is much more on the order of $2Y(x)$. For these reasons, entropy integral results cannot be used. We found no other way than to use the full force of majorizing measures coupled with Fernique's zero-one laws to prove (a). Moreover we found no easy way of applying [11, Theorem 17, part (b)] directly, because in this general lower bound result, the various correction terms were too big for our purposes; this is why we have provided in Lemma 1 below a sharper version of [11, Theorem 17, part (b)], valid for the simple situation of the circle.

2.2. *Proof of Theorem 1*

2.2.1. *Proof of the lower bound of Theorem 1*

Let $X = \{X(t)\colon t \in I\}$ be a bounded Gaussian field on an index set I. Let δ be its canonical metric, and $B(x,\varepsilon)$ be the ball of radius ε centered at x in this metric. In this general situation, we introduce some notation. For a fixed measure m on I, let

$$\gamma_m(\eta) = \sup_{x\in I} \int_0^\eta \sqrt{\log(1/m(B(x,\varepsilon)))}d\varepsilon,$$

$$\theta_m(\eta) = \sup_{x\in I} \int_0^\eta \sqrt{\log(1/\sup\{m(\{u\}) : u\in B(x,\varepsilon)\})}d\varepsilon,$$

$$\phi_\delta(\eta) = E[\sup\{X(x) - X(y) : x,y\in I; \delta(x,y)<\eta\}]$$

$$\beta_\delta(\eta) = \sup_{x\in I} E[\sup\{|X(x) - X(y)| : y\in I; \delta(x,y)<\eta\}].$$

We note that $\theta_m \geqslant \gamma_m$. Since X is centered, we have $\beta_\delta \leqslant 2\phi_\delta$. Also we introduce the metric entropy of δ: $N(\varepsilon)$ is the smallest number of balls of radius ε in the metric δ that are needed to cover I. Let D be the diameter of I in the metric δ. Recall the following result from Fernique's general theory of suprema for Gaussian processes.

Proposition 1 (Talagrand [15, Theorem 17, part (a)]). *There exists a universal constant K (not dependent on X) such that with the notation as above, for any probability measure m on I,*

$$\phi_\delta(\eta) \leqslant K\gamma_m(\eta).$$

Theorem 17 in [11] also establishes the following lower bound which is original to Talagrand's paper: for some probability measure m on I,

$$\theta_m(\eta) \leqslant K\beta_\delta(\eta) + \eta(\log(2N(\eta)) + 2\log(2D/\eta)/\log 2)^{1/2}.$$

Yet this estimate is not sufficient for our purposes. In our specific situation however, we are able to bring a slight improvement to Talagrand's original lower bound proof, by assuming condition B.

Step 1: *A Talagrand-type lower bound.* We begin with a lemma inspired by Talagrand's lower bound [proof of Theorem 17 part (b) in [11]].

Lemma 1. *With the notation as above, with K denoting a universal constant (not dependent on X), let $\eta > 0$ be fixed. Let $\{B_j : j = 1, \ldots, N(\eta)\}$ be a covering of I with balls of radius no greater than η. There exists a probability measure m_j on B_j such that for all $x \in B_j$*

$$\int_0^{diam(B_j)} (\log[(\sup\{m_j(\{u\}) : \delta(x,u) \leqslant \varepsilon\})^{-1}])^{1/2} d\varepsilon \leqslant K\beta_\delta(\eta), \tag{10}$$

and moreover, defining the probability measure $m = N(\eta)^{-1} \sum_{i=1}^{N(\eta)} m_j$,

$$\theta_m(\eta) \leqslant \sup_{j \in \{1,\ldots,N(\eta)\}} \sup\nolimits_{x \in B_j} \int_0^{\eta} (\log N(\eta) + \log[(\sup\{m_j(\{u\}) : \delta(x,u) \leqslant \varepsilon\})^{-1}])^{1/2} d\varepsilon \tag{11}$$

Proof. Let u_j be the center of the ball B_j. On each ball $B_j : j = 1, \ldots, N(\eta)$ we apply Theorem 14 in [11] to obtain the existence of a probability measure m_j on B_j such that

$$\int_0^{diam(B_j)} (\log[(\sup\{m_j(\{u\}) : \delta(x,u) \leqslant \varepsilon\})^{-1}])^{1/2}\, d\varepsilon \leqslant K\mathbf{E}[\sup_{x \in B_j} X(x)]$$
$$= K\mathbf{E}[\sup_{x \in B_j} X(x) - X(u_j)] \leqslant K\mathbf{E}[\,|\sup_{x \in B_j} X(x) - X(u_j)|\,] \leqslant K\beta_\delta(\eta),$$

proving the lemma's first assertion. For the second, fix $x \in I$ and let j be such that $x \in B_j$; then for any $\varepsilon \leqslant \eta$,

$$\sup\{m(\{u\}) : \delta(x,u) \leqslant \varepsilon\} \geqslant \frac{1}{N(\eta)} \sup\{m_j(\{u\}) : \delta(x,u) \leqslant \varepsilon\},$$

so that

$$\int_0^{\eta} (\log(1/\sup\{m(\{u\}) : u \in B(x,\varepsilon)\}))^{1/2}\, d\varepsilon$$
$$\leqslant \int_0^{\eta} (\log N(\eta) + \log(1/\sup\{m_j(\{u\}) : u \in B(x,\varepsilon)\}))^{1/2}\, d\varepsilon,$$

and the result follows. $\square$

Step 2: *First majorizing measure integral estimation*. Assume I is a compact Lie group, let $|\cdot|$ and dx denote the Haar measure on I, and its differential, and assume X is homogeneous on I. Fix x_0 in I. Let $B(x_0, \varepsilon)$ be the ball in the metric δ centered at x_0 with radius ε. Let

$$\tau(\eta) = \int_0^{\eta} \sqrt{\log(1/|B(x_0,\varepsilon)|)}\, d\varepsilon.$$

Since X is homogeneous, we have $|B(y,\varepsilon)| = |B(x_0,\varepsilon)|$ for any $x_0, y \in I$. In particular, τ does not depend on x_0, and we have the following equalities for any fixed probability measure m on I:

$$\begin{aligned}\int_I m(B(x,\varepsilon))\,dx &= \int_I dx \int_I \mathbf{1}_{\{|x-y|\leqslant\varepsilon\}}\,m(dy)\\ &= \int_I m(dy)\int_I \mathbf{1}_{\{|x-y|\leqslant\varepsilon\}}\,dx\\ &= \int_I m(dy)|B(y,\varepsilon)| = |B(x_0,\varepsilon)|.\end{aligned}$$

Since the function $u(z) = \sqrt{\log(1/z)}$ is convex for $0<z<1/2$, for small ε, we can use

$$u\left(\int_I m(B(x,\varepsilon))\,dx\right) \leqslant \int_I u(m(B(x,\varepsilon)))\,dx.$$

Therefore

$$u(|B(x_0,\varepsilon)|) \leqslant \int_I u(m(B(x,\varepsilon)))\,dx,$$

which implies

$$\begin{aligned}\tau(\eta) &\leqslant \int_I dx \int_0^\eta u(m(B(x,\varepsilon)))\,d\varepsilon\\ &\leqslant \sup_{x\in I}\int_0^\eta u(m(B(x,\varepsilon)))\,d\varepsilon\\ &= \gamma_m(\eta).\end{aligned}$$

Step 3: *Using the Talagrand-type lower bound*. The last inequality, together with inequality (11), proves, with the measures m and $m_j : j = 1, \dots, N(\eta)$, identified in Lemma 1,

$$\begin{aligned}&\tau(\eta) \leqslant \gamma_m(\eta) \leqslant \theta_m(\eta)\\ &\quad\leqslant \sup_{j\in\{1,\dots,N(\eta)\}}\ \sup_{x\in B_j}\int_0^\eta (\log N(\eta) + \log[(\sup\{m_j(\{u\}) : \delta(x,u)\leqslant\varepsilon\})^{-1}])^{1/2}\,d\varepsilon. \quad (12)\end{aligned}$$

Since we have no control over the term involving m_j in the above expression, in comparison to $N(\eta)$, we have no choice, as did Talagrand himself, but to use the estimate $\sqrt{A+B} \leqslant \sqrt{A} + \sqrt{B}$. Then, with inequality (10), we obtain

$$\begin{aligned}\tau(\eta) &\leqslant \eta(\log N(\eta))^{1/2} + K\beta_\delta(\eta)\\ &\leqslant \eta(\log N(\eta))^{1/2} + K\phi_\delta(\eta). \quad (13)\end{aligned}$$

Step 4: *Calculation of the majorizing measure integral*. The next step in the proof is to calculate τ. Here we specialize to the case of δ on the circle $I = S^1$. Recall that we denote by $\check{\delta}$ the inverse function of δ. We have

$$|B(x_0, \varepsilon)| = |\{x : \delta(|x - x_0|) < \varepsilon\}| = |\{x : |x - x_0| < \check{\delta}(\varepsilon)\}| = 2\check{\delta}(\varepsilon).$$

Therefore τ becomes

$$\begin{aligned}\tau(\eta) &:= \int_0^\eta \sqrt{\log(1/|B(x_0, \varepsilon)|)}\, d\varepsilon \\ &= \int_0^\eta \sqrt{\log(1/(2\check{\delta}(\varepsilon)))}\, d\varepsilon. \end{aligned} \tag{14}$$

We also note that because we are working on a one-dimensional index set, we have

$$N(\eta) = \frac{1}{2\check{\delta}(\eta)} \tag{15}$$

for any value of η that yields an integer in this formula.

Step 5: *Using the hypothesis of a.s. modulus of continuity with a zero-one law of Fernique*. To complete the proof of the lower bound, we need to estimate ϕ_δ. To this end, we use a zero-one-type result due to Fernique. Let $C^f(I)$ be the space of continuous functions on I that have f as a uniform modulus of continuity, up to a multiplicative constant. For any $\alpha \leqslant 1$, for any function g defined on I, set

$$A_\alpha(g) = \sup_{|x-y| \leqslant \alpha} |g(x) - g(y)|, \quad N_f(g) = \sup_{\alpha \leqslant 1} \frac{A_\alpha(g)}{f(\alpha)}.$$

Then, following Fernique's definitions [7, Definition 1.2.1], N_f is a gauge on $C^f(I)$. Indeed, it suffices to see that N_f is lower-semi-continuous, that is, for every $M > 0$, the set $K_M = \{\Phi; N_f(\Phi) \leqslant M\}$ is closed. Let Φ_n be a sequence in K_M converging uniformly to Φ. Then, for every n and for every $\alpha \leqslant 1$ and for every x, y such that $|x - y| \leqslant \alpha$, we have

$$|\Phi_n(x) - \Phi_n(y)| \leqslant Mf(\alpha).$$

We obtain that $\Phi \in K_M$ when we let $n \to \infty$.

Recall the main assumption in statement (a) of the theorem: f is an almost-sure modulus of continuity for Y. This means that we have the existence of an almost surely positive random variable α_0 such that, if $\alpha < \alpha_0$ then

$$A_\alpha(Y) \leqslant f(\alpha).$$

Since Y is almost surely continuous, it is also almost surely bounded. This, together with the last inequality, implies $N_f(Y)$ is almost surely finite. A theorem of Fernique [7, Lemma 1.2.3] implies $E[N_f(Y)] := c < \infty$ where $c = c(f, Y)$ is a constant depending only on f and the law of Y (that is, depending only on the spatial covariance of B). Therefore

$$\begin{aligned}\phi_\delta(\eta) &\leqslant E \sup_{\delta(|x-y|)\leqslant\eta} |Y(x) - Y(y)| \\ &= E \sup_{|x-y|\leqslant\check{\delta}(\eta)} |Y(x) - Y(y)| \\ &\leqslant cf(\check{\delta}(\eta)). \end{aligned} \tag{16}$$

Step 6: *Conclusion*. Combining (13), (14), and (15), we obtain

$$\int_0^\eta \sqrt{\log(1/(2\check{\delta}(\varepsilon)))}d\varepsilon \leqslant K\phi_\delta(\eta) + \eta\left(\log\left(\frac{1}{2\check{\delta}(\eta)}\right)\right)^{1/2}$$

Now let α be defined by $\alpha = \check{\delta}$. Then for small α, with (16) and formula (5),

$$\int_0^\alpha \delta(r)\frac{dr}{2r\sqrt{\log(r^{-1})}} + \delta(\alpha)\sqrt{\log(\alpha^{-1}/2)} \leqslant \delta(\alpha)\sqrt{\log(\alpha^{-1}/2)} + cKf(\alpha),$$

or in other words

$$\int_0^\alpha \delta(r)\frac{dr}{2r\sqrt{\log(r^{-1})}} \leqslant cKf(\alpha). \tag{17}$$

Condition B and the formula for f_δ in (5) finish the proof of (a).

2.2.2. *Proof of the upper bound of Theorem 1*

Part (b) is a consequence of a well-known property of homogeneous Gaussian processes and the general theory of Gaussian regularity. Indeed, one only needs to apply Theorem 4.4 and Corollary 4.7 in [1]. The details are left to the reader.

2.2.3. *Proof of Corollary 1*

In the proof of Theorem 1, in the Hölder case $\delta(r) = r^\alpha$, all lower bound calculations are valid up to inequality (17). The conclusion of the corollary follows by calculating the left-hand side of (17) and comparing it to $f(\alpha)\log(1/\alpha)$.

2.3. *Summability result*

Our basic *summability* theorem translates the magnitude of δ—and thus, by Theorem 1 and Corollary 1, the regularity of Y—into a condition on the summability of the q_n's. This explains why it is the basis for our Type II characterization. The "sufficient condition" requires Condition C below, which is stated relative to an upper bound g on the canonical metric (see statement (c) in the theorem). One can think of Condition C as a condition on $g = \delta^2$.

Condition C: There exist constants $c, y_0 > 0$ such that for all $0 < x < y < y_0$

$$g(x)/x^2 - g(y)/y^2 \geqslant c(g(y) - g(x))/y^2.$$

Condition C essentially places no restriction on the regularity of the canonical metrics that can be used in Type II characterizations. Indeed, Condition C is satisfied for all the following basic examples:

- "Hölder" scale:

$$\delta(r)^2 \leqslant r^{2\alpha}$$

  for any $\alpha \in (0,1)$; up to logarithmic corrections, this scale corresponds to the Hölder scale of almost-sure uniform moduli of continuity $f(r) = r^\alpha$;
- logarithmic scale:

$$\delta(r)^2 \leqslant (\log(1/r))^{-1-2\varepsilon}$$

  for any $\varepsilon > 0$; this scale corresponds to the scale of moduli of continuity given by $f(r) = (\log(1/r))^{-\varepsilon}$;
- iterated logarithmic scale:

$$\delta(r)^2 \leqslant (\log(1/r))^{-1}(\log\log(1/r)\cdots\log_{(n-1)}(1/r))^{-2}(\log_{(n)}(1/r))^{-2-2\varepsilon}$$

  for any $n \in \{2, 3, \dots\}$ and any $\varepsilon > 0$; here $\log_{(n)}$ denotes the n-fold iterated logarithm; this scale corresponds to the scale of moduli of continuity given by $f(r) = (\log_{(n)}(1/r))^{-\varepsilon}$.

Condition C is even satisfied in a scale which yields a.s. discontinuous Y, although this scale cannot be used for our purposes:

- logarithmic scale for discontinuous processes:

$$\delta(r)^2 \leqslant (\log(1/r))^{-\varepsilon}$$

  for any $\varepsilon \in (0,1]$.

Our basic *summability* theorem is the following.

Theorem 2. *Let Y be a homogeneous Gaussian random field on S^1 with canonical metric function δ. Let $\{r_n\}_n$ be the sequence defined by the random Fourier series representation* (3) *for Y. Let g be a strictly increasing continuous function on $\mathbb{R}_+$, continuously differentiable on $(0,\infty)$, with $\lim_{0^+} g = 0$. Consider the following statements*:

(c) *There exists a constant $K > 0$ such that for all $r \geqslant 0$, $\delta(r) \leqslant K\sqrt{g(r)}$.*

(d) *For any strictly decreasing, positive function* h *on a neighborhood of* 0 *with* $\int_0 h(x)dx < \infty$:

$$\sum_n r_n h\left(g\left(\frac{1}{n}\right)\right) < \infty.$$

The implication (c) $\Rightarrow$ (d) always holds. The converse (d) $\Rightarrow$ (c) *holds if we assume Condition* C.

2.4. *Proof of Theorem 2*

2.4.1. *Proof of* (c) $\Rightarrow$ (d)

To exploit the hypothesis on δ in condition (c) note that we have for all n,

$$K^2 g\left(\frac{1}{n}\right) \geqslant \sum_{j=1}^{\infty} r_j(1-\cos(j/n)) \geqslant \frac{1}{3}\sum_{j=n}^{5n} r_j \tag{18}$$

since for $x \in [1,5]$, $1-\cos x > 1/3$. Fix an h as in condition (d). Fix an integer $k > 0$. Let

$$\begin{aligned} I_k &= \sum_{n=1}^{5^{k+1}} r_n h\left(g\left(\frac{1}{n}\right)\right) \\ &= \sum_{l=0}^{k} \sum_{n=5^l}^{5^{l+1}-1} r_n h\left(g\left(\frac{1}{n}\right)\right). \end{aligned}$$

We only need to show that I_k is bounded. Since h is decreasing and g is increasing, we have $h(g(\frac{1}{n})) \leqslant h(g(\frac{1}{5^{l+1}}))$ for any $n \in [5^l; 5^{l+1}]$. This, coupled with inequality (18), and the notation $F(x) = 1/(xh(x))$, yields

$$\begin{aligned} I_k &\leqslant \sum_{l=0}^{k}\left(\sum_{n=5^l}^{5^{l+1}-1} r_n\right) h\left(g\left(\frac{1}{5^{l+1}}\right)\right) \\ &\leqslant 3K^2 \sum_{l=0}^{k} g\left(\frac{1}{5^l}\right) h\left(g\left(\frac{1}{5^{l+1}}\right)\right) \\ &= 3K^2 \sum_{l=0}^{k} \frac{g(5^{-l})}{g(5^{-(l+1)})} \frac{1}{F(g(5^{-(l+1)}))} \end{aligned}$$

By Assumption A, without loss of generality, we can assume that $g(0) = 0$ and that $\tilde{g} = g^{1/2}$ is concave (near 0), and also that $g(x) \geqslant x^2$ (near 0). This implies that (near 0)

$$\frac{g(x)}{g(x/5)} \leqslant 25; \quad g(x) \geqslant x^2.$$

Also, without loss of generality, we can assume that $U(x) = xh(x)$ is decreasing near 0, as the following lemma shows.

Lemma 2. *Let h be a function satisfying the assumptions of condition* (d). *Then one can find a function ϕ satisfying the same assumptions, such that $\phi \geqslant h$ and $x\phi(x)$ is a decreasing function in a neighborhood of* 0.

Proof. Without loss of generality, one can assume that h is $C^1((0,1))$ and that $\lim_0 h = \infty$. Hence, in a neighborhood of 0, h is a decreasing convex function. Now, if there exists a constant c_1 such that, for $\beta > 1$,

$$h(x) \leqslant \frac{c_1}{x|\log(x)|^\beta} \equiv \psi_1(x),$$

just take $\phi = \psi_1$. Otherwise, if there exists a x_0 and a strictly positive constant c_2 such that, for $\beta > 1$,

$$h(x_0) \geqslant \frac{c_2}{x_0|\log(x_0)|^\beta} \equiv \psi_2(x_0),$$

then, by convexity of h, we have $h \geqslant \psi_2$ in a neighborhood of 0, and

$$\lim_{x\to 0} xh'(x) = -\infty. \tag{19}$$

On the other hand, if $\int_0 h < \infty$ and h is a decreasing convex function, then

$$xh(x) \leqslant c_3 \tag{20}$$

in a neighborhood of 0. Putting together (19) and (20), we get

$$h(x) + xh'(x) < 0$$

for x small enough. Thus, $xh(x)$ is a decreasing function in a neighborhood of 0. $\square$

The estimate of I_k now yields

$$\begin{aligned} I_k &\leqslant 75K^2 \sum_{l=0}^{k} U(g(5^{-(l+1)})) \\ &\leqslant 75K^2 \sum_{l=0}^{k} U(25^{-(l+1)}). \end{aligned}$$

The conclusion the proof of (c) $\Rightarrow$ (d) is now immediate from the following trivial calculation,

$$\infty > \int_0^{x_0} h = \int_0^{x_0} U(x)\frac{dx}{x} = \ln(25)\int_{y_0}^{\infty} U(25^{-y})dy \geqslant \ln(25)\sum_{n=n_0}^{\infty} U(25^{-(l+1)}),$$

where the values x_0, y_0, and n_0 are obvious notation signifying that we ignore the first terms in I_k to ensure the validity of the inequalities used above when x is close to 0.

2.4.2. Proof of (d) $\Rightarrow$ (c)

The following lemma on summation by parts will be useful.

Lemma 3. *Let $(A_n)_n$ and $(B_n)_n$ be sequences of real numbers. Let $a_1 = A_1$, $b_1 = B_1$, and for all $j \geqslant 2$, $a_j = A_j - A_{j-1}$ and $b_j = B_j - B_{j-1}$. Then*

$$A_n B_n = \sum_{j=1}^{n} A_j b_j + \sum_{j=2}^{n} B_{j-1} a_j.$$

Proof. Iterate the relation:

$$A_n B_n = A_n(B_n - B_{n-1}) + B_{n-1}(A_n - A_{n-1}) + A_{n-1}B_{n-1}.$$

from n to 1. $\square$

Step 1: *Space discretization.* We first show that it is sufficient to show the conclusion of (c) for the x's of the form $x = 1/n$ where n is an integer. Indeed assume that there exist $K > 0$ and $n_{\min}$ an integer such that for all $n \geqslant n_{\min}$:

$$\delta(1/n)^2 \leqslant Kg(1/n).$$

For an arbitrary $x \in (0; 1/n_{\min}]$, let n be such that $x \in (1/(n+1); 1/n]$. We have

$$\begin{aligned} \delta^2(x) &\leqslant \delta^2(1/n) \leqslant Kg(1/n) \\ &= Kg(x)\left(1 + \frac{g(1/n) - g(x)}{g(x)}\right). \end{aligned} \tag{21}$$

By Assumption A, without loss of generality, we can assume that $g(0) = 0$ and that $\tilde{g} = g^{1/2}$ is concave near 0. This implies that if $b > x > 0$, $[\tilde{g}(b) - \tilde{g}(x)]/[b - x] \leqslant g(x)/x$. Using this fact and the fact that $1/(n+1) < x$ implies $1/n < 2x$ as long as $x < 1/2$, we obtain:

$$\begin{aligned} \frac{\tilde{g}(1/n) - \tilde{g}(x)}{\tilde{g}(x)} &= \frac{\tilde{g}(1/n) - \tilde{g}(x)}{1/n - x} \frac{1/n - x}{\tilde{g}(x)} \\ &\leqslant \frac{\tilde{g}(x)}{x} \frac{1/n - x}{\tilde{g}(x)} \\ &= \frac{1/n - x}{x} < 1. \end{aligned}$$

Then we can estimate

$$\frac{g(1/n)-g(x)}{g(x)}=\frac{\tilde{g}(1/n)-\tilde{g}(x)}{\tilde{g}(x)}\frac{\tilde{g}(1/n)+\tilde{g}(x)}{\tilde{g}(x)}$$
$$=\frac{\tilde{g}(1/n)-\tilde{g}(x)}{\tilde{g}(x)}\left[\frac{\tilde{g}(1/n)-\tilde{g}(x)}{\tilde{g}(x)}+2\right]<3.$$

Returning to (21) we get

$$\delta^2(x)\leqslant 3Kg(x).$$

Step 2: *Separating the head and the tail of* δ. Let n_0 be a fixed integer larger than $n_{\min}$. We have

$$\delta^2(1/n_0)=\sum_{n=1}^{n_0-1} r_n(1-\cos(n/n_0))+\sum_{n=n_0}^{\infty} r_n(1-\cos(n/n_0))$$
$$\leqslant \sum_{n=1}^{n_0-1} r_n(n/n_0)^2+\sum_{n=n_0}^{\infty} r_n.$$

We only need to show that there exists $K>0$ such that for all $n_0>n_{\min}$, the following two inequalities hold:

$$\sum_{n=1}^{n_0} r_n(n/n_0)^2\leqslant Kg(1/n_0), \tag{22}$$

$$\sum_{n=n_0}^{\infty} r_n\leqslant Kg(1/n_0). \tag{23}$$

We will assume (d) holds and will assume successively that each of these two inequalities does not hold; we will obtain a contradiction in each case.

Step 3: *Controlling the tail.*

Step 3.1: *Assuming the tail is unbounded.* The negation of inequality (23) is equivalent to the existence of a sequence of integers $(N_m)_m$ that increases to $+\infty$, and a sequence of positive reals $(K_m)_m$ that increases to $+\infty$, satisfying for all $m\in\mathbf{N}$,

$$\sum_{n=N_m}^{\infty} r_n\geqslant g(1/N_m)K_m. \tag{24}$$

It will be convenient below to use the fact that without loss of generality, we can choose K_m to increase to infinity as slowly as we want, without effecting the sequence $(N_m)_m$. Let h be a function as in (d). Recall that $n\mapsto h(g(1/n))$ is strictly increasing.

We introduce the following notation:

$$B^{(m)} := \sum_{n=N_m}^{\infty} r_n,$$

$$-b^{(m+1)} := B^{(m)} - B^{(m+1)} = r_{N_m} + \cdots + r_{N_{m+1}-1},$$

$$A^{(m)} := h(g(1/N_m))$$

$$a^{(m+1)} := A^{(m+1)} - A^{(m)} > 0.$$

By hypothesis (d) the tail $\sum_{n=N_m}^{\infty} r_n h(g(1/n))$ converges to 0 as $m \to \infty$. We will calculate this tail using the summation-by-parts Lemma 3 with the A's and B's as above. This will enable us to use the hypothesis (24) on this tail, and another application of Lemma 3 will yield a contradiction thanks to an appropriately chosen h. Let m_0 be fixed. We have

$$\begin{aligned}\sum_{n=N_{m_0}}^{\infty} r_n h(g(1/n)) &= \sum_{m=m_0}^{\infty} \sum_{n=N_m}^{N_{m+1}-1} r_n h(g(1/n)) \\ &\geqslant \sum_{m=m_0}^{\infty} A^{(m)}(-b^{(m+1)}) \\ &= \sum_{m=m_0-1}^{\infty} a^{(m+1)} B^{(m+1)} - \lim_{m\to\infty} A^{(m)} B^{(m)},\end{aligned}$$

where the last equality is by Lemma 3. We can prove that the last limit does in fact exist and is equal to 0. Indeed

$$\begin{aligned}A^{(m)} B^{(m)} &= h(g(1/N_m)) \sum_{n=N_m}^{\infty} r_n, \\ &\leqslant \sum_{n=N_m}^{\infty} r_n h(g(1/n))\end{aligned}$$

and the last term converges to 0 as $m \to \infty$ by hypothesis (d). Now we use (24) on the first expression for the tail, which yields

$$\begin{aligned}\sum_{n=N_{m_0}}^{\infty} r_n h(g(1/n)) &\geqslant \sum_{m=m_0}^{\infty} a^{(m)} g(1/N_m) K_m \\ &\geqslant K_{m_0} \sum_{m=m_0}^{\infty} a^{(m)} g(1/N_m).\end{aligned}$$

We calculate this last series by Lemma 3 again, using the notation $C^{(m)} := g(1/N_m)$ and $-c^{(m)} := g(1/N_{m-1}) - g(1/N_m)$:

$$\sum_{m=m_0}^{\infty} a^{(m)} g(1/N_m) = \sum_{m=m_0+1}^{\infty} A^{(m-1)}(-c^{(m)}) + \lim_{m\to\infty} A^{(m)} C^{(m)}$$
$$= \sum_{m=m_0+1}^{\infty} h(g(1/N_{m-1}))[g(1/N_{m-1}) - g(1/N_m)],$$

where the fact that the last limit is zero is a trivial consequence of the fact that h is decreasing and integrable at 0.

To summarize we have proved:

$$\sum_{n=N_{m_0}}^{\infty} r_n h(g(1/n)) \geqslant K_{m_0} \sum_{m=m_0+1}^{\infty} h(g(1/N_{m-1}))[g(1/N_{m-1}) - g(1/N_m)]. \tag{25}$$

It is now sufficient to show that h can be chosen integrable at 0 and strictly decreasing, and such that for all m_0 large enough,

$$\sum_{m=m_0}^{\infty} h(g(1/N_{m-1}))[g(1/N_{m-1}) - g(1/N_m)] \geqslant \frac{1}{\sqrt{K_{m_0}}}. \tag{26}$$

Step 3.2: *Choosing h*. We let $g_m = g(1/N_{m-1})$ and introduce a arbitrary sequence $(k_m)_m$ such that $\lim k_m = 0$ and $k_m \geqslant (K_m)^{-1/2}$. First we show that we can reduce the problem of finding h as above to the problem of finding a strictly increasing sequence of positive numbers $(h_m)_m$ such that

$$\sum_{m=m_0}^{\infty} h_m[g_m - g_{m+1}] \geqslant k_{m_0}, \tag{27}$$

and such that the series on the left converges. Indeed define a function h as follows: for each fixed m, define h to be linear on the interval $(g_m; g_{m-1}]$, with endpoints set to

$$h(g_{m-1}) = h_{m-1},$$

$$\lim_{x\downarrow g_m} h(x) = \min(h_m; 2h_{m-1}).$$

Since $(h_m)_m$ is strictly increasing, this h is strictly decreasing. Moreover it is clear that

$$\begin{aligned}&\sum_{m=m_0}^{\infty} h_m[g_m - g_{m+1}]\\ &\quad= \sum_{m=m_0}^{\infty} h(g(1/N_{m-1}))[g(1/N_{m-1}) - g(1/N_m)]\\ &\quad\leqslant \int_0^{g_{m_0}} h(x)dx \leqslant 2 \sum_{m=m_0}^{\infty} h_m[g_m - g_{m+1}].\end{aligned}$$

This implies that (26) holds and also that h is integrable at 0, which is what we want. Thus we only need to find $(h_m)_m$ as in (27). Because of the flexibility we have on the choice of $(K_m)_m$ (being able to decrease all the values of K_m as long as the resulting sequence still converges to $+\infty$), there is no actual loss of generality in fixing the values of h_m and searching for new values of K_m such that (27) holds and $k_m \geqslant (K_m)^{-1/2}$. More precisely we *choose* $h_m = f(g_m)$ where f is any positive strictly decreasing integrable function (e.g. $f(x) = x^{-1/2}$). Then we can simply define $(k_m)_m$ by imposing that (27) hold as an equality. Note that we have

$$\int_0^{g_1} f(x)\,dx \geqslant \sum_{m=1}^{\infty} f(g_m)[g_m - g_{m+1}].$$

Therefore, k_m is the tail of this convergent series, and so it decreases to zero. Therefore there is no loss of generality in reassigning the values of K_m to satisfy for all $m \geqslant 1$:

$$K_m = (k_m)^{-2}.$$

Step 4: *Controlling the head*. This step follows a similar structure to Step 3.

Step 4.1. Negating the head bound. The negation of inequality (22) is equivalent to the existence of a sequence of integers $(N_m)_m$ that increases to $+\infty$, and a sequence of positive reals $(K_m)_m$ that increases to $+\infty$, satisfying for all $m \in \mathbf{N}$,

$$\sum_{n=1}^{N_m} n^2 r_n \geqslant (N_m)^2 g(1/N_m) K_m. \tag{28}$$

Let h be a function as in (d). We introduce the following notation:

$$\begin{aligned}B^{(m)} &:= \sum_{n=1}^{N_m} n^2 r_n,\\ b^{(m)} &:= B^{(m)} - B^{(m-1)} = \sum_{n=N_{m-1}+1}^{N_m} n^2 r_n,\end{aligned}$$

$$A^{(m)} := \frac{1}{(N_m)^2} h\left(g\left(\frac{1}{N_m}\right)\right),$$
$$a^{(m)} := A^{(m)} - A^{(m-1)}.$$

We will show in Step 4.3 that h can be chosen in such a way that with $A_n := n^{-2}h(g(1/n))$, the sequence $(A_n)_n$ is decreasing. This yields

$$\begin{aligned}\sum_{n=N_{m_0}}^{\infty} r_n h(g(1/n)) &= \sum_{n=N_{m_0}}^{\infty} n^2 r_n \frac{1}{n^2} h(g(1/n) \text{ 1a somme ci-dessous commence a } m = m_0 \\ &\geqslant \sum_{n=Nm_0}^{\infty} \frac{1}{(N_m)^2} h\left(g\left(\frac{1}{N_m}\right)\right) \sum_{n=N_{m-1}+1}^{N_m} n^2 r_n \\ &= \sum_{m=m_0}^{\infty} A^{(m)} b^{(m)}\end{aligned}$$

and then using summation by parts, this equals

$$\begin{aligned}&\sum_{m=m_0+1}^{\infty} (-a^{(m)}) B^{(m-1)} + \lim_{m\to\infty} A^{(m)} B^{(m)} \\ &\quad \geqslant \sum_{m=m_0+1}^{\infty} (-a^{(m)}) K_{m-1} g\left(\frac{1}{N_{m-1}}\right)(N_{m-1})^2 \\ &\quad \geqslant K_{m_0+1} \sum_{m=m_0+1}^{\infty} (-a^{(m)}) g\left(\frac{1}{N_{m-1}}\right)(N_{m-1})^2 \\ &\quad = K_{m_0+1} \sum_{m=m_0}^{\infty} A^{(m)} c^{(m)} - \lim_{m\to\infty} A^{(m)} C^{(m)},\end{aligned}$$

where $C^{(m)} = g(1/N_m)(N_m)^2$. We note that $A^{(m)}C^{(m)} = g(1/N_m)h(g(1/N_m))$. However, since we assumed that h is decreasing and $\int_0 h < \infty$, we get immediately that $\lim_{x\to 0} xh(x) = 0$, so that the last limit above is zero.

Step 4.2. Applying the method of Step 3. Thanks to the previous step we have:

$$\sum_{n=N_{m_0}}^{\infty} r_n h(g(1/n)) \geqslant K_{m_0} \sum_{m=m_0}^{\infty} h(g(1/N_m)) \left[g(1/N_m) - \frac{(N_{m-1})^2}{(N_m)^2} g(1/N_{m-1}) \right]$$

In fact this implies that inequality (25) holds. Indeed by Condition C, we get

$$\sum_{n=N_{m_0}}^{\infty} r_n h(g(1/n)) \geqslant c K_{m_0} \sum_{m=m_0}^{\infty} h(g(1/N_m))[g(1/N_{m-1}) - g(1/N_m)]$$

which is stronger than (25) since $h(g(1/N_m)) > h(g(1/N_{m-1}))$.

We now apply the strategy of Step 3 by saying that it is sufficient to find a function h integrable at 0 and strictly decreasing, and such that for all m_0 large enough, (26) holds. In addition to the methodology of Step 3, we also need to be sure that $A_n := n^{-2}h(g(1/n))$ is a decreasing sequence, since we needed this condition in Step 4.1.

Step 4.3. Choosing h. We choose $h(y) = 1 - y$, defined in a neighborhood of 0. Clearly this h is integrable at 0 and strictly decreasing. Now for each integer m_0 we define k_{m_0} by

$$\sum_{m=m_0}^{\infty} h(g(1/N_m))[g(1/N_{m-1}) - g(1/N_m)] = k_{m_0}.$$

Since h is bounded by 1, the left-hand side of this equality is the tail of a converging series. Therefore k_m decreases to 0, and (26) holds by invoking the reassignment of the values of $(K_m)_m$ described at the end of Step 3.2.

The only thing left to prove is that this h is consistent with the condition, announced at the beginning of Step 4.1, that $A_n = n^{-2}h(g(1/n))$ is a decreasing sequence for n large enough. That is, we want to show that for all n large enough,

$$\left(\frac{1}{n+1}\right)^2 (1 - g(1/(n+1))) \leqslant \left(\frac{1}{n}\right)^2 (1 - g(1/n))$$

which is equivalent to:

$$\frac{1 - g(1/(n+1))}{1 - g(1/n)} < 1 + 2/n + 1/n^2.$$

To see this we can assume without loss of generality that near 0, either g is concave or $g(x) \leqslant x$. In the first case, we have

$$\frac{1 - g(1/(n+1))}{1 - g(1/n)} = 1 + [1 + g(1/n) + o(g(1/n))]g'(\xi_n)\frac{1}{n(n+1)},$$

where $\xi \in (1/(n+1); 1/n)$. Since g is concave we have $g'(\xi_n) \leqslant g'(1/(n+1)))$, and also $g'(x) \leqslant g(x)/x < 1/x$ for small x. Thus we get:

$$\frac{1 - g(1/(n+1))}{1 - g(1/n)} < 1 + 2/n.$$

For the other case, $g(x) \leqslant x$, we can also assume without loss of generality that $g(x) \geqslant x^2$ because of Assumption A. Then we get for large n:

$$\frac{1 - g(1/(n+1))}{1 - g(1/n)} \leqslant \frac{1 - (n+1)^2}{1 - n^{-1}} = 1 + \frac{1}{n} + o\left(\frac{1}{n}\right) < 1 + \frac{2}{n}.$$

3. Sharp regularity for the fractional stochastic heat equation

3.1. The SHE with infinite-dimensional fractional Brownian noise

The metric $|\cdot|$ on the circle S^1 identified to $[0, 2\pi)$ coincides with the usual Euclidean distance on any subinterval of length π, normalized by the factor 2π, with an obvious extension to S^1 because of the identification of 0 and 2π. Let $\{e_n\} \cup \{\tilde{e}_n\}$ be the orthonormal basis of $L^2 = L^2(S^1)$ made of sine and cosine functions, i.e. the set of real-valued eigenfunctions of Δ on S^1, namely $\cos nx$ and $\sin nx$. As in [12], and as is often done in the theory of stochastic PDEs, let us write the SHE (1) in its weaker *evolution* form:

$$X(t,x) = \int_0^t P_{t-s}[B(ds,\cdot)](x) : t \in [0,1]; \quad x \in S^1. \tag{29}$$

Clearly, because the original equation had only an additive noise term, the evolution form of the equation gives the solution explicitly. Here $(P_t)_{t \geqslant 0}$ is the semigroup of operators generated by the Laplacian on S^1. The action of this semigroup on $L^2(S^1)$ is characterized by

$$P_t[e^{in\cdot}](x) = \exp(-tn)e^{inx}$$

which can of course be translated into a characterization using the trig functions e_n. In accordance with our announcement in the introduction, we assume, as was done in [12], and as is often done in the theory of stochastic PDEs with any sort of infinite-dimensional noise, that the random field B is an infinite-dimensional fBm, with values in $L^2(S^1)$, given by a random Fourier series, generalizing the decomposition (3): for $\{q_n\}_n$ a sequence of non-negative terms, we let

$$B(t,x) = B^H(t,x) = \beta_0^H(t)\sqrt{q_0} + \sum_{n=1}^{\infty} \sqrt{q_n}\{\beta_n^H(t)\cos(nx) + \tilde{\beta}_n^H(t)\sin(nx)\}, \tag{30}$$

where the processes β_n^H and $\tilde{\beta}_n^H$ are IID fBm's with constant Hurst parameter $H \in (0,1)$. The reader can refer to any recent article on fractional Brownian motion for a detailed treatment of how each process β_n^H can be defined. We suggest our own [12], or for a more detailed treatment, [2,5], or [3]. The following characteristic formulas will help fix the reader's ideas on fBm, although they will not be used explicitly in the sequel:

- fBm β^H is a centered Gaussian process with $\beta^H(0) = 0$ and

$$E[(\beta^H(t) - \beta^H(s))^2] = |t-s|^{2H}.$$

- the stochastic integral of deterministic functions with respect to fBm (Wiener integral) is easily defined as a centered Gaussian random variable by extending the above relation; a fundamental formula for $H > 1/2$ is

$$E\left[\int_0^1 f(u)\, d\beta^H(u) \int_0^1 g(u)\, d\beta^H(u)\right] = \int_{[0,1]^2} f(s)g(t)|t-s|^{2H-2}\, ds\, dt.$$

When $H < 1/2$, this scalar product can also be computed, but it requires the use of a singular integral kernel.
- fBm is almost surely γ-Hölder-continuous for all $\gamma < H$.
- when $H > 1/2$ the increments of β^H over small contiguous intervals are positively correlated; when $H < 1/2$, they are negatively correlated.

The random field B can be *generalized* in the x parameter—meaning that $X(t,\cdot)$ can be almost-surely a (Schwartz) distribution (a generalized function)—while still allowing for the solution X in (29) to be defined as a bonafide function of both t and x. This corresponds to any situation in which $\sum q_n = \infty$. In [12], the following (easy) random Fourier representation is given for X when it exists:

$$X(t,x) = \sum_{n=0}^{\infty} \sqrt{q_n}\cos(nx) \int_0^t e^{-n^2(t-s)} \beta_n^H(ds) + \sum_{n=1}^{\infty} \sqrt{q_n}\sin(nx) \int_0^t e^{-n^2(t-s)} \tilde{\beta}_n^H(ds). \tag{31}$$

Moreover it is proved that a necessary and sufficient condition for existence of X is

$$\sum_n \frac{q_n}{n^{4H}} < \infty. \tag{32}$$

Only under the stronger condition that q_n is summable can we guarantee that B is a true function in $L^2(S^1)$, but this is somewhat irrelevant to the purpose of studying X.

Note that both B and X are spatially homogeneous centered Gaussian random fields. In particular, for fixed t, they can be represented using the following random Fourier series

$$B(t,\cdot) = \sum_{n\in\mathbb{Z}} \sqrt{q_n(t)}e_n W_n,$$
$$X(t,\cdot) = \sum_{n\in\mathbb{Z}} \sqrt{s_n(t)}e_n G_n,$$

where for $n \geqslant 1$, $e_{-n} = \tilde{e}_n$ and $q_{-n}(t) = q_n(t)$, and where $(W_n)_n$ and (G_n) are I.I.D. sequences of standard normal r.v.'s. The sequences W and G are independent. Clearly $q_n(t) = q_n t^{2H}$ and s_n is another function of t. Also, for fixed t, $X(t,\cdot)$ is almost-surely in L^2 if and only if for each fixed x, $E[X(t,x)^2] < \infty$. The above facts are where condition (32) come from. More specifically, in the proof of Theorems 2 and 3 in [12], it is established that the variance of the centered Gaussian r.v.

$\int_0^t e^{-n^2(t-s)} B_n^H(ds)$ is commensurate with n^{-4H}, which means we have the following single fundamental estimate:

Lemma 4. *For fixed $t \in [0,1,]$ we have $s_n(t) \asymp q_n(t)/n^{4H}$, where the commensurability constants depend only on t, H, and on the original sequence $(q_n)_n$.*

Our purpose now is to seek a stronger condition on $(q_n)_n$ than (32) which characterizes the fact that the solution has a specified almost-sure spatial modulus of continuity.

3.2. Type I (pathwise) characterization

3.2.1. General result and examples

We now describe how to use the first theorem to compare the almost-sure regularities of B and X. In what follows t is a fixed positive value, and with Lemma 4 in mind, we abusively use the notation q_n and s_n for $q_n(t)$ and $s_n(t)$, omitting the t. It is reassuring to note that from the proof of Theorems 2 and 3 in [12], q_n and s_n (and similar time-dependent constant used below) are bounded away from 0 and ∞ as soon as the same holds for t.

Let $Y = (I - \Delta)_x^{-H} B$. The operator $(I - \Delta)_x^{-H}$ on L^2 is defined by saying that

$$(I - \Delta)_x^{-H}[e^{in\cdot}](x) = e^{inx}/(1+n^2)^H.$$

Y can be interpreted as an "antiderivative of order $2H$" for B. For fixed $t>0$, the expansion of Y is a random Fourier series

$$Y(t,\cdot) = \sum_{n \in \mathbb{Z}} \sqrt{r_n} e_n(\cdot) W_n'$$

where, by Lemma 4, $r_n = r_n(t)$ is commensurate with s_n:

$$s_n \asymp r_n. \tag{33}$$

Now assume that Y has for fixed t, an almost-sure uniform spatial modulus of continuity f. Let δ_Y be the canonical metric function for $Y(t,\cdot)$. Assume δ_Y satisfies Assumption A and Condition B. Then by Theorem 1 part (a), for some $K>0$, for all α small enough,

$$Kf(\alpha) \geqslant \int_0^\infty \delta_Y(\min(e^{-x^2}, \alpha))\, dx.$$

Because of formula (2) and the fact that $s_n \asymp r_n$ we get that for some (possibly different) constant K, for all small r,

$$\begin{aligned}\delta_Y(r) &= \sum_{n=1}^{\infty} \sqrt{r_n}(1-\cos(nr)) \\ &\geqslant K \sum_{n=1}^{\infty} \sqrt{s_n}(1-\cos(nr)) \\ &= K\delta_X(r),\end{aligned}$$

where δ_X is the canonical metric function for $X(t,\cdot)$. Thus for some constant K,

$$Kf(\alpha) \geqslant \int_0^{\infty} \delta_X(\min(e^{-x^2},\alpha))dx = f_{\delta_X}(\alpha),$$

Now use part (b) of Theorem 1: since $\lim_{0^+} f = 0$, the same holds for f_{δ_X}, and we get that f_{δ_X} is an almost-sure uniform modulus of continuity for $X(t,\cdot)$ up to a constant. Since $Kf \geqslant f_{\delta_X}$ we get that f itself is an almost-sure uniform modulus of continuity for $X(t,\cdot)$. Since $s_n \asymp r_n$, the roles of X and Y can be swapped, which proves the following theorem, modulo the statements in the Hölder case, which are clear given Corollary 1.

Theorem 3. *Let X, Y be as above, relative to B. Let the function δ_Y be defined by*

$$\delta_Y(r) = \sum_{n=1}^{\infty} q_n \frac{1}{n^{4H}}(1-\cos(nr))$$

We assume Condition B *hold for δ_Y. Let f be an increasing continuous function on $\mathbb{R}_+$ with $\lim_{0^+} f = 0$. For any fixed $t>0$, f is, up to a multiplicative constant, an almost-sure uniform modulus of continuity for $Y(t,\cdot)$ if and only if f is, up to a multiplicative constant, an almost-sure uniform modulus of continuity for $X(t,\cdot)$. Also, δ_Y is the canonical metric function of $Y(1,\cdot)$, and the function f_Y defined by*

$$f_Y(\alpha) = \int_0^{\infty} \delta_Y(\min(e^{-x^2},\alpha))\,dx$$

is also an almost-sure uniform modulus of continuity for both $Y(t,\cdot)$ and $X(t,\cdot)$, and is bounded above by a constant multiple of f.

In the Hölder case $\delta_Y(r) = r^{\alpha}$ for some $\alpha \in (0,1)$, Condition B *is not satisfied. However, we can assert that if f is an almost-sure uniform modulus of continuity for $Y(t,\cdot)$, then $\bar{f}(r) = f(r)\log(1/r)$ is, up to a multiplicative constant, an almost-sure uniform modulus of continuity for $X(t,\cdot)$, and f_Y is bounded above by a constant multiple of $f\log(1/\cdot)$; the same statements hold if one exchanges the roles of X and Y.*

We now consider some illustrative examples. In this development, we omit the appellation "almost-sure, uniform" when talking about spatial moduli of continuity.

- Assume that Z is a Gaussian field on $R_+ \times S^1$ that is fBm in time with parameter H and has fBm behavior in space with parameter H'. According to the summability Theorem 2, we see it is safe to assume that Z is of the form $\sum_n \sqrt{z_n} e_n(x) \beta_n^H(t)$ where $z_n \asymp n^{-2H'-1}$, or that at least z_n is of the order of $n^{-2H'-1}$ in the power scale. Let then $B^H = (I - \Delta)^{1/2} Z$. In this way, $q_n \asymp n^{1-2H'}$, and we have a generalized random field $\partial B / \partial t$ in the SHE (1) that is the first derivative of fBm both in time and in space (with different Hurst parameters), and can thus be considered as space-time fractional noise. Then we have $Y = (I - \Delta)^{1/2-H} Z$. From this (or directly from [12] or from Lemma 4) we see that there is existence of X if and only if $H' > 1 - 2H$. Certainly if $H' < 1 - 2H$, we do not have existence. But if we cannot guarantee that H' actually exceeds $1 - 2H$—that is, if in the Hölder scale it looks like $H' = 1 - 2H$—we can still have a solution with a certain amount of regularity. Indeed Theorem 3 asserts that only a spatial "derivative" of Z of order $1 - 2H$ needs to exist; for example, the spatial modulus of continuity of X is commensurate with $f_\alpha(r) = (\log(1/r))^{-\alpha}$ for some fixed $\alpha > 0$ if and only if the same holds for the spatial derivative or order $1 - 2H$ of Z. In this situation, Z is spatially more regular than fBm of parameter $1 - 2H$, but is not spatially fBm for any parameter $H' > 1 - 2H$.
- Consider now the case where Z is spatially fBm with parameter $H' > 1 - 2H$. One can check that a sharp spatial modulus of continuity for Y is $f(r) = r^{H'-1+2H} \log^{1/2}(1/r)$. Theorem 3 then asserts the following.
  - For Eq. (1) driven by space–time fractional noise with Hurst parameters H and H' respectively, the evolution solution X admits

$$f(r) = r^{H'-1+2H} \log^{1/2}(1/r)$$

    as a modulus of continuity. Note here that the full force of the characterization is being used because we start with a bound on the canonical metric of the potential, and can reprove Theorem 3 without needing to invoke the "lower bound" portion (a) of Theorem 1 and Corollary 1 (see Corollary 2).
  - If the evolution solution of Eq. (1) has fractional Brownian regularity in space, in the sense that for some $H'' \in (0,1)$, it admits $f(r) = r^{H''} \log^{1/2}(1/r)$ as a spatial modulus of continuity, then the equation's potential is the spatial derivative of a Gaussian field which admits $\bar{f}(r) = r^{H''+1-2H} \log^{3/2}(1/r)$ as a spatial modulus of continuity.
  - In the previous "necessary condition implication", we do not know if the logarithmic corrections can be disposed of, because we do not know whether Corollary 1 is sharp. However, in the Hölder scale, these corrections can be viewed as irrelevant.
- When $H = H' = 1 - 2H = 1/3$, since then Y cannot be Hölder continuous, we can try to invoke Theorem 1 without needing Corollary 1. We get the following.

- For Eq. (1) driven by space–time fractional noise with common Hurst parameters $1/3$ in time and space, the evolution solution X does not exist. This can be established using the results of [12] only. It can be considered as the limit of non-existence of X...
- However, in the case $H = 1/3$, assume Y admits $f(r) = r^{1/3}\tilde{f}(r)$ as a spatial modulus of continuity where $\lim_0 \tilde{f} = 0$ and $\tilde{f}(r) \gg r^\alpha$ for all $\alpha > 0$. Then $\tilde{f}(r)$ is a spatial modulus of continuity for X, and the converse holds, still assuming $H = 1/3$.

3.2.2. Condition B is morally not necessary for regularity

A slightly weaker version of Theorem 3 can be formulated without Condition B if one is willing to change from a *pathwise* to a *distributional* hypothesis. The distributional hypothesis we make here is that the function f_Y, which can be calculate directly from the law of B, is continuous at 0. The final conclusion of the corollary seems to be a pathwise statement, but we consider it a hybrid Type I–Type II characterization because f_Y is characterized by the distribution of B. In other words, an empirical measurement of the almost-sure modulus of continuity of X cannot be guaranteed to apply also to B without Condition B; the only almost sure modulus of continuity they are guaranteed to share is f_Y, which requires a precise knowledge of B's distribution.

Corollary 2. *Let B, X, Y, δ_Y, f_Y be as in Theorem* 3, *and let δ_X and f_X be defined similarly relative to X. We have*

$$\lim_{r \downarrow 0} f_Y(r) = 0 \Leftrightarrow \lim_{r \downarrow 0} f_X(r) = 0.$$

In that situation $X(1,\cdot)$ and $Y(1,\cdot)$ share both f_Y and f_X as a.s. uniform moduli of continuity. Consequently f_Y is an a.s. uniform spatial modulus of continuity of continuity for X if and only if the same holds for Y. This last statement holds as soon as one knows that either X or Y is a.s. continuous, or even merely a.s. bounded.

Proof. The proof follows from Theorem 3 part (b) and the fact, made trivial by relations (2), (6), and (33), that $f_Y \asymp f_X$. Note that the last statement is a well-known fact from the general theory of homogeneous Gaussian processes (see [1]). □

3.3. Type II characterization: summability interpretation

3.3.1. Statement of the summability results

The next lemma shows how to invert the formula that gives the almost-sure modulus of continuity from the canonical metric function. In the notation of Theorem 1, it is interesting to note that this lemma implies that $\delta \mapsto f_\delta$ is a bijective linear map.

Lemma 5. *Let δ be an increasing continuous function on $\mathbb{R}_+$ with* $\lim_{0^+}\delta = 0$. *Let*

$$f(\alpha) = f_\delta(\alpha) := \int_0^\infty \delta(\min(e^{-x^2}, \alpha))\,dx = \int_0^{\delta(\alpha)} \sqrt{\log 1/\check{\delta}(\varepsilon)}d\varepsilon. \tag{34}$$

Then

$$\begin{aligned}\delta(\alpha) = \delta_f(\alpha) &= \int_0^{f(\alpha)} (\log 1/\check{f}(\varepsilon))^{-1/2}\,d\varepsilon \\ &= \int_0^\alpha f'(r)(\log(1/r))^{-1/2}\,dr \end{aligned}\tag{35}$$

$$= f(\alpha)(\log(1/\alpha))^{-1/2} - \int_0^\alpha f(r)(\log(1/r))^{-3/2}(2r)^{-1}\,dr \tag{36}$$

Proof. Trivial. □

This lemma poses a difficulty in terms of monotonicity of δ with respect to f, because of the negative term in expression (36). However, by virtue of (35), some knowledge of f' may help exploit this expression in a monotone way.

Theorem 4. *Let f be an increasing continuous function on a neighborhood of* 0 *in* $\mathbb{R}_+$, *continuously differentiable everywhere except at* 0, *with* $\lim_{0^+} f = 0$. *Let δ be given by* (36). *Let B be defined by* (30), *and X, Y be as above relative to B. Let δ_Y be the canonical metric function of $Y(1,\cdot)$. Recall the following three statements, which are equivalent if Condition B holds for δ_Y*:

- (e) *for some fixed $t>0$, $X(t,\cdot)$ has a constant multiple of f as an almost-sure uniform modulus of continuity*;
- (e′) *for some fixed $t>0$, $Y(t,\cdot)$ has a constant multiple of f as an almost-sure uniform modulus of continuity*;
- (f) *for all $t>0$, $X(t,\cdot)$ and $Y(t,\cdot)$ both have a constant multiple of f as an almost-sure modulus of continuity*;

  If δ satisfies Condition C, *then* (e), (e′) *and* (f) *are implied by the following*:
- (g) *for any continuous, decreasing, differentiable function h on* $[0,1]$ *with* $\int_0^1 h(x)dx < \infty$,

$$\sum_n \frac{q_n}{n^{4H}} h\left(\delta\left(\frac{1}{n}\right)^2\right) < \infty. \tag{37}$$

For the converse, assume that $F(y) = f_Y \circ \check{f}$ is continuously differentiable at 0. *Then* (g) *is implied by* (e′). *If moreover δ_Y satisfies Condition* B *then* (g) *is implied by* (e). *If*

Condition B *is not satisfied*, (g) *is still implied by* (e) *provided one replaces*, *in* (37), *the expression* $\delta(1/n)$ *by* $\delta(1/n)\log(n)$.

Remark 4. In [14], a conjecture in the direction of Theorem 4 was formulated. The authors believed the above result would hold with $h(r) = r^{-1}$ in condition (g). This theorem shows that such a condition (g) would be too strong. In fact, one can say that the gap in regularity that is introduced by the stronger version of (g) translates into a factor of order $(\log(\delta^{-2}(r)))^{1/2}$; this factor is not visible in the Hölder scale, which explains why in [13], in which only the Hölder scale is considered, it had been possible to formulate necessary and sufficient conditions whose naive generalization would lead the authors to the slightly erroneous conjecture of [14].

Remark 5. In [13,14], the authors had formulated results similar to Theorems 3 and 4 in the belief that a Type II characterization was a necessary intermediate step in the proof of a Type I characterization. The proofs we propose here show that the two types of characterizations can be established independently of each other.

Remark 6. The differentiability of F at 0 essentially brings no restriction. Indeed, if the measured empirical modulus of continuity f is such that $f \gg f_Y$, then we can easily see that $F'(0) = 0$ with continuity at 0. On the other hand, if it turns out that $f \asymp f_Y$, then since our moduli of continuity are only defined up to a multiplicative constant, one may as well assume that $f_Y = f$, in which case, as seen in Corollary 2 above and Corollary 3 below, Condition B and the condition on F' can be dispensed with. A situation in which neither $f \gg f_Y$ nor $f \asymp f_Y$ hold can perhaps be considered as pathological, with a discontinuous F' as an even more of a stretch.

3.3.2. *Proof of Theorem 4*

Step 1 (*converse*). We begin by proving the "Converse" part.

Step 1.a: (e′) ⇒ (g) *and* [*Condition* B *and* (e)] ⇒ (g). Assume first that Condition B holds for δ_Y. Then since the equivalence of (e) and (e′) follows from Theorem 3, we only need to prove (e′) implies (g). First note by Theorem 3, under condition B, with δ_Y the canonical metric function of $Y(1,\cdot)$, (e′) implies

$$f_Y(\alpha) := \int_0^\infty \delta_Y(\min(e^{-x^2}, \alpha))\, dx \leqslant Kf(\alpha) \tag{38}$$

for some constant $K > 0$. We wish for a similar statement relating δ_Y and δ.

Consider $F(y) = f_Y(\check{f}(y))$. By hypothesis, this function is continuously differentiable everywhere in a neighborhood of 0 in $\mathbb{R}_+$. Also note by (38) that $F(y) \leqslant Ky$. The following argument shows that we have, in a neighborhood of 0 in $\mathbb{R}_+$,

$$f_Y'(y) \leqslant 2Kf'(y). \tag{39}$$

Indeed, if this were not the case, there would exist a sequence $\{y_n\}_n$ decreasing to 0 such that

$$f'_Y(y_n) > 2Kf'(y_n).$$

Since we have $F'(y) = f'_Y(\check{f}(y))/f'(\check{f}(y))$, for all y in a neighborhood of 0 in $\mathbb{R}_+$, this means that with $z_n = f(y_n)$, we get $F'(z_n) > 2K$ for all n. Now by the mean value theorem on each interval $[0, y_n]$, there exists a value $\xi_n \in (0, y_n)$ such that $F(y_n) = y_n F'(\xi_n)$. Therefore $F'(\xi_n) \leqslant K$ for all n. We thus have two sequences $\{z_n\}$ and $\{\xi_n\}$ that tend to 0 along which F' cannot tend to a common value, contradicting the hypothesis that F is continuously differentiable at 0.

Therefore, using (35) and (39), in a neighborhood of 0, we have,

$$\delta_Y(\alpha) \leqslant 2K\delta(\alpha). \tag{40}$$

Recall that $r_n = q_n n^{-4H}$ are the coefficients of the expansion of δ_Y in the form of (2). Inequality (40) allows us to use the implication "(c) implies (d)" from Theorem 2 with $g = \delta^2$ to conclude that for any decreasing, differentiable function h on $[0, 1]$ with $\int_0^1 h(x)\, dx < \infty$,

$$\sum_n \frac{q_n}{n^{4H}} h\left(\delta\left(\frac{1}{n}\right)^2\right) < \infty.$$

We have thus proved (e′) implies (g) using Condition B.

*Step*1.b: (e) $\Rightarrow$ (g) *with log correction factor*. If (e) is assumed without Condition B, the Hölder-case portion of Theorem 3 proves that $\bar{f}(r) := f(r)\log(1/r)$ exceeds f_Y up to a multiplicative constant. Then applying the implication (e′) $\Rightarrow$ (g), which we have just established, we obtain inequality (37) with δ replaced by $\delta_{\bar{f}}$:

$$\sum_n \frac{q_n}{n^{4H}} h\left(\delta_{\bar{f}}\left(\frac{1}{n}\right)^2\right) < \infty. \tag{41}$$

However we can write

$$\begin{aligned}
\delta_{\bar{f}}(\alpha) &:= \bar{f}(\alpha)(\log(1/\alpha))^{-1/2} - \int_0^\alpha \bar{f}(r)(\log(1/r))^{-3/2}(2r)^{-1}\, dr \\
&= \log(1/\alpha) f(\alpha)(\log(1/\alpha))^{-1/2} - \int_0^\alpha \log(1/r) f(r)(\log(1/r))^{-3/2}(2r)^{-1}\, dr \\
&\leqslant \log(1/\alpha) f(\alpha)(\log(1/\alpha))^{-1/2} - \log(1/\alpha)\int_0^\alpha f(r)(\log(1/r))^{-3/2}(2r)^{-1}\, dr \\
&= \delta(\alpha)\log(1/\alpha).
\end{aligned}$$

Since h in (41) is decreasing, we obtain

$$\sum_n \frac{q_n}{n^{4H}} h\left(\delta\left(\frac{1}{n}\right)^2 \log^2(n)\right) < \infty,$$

which completes Step 1.b.

Step 2: (g) $\Rightarrow$ (f): For the first statement of the theorem, assuming (g), and since $r_n = q_n n^{-4H}$ are still the coefficients of the expansion of δ_Y, assuming Condition C, the implication "(d) implies (c)" in Theorem 2 proves that there exists $K > 0$ such that the inequality $\delta_Y(\alpha) < K\delta(\alpha)$ holds for small α. Applying the transformation $\delta \mapsto f_\delta$ to this inequality yields for small α,

$$f_Y(\alpha) \leqslant K f_\delta(\alpha) = Kf(\alpha),$$

where the last equality is by the definition of δ. Theorem 3 could now be used directly to conclude on the moduli of continuity of X and Y. However, our claim is that condition B is not needed. To see this, note that by hypothesis $\lim_{0+} f = 0$, so that the previous inequality justifies invoking Corollary 2, which does not require any conditions, and implies here that both X and Y share both f and f_Y as a.s. uniform spatial moduli of continuity. The theorem is proved.

3.3.3. Condition B is morally not necessary for summability

The presence of Condition B and the differentiability condition on F in Theorem 4 masks the fact that the summability condition (g) is a necessary condition for continuity even when F is not differentiable at 0, and even when Condition B is not satisfied. We can state this by rephrasing Theorem 4 in a slightly weaker form, assuming only condition C: The resulting corollary operates in the same way as our hybrid theorem Corollary 2: the usage of the corollary below assumes that we can estimate f_Y sharply, i.e. that we have a handle on the distribution of B.

Corollary 3. *Let B be as in* (30), *and assume that $Y = (I - \Delta)^{-H} X$ has canonical metric function δ satisfying Condition* C. *Define $f = f_\delta$ as in* (34). *Then Conditions* (e′) *and* (g) *are equivalent. Moreover, they are equivalent to each of the following*:

(i) $Y(1,\cdot)$ *is almost-surely bounded*;
(ii) $Y(1,\cdot)$ *is almost-surely continuous*;
(iii) $\lim_{0+} f_\delta = 0$

All these conditions are also equivalent to (e) *and to* (f).

Proof. That (e′) is equivalent to (i), (ii) and (iii) is a well-known fact from the general theory of homogeneous Gaussian processes (see [1]). The only part that has not already been established is (e′) $\Rightarrow$ (g). This follows by the proof of the same implication in the proof of Theorem 4 because here since $\delta = \delta_Y$, inequality (40) holds automatically, so there is no need to invoke the differentiability of F, and for the same reason there is no need to use part (a) of Theorem 1, which makes Condition B superfluous. The last statement is obvious by Corollary 2. □

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