Journal of Functional Analysis **169**, 559–603 (1999)
Article ID jfan.1999.3486, available online at http://www.idealibrary.com on IDEAL®

On Space-Time Regularity for the Stochastic Heat Equation on Lie Groups

Samy Tindel

Département de Mathématiques, Institut Galilée-Université Paris 13, Avenue J. B. Clément, 93430 Villetaneuse, France
E-mail: tindel@zeus.math.univ-paris13.fr

and

Frederi Viens

Department of Mathematics, University of North Texas, P.O. Box 305118, Denton, Texas 76203-5118
E-mail: fviens@unt.edu

Communicated by Paul Malliavin

Received May 30, 1999; accepted July 20, 1999

We consider the stochastic heat equations on Lie groups, that is, equations of the form $\partial_t u = \Delta_x u + b(u) + F(u)\, \dot{W}$ on $\mathbb{R}_+ \times G$, where G is a compact Lie group, Δ is the Laplace–Beltrami operator on G, b and F are Lipschitz coefficients, and where $\dot{W}$ is a Gaussian space-correlated noise, which is white-noise in time. We find necessary and sufficient conditions on the space correlation of $\dot{W}$ such that u is an L^2 or Hölder-continuous function in the spatial variable x, using some basic tools of stochastic analysis and harmonic analysis on the Lie group G. © 1999 Academic Press

Key Words: parabolic stochastic partial differential equations; Lie groups; harmonic analysis; regularity of Gaussian processes.

1. INTRODUCTION

This article proposes to study the weakest conditions under which a stochastic parabolic partial differential equation on a compact Lie group has a unique function-valued solution, and under which this solution is continuous. We will study the *evolution* (or *mild*, or *semigroup*) form of the

0022-1236/99 $30.00

stochastic P.D.E.; that is, we seek the solution to the stochastic evolution equation

$$u(t,x)=\int_G u_0(y)\,H_t(x,y)\,dy+\int_0^t\int_G b(s,y,u(s,y))\,H_{t-s}(x,y)\,ds\,dy$$

$$+\int_0^t\int_G H_{t-s}(x,y)\,F(s,y,u(s,y))\,W(ds,dy). \tag{1}$$

Here G is a compact Lie group, dy denotes Haar measure on G. $H_t(x,y)$ is the heat kernel for the Laplace–Beltrami operator Δ on G (see Subsection 2.3), W is a real-valued centered Gaussian orthogonally scattered generalized function on $\mathbb{R}_+\times G$. We assume that it is white-noise in time, and spatially homogeneous, but it is not required even to be as regular as an L^2-measure in the variable y. In this sense it may be more spatially irregular than space-time white noise (see Subsection 2.5 for the precise definition of W).

Under the assumption that $W(s,dy)$ is a measure in the variable y with a density with respect to Haar measure, denoted by $\tilde{W}(s,y)$, if $\tilde{W}$ is P-a.s. of class C^2 in G and there exists a solution u to (1) that is P-a.s. of class C^2 in G, then it easy to show that it satisfies the following bona fide stochastic P.D.E.,

$$u(t,x)=u_0(x)+\int_0^t \Delta_x u(s,x)\,ds+\int_0^t F(s,x,u(s,x))\,\tilde{W}(ds,x)$$

$$+\int_0^t b(s,y,u(s,y))\,ds. \tag{2}$$

It is in this sense that a solution to equation (1) is a weak form of the above stochastic P.D.E. This being said, we will discuss the existence of a strong solution to (2) no further.

Our work fits into the general project of stochastic evolution equations, which have been studied in an abstract setting in the 1980s (see [5] and references therein). Recently, several authors ([19] for parabolic equations; [4, 18] for the wave equation) have taken up giving explicit sufficient conditions under which the evolution form of a stochastic P.D.E. in Euclidean space admits a function-valued solution. Their results also include explicit sufficient conditions for spatial Hölder-continuity of the solution. The conditions in the latter two papers are formulated in the case of covariance functions $\tilde{Q}$ given by $\tilde{Q}(dx,dy)=f(|x-y|)\,dx\,dy$. The condition for existence is proved to be sharp in [18].

Another popular approach to weakened forms of stochastic P.D.E.'s is the so-called *weak* formulation, in which equation (2) is integrated by parts against test functions: [22] (see also the references therein) reports the existence, depending on the dimension of space, of distribution-valued or function-valued *weak* solutions in the case of space-time white-noise (i.e., $EW(1, dx)\, W(1, dy) = \delta(dx, dy)$). The *weak* formulation has seen a recent regain of interest in the setting of measure-valued solutions, as in [16, 13]. Although in principle, *evolution* and *weak* formulations are morally equivalent (as evidenced for example by the fact that the construction of *weak* solutions in [22] uses the semigroup techniques of evolution equations), the techniques employed in [16, 13] show that the *weak* formulation is not as well-tailored to telling when a function-valued solution (i.e., a measure with a density) exists. Note however the very successful treatment of solutions in Sobolev and Hölder spaces in [14] by means of an analytic approach to weak solutions.

Some of our techniques and goals are similar to those followed in [18]. However, we have tried to delve deeper in the understanding of the conditions we impose. We consider all compact Lie groups in order to illustrate that despite the technical difficulties that non-commutativity entails, the phenomenon observed in [19, 4, 18] is not specific to Euclidean space. The use of compact Lie groups actually has two advantages: the lack of any boundary conditions in space, and the compactness, which implies that harmonic analysis in G takes the form of Fourier series. In order to make efficient use of stochastic calculus, we seek $L^2(G)$-valued solutions to (1). Instead of a condition of the form $\tilde{Q}(dx, dy) = f(|x-y|)\, dx\, dy$, which may be labelled as *isotropy*, we use the more general assumption of *spatial homogeneity* of W, i.e., that $\tilde{Q}(dx, dy)$ depends only on the product xy^{-1}. Since $\tilde{Q}$ need not be a measure, but merely a generalized bilinear function, we define a generalized notion of homogeneity. This alone is enough to exploit the theory of Fourier series. Thanks to necessary and sufficient conditions for the continuity of homogeneous Gaussian processes, we establish the sharpness of all the conditions we impose, and interpret them in terms of almost-sure spatial regularity for W.

We have received some preprints ([20, 12]) which deal with the wave and heat equation in flat space and uses a general spatially homogeneous noise, exploiting harmonic analysis on $\mathbb{R}^d$ and T^d, much like we have done in non-commutative space, including the consideration of a general (positive definite) distribution covariance. We have just been made aware of ongoing work by P. L. Chow [2] regarding the regularity of the stochastic wave equation, examining general (non-Gaussian) sufficient conditions under which Hölder continuity may be obtained.

This paper is organized as follows. A detailed review of relevant material from Lie Groups, including harmonic analysis and properties of the heat kernel H, is given in Section 2. Section 3 is devoted to proving existence and uniqueness of a function-valued solution under a sharp condition on the Fourier coefficients of Q. Hölder continuity of the solution is established in Section 4. In Section 5, we interpret the condition for existence as the existence of a pathwise spatial "antiderivative" of W; the condition for Hölder-continuity is interpreted as the fact that W's "antiderivative" be almost-surely Hölder-continuous. All conditions are proved to be sharp insofar as they are necessary in the linear additive case ($F=1, b=0$).

2. PRELIMINARIES ON COMPACT LIE GROUPS

2.1. *General Notations*

We shall consider here a connected compact Lie group G, that is a group with a C^∞-manifold structure, such that the multiplication (resp. the inverse operation) is a C^∞ function from $G \times G$ to G (resp. from G to G). Let us denote by e the identity element of G as a group. Then the Lie algebra of G, that is the set of left-invariant vector fields on G, is in one-to-one correspondence with T_eG (the tangent space of G at e), and we shall denote both of them by $\mathcal{G}$. We set $d=\dim(G)=\dim(\mathcal{G})$. Recall also that in the case of compact connected Lie group, the exponential map is defined on all of $\mathcal{G}$, and is onto from $\mathcal{G}$ to G.

For any C^∞-manifolds M and N, and for any differentiable function $f: M \to N$, we shall denote by $(df)_x$ the differential of f at a point $x \in M$. For a given $g \in G$, set

$$\begin{aligned} I_g\colon G &\to G \\ h &\mapsto ghg^{-1}. \end{aligned}$$

Then $(dI_g)_e\colon \mathcal{G} \to \mathcal{G}$ is an automorphism of $\mathcal{G}$, called the adjoint representation of g, and denoted by $\mathrm{Ad}(g)$.

A Riemannian structure can be given to G by the definition of a scalar product $\langle .; . \rangle$ on $\mathcal{G}$. In the case of a compact Lie group, this scalar product can be chosen to be Ad-invariant, which means that for every $X, Y \in \mathcal{G}$ and every $g \in G$, we have

$$\langle \mathrm{Ad}(g) X; \mathrm{Ad}(g) Y \rangle = \langle X; Y \rangle.$$

On the Riemannian manifold obtained, we can construct a unique volume element, called the Haar measure and denoted by dx, such that the following properties are satisfied:

(1) For any $f \in L^1(G, dx)$ and any $g \in G$,

$$\int_G f(x)\, dx = \int_G f(gx)\, dx = \int_G f(xg)\, dx = \int_G f(x^{-1})\, dx. \tag{3}$$

(2) The total volume of G is one, that is,

$$\int_G \mathbf{1}\, dx = 1.$$

We shall denote the Haar measure of any Borel subset A of G by $|A|$.

Moreover, for this specific Riemannian metric, the geodesics are the one-dimensional subgroups, and the manifold G is complete. Let us call $\rho\colon G\times G\to\mathbb{R}_+$ the metric associated to the scalar product we have just defined. Then ρ satisfies

(1) For any x, $y\in G$,

$$\rho(x, y)=\rho(e, y^{-1}x).$$

(2) If $x=\exp(X)$ for a $X\in\mathcal{G}$, then

$$\rho(e, x)=|X|_{\mathcal{G}}=\langle X; X\rangle^{1/2}.$$

(3) There exist constants c_1 and c_2 such that for all $r\geqslant 0$, the Haar measure $|B_\rho(e, r)|$ of the ρ-ball of radius r centered at the identity is estimated as

$$c_1 r^d\leqslant |B_\rho(e, r)|\leqslant c_2 r^d. \tag{4}$$

In the remainder of the paper, we shall also denote $\rho(e, x)$ by $\rho(x)$ and $|.|_{\mathcal{G}}$ by $|.|$ when this is not an ambiguous notation. Let us call D_G the diameter of G, $\Re c$ the real part of a complex number c, and $\Im c$ its imaginary part. For a given function $\phi\colon G\to\mathbb{R}$, we shall also call $\check{\phi}$ the function defined on G by

$$\check{\phi}(g)=\phi(g^{-1}). \tag{5}$$

2.2. *Representation Theory*

We only give here a brief survey of this subject. We refer to [8, 10, 21] for further details. We shall recall first the following fundamental definitions:

Definition 2.1. Let G be a compact connected Lie group.

(1) A *unitary representation* of G is a strongly continuous homomorphism π from G into the group $U(H_\pi)$ of unitary operators of some Hilbert space H_π.

(2) Two unitary representations π_1 and π_2 of G are called *equivalent*, denoted by $\pi_1\simeq\pi_2$, if there exists an isometry A of H_{π_1} onto H_{π_2} satisfying

$$A\pi_1(g)=\pi_2(g)A$$

for every $g \in G$. The relation $\simeq$ is in fact an equivalence relation.

(3) A unitary representation π is called *irreducible* if $H_\pi \neq 0$ and if the only invariant closed subspaces of H_π by π are H_π and $\{0\}$.

It is known that the irreducible representations of a compact Lie group G are finite-dimensional, and that any unitary representation of G is the direct sum of irreducible representations. The set of all equivalence classes of irreducible unitary representations of G is denoted by $\hat{G}$ and is called the dual of G. Within this set of equivalence classes, we denote by **1** the class of constant representation $\mathbf{1}: G \to \mathbb{R}$ defined by $\mathbf{1}(x) = 1$ for all $x \in G$. We shall put again π for the generic representative of an equivalence class in $\hat{G}$, and set $d_\pi = \dim(H_\pi)$, $\chi_\pi(g) = \operatorname{tr}(\pi(g))$ for any $g \in G$. The function χ_π is called the *character* of the representation π. Note that an irreducible representation π can be seen as a $d_\pi \times d_\pi$-matrix-valued function defined on G. The generic element of that matrix will be designated by π_{ij}. We will usually omit the words "irreducible" an "unitary" when referring to a representation in $\hat{G}$.

Let $d\pi_e$ be the differential of π at e, so that for $X \in \mathcal{G}$, $d\pi_e(X)$ is a $d_\pi \times d_\pi$ square matrix. Let T be a maximal torus of G, denote by $\mathcal{T}$ its Lie algebra, and by $\mathcal{T}^*$ the dual of $\mathcal{T}$. For the sake of simplicity, the duality relation between $\mathcal{T}$ and $\mathcal{T}^*$ will be denoted by $\langle .,. \rangle$, just like the scalar product in $\mathcal{T}$. The image of $\mathcal{T}$ by $d\pi_e$ is a commutative set of skew-Hermitian matrices, and hence, for any π in $\hat{G}$, there is a set of d_π vectors $\{\mu_\pi^{(k)}: k = 1, ..., d_\pi\}$ on $\mathcal{T}^*$ such that for some unitary matrix U,

$$U\, d\pi_e(V)\, U^{-1} = i \operatorname{Diag}(\langle \mu_\pi^{(1)}; V \rangle, ..., \langle \mu_\pi^{(d_\pi)}; V \rangle),$$

for all $V \in \mathcal{T}$. The vectors (linear forms) $\mu_1, ..., \mu_{d_\pi}$ are called the *weights* of π. The lattice of all weights for all the representations $\pi \in \hat{G}$ will be denoted by P. One of the fundamental theorem of representation theory is the following: there is a region delimited by hyperplanes in $\mathcal{T}^*$ called the *dominant chamber* and denoted by D such that each point of $P \cap D$ is in one-to-one correspondance with an element $\pi \in \hat{G}$. Let us call h_π such an element. Then h_π is called the *highest weight* of π. It is shown in [10] that h_π is of maximal norm among the weights of π.

For each element v of the maximal torus T, a more explicit expression can be given for the character $\chi_\pi(v)$, for a given representation $\pi \in \hat{G}$,

$$\Re \chi_\pi(v) = \sum_{k=1}^{d_\pi} \cos \langle \mu_\pi^{(k)}; V \rangle. \tag{6}$$

In particular, the next proposition easily follows.

PROPOSITION 2.2. *For any $\pi \in \hat{G}$, and any $v \in G$,*

$$d_\pi - \Re\chi_\pi(v) \leqslant \frac{d_\pi\,|h_\pi|^2}{2}\,\rho^2(v).$$

Proof. If $V \in \mathcal{T}$ and $v = \exp V \in T$, since $1-\cos(u) \leqslant u^2/2$, we get

$$d_\pi - \Re\chi_\pi(v) \leqslant \frac{1}{2}\sum_{k=1}^{d_\pi} \langle \mu_\pi^{(k)}; V\rangle^2 \leqslant \frac{d_\pi}{2}\,|h_\pi|^2\,|V|^2_{\mathcal{G}} = \frac{d_\pi}{2}\,|h_\pi|^2\,\rho^2(v).$$

We then get the general result for $v \in G$ noticing that v is always the conjugate of a point of the maximal torus T, since the conjugation is an isometry on G. ∎

Notice that in the remainder of the paper, when this does not lead to any confusion, we shall write χ_π instead of $\Re\chi_\pi$.

2.3. *Harmonic Analysis on Compact Lie Groups*

This quick overview is taken mainly from [8]. The main result concerning harmonic analysis on compact Lie groups is given by the following proposition.

PROPOSITION 2.3. *Let G be a compact Lie group. Let $L^2(G)$ be the Hilbert space of all square integrable functions on G against the Haar measure. Let $ZL^2(G)$ be the subspace of $L^2(G)$ formed of the central functions, i.e., those functions f such that $f(xy) = f(yx)$ for all $x, y \in G$. Then*

1. $\{d_\pi^{1/2}\pi_{ij};\, i, j = 1, ..., d_\pi, \pi \in \hat{G}\}$ *is an orthonormal basis for $L^2(G)$.*
2. *The characters $\{\chi_\pi : \pi \in \hat{G}\}$ form an orthonormal basis for $ZL^2(G)$.*

The functions π_{ij} and the characters χ_π are also related to the Laplace operator on G: let $X_1, ..., X_d$ be an orthonormal basis of $\mathcal{G}$. The Laplacian on G is given by

$$\Delta = \sum_{i=1}^{d} X_i^2. \tag{7}$$

Note that Δ is a self-adjoint operator on $L^2(G)$, and that the expression (7) is independent of the chosen orthonormal basis. The eigenvalue decomposition of Δ is as follows:

PROPOSITION 2.4. *There is a fixed vector ξ in $\mathcal{T}$ such that for any representation π in $\hat{G}$, and any $i, j \in \{1, ..., d_\pi\}$, the function π_{ij} and the character χ_π are eigenfunctions of Δ, associated with the eigenvalue*

$$\lambda_\pi = \langle h_\pi ; h_\pi + \xi \rangle.$$

The vector ξ in this proposition is known as the "half-sum of the positive roots of G." This proposition shows that for large $|h_\pi|$, λ_π is of the order of $|h_\pi|^2$. The integer k is the "number of positive roots of G." Note that this asymptotics is relevant since the set $\{|h_\pi|\}_{\pi \in \hat{G}}$ is unbounded. In fact, it has no accumulation point other than infinity.

2.4. *Heat Kernel on Compact Lie Groups*

We shall give here some estimations for the heat kernel H on compact Lie groups, taken from [3], and show a basic property of H we shall use all along the remainder of the paper.

Let G be a connected compact Lie group. The heat kernel H on G is defined as the fundamental solution of the heat equation on G, which means that for any function f in $L^2(G)$ and a given $T > 0$, the solution of the equation

$$\partial_t u(t, x) = \Delta u(t, x), \qquad (t, x) \in [0, T] \times G,$$

with initial condition f, is given by

$$u(t, x) = H_t f(x) \equiv \int_G H_t(x, y) f(y) \, dy.$$

By left invariance of the Laplacian on G, it easily seen that H_t admits in fact a convolution kernel, called again H_t, such that $H_t \in L^2(G)$ and

$$H_t f(x) = \int_G H_t(xy^{-1}) f(y) \, dy.$$

Moreover, the symmetries of the Laplacian in G imply that for any $t > 0$ and any $x, y \in G$, we have

$$H_t(x^{-1}) = H_t(x), \qquad H_t(xy) = H_t(yx). \tag{8}$$

Note that the semi-group property of H_t can be written

$$\int_G H_t(xy^{-1}) H_s(yz^{-1}) \, dy = H_{t+s}(xz^{-1}) \tag{9}$$

for any $t, s > 0$ and $x, z \in G$.

It is shown in [3] that the following estimates for H hold:

PROPOSITION 2.5. *Let G be a compact connected Lie group, and $T>0$. Then, for every $t\in(0,T]$, $x\in G$ and every $i\in\{1,...,d\}$, we have, for some constants $c_1, ..., c_4>0$ and $d_1, ..., d_4>0$,*

$$H_t(x)\geqslant\frac{c_1}{t^{d/2}}\exp\left(\frac{-\rho^2(x)}{d_1 t}\right)$$

$$H_t(x)\leqslant\frac{c_2}{t^{d/2}}\exp\left(\frac{-\rho^2(x)}{d_2 t}\right)$$

$$\partial_t H_t(x)\leqslant\frac{c_3}{t^{d/2+1}}\exp\left(\frac{-\rho^2(x)}{d_3 t}\right)$$

$$X_i H_t(x)\leqslant\frac{c_4}{t^{(d+1)/2}}\exp\left(\frac{-\rho^2(x)}{d_4 t}\right),$$

where $X_1, ..., X_d$ is an orthonormal basis of $\mathscr{G}$.

We shall need the following property in the remainder of the paper.

PROPOSITION 2.6. *Let $V\in L^2(G)$, and $R\in ZL^2(G)$. For a $t\in(0,T]$, and any $x, a, b\in G$, set $H_t^x(y)=H_t(xy^{-1})$ and*

$$J_{t,x}(V)=\int_G\int_G H_t^x(y)\, V(\hat{y}^{-1}y)\, H_t^x(\hat{y})\, dy\, d\hat{y}$$

$$K_{t,a,b}(R)=\int_G\int_G H_t(a^{-1}zy)\, H_t(b^{-1}y)\, R(z)\, dy\, dz.$$

Then

$$J_{t,x}(V)=\int_G H_{2t}(v)\, V(v)\, dv$$

$$K_{t,a,b}(R)=\int_G H_{2t}(va^{-1}b)\, R(v)\, dv.$$

Proof. Use change of variable $y^{-1}x=v$ and $\hat{y}^{-1}x=\hat{v}$. Then

$$J_{t,x}(V)=\int_G\int_G H_t(v)\, V(\hat{v}v^{-1})\, H_t(\hat{v})\, dv\, d\hat{v}.$$

Using equalities (3) and (8), we hence get

$$\int_G H_t(v)\, V(\hat{v}v^{-1})\, dv = \int_G H_t(v^{-1})\, V(\hat{v}v)\, dv$$

$$= \int_G H_t(v)\, V(\hat{v}v)\, dv$$

$$= \int_G H_t(\hat{v}^{-1}v)\, V(v)\, dv.$$

Since relation (9) holds, we get

$$J_{t,x}(V) = \int_G dv\, V(v) \int_G d\hat{v}\, H_t(\hat{v}^{-1}v)\, H_t(\hat{v})$$

$$= \int_G H_{2t}(v)\, V(v)\, dv,$$

which ends the proof of the first equality. The second is proved using a similar computation: first use the change of variable $z = az'$, then use the fact that R is central, then change the variable again using $v = z'a$, and finally use the semigroup property (9). ∎

2.5. *Random Fourier Series in H_{-p}*

Throughout this paper, we are going to make heavy use of the Fourier representation of the Gaussian noise W. The purpose of this section is to establish this representation, thereby showing that the random Fourier series we use for W cover a wide class of generalized Gaussian noises, those which are spatially translation- and inverse-invariant in law.

One may define the Fourier expansion in G for all functions in the space of tempered distributions $\mathcal{S}'(G)$. However, Theorems 3.1 and 3.6 show that the class of noises W which lead to a function-valued solution to (1) is included in the class of noises whose spatial covariance is in H_{-p} for some $p \leqslant 2$ (see definition below). We thus only develop the corresponding harmonic analysis.

The material here is presumably fairly standard, and some of the ideas presented below are similar to those found in [17]. For the sake of completeness, we have chosen to give a detailed treatment of this topic, as we could not find any existing works that clearly contain the results we need. The proof of this section's main result is in the Appendix.

The matrix elements $\pi_{k,l}(x)$ of the irreducible unitary representations of G, their characters $\chi_\pi(x) = \operatorname{tr} \pi(x)$, and the eigenvalues λ_π of the Laplace operator, are as defined in Subsections 2.2 and 2.3.

DEFINITION 2.7. Let $p \in \mathbb{R}_+$. Let H_p be the Sobolev space

$$W^{p,2}(G) = \{f\colon X_{i_1} X_{i_2} \cdots X_{i_k} f \in L^2(G);\ k = 0, ..., p;\ i_j = 1, ..., d\}.$$

Let H_{-p} be its dual, i.e., the space of all continuous linear functionals on H_p.

The reader will verify that these spaces can also be defined by their Fourier series expansion as follows.

PROPOSITION 2.8. *Let $p \in \mathbb{R}$. A sequence $f = (f_{\pi, i, j})_{\pi \in \hat{G};\, i, j = 1, ..., d_\pi}$ of complex numbers defines a* Fourier series *f in G by the formula*

$$f = \sum_{\pi \in \hat{G}} \sum_{i, j = 1}^{d_\pi} d_\pi^{1/2} \pi_{i, j} f_{\pi, i, j}.$$

H_p is the set of all Fourier series f whose coefficients satisfy

$$\sum_{\hat{G}} \sum_{i, j = 1}^{d_\pi} |f_{\pi, i, j}|^2 (1 + \lambda_\pi)^p < \infty.$$

A Fourier series $Q = (q_{\pi, i, j})_{\pi \in \hat{G};\, i, j = 1, ..., d_\pi}$ defines a linear functional if the action of Q on the Fourier series f is given by the formula

$$Q(f) = \sum_{\hat{G}} \sum_{i, j = 1}^{d_\pi} \overline{q_{\pi, i, j}}\, f_{\pi, i, j}.$$

H_{-p} is the set of all such linear functionals with coefficients satisfying

$$\sum_{\hat{G}} \sum_{i, j = 1}^{d_\pi} |q_{\pi, i, j}|^2 (1 + \lambda_\pi)^{-p} < \infty.$$

Let $\mathscr{B}(\mathbb{R}_+)$ denote the set of Borel sets of $\mathbb{R}_+$. For the remainder of the paper, we shall consider a complete stochastic basis $(\Omega, \mathscr{F}, \mathscr{F}_t, P)$.

DEFINITION 2.9. Let $p \in \mathbb{N}$. Let $\mathscr{H}_{-p}$ be the class of real-valued centered Gaussian random fields W linearly indexed by H_p; i.e., $\mathscr{H}_{-p}$ is formed of every centered Gaussian field W whose generalized covariance function $\tilde{Q}(\phi, \psi) = E[W(\phi)\, W(\psi)]$ is a real-valued symmetric bilinear functional on H_p such that $0 \leqslant \tilde{Q}(\phi, \phi) < \infty$ for all $\phi \in H_p$.

Such a random field is called *homogeneous* if its distribution is invariant under shifts, i.e., if W and $\{W(\phi(h\,\cdot)) \colon \phi \in H_p\}$ have the same distribution for any fixed $h \in G$. It is called *inverse-invariant* if W has the same distribution as $\{W(\phi((\cdot)^{-1}))\colon \phi \in H_p\}$.

A Gaussian random field indexed by $\mathscr{B}(\mathbb{R}_+) \times H_p$ is said to be *spatially* $\mathscr{H}_{-p}$ and *white-noise in time* if its covariance has the following tensor-product form: given any two times $s, t \in \mathbb{R}_+$ and any two test functions $\phi, \psi \in H_p$,

$$E[W(ds, \phi)\, W(dt, \psi)] = \delta(ds, dt)\, \tilde{Q}(\phi, \psi),$$

where δ is Lebesgue measure concentrated on the diagonal of $\mathbb{R}_+$, $\{s, t):$ $s = t\}$, and the spatial covariance $\tilde{Q}$ is as above. Such a random field is called *spatially homogeneous* and/or *inverse-invariant* if it has those properties in the space variable.

From now on we assume W to be spatially $\mathscr{H}_{-p}$ and white-noise in time as defined above. The notation $\int_G \int_0^t W(ds, dy)\, a(s, y)$, used to define equation (1), is an abusive notation. Indeed it suggests that W is an L^2-measure in both parameters s, ϕ. This need not be the case: the class of spatial covariance functions identified in Theorem 3.1 may be larger than $M(G)$. For example, in the case of the circle goupe S^1, the proof of Remark 3.3 in the next section (see the Appendix) shows that the the covariance $Q := \sum_n \sqrt{n}\, \chi$ satisfies the existence and uniqueness conditions of Theorem 3.1. However, Q is not a measure, as it can be thought of as a fractional (half) derivative of the distribution $\sum_n \chi_n$ which is the Dirac mass at the origin. The proper way to understand Eq. (1) is as

$$u(t, x) = \int_G u_0(y)\, H_t(x, y)\, dy + \int_0^t W(ds, H_{t-s}(x, \cdot)\, F(s, \cdot, u(s, \cdot)))$$
$$+ \int_0^t \int_G b(s, y, u(s, y))\, H_{t-s}(x, y)\, ds\, dy. \tag{10}$$

Indeed, for any $\phi \in H_p$, $s \to W([0, s], \phi)$ is a standard scalar Brownian motion. The stochastic integral on the right-hand side is of Itô type, or more precisely, is the $L^2(\Omega)$-limit of its Riemann sums; moreover, we can and will assume that filtration $(\mathscr{F}_t)_{t \geqslant 0}$ is common to all Brownian motions, so that one should expect to find a solution that is $(\mathscr{F}_t)_{t \geqslant 0}$-adapted.

The assumption of spatial homogeneity of W is made in order to use the harmonic analysis on H_{-p}. Although we believe that this assumption is not necessary to solve equation (1), it is the only way we (and others, see [4, 18–20]) have found to formulate sharp conditions. The same is true for W's inverse-invariance, which follows from homogeneity in the case of an Abelian group. Recall from (5) that for any function f on G we denote $\check{f} := f \circ (\cdot)^{-1}$.

DEFINITION 2.10. For a $p \geqslant 0$, consider $Q \in H_{-p}$, and $\phi, \psi \in H_p$.

(1) The *convolution* of ϕ and ψ is defined by

$$\phi * \psi(z) = \int_G \phi(zy)\, \psi(y^{-1})\, dy.$$

(2) Q is called *positive* or *of positive type* if for all $\phi \in H_p$

$$Q(\phi * \overline{(\check{\phi})}) \geqslant 0.$$

(3) Q is called *central* if for all $\phi, \psi \in H_p$

$$Q(\phi * \psi) = Q(\psi * \phi).$$

The following proposition is elementary.

PROPOSITION 2.11. *A centered Gaussian field in $\mathscr{H}_{-p}$ is homogeneous if and only if its covariance is given by*

$$\tilde{Q}(\phi, \psi) = Q(\phi * \check{\psi}) \tag{11}$$

for some $Q \in H_{-p}$.

THEOREM 2.12. *Let $p \geqslant 0$. Assume W is a real-valued Gaussian field linearly indexed by H_p with possibly infinite covariance $\tilde{Q}$. The following four conditions are equivalent.*

(i) *W is a homogeneous and inverse-invariant centered Gaussian field in $\mathscr{H}_{-p}$.*

(ii) *$\tilde{Q}$ is given by* (11) *and there exists a sequence of non-negative numbers $(q_\pi)_{\pi \in \hat{G}}$ such that*

$$\sum_{\pi \in \hat{G}} q_\pi^2 (1 + \lambda_\pi)^{-p} < \infty, \tag{12}$$

and for all $\phi \in H_p$

$$Q(\phi) = \sum_{\pi \in \hat{G}} q_\pi \int_G \chi_\pi(z)\, \phi(z)\, dz. \tag{13}$$

(iii) *There exists a sequence of non-negative numbers $(q_\pi)_{\pi \in \hat{G}}$ such that* (12) *holds and a family of standard normal real or complex random variables $(W^\pi_{k,l})^{\pi \in \hat{G}}_{k,l=1,\dots,d_\pi}$ such that for all $\phi \in H_p$, we have the equality in $L^2(\Omega)$*

$$W(\phi) = \sum_{\pi \in \hat{G}} q_\pi^{1/2} \sum_{k,l=1}^{d_\pi} W^\pi_{k,l} \int_G \phi(x)\, \pi_{k,l}(x)\, dx. \tag{14}$$

$W^{\pi}_{k,l}$ is real-valued iff $\pi \simeq \bar{\pi}$, i.e. iff π can be chosen real-valued. Moreover $W^{\pi}_{k,l}$ and $W^{\pi'}_{k',l'}$ are independent unless $k=k'$, $l=l'$ and $\pi' \simeq \pi$ or $\bar{\pi}$. When $\pi' \simeq \bar{\pi}$, these variables are conjugate.

(iv) *$\tilde{Q}$ is given by* (11) *and Q is central and of positive type in H_{-p}.*

Proof. See the Appendix. ▮

We have the immediate

COROLLARY 2.13. *Let $p \geqslant 0$. Assume that W is a real-valued Gaussian random field indexed by $\mathscr{B}(\mathbb{R}_+) \times H_p$ that is white-noise in the time variable, such that $W([0,t],\phi)$ is $\mathscr{F}_t$-measurable, with possibly infinite spatial covariance $\tilde{Q}$. The previous proposition holds, if we replace statement* (i) *and* (iii) *by*

(i)' *W is spatially $\mathscr{H}_{-p}$ and white-noise in time, and is spatially homogeneous and inverse-invariant.*

(iii)' *There exists a sequence of non-negative numbers $(q_\pi)_{\pi \in \hat{G}}$ such that* (12) *holds and a family of standard real- or complex-valued Brownian motions $\{W^{\pi}_{k,l}(t) : t \geqslant 0\}^{\pi \in \hat{G}}_{k,l=1,\ldots,d_\pi}$ such that we have the equality in $L^2(\Omega)$*

$$W(f) = \sum_{\pi \in \hat{G}} q_\pi^{1/2} \sum_{k,l=1}^{d_\pi} \int_{\mathbb{R}_+} W^{\pi}_{k,l}(ds) \int_G f(s,x)\, \pi_{k,l}(x)\, dx. \tag{15}$$

for all $f : \mathbb{R}_+ \times G \to \mathbb{R}$ in $L^2(\mathbb{R}_+; H_p)$. $W^{\pi}_{k,l}(t)$ is real-valued iff $\pi \simeq \bar{\pi}$, i.e., iff π can be chosen real-valued. Independence of $W^{\pi}_{k,l}(t)$ and $W^{\pi'}_{k',l'}(t)$ holds unless $k=k'$, $l=l'$ and $\pi' \simeq \pi$ or $\bar{\pi}$; when $\pi' \simeq \bar{\pi}$, these Brownian motions are conjugate.

The fact that W is real-valued implies that the purely imaginary terms in the series (14) or (15) vanish. In fact, we have the following real-valued equivalent form of (14). Let $\hat{G}_0$ be the subset of all π in $\hat{G}$ such that $\pi \simeq \bar{\pi}$. Since conjugation is an involution, $\hat{G} - \hat{G}_0$ can be partitioned into non-equivalent couples $\{\pi, \pi'\}$ such that $\pi' \simeq \bar{\pi}$. Let then $\hat{G}_1$ be a subset obtained by choosing one element of each couple.

COROLLARY 2.14. *Let $\phi \in H_p$.*

$$\begin{aligned} W(\phi) = &\sum_{\pi \in \hat{G} - \hat{G}_0} q_\pi^{1/2} \sum_{k,l=1}^{d_\pi} \int_G \phi(x)(U^{\pi}_{k,l} \Re \pi_{k,l}(x) + V^{\pi}_{k,l} \Im \pi_{k,l}(x))\, dx \\ &+ \sum_{\pi \in \hat{G}_0} q_\pi^{1/2} \sum_{k,l=1}^{d_\pi} W^{\pi}_{k,l} \int_G \phi(x)\, \pi_{k,l}(x)\, dx \end{aligned}$$

for some fixed families $\{U^{\pi}_{k,l}\}^{\pi \in \hat{G}-\hat{G}_0}_{k,l=1,...,d_\pi}$, $\{V^{\pi}_{k,l}\}^{\pi \in \hat{G}-\hat{G}_0}_{k,l=1,...,d_\pi}$, *and* $\{W^{\pi}_{k,l}\}^{\pi \in \hat{G}_0}_{k,l=1,...,d_\pi}$, *of real-valued centered Gaussian variables with variances equal to* 1/2, 1/2, *and* 1, *respectively.* $W^{\pi}_{k,l}$ *and* $W^{\pi'}_{k',l'}$ *are independent unless the indices are equal.* $U^{\pi}_{k,l}$ *and* $V^{\pi'}_{k',l'}$ *are always independent.* $U^{\pi}_{k,l}$ *and* $U^{\pi'}_{k',l'}$ *are independent unless* $k=k'$, $l=l'$ *and* $\pi' \simeq \pi$ *or* $\bar{\pi}$, *in which case they are equal.* $V^{\pi}_{k,l}$ *and* $V^{\pi'}_{k',l'}$ *are independent unless* $k=k'$, $l=l'$ *and* $\pi' \simeq \pi$ *or* $\bar{\pi}$, *in which case they are opposite. A similar real expansion can be given instead of* (15).

3. EXISTENCE AND UNIQUENESS

We assume that W, defined on $\mathscr{B}(\mathbb{R}_+) \times H_p$, is spatially-$\mathscr{H}_{-p}$, white-noise in time, and is spatially homogeneous and inverse-invariant. The last result shows we can assume that W is in the Fourier series form (15). We first prove existence and uniqueness of the solution to (1) (or rather (10)) in the case of linear additive noise, in which a complete characterization of the admissible W's is obtained. The general case requires an additional assumption on W of a technical nature, which does not appear to be necessary. In fact the result we present for general noise is included in the general theory of [5]; we only present enough details to show what the condition on the covariance of the cylindrical Brownian motion in [5] translates to in our setting.

3.1. *Additive Noise*

THEOREM 3.1. *Let* W *be spatially-*$\mathscr{H}_{-p}$, *white-noise in time, and spatially homogeneous and inverse invariant. Let* $Q = \sum_{\pi \in \hat{G}} q_\pi \chi_\pi$ *be its spatial covariance, as in* (11). *The following three conditions are equivalent.*

(a) *Let*

$$\sum_{\pi \in \hat{G}} \frac{q_\pi d_\pi}{1+\lambda_\pi} < \infty. \tag{16}$$

(b) *Let* $F \equiv 1$ *and* $b \equiv 0$. *For any bounded function* u_0 *in* G, *for any* $T > 0$, *there is a unique adapted solution* u *to Eq.* (10) *in* $[0, T] \times G$ *satisfying*

$$\sup_{x \in G,\, t \leqslant T} E\,|u(t,x)|^2 < \infty. \tag{17}$$

(c) *For all* $T > 0$,

$$\int_0^T Q(H_t)\, dt < \infty. \tag{18}$$

Proof. Since $F \equiv 1$ and $b \equiv 0$, the solution to (10) is given explicitly by the right-hand side of (10), in which u is absent. So we only need to prove that (a) and (c) are equivalent to (b) with

$$u(t, x) = \int H_t(y^{-1}x)\, u_0(y)\, dy + \int_0^t W(ds, H_{t-s}(x, \cdot)).$$

With bounded u_0, since $H_t(x, \cdot)\, dy$ is a probability measure for any x, t, the supremum in t, x of the first term in u, which is non-random, is always finite regardless of Q. Recall that we denote by H_s^x the function $H_s(x, \cdot)$ for $x \in G$, $t > 0$, and that $H^e = H$. Using the representation (15), the isometry property of Brownian motion, as well as the identities of Subsection 2.4 (including Proposition 2.6) and the fact that χ_π is central, we calculate

$$\begin{aligned} E\left[\left|\int_0^t W(ds, H_{t-s}^x)\right|^2\right] & \\ &= \int_0^t Q(H_{t-s}^x * \check{H}_{t-s}^x)\, ds \\ &= \sum_{\pi \in \hat{G}} q_\pi \int_0^t \left(\int_G \int_G \chi_\pi(yz^{-1})\, H_{t-s}(x^{-1}y)\, H_{t-s}(z^{-1}x)\, dy\, dz\right) ds \\ &= \sum_{\pi \in \hat{G}} q_\pi \int_0^t \int_G \chi_\pi(v)\, H_{2(t-s)}(v)\, dv\, ds \\ &= \int_0^t Q(H_{2s})\, ds. \end{aligned} \tag{19}$$

This shows that $\sup_{x \in G, t \leqslant T} E\,|u(t, x)|^2$ is finite if and only if $\int_0^T Q(H_t(\cdot))\, dt$ is finite, so (b) $\Leftrightarrow$ (c).

To finish the proof, we use the fact that, by definition, the linear operator $\phi \to \int_G H_t(v^{-1}x)\, \phi(v)\, dv$ can be written as $e^{s\Delta}\phi$, so that since χ_π is an eigenfunction of Δ with eigenvalue $-\lambda_\pi$, we get

$$\int_G H_t(v^{-1}x)\, \chi_\pi(v)\, dv = \chi_\pi(x) \exp(-t\lambda_\pi).$$

Therefore

$$\begin{aligned} \int_0^T Q(H_t)\, dt &= \sum_{\pi \in \hat{G}} q_\pi \chi_\pi(e) \int_0^T \exp(-t\lambda_\pi)\, dt \\ &= q_{\mathbf{1}} d_{\mathbf{1}} T + \sum_{\pi \in \hat{G} \setminus \{\mathbf{1}\}} q_\pi\, d_\pi \frac{1}{\lambda_\pi} (1 - \exp(-T\lambda_\pi)). \end{aligned} \tag{20}$$

Since $-\Delta$ is a positive operator and has a spectral gap, the set of values $(\lambda_\pi)_{\pi \neq \mathbf{1}}$ has a strictly positive lower bound, so that the above series converges if and only if $\sum_{\pi \in \hat{G}} q_\pi d_\pi/(1+\lambda_\pi)$ converges. ∎

COROLLARY 3.2. *The three conditions in Theorem* 3.1 *are equivalent to*

(d) *Let* $F \equiv 1$ *and* $b \equiv 0$. *For any function* $u_0 \in L^2(G)$, *for any* $T > 0$, *there is a unique solution* u *to equation* (10) *in* $[0, T] \times G$ *satisfying*

$$\sup_{t \leqslant T} E \|u(t)\|^2_{L^2(G)} < \infty.$$

Proof. We only need to show (d) $\Leftrightarrow$ (c), which is established by the immediate calculation (see proof of Theorem 3.1)

$$E \int_G \left| \int_0^t W(ds, H^x_{t-s}) \right|^2 dx = \int_G \left[\int_0^t Q(H^x_{t-s} * \check{H}^x_{t-s})\, ds \right] dx$$

$$= \int_G \left(\int_0^t Q(H_{2s})\, ds \right) dx = \int_0^t Q(H_{2s})\, ds. \quad ∎$$

We now record some remarks and examples regarding the equivalent conditions of Theorem 3.1. Proofs may be found in the appendix. Denote by $\mathscr{H}$ the set of all formal series $Q = \sum_{\pi \in \hat{G}} q_\pi \chi_\pi$ such that $\sum_{\pi \in \hat{G}} q_\pi d_\pi/(1+\lambda_\pi)$ converges. Denote by $M(G)$ the set of all finite signed measures on G.

Remark 3.3. Arguably the most important and/or basic example of a Lie group is the circle S^1. In this case, not only is $\mathscr{H}$ not included in $M(S^1)$, it is not even included in H_{-1}. However, it is included in H_{-2}. In fact, no matter what G is, $\mathscr{H} \subset H_{-2}$.

Remark 3.4. Nevertheless, the case of S^1 is somewhat exotic, since it is possible to show that for many NON-commutative compact Lie groups, and for all the tori $(S^1)^d$, $d \geqslant 2$, $\mathscr{H}$ is included in $M(G)$.

Remark 3.5. Generally speaking, the regularity of $\mathscr{H}$ is determined by the relationship between λ_π, which is of order $|h_\pi|^2$, and d_π, which, according to the Weyl dimension formula (see appendix) is a polynomial of degree k in h_π, and should therefore be expected to be of order $|h_\pi|^k$. The higher the value of k, the more regular the elements in $\mathscr{H}$ are required to be. Modest values of k already command relatively strict regularity, since it can be shown that if $k \geqslant 2$, any Q in $\mathscr{H}$ will be a function in $L^2(G)$.

3.2. *General Case*

To show that Eq. (10) has a unique solution under condition (16) in the general nonlinear case, we need to make the additional assumption that Q is a non-negative measure. We hope to be able to weaken, or even remove, this condition, although none of the work to date [4, 18–20] shows any evidence that this may be done in a general setting even in flat space.

Although Q is a measure and $W(ds, \cdot)$ is an $L^2(\Omega)$-measure, we continue to use the notation $Q(f)$ and $W(ds, f)$ for the integrals $\int f\,dQ$ and $\int f(x)\,W(ds, dx)$.

THEOREM 3.6. *Let W be spatially-$\mathcal{H}_{-p}$, white-noise in time, and spatially homogeneous and inverse invariant, with covariance $Q = \sum_{\pi \in \hat{G}} q_\pi \chi_\pi$. Assume condition* (a). *Assume moreover that Q is a non-negative measure. Let $F(s, y, r)$ and $b(s, y, r)$ be $\mathcal{F}_t$-adapted functions that are globally Lipschitz in r uniformly in the other variables* (*including the random one*). *Assume that u_0 is a random field on G, independent of W, and with $E \int dx\, |u_0(x)|^2 < \infty$. Then there is a unique $\mathcal{F}_t$-adapted random field u on $\mathbb{R}_+ \times G$ satisfying equation* (10) *and which is bounded on compacts as a function from $\mathbb{R}_+$ to $L^2(G \times \Omega)$, and there is a constant A such that for all T*

$$\sup_{t \leqslant T} \int_G E\, |u(t, x)|^2 \, dx \leqslant A e^{AT} E\, \|u_0\|^2_{L^2(G)}.$$

Proof. The proof follows the standard use of the Banach fixed point theorem, as in [5]. We outline the proof to show where condition (a) comes in. Using the isometry property of Brownian motion, the definition of Q, the properties of the heat kernel and the Hölder's and Jensen's inequalities for the positive measure Q, we obtain that, defining the map $\mathcal{L}_T$ by

$$\mathcal{L}_T(\sigma)(t, x) = \int_0^t W(ds, F(s, \cdot, \sigma_s(\cdot))\, H_{t-s}(x, \cdot))$$

$$+ \int_0^t ds \int_G b(s, y, \sigma_s(y))\, H_{t-s}(x, y)\, dy$$

for all adapted random fields σ on $\mathbb{R}_+ \times G$, $\mathcal{L}_T$ is a Lipschitz map in the Banach space $\mathcal{X}_T$ defined by its norm

$$\|\sigma\|^2_{\mathcal{X}_T} = \sup_{s \leqslant T} E\, \|\sigma_s\|^2_{L^2(G)}.$$

with Lipschitz constant $c(\int_0^T ds\, Q(H_{2s}) + T^2)^{1/2}$. To show that for T small enough, $\mathcal{L}_T$ is a contraction, we only need to prove that $\int_0^T ds\, Q(H_{2s})$

converges to 0 as T goes to 0. This follows from condition (c), which is equivalent to assumption (a). The fixed point theorem then guarantees the existence of a unique solution to Eq. (10) for all $t \leqslant T$ as soon as we remark that this equation can be written as $u = \mathscr{L}_T(u) + U_0$ where $U_0(t, x) = \int H_t(x, y)\, u_0(y)\, dy$, and that it is immediate to check that $U_0 \in \mathscr{X}_T$. The construction of the fixed point solution shows that it is adapted, and that the estimate of the theorem is satisfied. In order to obtain a solution defined for all $t \geqslant 0$, the usual piecing procedure applies, and yields a unique adapted solution. ∎

COROLLARY 3.7. *If Q is a non-negative measure, then condition* (c) *is equivalent to*:

(e) $Q(h) < \infty$, *where for* $d \geqslant 3$, $h(x) = \rho^{-d+2}(x)$, *and for* $d = 2$, $h(x) = \log \rho^{-1}(x)$.

Consequently, Theorem 3.6 *remains true if condition* (a) *is replaced by condition* (e).

Proof. Set

$$J \equiv \int_0^T Q(H_t)\, dt = Q\left(\int_0^T H_t\, dt\right).$$

Here, Fubini's theorem is justified by Q's σ-finiteness. For any $u \in G$, > 0, we have, by Proposition 2.5,

$$H_t(x) \geqslant c_1 t^{-d/2} \exp\left(-\frac{\rho^2(x)}{d_1 t}\right)$$

for some constant $c_1, d_1 > 0$. Hence, for $u \in G$ and a constant $K_1 > 0$,

$$\int_0^T H_t(u)\, dt \geqslant c_1 \int_0^T t^{-d/2} \exp\left(-\frac{\rho^2(x)}{d_1 t}\right) dt \geqslant K_1 h(u), \tag{21}$$

where h is defined in condition (e). Thus $J \geqslant K_1 Q(h)$. The upper bounds of Proposition 2.5 allow to reverse the inequalities in (21), so we also get $J \leqslant K_2 Q(h)$ for a constant $K_2 > 0$, which ends the proof. ∎

This last result shows that our conditions coincide with the conditions given in [4, 18] (resp. [19, 20]) for existence and uniqueness of the stochastic wave solution in compact flat space for $d = 2$ (resp. for the stochastic heat and wave equation in $\mathbb{R}^d$).

THEOREM 3.8. *Theorem 3.6 and its corollary remain true if we replace the $L^2(G)$-norm by the $L^q(G)$-norm for any $q>1$, and we assume that $E[\|u_0\|^q_{L^q(G)}]$ is finite. Assuming that u_0 is a bounded function on G, we also have*

$$\sup_{t\leqslant T,\, x\in G} E[|u(t,x)|^q] < \infty.$$

Proof. We use the same argument as in the proof of Theorem 3.6, and make use of Burkholder's inequality and the Jensen inequality for Q. ∎

4. HÖLDER CONTINUITY

We still assume here that W has the form given by (15). In this section, we state some general sufficient assumptions on the correlation Q under which the solution u to equation (10) is almost surely Hölder-continuous; the continuity exponent depends on the regularity of Q. In Section 5, we will interpret these assumptions, and show that, in the case of additive noise, they are necessary.

THEOREM 4.1. *Let W be spatially $\mathscr{H}_{-p}$, white noise in time, and spatially homogeneous and inverse invariant. Assume W's covariance function $Q=\sum_{\pi\in\hat{G}} q_\pi\chi_\pi$ is a non-negative measure. Let $T>0$ and F, b: $[0,T]\times G\times\mathbb{R}\to\mathbb{R}$ two Lipschitz functions in the last variable, uniformly in the other ones. Suppose that for an $\varepsilon>0$, Q satisfies*

(f) *For any $\gamma\in(0,\varepsilon)$,*

$$\sum_{\pi\in\hat{G}} \frac{q_\pi d_\pi}{(1+\lambda_\pi)^{1-\gamma}} < \infty.$$

Assume also that u_0 is γ-Hölder continuous on G for any $\gamma<\varepsilon$. Then there exists a version of the solution $u=\{u(t,x);\,(t,x)\in[0,T]\times G\}$ to (10) which is $(\gamma/2,\gamma)$-Hölder continuous on $[0,T]\times G$, for any exponent $\gamma\in(0,\varepsilon)$.

Proof. We shall use the Kolmogorov criterion. We shall show that for any $p>2$, and any (t_1,x_1), $(t_2,x_2)\in[0,T]\times G$, we have, for a constant $c_p>0$, and for all $\gamma<\varepsilon$

$$E[|u(t_1,x_1)-u(t_2,x_2)|^p] \leqslant c_p(|t_2-t_1|^{p\gamma/2}+\rho^{p\gamma}(x_2,x_1)).$$

Then, since G is locally diffeomorphic to $\mathbb{R}^d$, the Kolmogorov criterion (e.g., [11, Problem 2.2.9]) would imply that u has a version which is (α,β)-Hölder continuous in (t,x) for all $\alpha<(p\gamma/2-(d+1))/p$ and for all $\beta<$

$(p\gamma-(d+1))/p$. Since p is arbitrarily large and γ is arbitrarily close to ε, the conclusion of the theorem would follow.

In the remainder of the proof, we shall denote all constants by c, although they may change from line to line. For any $(t, x) \in [0, T] \times G$, $F(t, x, u(t, x))$ will be denoted by $\sigma_t(x)$, $b(t, x, u(t, x))$ by $\beta_t(x)$, and the function $H_t(x, .)$ by H_t^x, like in the proof of Theorem 3.1. Since Q is a measure, the notation $Q(f)$ means again $\int f\,dQ$, and likewise for $W(ds, f)$. We shall divide the proof into several steps.

Step 1. For $0 \leqslant t_1 < t_2 \leqslant T$ and $x \in G$, let us study the quantity $u(t_2, x)-u(t_1, x)$. We have

$$u(t_2, x)-u(t_1, x)=\sum_{i=1}^{5} J_i$$

with

$$J_1=\int_G u_0(y)[H_{t_1}^x(y)-H_{t_2}^x(y)]\,dy$$

$$J_2=\int_{t_1}^{t_2} W(ds, H_{t_2-s}^x \sigma_s)$$

$$J_3=\int_0^{t_1} W(ds, [H_{t_2-s}^x-H_{t_1-s}^x]\,\sigma_s)$$

$$J_4=\int_{t_1}^{t_2}\int_G H_{t_2-s}^x(y)\;\beta_s(y)\,ds\,dy$$

$$J_5=\int_0^{t_1} [H_{t_2-s}^x(y)-H_{t_1-s}^x(y)]\;\beta_s(y)\,ds\,dy.$$

Then, for every $p>2$,

$$E[\,|u(t_1, x)-u(t_2, x)|^p] \leqslant c\left(|J_1|^p+\sum_{i=2}^{5} E[\,|J_i|^p]\right).$$

Step 2. On a probability space $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{P})$, let us consider the left-invariant Brownian motion p starting at $x \in G$, that is the solution to the Stratonovich differential equation

$$dp_t=\sum_{i=1}^{d} X_i(p_t) \circ d\tilde{B}_t^i$$

with initial condition $p_0 = x$, where $\tilde{B}$ is a $\mathbb{R}^d$-valued Brownian motion and $X_1, ..., X_d$ an orthonormal basis of $\mathcal{G}$. Denote by $\tilde{E}$ the expectation on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$. Then it is well known that for any $t \geqslant 0$,

$$\int_G H_t^x(y)\, u_0(y)\, dy = \tilde{E}[u_0(p_t)],$$

and since u_0 is γ-Hölder continuous for any $\gamma < \varepsilon$, we have for any $\delta < 1/2$,

$$J_1 \leqslant c\tilde{E}[\rho^\gamma(p_{t_1}, p_{t_2})] \leqslant c\, |t_2 - t_1|^{\gamma\delta},$$

where we have used some classical estimates on the modulus of continuity of the left invariant Brownian motion. Since $\gamma\delta$ is arbitrarily close to $\varepsilon/2$, we get $|J_1|^p \leqslant c\, |t_2 - t_1|^{\gamma p/2}$ for any $\gamma < \varepsilon$.

Step 3. Suppose $F \equiv 1$. Using Burkholder's inequality for our noise W (see [22]), we have

$$E[\,|J_2|^p] \leqslant c \left(\int_{t_1}^{t_2} Q(H_{t_2-s}^x * \check{H}_{t_2-s}^x)\, ds \right)^{p/2}.$$

Just like in the series of equalities (19) and (20), we get, for all $\gamma < \varepsilon$,

$$\begin{aligned}
\int_{t_1}^{t_2} Q(H_{t_2-s}^x * \check{H}_{t_2-s}^x)\, ds &= \int_{t_1}^{t_2} Q(H_{2(t_2-s)})\, ds \\
&= \sum_{\pi \in \hat{G}} q_\pi d_\pi \int_{t_1}^{t_2} \exp(-2\lambda_\pi(t_2 - s))\, ds \\
&\leqslant c \sum_{\pi \in \hat{G}} \frac{q_\pi d_\pi}{1 + \lambda_\pi} [1 - \exp(-2\lambda_\pi(t_2 - t_1))] \\
&\leqslant c\, |t_2 - t_1|^\gamma \sum_{\pi \in \hat{G}} \frac{q_\pi d_\pi}{(1 + \lambda_\pi)^{1-\varepsilon}} \phi_\gamma(\lambda_\pi(t_2 - t_1)),
\end{aligned}$$

where $\phi_\gamma \colon \mathbb{R} \to \mathbb{R}$ is defined by $\phi_\gamma(r) = r^{-\gamma}(1 - \exp(-r))$ for any $\gamma \in (0, \varepsilon)$. Since ϕ_γ is a bounded function on $\mathbb{R}$, for $\gamma < \varepsilon$,

$$\int_{t_1}^{t_2} Q(H_{t_2-s}^x * \check{H}_{t_2-s}^x)\, ds \leqslant c\, \|\phi_\gamma\|_\infty \sum_{\pi \in \hat{G}} \frac{q_\pi d_\pi}{(1 + \lambda_\pi)^\gamma} |t_2 - t_1|^\gamma \leqslant c\, |t_2 - t_1|^\gamma.$$

Therefore,

$$E[\,|J_2|^p] \leqslant c\, |t_2 - t_1|^{\gamma p/2}.$$

Step 4. Suppose now F satisfies our general assumptions, and let us show that this case can be reduced easily to the case $F \equiv 1$. By Burkholder's inequalities,

$$E[|J_2|^p] \leqslant cE\left[\left|\int_{t_1}^{t_2} Q((H^x_{t_2-s}\sigma_s) * (\check{H}^x_{t_2-s}\check{\sigma}_s))\, ds\right|^{p/2}\right].$$

Note that, for any $y \in G$,

$$\begin{aligned}
&[(H^x_{t_2-s}\sigma_s) * (\check{H}^x_{t_1-s}\check{\sigma}_s)](y)\\
&\quad = \int_G H_{t_2-s}(x, yz)\, \sigma_s(yz)\, H_{t_2-s}(x, z)\, \sigma_s(z)\, dz\\
&\quad = \int_G [H^x_{t_2-s} \circ R_z](y)[\sigma_s \circ R_z](y)\, H^x_{t_2-s}(z)\, \sigma_s(z)\, dz,
\end{aligned}$$

where R_z denotes the right translation by z. Let us write then

$$(H^x_{t_2-s}\sigma_s) * (\check{H}^x_{t_1-s}\check{\sigma}_s) = \int_G [H^x_{t_2-s} \circ R_z][\sigma_s \circ R_z]\, H^x_{t_2-s}(z)\, \sigma_s(z)\, dz,$$

and note that by Fubini's theorem,

$$E[|J_2|^p] \leqslant cE\left[\left|\int_{t_1}^{t_2}\int_G Q([\sigma_s \circ R_z][H^x_{t_2-s} \circ R_z]\, H^x_{t_2-s}(z)\, \sigma_s(z))\, dz\, ds\right|^{p/2}\right].$$

Then Hölder's inequality for $q = p/2$, $q' = p/p-2$ gives

$$\begin{aligned}
E[|J_2|^p] &\leqslant c\left|\int_{t_1}^{t_2}\int_G Q([H^x_{t_2-s} \circ R_z]\, H^x_{t_2-s}(z))\, dz\, ds\right|^{(p-2)/2}\\
&\qquad \times E\left[\int_{t_1}^{t_2}\int_G Q(|[\sigma_s \circ R_z]\, \sigma_s(z)|^{p/2}\, [H^x_{t_2-s} \circ R_z]\, H^x_{t_2-s}(z))\, dz\, ds\right]\\
&= c\left|\int_{t_1}^{t_2}\int_G Q([H^x_{t_2-s} \circ R_z]\, H^x_{t_2-s}(z))\, dz\, ds\right|^{(p-2)/2}\\
&\qquad \times \int_{t_1}^{t_2}\int_G Q(E[|[\sigma_s \circ R_z]\, \sigma_s(z)|^{p/2}][H^x_{t_2-s} \circ R_z]\, H^x_{t_2-s}(z))\, dz\, ds.
\end{aligned}$$

As stated in Theorem 3.6, $\sup\{E[|\sigma_s(x)|^p]; (s,x)\in[0,T]\times G\}$ is finite. Therefore,

$$E[|J_2|^p]\leqslant c\left|\int_{t_1}^{t_2}\int_G Q([H^x_{t_2-s}\circ R_z]\,H^x_{t_2-s}(z))\,dz\,ds\right|^{p/2}$$
$$=c\left(\int_{t_1}^{t_2} Q(H^x_{t_2-s}*\check{H}^x_{t_1-s})\,ds\right)^{p/2},$$

which is the quantity studied in Step 3.

Step 5. Let us give an estimate for $E[|J_3|^p]$. Using the same trick as in Step 4, we can suppose $F\equiv 1$. Thus,

$$E[|J_3|^p]\leqslant c\left|\int_0^{t_1} Q((H^x_{t_2-s}-H^x_{t_1-s})*(\check{H}^x_{t_2-s}-\check{H}^x_{t_1-s}))\,ds\right|^{p/2}.$$

But

$$\int_0^{t_1} Q((H^x_{t_2-s}-H^x_{t_1-s})*(\check{H}^x_{t_2-s}-\check{H}^x_{t_1-s}))\,ds$$
$$=\int_0^{t_1} Q(H_{2(t_2-s)}+H_{2(t_1-s)}-2H_{t_1+t_2-2s})\,ds$$
$$=\sum_{\pi\in\hat{G}} q_\pi\,d_\pi\int_0^{t_1} ds\,[\exp(-2\lambda_\pi(t_2-s))+\exp(-2\lambda_\pi(t_1-s))$$
$$-2\exp(-2\lambda_\pi(t_1+t_2-2s))]$$
$$=0+\frac{1}{2}\sum_{\pi\in\hat{G}-\{\mathbf{1}\}}\frac{q_\pi d_\pi}{\lambda_\pi}[1-\exp(-\lambda_\pi(t_2-t_1))]^2\,[1-\exp(-2\lambda_\pi t_1)]$$
$$\leqslant c\,\|\phi_{\gamma/2}\|_\infty^2\sum_{\pi\in\hat{G}}\frac{q_\pi d_\pi}{(1+\lambda_\pi)^{1-\gamma}}|t_2-t_1|^\gamma$$
$$\leqslant c\,|t_2-t_1|^\gamma,$$

where ϕ_γ has been defined in Step 3. Hence

$$E[|J_3|^p]\leqslant c\,|t_2-t_1|^{\gamma p/2}.$$

Step 6. In order to give an estimate for $E[|J_4|^p]$, let us first use Jensen's inequality for the finite measure $H_t(u)\,dt\,du$,

$$E[|J_4|^p]\leqslant|t_2-t_1|^{p-1}\int_{t_1}^{t_2}\int_G H_{t_2-s}(xy^{-1})\,E[|\beta_s|^p]\,dy\,ds,$$

and since $\sup\{E[|\beta_s(x)|^p]; (s,x)\in[0,T]\times G\}$ is finite,

$$E[|J_4|^p] \leqslant c\,|t_2-t_1|^{p-1}\int_{t_1}^{t_2}\int_G H_{t_2-s}(y)\,dy\,ds = c\,|t_2-t_1|^{p-1}.$$

Using again Hölder's inequality, the computation of $E[|J_5|^p]$ can be reduced to the case $\beta\equiv 1$, for which

$$E[|J_5|^p] \leqslant \left(\int_0^{t_1}\int_G |H_{t_2-s}(xy^{-1})-H_{t_1-s}(xy^{-1})|\,dy\,ds\right)^p$$
$$= \left(\int_0^{t_1}\int_G |H_{t_2-t_1+r}(y)-H_r(y)|\,dy\,dr\right)^p.$$

But for $z\in G-\{e\}$,

$$\int_0^{t_1} |H_{t_2-t_1+r}(z)-H_r(z)|\,dr \leqslant \left(\int_0^{t_1} |H_{t_2-t_1+r}(z)-H_r(z)|\,dr\right)^{\gamma}$$
$$\times\left(\int_0^{t_1} (H_{t_2-t_1+r}(z)+H_r(z))\,dr\right)^{1-\gamma}.$$

Using the results of Proposition 2.5, we get

$$\int_0^{t_1} (H_{t_2-t_1+r}(z)+H_r(z))\,dr \leqslant c\int_0^{t_1} r^{d/2}\exp\left(-\frac{\rho^2(z)}{cr}\right)dr \leqslant c(\rho(z))^{-(d-2)},$$

and

$$\int_0^{t_1} |H_{t_2-t_1+r}(z)-H_r(z)|\,dr \leqslant \int_0^{t_1}\int_r^{t_2-t_1+r}\left|\frac{\partial H_s}{\partial s}(z)\right|ds\,dr$$
$$\leqslant \int_0^{t_1}(t_2-t_1)\left|\frac{\partial H_s}{\partial s}(z)\right|ds$$
$$\leqslant c(t_2-t_1)\int_0^{t_1} s^{-(d/2+1)}\exp\left(-\frac{\rho^2(z)}{cs}\right)ds$$
$$\leqslant c(t_2-t_1)(\rho(z))^{-d}.$$

Thus

$$\int_0^{t_1} |H_{t_2-t_1+r}(z)-H_r(z)|\,dr \leqslant c(t_2-t_1)^{\gamma}\,(\rho(z))^{-(d-2(1-\gamma))}.$$

The function ρ^{-q} being integrable on G for $q<d$, we get for any $\gamma<1$

$$E[|J_5|^p] \leqslant c\,|t_2-t_1|^{\gamma p}.$$

Step 7. For a $t\in[0,T]$ and $x_1, x_2\in G$, let us give now some estimates for $u(t,x_2)-u(t,x_1)$. We shall write again

$$u(t,x_2)-u(t,x_1)=\sum_{i=1}^{3} K_i$$

with

$$K_1=\int_G [H_t^{x_2}(y)-H_t^{x_1}(y)]\,u_0(y)\,dy$$

$$K_2=\int_0^t W(ds, [H_{t-s}^{x_2}-H_{t-s}^{x_1}]\,\sigma_s)$$

$$K_3=\int_0^t\int_G [H_{t-s}^{x_2}(y)-H_{t-s}^{x_1}(y)]\,\beta_s(y)\,dy\,ds,$$

and for every $p>2$,

$$E[|u(t,x_1)-u(t,x_1)|^p]\leqslant c\left(|K_1|^p+\sum_{i=2}^{3} E[|K_i|^p]\right).$$

Moreover, using again the probabilistic representation of $H_t(u_0)$ as in Step 2, it is easily seen that if u_0 is γ-Hölder for any $\gamma<\varepsilon$, then also for any $\gamma<\varepsilon$

$$|K_1|^p\leqslant c[\rho(x_1,x_2)]^{\gamma p}.$$

Step 8. We can assume $F\equiv 1$ for the estimation of $E[|K_2|^p]$, using the same kind of computations as in Step 4. In that case,

$$E[|K_2|^p]\leqslant c\left(\int_0^t Q((H_s^{x_2}-H_s^{x_1}) * (\check{H}_s^{x_2}-\check{H}_s^{x_1}))\,ds\right)^{p/2}. \tag{22}$$

By relations (19), we have

$$\int_0^t Q(H_s^{x_2} * \check{H}_s^{x_2})\,ds=\int_0^t Q(H_s^{x_1} * \check{H}_s^{x_1})\,ds=\int_0^t Q(H_{2s})\,ds.$$

Moreover, using the second part of Proposition 2.6, for $s \in [0, T]$,

$$\int_G \chi_\pi(z)[H_s^{x_2} * \check{H}_s^{x_1}](z)\, dz = \int_G \chi_\pi(z)\, H_{2s}(zx_2^{-1}x_1)\, dz$$
$$= \exp(-2s\lambda_\pi)\, \chi_\pi(x_1^{-1}x_2),$$

where we use the fact that χ_π is an eigenvector of Δ associated to the eigenvalue $-\lambda_\pi$. Hence, since $Q = \sum_{\pi \in \hat{G}} q_\pi \chi_\pi$, for any $s \in [0, T]$,

$$\int_0^t Q((H_s^{x_2} - H_s^{x_1}) * (\check{H}_s^{x_2} - \check{H}_s^{x_1}))\, ds$$
$$= \int_0^t [Q(H_{2s} - H_{2s}^{x_1^{-1}x_2}) + Q(H_{2s} - H_{2s}^{x_2^{-1}x_1})]\, ds$$
$$= 2\left\{0 + \frac{1}{2} \sum_{\pi \neq \mathbf{1}} \frac{q_\pi}{\lambda_\pi} (1 - \exp(-2t\lambda_\pi))(d_\pi - \chi_\pi(x_2^{-1}x_1))\right\}$$
$$\leqslant c \sum_{\pi \in \hat{G}} \frac{q_\pi}{1 + \lambda_\pi} (1 - \exp(-2t\lambda_\pi))(d_\pi - \chi_\pi(x_2^{-1}x_1)).$$

Notice that, using Proposition 2.2 and 2.4, we have

$$d_\pi - \chi_\pi(v) \leqslant c d_\pi h_\pi^2 [\rho(v)]^2 \leqslant c d_\pi \lambda_\pi [\rho(v)]^2,$$

and moreover, $d_\pi - \chi_\pi(v) \leqslant d_\pi$. We thus get, for all $\gamma \in (0, 1)$,

$$d_\pi - \chi_\pi(v) \leqslant c d_\pi \lambda_\pi^\gamma [\rho(v)]^{2\gamma}$$

and

$$\int_0^t Q((H_s^{x_2} - H_s^{x_1}) * (\check{H}_s^{x_2} - \check{H}_s^{x_1}))\, ds$$
$$\leqslant c[\rho(x_2^{-1}x_1)]^{2\gamma} \sum_{\pi \neq \mathbf{1}} \frac{q_\pi d_\pi}{\lambda_\pi} \lambda_\pi^\gamma$$
$$\leqslant c[\rho(x_1; x_2)]^{2\gamma} \sum_{\pi \in \hat{G}} \frac{q_\pi d_\pi}{(1 + \lambda_\pi)^{1-\gamma}} \leqslant c[\rho(x_1; x_2)]^{2\gamma}$$

since $\{\lambda_\pi : \pi \neq \mathbf{1}\}$ is bounded away from zero. Plugging into (22), this yields

$$E[|K_2|^p] \leqslant c[\rho(x_1, x_2)]^{\gamma p}.$$

Step 9. The computation of $E[|K_3|^p]$ can be reduced by Hölder's inequality, as in Step 6, to the case $b \equiv 1$, and hence

$$E[|K_3|^p] \leqslant c\left(\int_0^t \int_G |H_{t-s}(x_2, y) - H_{t-s}(x_1, y)|\, dy\right)^p$$
$$= c\left(\int_0^t \int_G |H_s(yx) - H_s(y)|\, dy\right)^p,$$

where we have set $x = x_1^{-1} x_2$. Let γ be a geodesic joining e and x, determined by the unit vector $X \in \mathcal{G}$, of length $\rho(x)$. Then

$$\int_G |H_s(yx) - H_s(y)|\, dy \leqslant \int_G \int_0^{\rho(x)} |XH_s(y\gamma(r))|\, dr\, dy$$
$$= \int_0^{\rho(x)} \int_G |XH_s(y\gamma(r))|\, dy\, dr$$
$$= \rho(x) \int_G |XH_s(y)|\, dy,$$

and by Proposition 2.5,

$$\int_0^t \int_G |H_s(yx) - H_s(y)|\, dy\, ds \leqslant c\rho(x) \int_0^t \int_G |XH_s(y)|\, dy\, ds$$
$$\leqslant c\rho(x) \int_G \int_0^t s^{-(d+1)/2} \exp\left(\frac{\rho(y)^2}{cs}\right) ds\, dy$$
$$\leqslant c\rho(x) \int_G [\rho(y)]^{-(d-1)}\, dy$$
$$\leqslant c\rho(x),$$

which gives

$$E[|K_3|^p] \leqslant c[\rho(x)]^p. \quad \blacksquare$$

COROLLARY 4.2. *The previous theorem holds in the case* $F \equiv 1$, $b \equiv 0$, *even if* Q *is not a measure.*

Proof. Parts (3), (5), and (8) of the previous proof do not need the assumption that Q is a measure because Holder's, Jensen's, and Fubini's theorems on Q are not invoked there if $F \equiv 1$, $b \equiv 0$.

5. INTERPRETATION OF THE EXISTENCE AND CONTINUITY CONDITIONS

5.1. *The Results*

We first note that, for any function $f = \sum_{\pi \in \hat{G}} \sum_{i,j=1}^{d_\pi} f_{\pi,i,j} \pi_{i,j}$ in H_{-p}, the operator $(I - \Delta)^{-1/2}$ applied to f can be written as

$$(I - \Delta)^{-1/2} f = \sum_{\pi \in \hat{G}} \sum_{i,j=1}^{d_\pi} f_{\pi,i,j} (1 + \lambda_\pi)^{-1/2} \pi_{i,j}.$$

PROPOSITION 5.1. *Let W be given by* (15). *The equivalent conditions* (a)–(e) *for existence and uniqueness of an adapted solution to* (10) *are equivalent to requiring that the Gaussian field* $Y = (I - \Delta)^{-1/2} W$ *be function-valued, i.e., that*

$$Y(t, x) = \sum_{\pi \in \hat{G}} \sqrt{q_\pi} \, (1 + \lambda_\pi)^{-1/2} \sum_{i,j=1}^{d_\pi} \pi_{i,j}(x) \, W_{k,l}^{\pi}(t)$$

is an a.s. finite random variable.

Proof. Since $Y(t, x)$ is a centered Gaussian random variable, it will exist if and only if its variance is finite. This variance calculates out to be $t \sum_{\pi \in \hat{G}} q_\pi d_\pi (1 + \lambda_\pi)^{-1}$, and the finiteness of this is exactly condition (a). ∎

This proposition shows that the conditions for existence and uniqueness can be expressed intrinsically as a spatial regularity condition on W, and more specifically, as the existence of a functional "antiderivative" Y for W.

We now turn to a characterization of condition (f) for Hölder-continuity, which is also expressed intrinsically in terms of Y. Referring to Subsection 2.2, for any $\pi \in \hat{G}$, we let $\{\mu_\pi^1, ..., \mu_\pi^{d_\pi}\}$ be the set of all d_π weights of π, which are vectors in the dual of the maximal torus algebra $\mathcal{T}^*$, and we let h_π be the maximal one. For any real numbers $K, \eta > 0$, let us call $\hat{G}_{K,\eta}$ the set of irreducible representations π such that $K \leqslant |h_\pi| \leqslant (1 + \eta) K$.

THEOREM 5.2. *Let W be given by* (15). *Condition* (f) *for* $(\gamma/2, \gamma)$*-Hölder-continuity of the solution to* (10) *for any* $\gamma < \varepsilon$, *implies that the Gaussian field* $Y = (I - \Delta)^{-1/2} W$ *is almost-surely* γ*-Hölder-continuous in the space variable for all* $\gamma < \varepsilon$.

The converse also is true, i.e., (f) *follows from the a.s.* γ*-Hölder continuity of Y for all* $\gamma < \varepsilon$, *assuming the following structural hypotheses on the group G*:

[G] *There is an integer* K_0 *and constants* $c_1, \xi \in (0, 1)$ *such that for any* $\pi \in \hat{G}$ *satisfying* $|h_\pi| \geqslant K_0$,

$$Card\{k: k = 1, ..., d_\pi; |\mu_\pi^k| > \xi |h_\pi|\} \geqslant c_1 d_\pi.$$

[H] *Let* $\{h(n)\}_{n=0}^{\infty}$ *be a numbering of* $\{h_\pi\}_{\pi \in \hat{G}}$ *in order of increasing* $|h_\pi|$. *Then* $\{|h(n+1)|/|h(n)|\}_{n=1}^{\infty}$ *is bounded.*

We shall verify in the Appendix that conditions [G] and [H] are satisfied for the classical compact and connected Lie groups:

PROPOSITION 5.3. *Hypotheses* [G] *and* [H] *are satisfied for the following Lie groups*: $(S^1)^d$, $SU(n)$, *and* $SO(n)$ *for* $d, n \geqslant 1$.

In order to prove Theorem 5.2, we shall use the following technical assumption implied by condition [G].

PROPOSITION 5.4. *Under condition* [G], *the following condition* [$\tilde{\mathrm{G}}$] *holds*:

[$\tilde{\mathrm{G}}$] *There are integers* m, K_0 *and constants* $c_1, c_2, a \in (0, 1)$, $\eta > 0$ *such that*

(1) $(1+\eta)/c_2 < \pi/2$.

(2) *For any* $K > K_0$, *there are* m *weights* $v_K^1, ..., v_K^m$ *belonging to the representations in* $\hat{G}_{K,\eta}$ *satisfying* $|v_K^i| \geqslant c_2 K$, *for* $i = 1, ..., m$.

(3) *For any* $\pi \in \hat{G}_{K,\eta}$,

$$Card\{k: k = 1, ..., d_\pi; \exists 1 \leqslant i \leqslant m \text{ s.t. } |\langle \mu_\pi^k; v_K^i \rangle| > a |v_K^i|^2\} \geqslant c_1 d_\pi.$$

The proof of the last proposition is also left to the Appendix.

The following theorem shows that Condition (f) is optimal.

THEOREM 5.5. *Under Hypotheses* [G] *and* [H], *Condition* (f) *is equivalent to*:

(g) *Let* $F \equiv 1$ *and* $b \equiv 0$. *If* u_0 *is* γ-*Hölder-continuous in* G *for all* $\gamma < \varepsilon$, *the unique adapted solution* u *to Eq.* (10) *in* $\mathbb{R}_+ \times G$ *is a.s.* γ-*Hölder-continuous in the space variable for all* $\gamma < \varepsilon$.

Proof. That (f) is sufficient for (g), even without Hypothesis [G] or [H], is the result of Theorem 4.1. When $F \equiv 1$ and $b \equiv 0$, the solution to (10) is given explicitly as the sum of a deterministic Hölder-continuous function and of the random field

$$U(t, x) = \int_0^t W(ds, H_{t-s}(x, \cdot)).$$

This random field is Gaussian, since W is Gaussian and H is deterministic. We can calculate its covariance just as we calculated the variances in (19) and (20), or in the proof of Theorem 4.1:

$$E[U(t,x)\,U(s,z)]=q_{\mathbf{1}}(s\wedge t)+\sum_{\pi\neq\mathbf{1}}\frac{q_\pi}{2\lambda_\pi}\left[e^{-|t-s|\,\lambda_\pi}-e^{-(t+s)\,\lambda_\pi}\right]\chi_\pi(xz^{-1}).$$

This proves that, in the space variable, U is an inverse-invariant homogeneous Gaussian field on G. Thus the same is true for $\{U(1,x): x\in G\}$. If U is a.s γ-Hölder-continuous in x, then so is $U(1,\cdot)$. Let $R=\sum_{\pi\in\hat{G}} r_\pi\chi_\pi$ be the homogeneous covariance function of $U(1,\cdot)$. The "converse" portion of Theorem 5.2 asserts that condition (f) holds for the coefficients of the covariance function of $(1-\Delta)^{1/2}\,U(1,\cdot)$; as pointed out in the proof of Theorem 5.2, this is equivalent to condition (23), i.e., we must have for all $\gamma<\varepsilon$, $\sum_{\pi\in\hat{G}} r_\pi d_\pi(1+\lambda_\pi)^\gamma<\infty$. The covariance of U yields that

$$r_\pi=\frac{q_\pi}{2\lambda_\pi}\left[1-\exp(-2\lambda_\pi)\right];\ r_{\mathbf{1}}=q_{\mathbf{1}}.$$

Since the infemum of all eigenvalues λ_π for $\pi\neq\mathbf{1}$ is a value $c>0$, we get

$$\begin{aligned}\infty &> \sum_{\pi\in\hat{G}-\{\mathbf{1}\}}\frac{q_\pi d_\pi}{2\lambda_\pi}\left[1-\exp(-2\lambda_\pi)\right](1+\lambda_\pi)^\gamma\\ &\geqslant (1-\exp(-2c))\sum_{\pi\neq\mathbf{1}}\frac{q_\pi d_\pi}{(1+\lambda_\pi)^{1-\gamma}},\end{aligned}$$

which finishes the proof. ∎

5.2. *Proof of Theorem* 5.2

In order to illustrate the difference between the proof of (f) ⇒ "Y Hölder" and the proof of "Y Hölder" ⇒ (f), we will use a straightforward application of the Kolmogorov lemma to show the first implication, while for the second one, we will need to use specifically Gaussian tools for characterizing the boundedness and continuity of a Gaussian field (see [1]). It should be noted that such tools can also be used to show the first implication; however, doing so would arguably be an overkill.

Proof of (f) ⇒ "*Y Hölder.*" For the homogeneous spatial covariance $R=\sum_{\pi\in\hat{G}} r_\pi\chi_\pi$ of the process $Y=(I-\Delta)^{-1/2}\,W$, condition (f) translates into that for all $\gamma<\varepsilon$

$$K:=\sum_{\pi\in\hat{G}} r_\pi d_\pi(1+\lambda_\pi)^\gamma<\infty. \tag{23}$$

We calculate

$$E[(Y(t,x)-Y(t,y))^2]=2t(R(e)-R(xy^{-1}))$$
$$=2t\sum_{\pi\in\hat{G}} r_\pi(d_\pi-\chi_\pi(xy^{-1})).$$

Using Propositions 2.2 and 2.4, and the fact that $0\leqslant d_\pi-\chi_\pi(z)\leqslant 2d_\pi$,

$$\sum_{\pi\in\hat{G}} r_\pi(d_\pi-\chi_\pi(xy^{-1}))\leqslant c\sum_{\pi\in\hat{G}} r_\pi[d_\pi\lambda_\pi\rho^2(x,y)]^\gamma\,[d_\pi]^{1-\gamma}$$
$$\leqslant c\sum_{\pi\in\hat{G}} r_\pi d_\pi(1+\lambda_\pi)^\gamma\,[\rho(x,y)]^{2\gamma}.$$

This implies that for any integer $p\geqslant 2$, there is a constant C depending on c, K, p such that

$$E[|Y(t,x)-Y(t,y)|^p]=c_pE(Y(t,x)-Y(t,y))^2$$
$$\leqslant Ct^{p/2}\rho(x,y)^{2p\gamma/2},$$

which, by Kolmogorov's lemma, implies that Y has a version which is spatially a.s.-Hölder continuous with any exponent $\gamma'<(p\gamma-d)/p$. Since γ can be arbitrarily close to ε, so can γ'.

5.2.2. *Using Characterizations for Boundedness and Continuity of Gaussian Processes.* As a tool to prove "Y Hölder" $\Rightarrow$ (f), we use the intermediate step provided by the following general proposition. Only the direct implication part of this proposition is needed for our purposes.

PROPOSITION 5.6. *Let $\{Y(x): x\in G\}$ be a homogeneous Gaussian process indexed by a compact Lie group G. The metric on G is denoted by ρ. The* canonical metric *of Y is the pseudo-metric defined by*

$$\delta(x,y)=E^{1/2}[(Y(x)-Y(y))^2].$$

Then Y is almost-surely γ-Hölder-continuous for all $\gamma<\varepsilon$ if and only if

$$\forall\gamma<\varepsilon,\qquad \exists\alpha_0>0:\forall x\in G,\qquad \rho(x,e)<\alpha_0\Rightarrow\delta(x,e)\leqslant\rho(x,e)^\gamma. \tag{24}$$

Proof. (i) Proof that "Y Hölder" $\Rightarrow$ (24).

Let $C^\theta(G)$ be the space of θ-Hölder continuous functions on G. For any $\theta\in(0,1)$, $\alpha\leqslant D_G$, and any function f defined on G, set

$$A_\alpha(f)=\sup_{\rho(x,y)\leqslant\alpha}|f(x)-f(y)|,\qquad N_\theta(f)=\sup_{\alpha\leqslant D_G}\frac{A_\alpha(f)}{\alpha^\theta},$$

where D_G designates the diameter of G. Then, following Fernique's notations [7, Definition 1.2.1], N_θ is a gauge on $C^\theta(G)$. For any $0<\theta<\varepsilon$, we have supposed that $P(Y\in C^\theta(G))=1$. Then, using a lemma of Fernique [7, Lemma 1.2.3], we have $E[N_\theta(Y)]\leqslant c_\theta$, and hence $E[A_\alpha(Y)]\leqslant c_\theta\alpha^\theta$ for any $\alpha\leqslant D_G$. Take now $0<\gamma<\theta<\varepsilon$. Then

$$E[A_\alpha(Y)]\leqslant (c_\theta\alpha^{\theta-\gamma})\,\alpha^\gamma,$$

and choosing α small enough, we get

$$E[A_\alpha(Y)]\leqslant \alpha^\gamma. \tag{25}$$

Now assume that (24) is not satisfied. Therefore, there exists a $\gamma_0<\varepsilon$ and there exists a sequence $\{x_n\}_n$ in G that converges to e, such that

$$\delta(x_n, e)>\rho(x_n, e)^{\gamma_0}.$$

Suppose moreover that $x\in S$ with $S=\{x\in G;\ \rho(e, x)\leqslant\alpha\}$. Let $\beta_n:=\delta(x_n, e)$. In particular, $\beta_n>0$, and $\rho(x_n, e)<\beta_n^{1/\gamma_0}$, so that x_n is in the set $\{x\in S;\ \rho(x, e)<\beta_n^{1/\gamma_0}\}$. Since $Y(x_n)-Y(e)$ is a centered Gaussian variable with standard deviation β_n, we have the exact formula $E\,|Y(x_n)-Y(e)|=c\beta_n$ where the universal constant $c=(\pi/2)^{-1/2}$. This proves that

$$EA_{\beta_n^{1/\gamma_0}}\geqslant c\beta_n.$$

Let $\alpha_n:=\beta_n^{1/\gamma_0}>0$. By almost-sure continuity of Y at e, and since $\sup_n|Y(x_n)-Y(e)|$ is integrable (Theorem 2.1 and Lemma 3.1 in [1]), dominated convergence implies that β_n, and consequently α_n, converge to zero. Since $EA_{\alpha_n}\geqslant c\alpha_n^{\gamma_0}$, this contradicts (25), and part (i) of this proof is completed.

(ii) Proof that "Y Hölder" $\Leftarrow$ (24). Let $B_\delta(x, \beta)$ be the ball $\{y\in G: \delta(x, y)<\beta\}$. For any probability measure m on G, define

$$\gamma_m(\eta):=\sup_{x\in G}\int_0^\eta\sqrt{\log\frac{1}{m(B_\delta(x, \beta))}}\,d\beta.$$

A measure m such that $\gamma_m(\infty)$ is finite is called a majorizing measure for Y. It turns out that for any bounded homogeneous Gaussian process, the Haar measure is a majorizing measure (Theorem 4.4 in [1], which also holds for non-abelian compact groups). A result of Talagrand (Corollary 4.7 in [1]) states that there is a universal constant K such that, for any a.s. bounded Gaussian process Y with canonical metric δ, the function $K\gamma_m$

serves as an a.s. uniform modulus of continuity for Y relatively to δ, if m is a majorizing measure. This yields in our homogeneous situation that there exists an almost-surely positive random number η_0 such that if $\eta < \eta_0$

$$\sup_{x,\, y:\, \delta(x,\, y) < \eta} |Y(x) - Y(y)| \leqslant K \int_0^\eta \sqrt{\log |B_\delta(e, \beta)|^{-1}}\, d\beta := K\omega(\eta).$$

If we assume (24) holds, for fixed $\gamma < \varepsilon$, for α small enough, it follows that

$$\sup_{x,\, y:\, \rho(x,\, y) < \alpha} |Y(x) - Y(y)| \leqslant K\omega(\alpha^\gamma).$$

We now estimate ω using (24) and Property (4) that for some constant $c_1 > 0$, $|B_\rho(e, r)| \geqslant c_1 r^d$: for small enough η,

$$\begin{aligned}
\omega(\eta) &\leqslant \int_0^\eta \sqrt{\log |B_\rho(e, \beta^{1/\gamma})|}\, d\beta \\
&\leqslant \int_0^\eta \sqrt{\log \left(\frac{c_1}{\beta^{d/\gamma}} \right)}\, d\beta. \\
&\leqslant c' \int_0^{\eta/c''} \sqrt{\log \tilde{\beta}^{-1}}\, d\tilde{\beta},
\end{aligned}$$

where c', c'' are constants depending on c_1 and d/γ. We have the following elementary inequality

$$\int_0^\eta \sqrt{\log a^{-1}}\, da \leqslant \eta(\sqrt{\pi} + \sqrt{\log \eta^{-1}}),$$

whose proof is left to the reader. This shows that $\omega(\eta) = o(\eta^{1-\theta})$ for any $\theta \in (0, 1)$, proving part (ii), and the proposition. ∎

It is remarkable to note that in the proof of (i), the notion of majorizing measures is not needed. Such measures provide lower bounds (the Fernique–Talagrand lower bound in Theorem 4.1 in [1]) which do go in the direction of the second part of the proof of (i), but the much weaker fact that $E\,|N_\sigma| = c\sigma$ for a Gaussian r.v. with variance σ^2 is the only quantitative "lower bound" we need.

5.2.3. *Proof of "Y Hölder"* $\Rightarrow$ (f). According to the last proposition, we only need to show that (f) holds under assumptions [G], [H], and the assumption that for all $\gamma < \varepsilon$, if x is close enough to e, $\delta(x, e) \leqslant \rho(x, e)^\gamma$. Under these two assumptions, we have the following:

LEMMA 5.7. *If K is large enough,*

$$\sum_{\pi \in \hat{G}_{K,\eta}} r_\pi d_\pi \leqslant \frac{L}{K^{2\gamma}},$$

where $L = mc_1^{-1} c_2^{-2\gamma}(1-\cos a)^{-1}$ and m, η, c_1, c_2, a are as in Assumption $[\tilde{\mathrm{G}}]$.

Proof. For $i = 1, ..., m$, set $\sigma_K^i = v_K^i \, |v_K^i|^{-2}$. Then σ_K^i is an element of $\mathscr{T}^*$ that can be identified with an element of $\mathscr{T}$ by usual techniques. Set then $x_K^i = \exp(\sigma_K^i)$.

Since $E[Y(x)\, Y(e)] = \sum_{\pi \in \hat{G}} r_\pi \chi_\pi$, we have

$$\delta(x, e)^2 = \sum_{\pi \in \hat{G}} r_\pi (d_\pi - \chi_\pi(x)).$$

Moreover, formula (6) yields for any $x = \exp \sigma$ in T,

$$d_\pi - \chi_\pi(x) = \sum_{k=1}^{d_\pi} [1 - \cos \langle \mu_\pi^{(k)}; \sigma \rangle].$$

Let $\pi \in \hat{G}_{K,\eta}$. For each fixed i in $\{1, ..., m\}$, let $A_{i,\pi}$ be the set of k's, given in assumption $[\tilde{\mathrm{G}}]$, for which $|\langle \mu_\pi^{(k)}; v_K^i \rangle| > a\, |v_K^i|^2$. Therefore for $k \in A_{i,\pi}$,

$$|\langle \mu_\pi^{(k)}; \sigma_K^i \rangle| > a.$$

Also, since $|\mu_\pi^{(k)}| \leqslant (1+\eta) K$ and $|v_K^i| \geqslant c_2 K$ we get

$$|\langle \mu_\pi^{(k)}; \sigma_K^i \rangle| \leqslant (1+\eta) K\, |v_K^i|^{-1} \leqslant \frac{1+\eta}{c_2} < \frac{\pi}{2}.$$

Therefore

$$\cos \langle \mu_\pi^{(k)}; \sigma_K^i \rangle \leqslant \cos a < 1,$$

and using condition $[\tilde{\mathrm{G}}]$

$$\begin{aligned} \sum_{i=1}^m [d_\pi - \chi_\pi(x_K^i)] &\geqslant \sum_{i=1}^m \sum_{k \in A_{i,\pi}} [1 - \cos \langle \mu_\pi^{(k)}; \sigma_K^i \rangle] \\ &\geqslant \sum_{i=1}^m (1 - \cos a)\, Card(A_{i,\pi}) \\ &\geqslant c_1 (1 - \cos a)\, d_\pi. \end{aligned}$$

Using this and the fact that $\delta(x, e) \leqslant \rho(x, e)^{\gamma}$ near e, we obtain that for K large enough,

$$\begin{aligned}
\sum_{\pi \in \hat{G}_{K,\eta}} r_\pi d_\pi &\leqslant \frac{1}{c_1(1-\cos a)} \sum_{\pi \in \hat{G}_{K,\eta}} r_\pi \sum_{i=1}^{m} [d_\pi - \chi_\pi(x_K^i)] \\
&\leqslant \frac{1}{c_1(1-\cos a)} \sum_{i=1}^{m} \sum_{\pi \in \hat{G}} [d_\pi - \chi_\pi(x_K^i)] \\
&= \frac{1}{c_1(1-\cos a)} \sum_{i=1}^{m} \delta(e, x_K^i)^2 \\
&\leqslant \frac{1}{c_1(1-\cos a)} \sum_{i=1}^{m} |\sigma_K^i|^{-2\gamma} \\
&\leqslant \frac{m}{c_1 c_2^{2\gamma}(1-\cos a)\, K^{2\gamma}}.
\end{aligned}$$

and the lemma follows. ∎

Therefore we have proved that for K large enough,

$$\begin{aligned}
\sum_{|h_\pi| \geqslant K} r_\pi d_\pi &= \sum_{l=0}^{\infty} \sum_{K(1+\eta)^l \leqslant |h_\pi| < K(1+\eta)^{l+1}} r_\pi d_\pi \\
&\leqslant LK^{-2\gamma} \sum_{l=0}^{\infty} (1+\eta)^{-2\gamma l} = \frac{M}{K^{2\gamma}},
\end{aligned}$$

where M is a constant depending only on m, a, c, η, γ. Now let $\{\pi_m\}_{m=0}^{\infty}$ be a numbering of $\hat{G}$ in order of increasing $|h_{\pi_m}|$. This is possible because the maximal weights' moduli do not accumulate before infinity in a compact Lie group. Therefore we have, for $K = |h_{\pi_n}|$,

$$\sum_{m=n}^{\infty} r_{\pi_m} d_{\pi_m} \leqslant \frac{M}{|h_{\pi_n}|^{2\gamma}} \leqslant \frac{2M}{(\lambda_{\pi_n})^{\gamma}}$$

since, by Proposition 2.4, λ_π and $|h_\pi|^2$ are equivalent when they tend to infinity. This proposition also implies that λ_{π_n} is increasing for large n. Assumption [H] and the following lemma immediately yield condition (f), which finishes the proof of Theorem 5.2.

LEMMA 5.8. *Let f_n and λ_n be positive numerical sequences. Assume that λ is increasing, and that there is a $\gamma \in (0, 1)$ such that for all n large enough,*

$$\sum_{m=n}^{\infty} f_m \leqslant \lambda_n^{-\gamma}.$$

Then, for all $\theta < \gamma$, the series $\sum_{m=0}^{\infty} f_{m+1} \lambda_m^{\theta}$ converges.

Proof. It is convenient to introduce the following functions defined on $[0, \infty)$: $f(x) = f_{[x]}$ and $\lambda(x)$ is the function that is linear on each $[n, n+1]$ and coincides with λ_m for $x = m$. Since, by concavity of $\lambda(x)^{\theta}$, $\int_n^{n+1} f(x+1) \lambda(x)^{\theta}\, dx \geqslant f(n+1)[(\lambda(n) + \lambda(n+1))/2]^{\theta} \geqslant f(n+1)(\lambda(n))^{\theta}$, we also have

$$\sum_{m=0}^{\infty} f_{m+1} \lambda_m^{\theta} \leqslant \int_0^{\infty} f(x+1)\, \lambda(x)^{\theta}\, dx.$$

With the notation $F(x) = \int_x^B f$, we have by hypothesis, for integer n and $x \in [n, n+1)$, $F(x+1) \leqslant \lambda(n+1)^{-\gamma} \leqslant \lambda(x)^{-\gamma}$. We get

$$\begin{aligned}
&\int_A^B f(x+1)\, \lambda(x)^{\theta}\, dx \\
&\qquad = F(A+1)\, \lambda(A)^{\theta} + \int_A^B F(x+1)\, \lambda'(x)\, \lambda(x)^{\theta - 1}\, dx \\
&\qquad \leqslant \lambda(A+1)^{-\gamma}\, \lambda(A)^{\theta} + \int_A^B \lambda'(x)\, \lambda(x)^{-1-(\gamma-\theta)}\, dx.
\end{aligned}$$

By the change of variable $y = \lambda(x)$ in the last integral on the intervals on which λ is strictly increasing, that integral can be written as a telescoping sum which converges as $B \to \infty$, proving the lemma. ∎

6. APPENDIX

We are going to prove here those results in our paper requiring some knowledge of representation theory of Lie groups, that is, Remarks 3.3, 3.4, and 3.5, and Propositions 5.3 and 5.4. We also include the proof of the random Fourier structure Theorem 2.12. Let us recall first some basic notations of representation theory: for a compact and connex Lie group G with maximal torus T having a Lie algebra $\mathscr{T}$, we shall call D the dominant chamber, P the lattice of weights, Q the lattice of roots and Q^+ the lattice of positive roots, all those objects being included in $\mathscr{T}^*$, the dual space of $\mathscr{T}$ (see, e.g., [6, 10] for the exact definition of those notions). Set also $\xi = \frac{1}{2} \sum_{\alpha \in Q^+} \alpha$. The Weyl dimension formula gives the dimension of an irreducible representation in terms of its highest weight:

$$d_{\pi} = \prod_{\alpha \in Q^+} \frac{\langle h_{\pi} + \xi; \alpha \rangle}{\langle \xi; \alpha \rangle}.$$

6.1. *Proof of Remark* 3.3

For the abelian group S^1, we have $\hat{G}=\mathbb{Z}$, $d_n=1$, and $\lambda_n=n^2$. Therefore, with $q_n=n^{1/2}$, we have $\sum_n q_n(1+\lambda_n)^{-1}<\infty$, which means $Q=\sum_n q_n\chi_n$ is in $\mathscr{H}$. However, $\sum_n q_n^2(1+\lambda_n)^{-1}$ diverges, which means Q is not in H_{-1}.

More generally, however, we can prove that $\mathscr{H}\subset H_{-2}$ in all cases as follows. Order the elements of $\hat{G}$ in the order of increasing $q_\pi d_\pi/(1+\lambda_\pi)$. Then for $\sum_{n\in\mathbb{N}} q_n d_n/(1+\lambda_n)<\infty$ we must have $q_n=o((1+\lambda_n)/nd_n)$ and therefore $q_n^2/(1+\lambda_n)^2=o(n^{-2}d_n^{-2})$, which is summable since d_n is always $\geqslant 1$.

6.2. *Proof of Remark* 3.4

Let G be any non-abelian compact Lie group, so that all d_π are greater than 1. Weyl's dimension formula implies that d_π is a polynomial in h_π of degree $\geqslant 1$, which suggests that for some constant c, numbering h_π, d_π and λ_π in order of increasing $|h_\pi|$, we have $d_n\geqslant c\,|h_n|$. We assume this inequality holds; it does in all classical examples. We also have $\lambda_n\leqslant c'\,|h_n|^2$ for some constant c', by Proposition 2.4. Therefore, for $Q=\sum q_n\chi_n$ in $\mathscr{H}$,

$$\infty>\sum_n \frac{q_n d_n}{1+\lambda_n}\geqslant c''\sum_n \frac{q_n}{|h_n|}$$

and thus, $q_n\,|h_n|^{-1}$ is bounded.

To prove that Q is a measure, it is sufficient to show that it is finite on all bounded functions. If f is bounded, it is in $L^2(G)$, and can be written as $\sum_n\sum_{i,j}\hat{f}_{n,i,j}\sqrt{d_n}\,\pi_{i,j}$. Since $\int_G\tilde{\pi}_{k,l}\chi_\pi=0$ unless $\tilde{\pi}=\pi$ and $k=l$, in which case it equals d_π^{-1}, we get

$$Q(f)=\sum_{\pi\in\hat{G}}\sum_{i=1}^{d_\pi} q_\pi d_\pi^{-1/2}\,\hat{f}_{\pi,i,i}.$$

Since f is bounded, $f(e)$ is finite. Moreover, $\pi(e)=Id$, so $f(e)=\sum_{\pi,i}\hat{f}_{\pi,i,i}\sqrt{d_\pi}<\infty$. In the previous paragraph we showed that $q_\pi d_\pi^{-1}$ is bounded. Therefore $q_\pi d_\pi^{-1/2}\,\hat{f}_{\pi,i,i}$ is summable, proving Q is a measure, as long as $d_n\geqslant c\,|h_n|$ holds.

We leave it to the reader to check that $\mathscr{H}\subset M((S^1)^d)$ for $d\geqslant 2$.

6.3. *Proof of Remark* 3.5

Assume that G is a group such that asymptotically, for $|h_\pi|$ numbered in increasing order, $d_n\geqslant c\,|h_n|^2$ for some constant c. Then for $Q=\sum_n q_n\chi_n$ in $\mathscr{H}$, by Proposition 2.4,

$$\infty>\sum_n q_n d_n(1+\lambda_n)^{-1}\geqslant c\sum_n q_n.$$

In particular, $\sum_n q_n^2 < \infty$, which means Q is a function in $L^2(G)$.

It is worth mentioning that the mere fact that d_π is a polynimial of degree $k \geqslant 2$ does not always imply that $d_n \geqslant c\,|h_n|^2$. In the case of $SO(4)$ for example, we have $k=2$ but d_π only of order $|h_\pi|$.

6.4. *Proof of Proposition* 5.3

Condition [G] is trivially satisfied in the case of $(S^1)^d$, since each irreducible representation is of dimension 1. We shall concentrate now on the case of $SO(2n)$ for $n \geqslant 1$, the cases of $SU(n)$ and $SO(2n+1)$ being very similar.

Let us verify that condition [G] is satisfied in the case of $SO(2n)$. The maximal torus T is then composed of all $2n \times 2n$ matrices that can be divided in 2×2 blocks $B_1, ..., B_n$ around the diagonal, with

$$B_i = \begin{pmatrix} \cos(\theta_i) & -\sin(\theta_i) \\ \sin(\theta_i) & \cos(\theta_i) \end{pmatrix}.$$

The algebra $\mathscr{T}$ of the torus T can be identified then with $\mathbb{R}^n$, and we shall denote by $\theta_1, ..., \theta_n$ an element of $\mathscr{T}$. Let $(\theta_1^*, ..., \theta_n^*)$ be the canonical basis of $\mathscr{T}^*$. The lattices Q, Q^+, P and D are defined by (see, e.g., [6] for more details): $P = \mathbb{Z}^n$ and

$$Q = \left\{ \sum_{i,\,j=1}^{n} c_{i,\,j}(\theta_i^* \pm \theta_j^*);\; c_{i,\,j} \in \mathbb{Z} \right\}$$

$$Q^+ = \left\{ \sum_{1 \leqslant i < j \leqslant n} c_{i,\,j}(\theta_i^* \pm \theta_j^*);\; c_{i,\,j} \in \mathbb{N} \right\}$$

$$D = \left\{ \tau = \sum_{i=1}^{n} a_i \theta_i^*;\; a_i \in \mathbb{N},\, a_1 \geqslant \cdots \geqslant a_n \right\}.$$

For a given $\tau \in D$, there is a unique $\pi \in \hat{G}$ such that $\tau = h_\pi$, and inversely, each representation $\pi \in \hat{G}$ is determined by an element $\tau \in D$. It is known that all the weights of π which are also elements of D can be obtained from τ descending along the lattice $-Q^+$. If $\mu \in D$ is a weight of π, denote by n_μ the multiplicity of μ as an eigenvalue of $d\pi_e$. The Freudenthal multiplicity formula (see, e.g., [10, Proposition 25.1, p. 416] gives an iterative formula, starting from τ, for the different values n_μ,

$$c(\mu)\, n_\mu = 2 \sum_{\alpha \in Q^+} \sum_{k \geqslant 1} \langle \mu + k\alpha; \alpha \rangle\, n_{\mu + k\alpha}, \tag{26}$$

with $c(\mu) = |\tau + \xi|^2 - |\mu + \xi|^2$.

Notice that, starting from $\tau \in D$, it is always possible to construct a succession $\tau^0, ..., \tau^{n(n-1)}$, such that

(1) $\tau^0 = \tau$ and $\tau^i \in D$ for every $i \in \{1, ..., n(n-1)\}$,

(2) $\tau^i = \tau^{i-1} - \alpha_i$ for an element $\alpha_i \in Q^+$, with $\alpha_i \neq \alpha_j$ if $i \neq j$.

Then, for $i = \{1, ..., n(n-1)\}$, we have $c(\tau^i) \leqslant \sqrt{2}\, n(n-1)$, and therefore, according to (26),

$$n_{\tau^i} \geqslant K_1 \langle \tau^{i-1}; \alpha_i \rangle\, n_{\tau^{i-1}},$$

where $K_1 = \sqrt{2}/n(n-1)$. Using this iterative relation, and the fact that $d_\pi \leqslant K_2 \prod_{\alpha \in Q^+} (1 + \langle \tau, \alpha \rangle)$ for a constant $K_2 > 0$, it is easily seen that

$$\sum_{i=1}^{n(n-1)} n_{\tau^i} \geqslant c_1 d_\pi,$$

for a constant $c_1 > 0$. Moreover, for any $i \in \{1, ..., n(n-1)\}$, we have $|\tau^i| \geqslant |\tau| - \sqrt{2}\, n(n-1)$, and for any constant $\xi \in (0, 1)$, if $|\tau|$ is large enough, we get $|\tau^i| \geqslant \xi\, |\tau|$, which shows that condition [G] is satisfied.

As far as hypothesis [H] is concerned, it is known that if $\{h(n)\}_{n=0}^{\infty}$ is a numbering of $\{|h_\pi|\}_{\pi \in \hat{G}}$ in order of increasing value, then $h(n)$ is a function of n with polynomial growth, which gives the boundedness of $\{h(n+1)/h(n)\}_{n=1}^{\infty}$.

6.5. *Proof of Proposition* 5.4

We shall prove here that condition [$\tilde{\mathrm{G}}$] is implied by condition [G]. Consider a constant $\zeta \in (0, 1)$ and the set P_ζ of weights τ of the representations $\pi \in \hat{G}_{K,\eta}$ such that $\tau \in D$ and $(1-\zeta) K \leqslant |\tau| \leqslant (1+\eta) K$. The chamber D can be split into m disjoint subsets $D_1, ..., D_m$ such that, for any $i = 1, ..., m$,

(i) $P_\zeta \cap D_i$ is not empty.

(ii) For any pair of vectors $(v_1, v_2) \in D_i^2$, we have $\langle v_1; v_2 \rangle \geqslant \kappa\, |v_1|\, |v_2|$,

for a fixed constant $\kappa \in (0, 1)$ (in fact this can be done for any region of the vector space $\mathscr{T}$, splitting for example by a finite number of hyperplanes). Pick a certain representative v_K^i of the sub-lattice $P_\zeta \cap D_i$. By definition, for any $\tau \in P_\zeta$, we have, for a $i \in \{1, ..., m\}$,

$$|\langle \tau, v_K^i \rangle| \geqslant \kappa\, |\tau|\, |v_K^i| \geqslant \kappa \frac{1-\zeta}{1+\eta} |v_K^i|^2.$$

Hence, setting $a = \kappa(1 - \zeta/1 + \eta)$, we get, for any $\pi \in \hat{G}_{K,\eta}$,

$$\begin{aligned}\operatorname{Card}\{k: k = 1, ..., d_\pi; \exists 1 \leqslant i \leqslant m \text{ s.t. } |\langle \mu_\pi^k; v_K^i \rangle| > a\, |v_K^i|^2\}\\ \geqslant \operatorname{Card}\{k: k = 1, ..., d_\pi; \mu_\pi^k \in P_\zeta\}.\end{aligned}$$

But using assumption [G], and since the global geometric distribution of the weights μ_π^k of any $\pi \in \hat{G}$ is obtained with a finite number of symmetries with respect to the boundaries of the Weyl chambers, we have

$$\operatorname{Card}\{k: k = 1, ..., d_\pi; \mu_\pi^k \in P_\zeta\} \geqslant c_1\, |d_\pi|,$$

which ends the proof.

6.6. *Proof of Theorem* 18 *and Corollaries* 2.13 *and* 2.14

(iii) $\Rightarrow$ (ii). The fact that under condition (iii), W is real valued, is easily established using the same argument as in the proof of Corollary 2.14 (see below). Then, condition (ii) is a consequence of the following calculation, which uses the fact that each π in $\hat{G}$ is a representation, so that $\pi(x)\, \overline{\pi(y)}^T = \pi(xy^{-1})$:

$$\begin{aligned}E[W(\phi)\, W(\psi)] &= \sum_{\pi \in \hat{G}} q_\pi \iint_{G^2} \phi(x)\, \psi(y) \sum_{k,l=1}^{d_\pi} \pi_{k,l}(x)\, \overline{\pi_{k,l}(y)}\, dx\, dy\\ &= \sum_{\pi \in \hat{G}} q_\pi \iint_{G^2} \phi(x)\, \psi(y) \sum_{k,l=1}^{d_\pi} \pi_{k,l}(x)\, \pi_{l,k}(y^{-1})\, dx\, dy\\ &= \sum_{\pi \in \hat{G}} q_\pi \iint_{G^2} \phi(x)\, \psi(y) \sum_{k=1}^{d_\pi} \pi_{k,k}(xy^{-1})\, dx\, dy\\ &= \sum_{\pi \in \hat{G}} q_\pi \iint_{G^2} \phi(x)\, \psi(y)\, \chi_\pi(xy^{-1})\, dx\, dy\\ &= \sum_{\pi \in \hat{G}} q_\pi \iint_{G^2} \phi(zy)\, \psi(y)\, \chi_\pi(z)\, dy\, dz\\ &= \sum_{\pi \in \hat{G}} q_\pi \int_G [\phi * \check{\psi}](z)\, \chi_\pi(z)\, dz.\end{aligned}$$

(ii) $\Rightarrow$ (i). For this implication, we only need to check that the $\tilde{Q}$ defined by (13) is indeed finite on all test functions in H_p, which is trivial by the integrability of the coefficients q_π.

(i) $\Rightarrow$ (iii). Note that by compactness, the constant functions are in H_p; they are the functions whose coefficients f_π in Proposition 2.8 are all zero except the one corresponding to the trivial representation $\pi \equiv 1$. Therefore, since W is defined on all of H_p,

$$\begin{aligned}|Q(\phi)| &= |Q(\phi * \mathbf{1})| \\ &= |E[W(\phi)\, W(\mathbf{1})]| \leqslant E^{1/2}[|W(\phi)|^2]\, E^{1/2}[|W(\mathbf{1})|^2] < \infty.\end{aligned}$$

which proves that Q is in H_{-p}. Thus Q has a representation as in Proposition 2.8,

$$Q(\cdot) = \sum_{\pi \in \hat{G}} \sum_{i,\, j=1}^{d_\pi} q_{\pi,\, i,\, j} d_\pi^{1/2} \pi_{i,\, j}(\cdot),$$

with

$$\sum_{\pi \in \hat{G}} \sum_{i,\, j=1}^{d_\pi} q_{\pi,\, i,\, j}^2 (1+\lambda_\pi)^{-p} < \infty. \tag{27}$$

By assumption, W is real valued. Therefore, $\bar{W}(\phi) = W(\bar{\phi})$. Using this fact and a calculation similar to that of the proof of (iii) $\Rightarrow$ (ii), one then checks that $q_{\pi,\, k,\, k} = d_\pi^{1/2} E\, |W(d_\pi \pi_{k,\, l})|^2 \geqslant 0$.

We now consider the Gaussian family of complex variables $B_{k,\, l}^\pi := W(d_\pi^{1/2} \pi_{k,\, l})$. We can compute

$$\begin{aligned}E\overline{B_{k,\, l}^\pi} B_{k',\, l'}^{\pi} &= \sum_{\sigma \in \hat{G}} \sum_{i,\, j=1}^{d_\sigma} q_{\sigma,\, i,\, j} d_\sigma^{1/2} \int \sigma_{i,\, j}(zx^{-1})\, d_\pi^{1/2} d_{\pi'}^{1/2} \bar{\pi}_{k,\, l}(z)\, \pi'_{k',\, l'}(x)\, dx\, dz \\ &= \sum_{\sigma \in \hat{G}} \sum_{i,\, j,\, m=1}^{d_\sigma} q_{\sigma,\, i,\, j} d_\sigma^{1/2} \\ &\quad \times \int \sigma_{i,\, m}(z)\, d_\pi^{1/2} \bar{\pi}_{k,\, l}(z)\, dz \int \bar{\sigma}_{j,\, m}(x)\, d_{\pi'}^{1/2} \pi'_{k',\, l'}(x)\, dx \\ &= q_{\pi,\, k,\, k'} d_\pi^{-1/2} \delta_{l,\, l'} \delta_{\pi,\, \pi'}.\end{aligned}$$

Call $\tilde{B}$ the same family corresponding to $\tilde{W} = W(\cdot \circ (\cdot)^{-1})$. A similar computation shows that $\tilde{B}$ satisfies

$$E\overline{\tilde{B}_{k,\, l}^\pi} \tilde{B}_{k',\, l'}^{\pi} = q_{\pi,\, l,\, l'} \delta_{k,\, k'} d_\pi^{-1/2} \delta_{\pi,\, \pi'}.$$

Therefore, since by assumption W and $\tilde{W}$ have the same distribution, we get that $q_{\pi,\, k,\, l} = 0$ unless $k = l$, and for all k, l, $q_{\pi,\, k,\, k} = q_{\pi,\, l,\, l}$. We call s_π this common value.

We can calculate the covariances of the real and imaginary parts of $B_{k,\, l}^\pi$. We will use the Schur orthogonality relations, which imply that the components of non equilavent $\pi, \bar{\pi}$ in $\hat{G}$ are orthogonal in $L^2(G)$, and that

$\{d_\pi^{1/2}\pi_{k,l}: k, l=1, ..., d_\pi\}$ is an orthonormal set. We will show how to calculate $E[\Re B^\pi_{k,l} \Re B^{\pi'}_{k',l'}]$ when neither π nor π' are real-valued. The other calculations are similar.

Since W is real-valued, $\Re B^\pi_{k,l} = W(\Re d_\pi^{1/2}\pi_{k,l})$. Thus

$$\begin{aligned} E[\Re B^\pi_{k,l} \Re B^{\pi'}_{k',l'}] = {} & \tfrac{1}{4}\, EW(d_\pi^{1/2}\pi_{k,l})\, W(d_{\pi'}^{1/2}\pi'_{k',l'}) \\ & + \tfrac{1}{4} E\bar{W}(d_\pi^{1/2}\pi_{k,l})\, \bar{W}(d_{\pi'}^{1/2}\pi'_{k',l'}) \\ & + \tfrac{1}{4} EW(d_\pi^{1/2}\pi_{k,l})\, W(d_{\pi'}^{1/2}\overline{\pi'_{k',l'}}) \\ & + \tfrac{1}{4}\, E\bar{W}(d_\pi^{1/2}\pi_{k,l})\, \bar{W}(d_{\pi'}^{1/2}\bar{\pi}'_{k',l'}). \end{aligned} \tag{28}$$

Now calculate

$$\begin{aligned} & EW(d_\pi^{1/2}\pi_{k,l})\, W(d_{\pi'}^{1/2}\pi'_{k',l'}) \\ & = d_\pi^{1/2} d_{\pi'}^{1/2}\, Q(\pi_{k,l} * \check{\pi}'_{k',l'}) \\ & = d_\pi^{1/2} d_{\pi'}^{1/2}\, Q\left(\sum_{j=1}^{d_\pi} \int \pi_{k,j}(\cdot)\, \pi_{j,l}(y)\, \pi'_{k',l'}(y)\, dy\right) \\ & = Q(\pi_{k,j})\, \delta_{j,k'} \delta_{l,l'} \delta_{[\pi],[\overline{\pi'}]} \\ & = d_\pi^{-1/2} s_\pi \delta_{k,k'} \delta_{l,l'} \delta_{[\pi],[\overline{\pi'}]}. \end{aligned}$$

This shows that if π is not equivalent to π' or $\overline{\pi'}$, all terms in (28) drop out, while if $\pi' \simeq \bar{\pi}$, the first two terms add up $2^{-1} d_\pi^{-1/2} s_\pi$ and the last two are still zero, and if $\pi' \simeq \pi$, the last two terms ad up to $2^{-1} d_\pi^{-1/2} s_\pi$ and the first two are zero.

This proves that, modulo the trivial fact that $\mathfrak{B}^\pi_{k,l} = \mathfrak{B}^{\bar{\pi}}_{k,l}$ and $\mathfrak{B}^\pi_{k,l} = -\mathfrak{B}^{\bar{\pi}}_{k,l}$, the family $\{\Re B^\pi_{k,l}, \Im B^\pi_{k,l}; \pi \in \hat{G};\ k, l=1, ..., d_\pi\}$ is formed of independent real centered Gaussian variables, with variances $d_\pi^{-1/2} s_\pi$ and zero when $\pi \simeq \bar{\pi}$, and variances $2^{-1} d_\pi^{-1/2} s_\pi$ and $2^{-1} d_\pi^{-1/2} s_\pi$ otherwise. It is now easy to check that W can be written as

$$W(\phi) = \sum_{\pi \in \hat{G}} \sum_{i,j=1}^{d_\pi} d_\pi^{1/2} \pi_{i,j}(\phi)\, B^\pi_{i,j};$$

indeed call $X(\phi)$ the right-hand side; then a direct computation shows that $E\,|W(\phi)|^2 = EW(\phi)\,\overline{X(\phi)}$. If we rewrite $d_\pi^{1/2} B^\pi_{i,j}$ as $(d_\pi^{1/2} s_\pi)^{1/2}\, W^\pi_{i,j}$, the previous computation shows that modulo the fact that $W^{\bar{\pi}}_{i,j} = \overline{W^\pi_{i,j}}$, $\{W^\pi_{i,j}: \pi \in \hat{G},\ i, j=1, ..., d_\pi\}$ is a family of independent standard Gaussian variable which are real when $\pi \simeq \bar{\pi}$ and complex otherwise. (Such variables are of the form $x+iy$ where x and y are independent real centered Gaussian variables with variances 1 and 0, or 1/2 and 1/2.)

Finally, it is easy to check that the class of elements of H_{-p} that is identified in equation (13) coincides with the subspace of positive and central generalized functions as defined above. This shows that (ii) $\Leftrightarrow$ (iv), and finishes the proof of Theorem 2.12.

Corollary 2.13 is obvious. Corollary 2.14 can be proved as follows. From formula (14) and the above calculations we obtain, setting $N_0 = \sum_{\pi \in \hat{G}_0} \sqrt{q_\pi} \sum_{k,l=1}^{d_\pi} W_{k,l}^{\pi} \int \pi_{k,l}\phi$,

$$\begin{aligned} W(\phi) &= \sum_{\pi \in \hat{G}_1} \sqrt{q_\pi} \sum_{k,l=1}^{d_\pi} \left(W_{k,l}^{\pi} \int \pi_{k,l}\phi + \overline{W_{k,l}^{\pi}} \int \overline{\pi_{k,l}}\phi \right) + N_0 \\ &= \sum_{\pi \in \hat{G}_1} \sqrt{q_\pi} \sum_{k,l=1}^{d_\pi} \left(2\Re W_{k,l}^{\pi} \int \Re\pi_{k,l}\phi - 2\Im W_{k,l}^{\pi} \int \Im\pi_{k,l}\phi \right) + N_0 \\ &= \sum_{\pi \in \hat{G}_1} \sqrt{q_\pi} \sum_{k,l=1}^{d_\pi} \left(U_{k,l}^{\pi} \int \Re\pi_{k,l}\phi + V_{k,l}^{\pi} \int \Im_{k,l}\phi \right) + N_0, \end{aligned}$$

where, by the independence properties of the families $\Re W$ and $\Im W$, the families U and V are as prescribed in the corollary, finishing its proof.

REFERENCES

1. R. Adler, “An Introduction to Continuity, Extrema, and Related Topics for General Gaussian Processes,” Inst. Math. Stat., Hayward, CA, 1990.
2. P. L. Chow, On solutions of the stochastic wave equation, Communication at the Fourth Joint AMS-SMM meeting, UNT, May 1999.
3. Y. Coulhon, L. Saloff-Coste, and N. Varopoulos, “Analysis and Geometry on Groups,” Cambridge Univ. Press, Cambridge, UK, 1992.
4. R. Dalang and N. Frangos, The stochastic wave equation in two spatial dimensions, preprint, 1997.
5. G. DaPrato and J. Zabczyk, “Stochastic Equations in Infinite Dimensions,” Cambridge Univ. Press, Cambridge, UK, 1992.
6. H. Fegan, “Introduction to Compact Lie Groups,” Prenctice Hall, Englewood Cliffs, NJ, 1991.
7. X. Fernique, “Fonctions Aléatoires Gaussiennes, Vecteurs Aléatoires Gaussiens,” Les Publications CRM, Montréal, 1997.
8. G. Folland, “A Course in Abstract Harmonic Analysis,” Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1995.
9. A. Friedman, “Partial Differential Equations of Parabolic Type,” Prentice Hall, Englewood Cliffs, NJ, 1964.
10. W. Fulton and J. Harris, “Representation Theory,” Springer-Verlag, New York/Berlin, 1991.
11. I. Karatzas and S. Shreve, “Brownian Motion and Stochastic Calculus,” 2nd ed., Springer-Verlag, New York/Berlin, 1991.
12. A. Karczewska and J. Zabczyk, A note on stochastic wave equation, preprint.

13. P. Kotelenez, A class of quasilinear stochastic partial differential equations of Mc-Kean–Vlasov type with mass conservation, *Probab. Theory Related Fields* **102** (1995), 159–188.
14. N. V. Krylov, An analytic approach to SPDEs, stochastic partial differential equations: Six perspectives, Math. Surveys Monogr., Vol. 64, pp. 185–242, Amer. Math. Soc., Providence, RI, 1999.
15. H. Kunita, "Stochastic Flows and Stochastic Differential Equations," Cambridge Univ. Press, Cambridge, UK, 1990.
16. T. Kurtz and P. Protter, Weak convergence of stochastic integrals and differential equations. II. Infinite dimensional case, lecture notes, CIME School in Probab., 1996.
17. P. Marcus and G. Pisier, "Random Fourier Series with Applications to Harmonic Analysis," Princeton Univ. Press, Princeton, NJ, 1981.
18. A. Millet and M. Sanz-Sole, The stochastic heat equation in two spatial dimensions, preprint.
19. S. Peszat and J. Zabczyk, Stochastic evolution equations with a spatially homogeneous Wiener process, *Stochastic Process. Appl.* **72**, No. 2 (1997), 187–204.
20. S. Peszat and J. Zabczyk, Nonlinear stochastic heat and wave equations, preprint.
21. M. Sugiura, Fourier series of smooth functions on compact Lie groups, *Osaka J. Math.* **8** (1971), 33–47.
22. J. B. Walsh, An introduction to stochastic partial differential equations, *in* "Ecole d'Eté de Probabilités de Saint Flour, XIV," Lecture Notes in Math., Vol. 1180, pp. 265–438, Springer-Verlag, New York/Berlin, 1986.