Parameter Estimation for a partially observed Ornstein-Uhlenbeck process with long-memory noise

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Abstract: We consider the parameter estimation problem for the Ornstein-Uhlenbeck process X driven by a fractional Ornstein-Uhlenbeck process V, i.e. the pair of processes defined by the non-Markovian continuous-time long-memory dynamics $dX_t = -\theta X_t dt + dV_t;\ t \geqslant 0$, with $dV_t = -\rho V_t dt + dB_t^H;\ t \geqslant 0$, where $\theta > 0$ and $\rho > 0$ are unknown parameters, and B^H is a fractional Brownian motion of Hurst index $H \in (\frac{1}{2}, 1)$. We study the strong consistency as well as the asymptotic normality of the joint least squares estimator $\left(\widehat{\theta}_T, \widehat{\rho}_T\right)$ of the pair (θ, ρ), based either on continuous or discrete observations of $\{X_s;\ s \in [0, T]\}$ as the horizon T increases to $+\infty$. Both cases qualify formally as partial-observation questions since V is unobserved. In the latter case, several discretization options are considered. Our proofs of asymptotic normality based on discrete data, rely on increasingly strict restrictions on the sampling frequency as one reduces the extent of sources of observation. The strategy for proving the asymptotic properties is to study the case of continuous-time observations using the Malliavin calculus, and then to exploit the fact that each discrete-data estimator can be considered as a perturbation of the continuous one in a mathematically precise way, despite the fact that the implementation of the discrete-time estimators is distant from the continuous estimator. In this sense, we contend that the continuous-time estimator cannot be implemented in practice in any naïve way, and serves only as a mathematical tool in the study of the discrete-time estimators' asymptotics.

Key words: Least squares estimator; fractional Ornstein Uhlenbeck process; Multiple integral; Malliavin calculus; Central limit theorem.
2010 Mathematics Subject Classification: 60F05; 60G15; 60H05; 60H07.

1 Introduction

1.1 Context and background

The question of drift parameter estimation for solutions of stochastic differential equations driven by fractional Brownian noise goes back at least as far as the seminar work of Kleptsyna and Le Breton in [17] (also see prior references therein), where a maximum likelihood estimator (MLE) was proposed. This work was a first genuine attempt to show how to compute the MLE in practice in the regular case (self-similarity 'Hurst' parameter $H > 1/2$), by relying on continuous-time observation of a single path over a finite time interval, with strong consistency and asymptotic normality as the horizon increases to infinity. This work was itself motivated by – and is an extension of – now classical ideas of how to use the Girsanov theorem to compute

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the MLE in the case of white noise, as presented in the 1978 treatise [19]. The continuous-time data was also invoked in [17] to justify that any diffusion-type parameter (any constant multiplicative term in front of the equation's driving noise) would then be directly observable. That observation remains generally true in many contexts for continuous-time data, including when the noise is fractional Brownian. It explains why so many authors since [17] have continued to study the estimation of drift parameters for fractional-noise-driven problems. We have listed some of these works below. Our paper inscribes itself in this line of work, and can be viewed as a study of partial observation questions, as we will explain shortly.

Generally speaking, this effort to understand drift estimation for ergodic diffusions, even Gaussian ones, is of fundamental importance for any quantitative study where mean-reverting quantities are believed to be asymptotically stationary, and are either observed with noise or are intrinsically stochastic. The best known class of examples, which also encapsulates the question of whether or not the stochastic process of interest is observed directly or indirectly, is that of stochastic volatility in quantitative finance. Some of the original ideas on how to estimate this volatility's drift parameters is given in the 2000 research monograph [12]. Similar models with fractional noise were introduced in the late 1990's, as in [10] for continuous time, but to our knowledge, their statistical estimation was left unexplored for more than ten years. The broadest question, applicable in finance and other fields, is to estimate jointly the drift, diffusion, and memory parameters for fractional-noise-driven equations; it is typically non-trivial, and with the exception of one study in [4], largely unresolved by bona fide statistical means for non-self-similar discretely observed continuous-time processes. We will not address this issue here. In the case of partially observed data, we refer the interested reader to a study in the context of high-frequency financial data, where a sequential Bayesian methodology is combined with classical estimation techniques and a calibration method to find H: see [9]. We also mention a method described in [27] for a similar model with partial observation, where it is shown that the minimax-sense optimal estimator has a very slow convergence rate, and relies on high-frequency data; it can be argued that when applied to bona fide financial data, this estimator cannot be implemented without leaving the realm where continuous-time semi-martingale models with long-memory volatility are appropriate.

The present paper looks specifically at a difficulty which arises when one single path is used to estimate more than one real drift-type parameter. Questions of identifiability can arise in this context (see for example the treatment of the Generalized Method of Moments (GMM) as described in [21]). Once such a question has been resolved, the main practical obstacle to implementation is typically that of discretizing the data given by the path, i.e. using only observations in discrete time. For the sake of conciseness, we look at a specific situation where identifiability is resolved explicitly and in a natural way, within the task of deriving the continuous-time and discrete-time estimators, without having to rely on abstract conditions which would provide a priori identifiability of the vector of parameters. Our objective is to describe conditions under which strongly consistent and asymptotically normal estimation can be established quantitatively, based on discretely-observed data alone. Since access to and analysis of continuous-time data is not typically a realistic assumption, we will view any estimators based on continuous-time data as tools in the task of deriving strong consistency and asymptotic normality for the discrete-time estimators. Our strategy is to attempt to discretize the former, which includes a need to approximate Riemann and stochastic integrals. In the process, we discover some fundamental differences between the discrete and continuous time estimators. In particular, we find that while least squares (LS) estimation appears to be the best tool based on continuous-time observation of a path, when converted to discrete time data, the estimator's interpretation as a LS optimizer is lost, and a GMM interpretation seems more appropriate. This introduction, including our summary of results in Section 1.2, contains specific details supporting these ideas.

We are largely motivated by the paper [6], which studies the drift-estimation problem for the Ornstein-Uhlenbeck process driven itself by another, unobserved Ornstein-Uhlenbeck process (OU-OU). Their work only deals with an underlying white noise. Specifically, let W be a standard Brownian motion and let θ and ρ be positive real parameters. The OU-OU process is the solution of the following system

$$\begin{cases} X_0 = 0; \quad dX_t = -\theta X_t dt + dV_t, \quad t \geqslant 0; \\ V_0 = 0; \quad dV_t = -\rho V_t dt + dW_t, \quad t \geqslant 0. \end{cases} \tag{1}$$

Here one may consider that X is observed, while V is not, or conversely, or that both processes are observed; which interpretation is used makes a crucial difference, as we will see.

Since the quadratic variation of V is t, the classical Girsanov theorem implies that a natural candidate to estimate θ is the MLE (recall that this idea was already contained in [19]), which can be easily computed for this model: one gets

$$\widehat{\theta}_T = \frac{-\int_0^T X_t \delta X_t}{\int_0^T X_t^2 dt} \tag{2}$$

$$= \frac{-X_T^2 + T}{2\int_0^T X_t^2 dt}, \tag{3}$$

In (2), the integral with respect to X must be understood in the Itô sense. Consequently, line (3) follows from Itô's formula, and the fact that X too has quadratic variation equal to t. One also notes that this estimator's construction is in fact non-dependent on the form of the bounded-variation part in the definition of V, which can be interpreted as a form of robustness of $\widehat{\theta}_T$ with respect to model misspecification, although we are about to see that the behavior of $\widehat{\theta}_T$ depends heavily on V's drift specification.

On the other hand, it is worth noticing that $\widehat{\theta}_T$ coincides formally with a least squares estimator (LSE). Indeed, by interpreting $\int_0^T X_t \dot{X}_t dt$ as the Itô integral in (2), $\widehat{\theta}_T$ formally minimizes

$$\theta \longrightarrow \int_0^T \left| \dot{X}_t + \theta X_t \right|^2 dt.$$

By reversing the roles of V and X, one can obtain an estimator for ρ similar to (3). However, if V is unobserved, one may recast that estimator by using the estimated value of V_t based on $\widehat{\theta}_T$ and the observed path of X, based on (1). In other words, one defines

$$\widehat{\rho}_T = \frac{-\widehat{V}_T^2 + T}{2\int_0^T \widehat{V}_t^2 dt}$$

where $\widehat{V}_t = X_t + \widehat{\theta}_T \int_0^t X_t dt$ for every $t \leqslant T$. As it turns out, this joint estimator $\left(\widehat{\theta}_T, \widehat{\rho}_T\right)$, based on continuous observations of X alone, *does not* converge to (θ, ρ), but rather to an explicit rational function of the pair of unknown parameters (θ, ρ). This was proposed and proved in [6], wherein a semimartingale approach was used to study the asymptotic behavior. Specifically they showed

- *strong consistency*: as $T \longrightarrow +\infty$, almost surely,

$$\widehat{\theta}_T \longrightarrow \theta + \rho \tag{4}$$

$$\widehat{\rho}_T \longrightarrow \frac{\theta\rho(\theta+\rho)}{(\theta+\rho)^2+\theta\rho} \tag{5}$$

- *asymtpotic normality*: as $T \longrightarrow +\infty$,

$$\sqrt{T}\left(\widehat{\theta}_T - (\theta+\rho), \widehat{\rho}_T - \frac{\theta\rho(\theta+\rho)}{(\theta+\rho)^2+\theta\rho}\right) \xrightarrow{law} N(0,\Gamma)$$

where Γ is a covariance matrix which has a explicit form as a function of θ and ρ. While intuition gathered from the full-observation case is in fact erroneous when V is unobserved (the naïve candidates for $\widehat{\theta}_T$ and $\widehat{\rho}_T$ lead to modified limits rather than $\left(\widehat{\theta}_T, \widehat{\rho}_T\right) \longrightarrow (\theta,\rho)$), nevertheless we have the full picture for the asymptotic behavior of the MLEs/LSEs associated with (1), at the minor cost of having to solve a non-linear system of two equations to obtain consistent estimates of (θ,ρ).

In the present paper, our goal is to investigate what happens when, in (1), the standard Brownian motion W is replaced by a fractional Brownian motion B^H. Thus we assume from now on that X is an Ornstein-Uhlenbeck process driven by a fractional Ornstein-Uhlenbeck process V : this means the pair (X,V) is the unique solution of the system of linear stochastic differential equations

$$\begin{cases} X_0 = 0; \quad dX_t = -\theta X_t dt + dV_t, \quad t \geqslant 0 \\ V_0 = 0; \quad dV_t = -\rho V_t dt + dB_t^H, \quad t \geqslant 0, \end{cases} \tag{6}$$

where $B^H = \left\{B_t^H, t \geqslant 0\right\}$ is a fractional Brownian motion (fBm) with Hurst index $H \in (\frac{1}{2},1)$, and where $\theta > 0$ and $\rho > 0$ are unknown parameters. Though X is defined for all H in $(0,1)$, to keep technical difficulties to a reasonable level, we restrict ourselves to the case $H \in (1/2,1)$. It turns out that we need the condition $\theta \neq \rho$ for identifiability; remarkably, this is not needed when $H = 1/2$, as we saw in the system (4), (5). Details of our results are summarized in Section 1.2.

We now provide some references to estimation with fBm noise, which are further motivations for our work. We mentioned that the single-drift parameter estimation problem for fractional diffusion processes based on continuous-time observations was originally studied in [17] via maximum likelihood; more recent work on this question includes, e.g., [28, 26]. Recently, the LSE for the fractional Ornstein-Uhlenbeck (fOU) process, i.e. the process V in (6) was proposed in [14]: assuming V is fully observed in continuous time, the LSE for ρ is defined by

$$\overline{\rho}_T := -\frac{\int_0^T V_t \delta V_t}{\int_0^T V_t^2 dt},$$

where the integral $\int_0^T V_t \delta V_t$ is interpreted in the sense of Skorokhod. This integral is the extension to fBm of Itô's integral for Brownian motion. In the case $\rho > 0$, [14] proved that $\overline{\rho}_T$ is strongly consistent and asymptotically normal as $T \to \infty$. Unlike in the case of Brownian motion ($H = 1/2$) discussed above, $\overline{\rho}_T$ does not coincide with the MLE given in [17], because the Girsanov theorem for fBm takes a different form than in the case $H = 1/2$. Given the notorious fact that Skorohod integrals are difficult to interpret in practical terms for fBm, the authors of [14] proposed in addition the following alternate estimator, which is arguably a method of moments :

$$\left(\frac{1}{H\Gamma(2H)T}\int_0^T V_t^2 dt\right)^{-\frac{1}{2H}}; \tag{7}$$

they proved it is strongly consistent and asymptotically normal. In the case $\rho < 0$, [5] established that $\overline{\rho}_T$ of ρ is strongly consistent and asymptotically Cauchy-distributed.

The alternate choice of estimator (7) does not, however, avoid the use of continuous observations over discrete ones; this is a problem with many works on fBm-driven models, and is an additional motivation for us to study the asymptotics of estimation for fBm-driven processes based on discrete observations. There exists a rich literature on this practical problem for ordinary diffusions driven by Brownian motions; we refer for instance to [26]. A handful of authors are beginning to look at these questions with various fBm-driven models, starting with [28] in 2007, and more recently [3, 7, 11, 20]. In particular, for the fOU process V, motivated by the estimator given in (7), [15] studied its natural Riemann-sum-type time discretization

$$\left(\frac{1}{nH\Gamma(2H)}\sum_{i=1}^{n} X_i^2\right)^{-\frac{1}{2H}} \tag{8}$$

providing strongly consistency and Berry-Esséen-type theorems for it. While we have no doubt that this estimator is indeed strongly consistent and asymptotically normal, the proofs in [15] rely on a possibly flawed technique, since the passage from line -7 to -6 on page 434 therein requires the condition $H > 3/4$, while one expects normal asymptotics only for the case $H \leqslant 3/4$.

In our paper, we focus our discussion on estimators which are derived from a basic LSE, since that technique is known, at least in the Brownian case described in [6], to allow for a straightforward bivariate extension, as mentioned previously. The LSE has also given rise to a number of successful studies in the univariate case with fractional processes: we have already cited [3, 7, 11, 20], while [2] is the continuous-time version of [3].

Herein, specifically, we begin our study of LSE for (θ, ρ) in (6) by using the formal least-squares interpretation mentioned above, i.e. looking for the minimizer of $\theta \longrightarrow \int_0^T \left|\dot{X}_t + \theta X_t\right|^2 dt$; this leads formally to the following estimator for θ

$$\widehat{\theta}_T = -\frac{\int_0^T X_t \delta X_t}{\int_0^T X_t^2 dt} \tag{9}$$

and to the similar estimator

$$\widehat{\rho}_T = -\frac{\int_0^T \widehat{V}_t \delta \widehat{V}_t}{\int_0^T \widehat{V}_t^2 dt} \tag{10}$$

for ρ, where, because of the fact that V is unobserved, one uses $\widehat{V}_t = X_t + \widehat{\theta}_T \int_0^t X_t dt$ for every $t \leqslant T$, instead of relying on the unobserved V_t in the construction of $\widehat{\rho}_T$. These estimators $\widehat{\theta}_T, \widehat{\rho}_T$ are no longer the MLEs, since, as we mentioned, the Girsanov theorem for fBm does not have the same form as for Brownian motion, but they are still formally LSEs. Nevertheless, there is a major difference with respect to the Brownian motion case. Indeed, since the process

X is no longer a semimartingale, in (9) and (10) one cannot interpret the numerators using the Itô integral; the Skorohod integral turns out to be the correct notion to use here. We mentioned above that Skorohod integrals are difficult to use in practice, but since our Hurst parameter H exceeds $1/2$, it is possible to reinterpret the Skorohod integrals as so-called Young integrals, a pathwise notion, modulo a correction term which we will be able to compute explicitly thanks to the Malliavin calculus.

Having succeeded in correctly interpreting the stochastic integrals in (9) and (10), the issue of how to discretize them becomes paramount to practical implementation, and herein we will propose several different options, some of which allow for strong consistency and asymptotic normality under broader conditions than others.

Our discrete-observation study also applies to the case $H = 1/2$ as a limiting case. The article [6] treats this case solely with continuous observations; our work thus covers an extension of their work to discrete observations. Checking the validity of this statement rigorously is straightforward; for the sake of conciseness, we leave it to the interested reader.

1.2 Summary of results and heuristics

We now summarize our results, the structure of our article, and our main proof elements, including useful heuristics when available.

- In Section 2 and in the Appendix we introduce the needed mathematical background material for our study, including elements of the Malliavin calculus, a convenient criterion for establishing normal convergence on Wiener chaos, and the relation between Skorohod integrals and Young integrals with respect to fBm when $H > 1/2$.

- In Section 3, we concentrate on proving strong consistency and asymptotic normality for the estimators $\widehat{\theta}_T$ and $\widehat{\rho}_T$ with continuous observations.

  - We first prove the following almost surely convergences:

$$\widehat{\theta}_T \quad \to \quad \theta^* := \rho + \theta, \tag{11}$$

$$\widehat{\rho}_T \quad \to \quad \rho^* := \frac{\rho\theta\left(\theta+\rho\right)}{\frac{\rho^{2-2H}-\theta^{2-2H}}{\theta^{-2H}-\rho^{-2H}} + (\theta+\rho)^2}. \tag{12}$$

    The proof relies on studying the numerator and the denominator of the expressions for $\widehat{\theta}_T$ and $\widehat{\rho}_T$ separately. For the denominators, we rely on Birkhoff's ergodic theorem, and elementary covariance estimations for exponential convolutions with fBm. For the numerators, we express the Skorohod integrals as Young integrals plus their correction terms involving Malliavin derivatives which are explicit deterministic functions since our processes are Gaussian.

  - The expression $\theta^* = \rho + \theta$ in (11) is easy to explain: as noted below in line (16), X satisfies a stochastic integro-differential equation in which the zero-mean-reversion term $-\left(\rho+\theta\right) X\left(t\right) dt$ appears, and thus a natural candidate for a consistent estimator of $\rho + \theta$ is the LSE $\widehat{\theta}$, whether one adds merely one mean-zero noise term dB^H or another term $\left(\int_0^t X_s ds\right)$ which is asymptotically small. This is why the limiting behavoir of $\widehat{\theta}$ remains the same for us as in (4), which is the Brownian case ($H = 1/2$) studied in [6]. The details of this heuristic are omitted, since the full proof we present herein is needed to be convincing. On the other hand, the expression in (12) for our ρ^* is more opaque; there does not seem to be a direct heuristic to explain it, beyond our computations. When one compares our ρ^* in (12) with the ρ^* in (5) found in [6], one sees that the term $\rho\theta$ in (5) is replaced by the expression $\left(\rho^{2-2H} - \theta^{2-2H}\right) / \left(\theta^{-2H} - \rho^{-2H}\right)$, which can help identify how the case of fBm deviates from the case $H = 1/2$.[4] The expression for ρ^* is analyzed further in the context of discretizing $\left(\widehat{\theta}, \widehat{\rho}\right)$, which helps explain to some extent why this complicated expression arises, as the reader will find out in the first paragraph of Section 4.2.

- We prove asymptotic normality of $\left(\widehat{\theta}_T, \widehat{\rho}_T\right)$ by expressing the Skorohod integrals as iterated Wiener integrals, identifying dominant portions of these integrals, relying on a criterion for normal convergence in law in Wiener chaos, combined with a number of almost sure convergences. Our main asymptotics normality result is a central limit theorem that holds for $\sqrt{T}\left(\widehat{\theta}_T - \theta^*, \widehat{\rho}_T - \rho^*\right)$ as $T \to \infty$, as soon as $H \in [1/2, 3/4)$. The asymptotic covariance is given explicitly. See Theorem 8. The upper limit of validity of this theorem is a typical threshold in normal convergence theorems in the second Wiener chaos. See for example a classical instance of this situation in the Breuer-Major central limit theorem, as presented in [22, Chapter 7]. For $H > 3/4$, we conjecture that the estimators are asymptotically Rosenblatt-distributed (again see [22, Chapter 7] for a classical example of such a phenomenon), and that the convergence occurs almost surely; this point is not discussed further, for the sake of conciseness.

- The topic of Section 4 is to construct estimators based solely on discrete observations. The asymptotic results we prove still require increasing horizon. We also assume that X is observed at evenly spaced intervals, with a time step Δ_n, and we set the time horizon to be $T_n = n\Delta_n \to \infty$ as $n \to \infty$. Let $t_k = k\Delta_n$ be the kth observation time. For instance, the case $\Delta_n = 1$ corresponds to a fixed observation frequency; other conditions on Δ_n will include requiring Δ_n to tend to 0 as fast as a certain negative power of n, i.e. the observation frequency increases as the horizon increases. For some strong consistency results, it will even be possible for us to relax conditions on Δ_n where it is allowed to tend to infinity like a power of n, i.e. with decreasing frequency as the horizon increases.

  Arguably, to be consistent with the assumption that only X is observed, the only estimators which are of practical use are those which rely solely on the values $\{X_{t_k} : k = 1, \ldots, n\}$. Designing such an estimator by discretizing $\left(\widehat{\theta}_T, \widehat{\rho}_T\right)$ turns out to be a difficult task, in which the final expression solves a non-linear system in the spirit of that which would follow from (11) and (12), but is rather distinct from this system because of the difficulty in how to interpret the discretizations of the Skorohod or Young integrals.

  The method we have chosen moves through several intermediate steps, where we gradually increase the number of terms in the estimators which are replaced by discretized versions. This method has the advantage of clearly showing where the restrictions on the

[4] As a way to compare these terms, which do coincide when $H = 1/2$, we can see that if ρ tends to θ, the aforementioned expression in (5) tends to $\theta^2(1-H)/H$, which thus deviates significantly from the case $H = 1/2$ quantitatively, particularly for H close to 1.

observation frequency Δ_n come into play. Each intermediate estimator can be considered as a perturbation of the previous one, starting with the continuous-time estimator of Section 3. Thus arguably all these estimators can be considered as tools used for the final objective, attained in Section 4.3, of constructing a strongly consistent and asymptotically normal estimator of (θ, ρ) based only on the data $\{X_{t_k} : k = 1, \ldots, n\}$. Nonetheless, some of the other estimators are relevant in their own right, as they might correspond to realistic partial or full observation cases.

– The main technical estimates which allow our discretization are given at the beginning of Section 4. These are Lemmas 10 and 11, based on applications of the Borel-Cantelli lemma. For Z any stochastic process, we let

$$Q_n(Z) := n^{-1} \sum_{k=1}^{n} (Z_{t_k})^2 .$$

Let $S_t := \int_0^t X_s^2 ds$ and $\Sigma_t := \int_0^t X_s ds$. We show that the discrepancy between S_{T_n}/T_n and its discrete version $Q_n(X)$ is $= o\left(1/\sqrt{T_n}\right)$ almost surely. We then compute three different discrepancies related to Σ: first we show that the difference between $T_n^{-1} \int_0^{T_n} \Sigma_t^2 dt$ and its discrete version $Q_n(\Sigma)$ is also $= o\left(1/\sqrt{T_n}\right)$ almost surely. Then we show that with $\hat{\Sigma}$ the version of Σ which depends only on X observations, i.e. $\widehat{\Sigma}_{t_k} := \Delta_n \sum_{i=1}^{k} X_{t_{i-1}}^2$, we get that $Q_n\left(\hat{\Sigma}\right) - Q_n(\Sigma)$ tend to 0 almost surely. This is helpful to prove strong consistency of discrete estimators. To prove asymptotic normality, we need that $Q_n\left(\hat{\Sigma}\right) - Q_n(\Sigma) = o\left(1/\sqrt{T_n}\right)$, which we prove holds almost surely. Increasingly restrictive conditions on Δ_n are needed for these successive results.

– We first concentrate on discretizing the denominators of $\left(\widehat{\theta}_{T_n}, \widehat{\rho}_{T_n}\right)$.

  * We replace the denominator of $\widehat{\theta}_{T_n}$ by $Q_n(X)$, yielding an estimator $\tilde{\theta}_n$, and we then replace the denominator of $\widehat{\rho}_T$ by $Q_n(X) + (\tilde{\theta}_n)^2 Q_n(\Sigma)$, yielding an estimator $\tilde{\rho}_n$, because, as it turns out, $\int_0^T \widehat{V}_t^2 dt$ is asymptotically equivalent, almost surely, to $Q_n(X) + (\tilde{\theta}_n)^2 Q_n(\Sigma)$. Thanks to this, to the almost sure equivalence of $Q_n(X)$ with S_{T_n}/T_n, and similarly for $Q_n(\Sigma)$, coming from Lemmas 10 and 11, the strong consistency and asymptotic normality of $\left(\tilde{\theta}_n, \tilde{\rho}_n\right)$ follows from that of $\left(\widehat{\theta}_{T_n}, \widehat{\rho}_{T_n}\right)$ proved in Section 3. Here it is sufficient to assume $H \in (1/2, 1)$ and $\Delta_n \leqslant n^{\alpha}$ for some $\alpha \in (-\infty, 1/H)$ for the strong consistency; note that Δ_n is allowed to remain constant or even increase like a moderate power in this case. For the asymptotic normality, it is sufficient that $H \in (1/2, 3/4)$ and $n\Delta_n^{H+1} \to 0$.
  * A second result is obtained in which we forego having access to the process Σ itself, relying instead on its discrete version $\widehat{\Sigma}_{t_k} = \Delta_n \sum_{i=1}^{k} X_{t_{i-1}}^2$; in this case, almost-sure converge of $\tilde{\rho}_n$ requires $n^{\alpha+1}\Delta_n^{H+1} \to 0$ for some $\alpha > 0$, and the central-limit result for $\tilde{\rho}_n$ requires $n^3 \Delta_n^{2H+3} \to 0$.

– We are then able to define and study a bonafide estimator based on discrete data alone.

- * We begin with assuming that we have access to both X_{t_k} and Σ_{t_k} for all $k=1,\ldots,n$. The stochastic integrals in $\tilde{\theta}_n$ and $\tilde{\rho}_n$ were analyzed in Section 3, and were found, under scaling by T_n^{-1}, to be asymptotically constant, where the explicit constants depend on the parameters. By using these limits and a discretization of the Riemann integrals in the denominators of $\tilde{\theta}_n$ and $\tilde{\rho}_n$, this allows us, at the beginning of Section 4.2, to motivate the definition of a pair of estimators $\left(\check{\theta}_n, \check{\rho}_n\right)$ as solution of the non-linear system

$$F\left(\check{\theta}_n, \check{\rho}_n\right) = \left(Q_n(X), Q_n(\Sigma)\right)$$

  where F is a positive function of the variables (x,y) in $(0,+\infty)^2$ defined by: for every $(x,y)\in(0,+\infty)^2$

$$F(x,y) = H\Gamma(2H)\times\begin{cases} \frac{1}{y^2-x^2}\left(y^{2-2H}-x^{2-2H}, x^{-2H}-y^{-2H}\right) & \text{if } x\neq y \\ \left((1-H)x^{-2H}, Hx^{-2H-2}\right) & \text{if } y=x, \end{cases}$$

  and the data statistics used in the system are $Q_n(X)$ and $Q_n(\Sigma)$. Strong consistency and asymptotic normality follow for the uniquely defined $\left(\check{\theta}_n, \check{\rho}_n\right)$. The delicate computation of the asymptotic covariance is given. The parameter restrictions remain the same as for $\left(\tilde{\theta}_n, \tilde{\rho}_n\right)$, namely strong consistency if $H\in(1/2,1)$ and $\Delta_n\leqslant n^{\alpha}$ for some $\alpha\in(-\infty,1/H)$, and asymptotic normality if $H\in(1/2,3/4)$ and $n\Delta_n^{H+1}\to 0$.
- * By redefining the estimators $\left(\check{\theta}_n, \check{\rho}_n\right)$ using $\hat{\Sigma}$ instead of Σ, one ensures that only the data $\{X_{t_k}: k=1,\ldots,n\}$ is used. Here, the results from the previous case can be applied directly with the auxiliary results on how $\hat{\Sigma}$ perturbs Σ (Lemma 11), to obtain the same almost-sure convergence and central-limit result, but these are now restricted respectively to the aforementioned observation frequency parameter ranges $n^{\alpha+1}\Delta_n^{H+1}\to 0$ for some $\alpha>0$, and $n^3\Delta_n^{2H+3}\to 0$.

- Finally, to illustrate how the complexity of the nonlinearities in the definition of $\left(\check{\theta}_n, \check{\rho}_n\right)$ may be attributable to the partial-observation problem, we define a pair of estimators $\left(\underline{\theta}_n, \underline{\rho}_n\right)$ under the assumption that both $\{X_{t_k}: k=1,\ldots,n\}$ and $\{V_{t_k}: k=1,\ldots,n\}$ are available. The $\underline{\rho}_n$ is explicit given $\{V_{t_k}: k=1,\ldots,n\}$, and is identical to the one given in [15], i.e. (8), as it should be. The $\underline{\theta}_n$ satisfies the following straightforward non-linear equation given $\underline{\rho}_n$ and $\{X_{t_k}: k=1,\ldots,n\}$:

$$\left(\underline{\theta}_n\right)^{2-2H} - \left(\frac{Q_n(X)}{H\Gamma(2H)}\right)\left(\underline{\theta}_n\right)^2 = \left(\underline{\rho}_n\right)^{2-2H} - \left(\frac{Q_n(X)}{H\Gamma(2H)}\right)\left(\underline{\rho}_n\right)^2.$$

  The parameter restrictions are the same as when X and Σ are discretely observed: strong consistency holds for $\left(\underline{\theta}_n, \underline{\rho}_n\right)$ if $H\in(1/2,1)$ and $\Delta_n\leqslant n^{\alpha}$ for some $\alpha\in(-\infty,1/H)$, and asymptotic normality holds if $H\in(1/2,3/4)$ and $n\Delta_n^{H+1}\to 0$.

Before we proceed with the details of our study, we provide needed mathematical tools in Section 2.

2 Preliminaries

In this section we describe some basic facts on the stochastic calculus with respect to a fractional Brownian motion. For a more complete presentation on the subject, see [23] and [1].

The fractional Brownian motion $(B_t^H, t \geqslant 0)$ with Hurst parameter $H \in (0,1)$, is defined as a centered Gaussian process starting from zero with covariance

$$R_H(t,s) = E(B_t^H B_s^H) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right);\ s,\ t \geqslant\ 0,$$

We assume that B^H is defined on a complete probability space $(\Omega, \mathcal{F}, P)$ such that $\mathcal{F}$ is the sigma-field generated by B^H. By Kolmogorov's continuity criterion and the fact $E\left(B_t^H - B_s^H\right)^2 = |s-t|^{2H}$, we deduce that B^H admits a version which has Hölder continuous paths of any order $\gamma < H$.

Fix a time interval $[0,T]$. We denote by $\mathcal{H}$ the canonical Hilbert space associated to the fractional Brownian motion B^H; the book [23], among many other references, can be consulted for the construction and properties of $\mathcal{H}$. We use the following convenient notation for Wiener integrals with respect to B^H:

$$B^H(\varphi) := \int_0^T \varphi(s)\, dB^H.$$

Of interest to us is the fact that, with $H > 1/2$, for a pair of (non-random) functional elements φ, ψ of $\mathcal{H}$, its inner product satisfies

$$\langle \varphi, \psi \rangle_{\mathcal{H}} = E\left(B^H(\varphi)\, B^H(\psi)\right) = H(2H-1)\int_0^T \int_0^T \varphi(u)\psi(v)|u-v|^{2H-2} du dv.$$

It follows from [25] that the set $|\mathcal{H}|$ of functional elements in $\mathcal{H}$ is Banach and actually contains $L^{\frac{1}{H}}([0,T])$.

The Malliavin derivative D w.r.t. B^H, which is an $\mathcal{H}$-values operator, is defined first by setting that

$$DB^H(\varphi) = \varphi$$

for any $\varphi \in \mathcal{H}$, and then by requiring that it satisfy a multi-parameter chain rule: for any $f \in C_b^\infty(\mathbb{R}^n, \mathbb{R})$ (infinitely differentiable functions from $\mathbb{R}^n$ to $\mathbb{R}$ with bounded partial derivatives) and any $\varphi_1, ..., \varphi_n \in \mathcal{H}$, D operates on the cylinder r.v. $F := f(B^H(\varphi_1), ..., B^H(\varphi_n))$ as

$$DF = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(B^H(\varphi_1), ..., B^H(\varphi_n))\varphi_i.$$

The domain $D^{1,2}$ of D is then the the closure of the set of cylinder r.v.'s F with respect to the norm

$$\|F\|_{1,2}^2 := E(F^2) + E(\|DF\|_{\mathcal{H}}^2).$$

The divergence operator δ is the adjoint of the derivative operator D : an $\mathcal{H}$-valued r.v. $u \in L^2(\Omega; \mathcal{H})$ belongs to its domain $Dom\delta$ if

$$E\left|\langle DF, u\rangle_{\mathcal{H}}\right| \leqslant c_u \|F\|_{L^2(\Omega)}$$

for some constant c_u and every cylinder r.v. F. In this case $\delta(u)$ is uniquely defined by the duality

$$E(F\delta(u)) = E\langle DF, u\rangle_{\mathcal{H}}$$

for any $F \in D^{1,2}$. We will make use of the notation

$$\delta(u) = \int_0^T u_s \delta B_s^H, \quad u \in Dom\delta.$$

In particular, δ extends the Wiener integral: for $h \in |\mathcal{H}|$, $B^H(h) = \delta(h) = \int_0^T h_s \delta B_s^H$.
For every $n \geqslant 1$, let $\mathcal{H}_n$ be the nth Wiener chaos of B^H, that is, the closed linear subspace of $L^2(\Omega)$ generated by the random variables $\{H_n(B^H(h)), h \in \mathcal{H}, \|h\|_{\mathcal{H}} = 1\}$ where H_n is the nth Hermite polynomial. The mapping $I_n(h^{\otimes n}) = n! H_n(B^H(h))$ provides a linear isometry between the symmetric tensor product $\mathcal{H}^{\odot n}$ (equipped with the modified norm $\|.\|_{\mathcal{H}^{\odot n}} = \frac{1}{\sqrt{n!}}\|.\|_{\mathcal{H}^{\otimes n}}$) and $\mathcal{H}_n$. It also turns out that $I_n(h^{\otimes n})$ is the multiple Wiener integral of $h^{\otimes n}$ w.r.t. B^H. For every $f, g \in \mathcal{H}^{\odot n}$ the following product formula holds

$$E\left(I_n(f) I_n(g)\right) = n! \langle f, g\rangle_{\mathcal{H}^{\otimes n}}.$$

For $h \in \mathcal{H}^{\otimes n}$, the multiple Wiener integrals $I_q(f)$, which exhaust the set $\mathcal{H}_q$, satisfy a hypercontractivity property (equivalence in $\mathcal{H}_q$ of all L^p norms for all $p \geqslant 2$), which implies that for any $F \in \oplus_{l=1}^q \mathcal{H}_l$, we have

$$\left(E\big[|F|^p\big]\right)^{1/p} \leqslant c_{p,q}\left(E\big[|F|^2\big]\right)^{1/2} \quad \text{for any } p \geqslant 2. \tag{13}$$

It is well-known that $L^2(\Omega)$ can be decomposed into the infinite orthogonal sum of the spaces $\mathcal{H}_n$. That is, any square integrable random variable $F \in L^2(\Omega)$ admits the following "Wiener chaos" expansion

$$F = E(F) + \sum_{n=1}^{\infty} I_n(f_n),$$

where the $f_n \in \mathcal{H}^{\odot n}$ are uniquely determined by F.
Finally, we will use the following central limit theorem for multiple stochastic integrals (see [24]).

Theorem 1 *Let $\{F_n\,, n \geqslant 1\}$ be a sequence of random variables in the q-th Wiener chaos $\mathcal{H}_q$, $q \geqslant 2$, such that $\lim_{n\to\infty} E(F_n^2) = \sigma^2$. Then the following conditions are equivalent:*

(i) F_n converges in law to $\mathcal{N}(0, \sigma^2)$ as n tends to infinity.

(ii) $\|DF_n\|_{\mathcal{H}}^2$ converges in L^2 to a constant as n tends to infinity.

3 Asymptotic behavior of LSEs

Throughout the paper we assume that $H \in (\frac{1}{2}, 1)$, $\theta > 0$ and $\rho > 0$ such that $\theta \neq \rho$.
It is readily checked that we have the following explicit expression for X_t:

$$X_t = \frac{\rho}{\rho - \theta} X_t^{\rho} + \frac{\theta}{\theta - \rho} X_t^{\theta} \tag{14}$$

where for $m > 0$

$$X_t^m = \int_0^t e^{-m(t-s)} dB_s^H. \tag{15}$$

On the other hand, we can also write that the system (6) implies that X solves the following stochastic integro-differential equation

$$dX_t = -(\theta + \rho) X_t dt - \rho\theta \left(\int_0^t X_s ds \right) dt + dB_t^H. \tag{16}$$

For convenience, and because it will play an important role in the forthcoming computations, we introduce the following processes related to X_t:

$$S_T = \int_0^T X_t^2 dt; \quad \Sigma_T = \int_0^T X_t dt; \quad L_T = \int_0^T V_t^2 dt; \quad P_T = \int_0^T X_t V_t dt;$$

and

$$\widehat{L}_T = \int_0^T \widehat{V}_t^2 dt$$

where for $0 \leqslant t \leqslant T$

$$\widehat{V}_t = X_t + \widehat{\theta}_T \Sigma_t, \tag{17}$$

and $\widehat{\theta}_T$ is our continuous LSE for θ as given in (9). We will need the following lemmas.

Lemma 2 *Assume $H \in \left(\frac{1}{2}, 1\right)$. Then, as $T \to \infty$*

$$\frac{1}{T} \int_0^T X_t^2 dt \longrightarrow \eta^X, \tag{18}$$

$$\frac{1}{T} \int_0^T \Sigma_t^2 dt \longrightarrow \eta^\Sigma, \tag{19}$$

$$\frac{1}{T} \int_0^T \Sigma_t X_t dt \longrightarrow 0 \tag{20}$$

almost surely, where

$$\eta^X = \frac{H\Gamma(2H)}{\rho^2 - \theta^2} [\rho^{2-2H} - \theta^{2-2H}],$$

and

$$\eta^\Sigma = \frac{H\Gamma(2H)}{\rho^2 - \theta^2} [\theta^{-2H} - \rho^{-2H}].$$

Proof. From (6) we can write

$$d \begin{pmatrix} X_t \\ \Sigma_t \end{pmatrix} = A \begin{pmatrix} X_t \\ \Sigma_t \end{pmatrix} dt + d \begin{pmatrix} B_t^H \\ 0 \end{pmatrix}$$

where $A = \begin{pmatrix} \theta + \rho & -\theta\rho \\ 1 & 0 \end{pmatrix}$. The process $\begin{pmatrix} X_t \\ \Sigma_t \end{pmatrix}$ is geometrically ergodic because the largest eigenvalue of A is negative. Then to prove Lemma 2, using Birkhoff's ergodic theorem (for instance see [13]), it is sufficient to study the convergence of $\mathbf{E}[X_t^2]$, $\mathbf{E}[\Sigma_t^2]$ and $\mathbf{E}[\Sigma_t X_t]$ as

$t \longrightarrow \infty$.
For the convergence of $\mathbf{E}[X_t^2]$, (14) leads to

$$\mathbf{E}[X_t^2] = \left(\frac{\rho}{\rho-\theta}\right)^2 \mathbf{E}[(X_t^\rho)^2] + \left(\frac{\theta}{\theta-\rho}\right)^2 \mathbf{E}[(X_t^\theta)^2] - \frac{2\theta\rho}{(\theta-\rho)^2}\mathbf{E}[(X^\theta)_t(X_t^\rho)].$$

Since

$$\eta^X = \left(\frac{\rho}{\rho-\theta}\right)^2 \lambda(\rho,\rho) + \left(\frac{\theta}{\theta-\rho}\right)^2 \lambda(\theta,\theta) - \frac{2\theta\rho}{(\theta-\rho)^2}\lambda(\theta,\rho)$$

then by using 1) of Lemma 21 we obtain

$$\left|\eta^X - \mathbf{E}[X_t^2]\right| \leqslant c(H,\theta,\rho)e^{-t/2}.$$

Thus we deduce the convergence (18).
Using the same argument and the fact that

$$\Sigma_t = \frac{V_t - X_t}{\theta} = \frac{X_t^\theta - X_t^\rho}{\rho-\theta} \tag{21}$$

we deduce the convergence (19).
Finally, the convergence (20) is satisfied by using $\int_0^T \Sigma_t X_t dt = \frac{\Sigma_T}{2}$ and point 5) of Lemma 21.
■

Lemma 3 *We have*

$$\frac{1}{T}\int_0^T X_t \delta X_t \longrightarrow -(\rho+\theta)\eta^X \tag{22}$$

almost surely as $T \longrightarrow \infty$.

Proof. From (6) and (A-2) we can write

$$\begin{aligned}\int_0^T X_t\delta X_t &= -\theta\int_0^T X_t^2 dt - \rho\int_0^T X_t V_t dt + \int_0^T X_t \delta B_t^H \\ &= -\theta\int_0^T X_t^2 dt - \theta\rho\int_0^T X_t\Sigma_t dt - \rho\int_0^T X_t^2 dt + \int_0^T X_t dB_t^H \\ &\quad -\alpha_H \int_0^T\int_0^t D_s X_t (t-s)^{2H-2} ds dt\end{aligned}$$

where $\alpha_H = 2H(2H-1)$. Moreover,

$$\begin{aligned}\int_0^T X_t dB_t^H &= \int_0^T X_t dX_t + (\theta+\rho)\int_0^T X_t^2 dt + \theta\rho\int_0^T X_t\Sigma_t dt \\ &= \frac{X_T^2}{2} + (\theta+\rho)\int_0^T X_t^2 dt + \theta\rho\int_0^T X_t\Sigma_t dt.\end{aligned}$$

Thus

$$\int_0^T X_t\delta X_t = \frac{X_T^2}{2} - \alpha_H\int_0^T\int_0^t D_s X_t(t-s)^{2H-2} ds dt.$$

Since

$$D_s X_t^m = e^{-m(t-s)} 1_{[0,t]}(s)$$

we deduce that

$$\begin{aligned}
\frac{1}{T}\int_0^T X_t \delta X_t &= \frac{X_T^2}{2T} - \frac{\alpha_H}{T}\int_0^T \int_0^t \frac{1}{\rho-\theta}(\rho e^{-\rho(t-s)} - \theta e^{-\theta(t-s)})(t-s)^{2H-2} ds dt \\
&= \frac{X_T^2}{2T} - \frac{\alpha_H}{T}\int_0^T \int_0^t \frac{1}{\rho-\theta}\left(\rho e^{-\rho r} - \theta e^{-\theta r}\right) r^{2H-2} dr dt. \qquad (23)
\end{aligned}$$

Thanks to l'Hôpital's rule, as $T \longrightarrow \infty$

$$\begin{aligned}
\frac{\alpha_H}{T}\int_0^T \int_0^t \frac{1}{\rho-\theta}\left(\rho e^{\rho r} - \theta e^{\theta r}\right) r^{2H-2} dr dt &\longrightarrow \frac{H\Gamma(2H)}{\rho-\theta}[\rho^{2-2H} - \theta^{2-2H}] \\
&= \frac{(\rho+\theta)\eta^X}{\alpha_H}.
\end{aligned}$$

Finally, combining this last convergence and point 5) of Lemma 21, the proof of Lemma 3 is done. ■

We now have all the elements to obtain our strong consistency result for $\widehat{\theta}_T$.

Theorem 4 *We have*

$$\widehat{\theta}_T \longrightarrow \theta^*$$

almost surely as $T \longrightarrow \infty$, where $\theta^* = \theta + \rho$.

Proof. The proof follows directly from the convergence (18) and Lemma 3. ■

The next lemmas are additional elements needed to prove the strong consistency of $\widehat{\rho}_T$.

Lemma 5 *We have*

$$\frac{\widehat{L}_T}{T} \longrightarrow \eta^X + (\rho+\theta)^2 \eta^\Sigma$$

almost surely as $T \longrightarrow \infty$.

Proof. The equation (17) ensures

$$\widehat{L}_T = \int_0^T X_t^2 dt + 2\widehat{\theta}_T \int_0^T X_t \Sigma_t dt + \widehat{\theta}_T^2 \int_0^T \Sigma_t^2 dt,$$

and the desired conclusion follows by using Lemma 2 and Theorem 4. ■

Lemma 6 *We have*

$$\frac{1}{T}\int_0^T \widehat{V}_t \delta \widehat{V}_t \longrightarrow -\rho\theta(\rho+\theta)\eta^\Sigma$$

almost surely as $T \longrightarrow \infty$.

Proof. From (6) and (17) we can write

$$\begin{aligned}
\int_0^T \widehat{V}_t \delta \widehat{V}_t &= \int_0^T X_t \delta X_t + \widehat{\theta}_T \int_0^T X_t^2 dt - \widehat{\theta}_T(\theta+\rho)\int_0^T \Sigma_t X_t dt - \rho\theta\widehat{\theta}_T \int_0^T \Sigma_t^2 dt \\
&+ \int_0^T \Sigma_t \widehat{\theta}_T \delta B_t^H + \widehat{\theta}_T^2 \int_0^T \Sigma_t X_t dt.
\end{aligned}$$

On the other hand

$$
\begin{aligned}
& \int_0^T \widehat{\theta}_T \Sigma_t dB_t^H \\
= \; & \int_0^T \widehat{\theta}_T \Sigma_t dV_t + \rho \widehat{\theta}_T \int_0^T \Sigma_t V_t dt \\
= \; & -\theta^{-1} \int_0^T \widehat{\theta}_T X_t dV_t + \theta^{-1} \widehat{\theta}_T \int_0^T V_t dV_t + \rho \widehat{\theta}_T \int_0^T \Sigma_t X_t dt + \rho\theta \widehat{\theta}_T \int_0^T \Sigma_t^2 dt \\
= \; & -\theta^{-1} \int_0^T \widehat{\theta}_T X_t dX_t - \widehat{\theta}_T \int_0^T X_t^2 dt + \theta^{-1} \widehat{\theta}_T \int_0^T V_t dV_t + \rho \widehat{\theta}_T \int_0^T \Sigma_t X_t dt + \rho\theta \widehat{\theta}_T \int_0^T \Sigma_t^2 dt \\
= \; & \frac{-1}{2\theta} X_T^2 - \widehat{\theta}_T \int_0^T X_t^2 dt + \frac{\widehat{\theta}_T}{2\theta} V_T^2 + \rho \widehat{\theta}_T \int_0^T \Sigma_t X_t dt + \rho\theta \widehat{\theta}_T \int_0^T \Sigma_t^2 dt.
\end{aligned}
$$

Now, applying (A-2), we obtain

$$
\begin{aligned}
\int_0^T \widehat{V}_t \delta \widehat{V}_t \;=\; & \int_0^T X_t \delta X_t - \frac{1}{2\theta} X_T^2 + \frac{\widehat{\theta}_T}{2\theta} V_T^2 - \widehat{\theta}_T \theta \int_0^T \Sigma_t X_t dt + \widehat{\theta}_T^2 \int_0^T \Sigma_t X_t dt \\
& -\alpha_H \int_0^T \int_0^t D_s(\widehat{\theta}_T \Sigma_t)(t-s)^{2H-2} ds dt. \qquad (24)
\end{aligned}
$$

On the other hand

$$
D_s(\widehat{\theta}_T \Sigma_t) = \Sigma_t D_s \widehat{\theta}_T + \widehat{\theta}_T D_s \Sigma_t.
$$

It follows from (23) that

$$
\begin{aligned}
\widehat{\theta}_T &= \frac{\int_0^T X_t \delta X_t}{S_T} \\
&= \frac{\frac{1}{2} X_T^2 - \alpha_H \int_0^T \int_0^t \frac{1}{\rho-\theta} \left(\rho e^{-\rho r} - \theta e^{-\theta r} \right) r^{2H-2} dr dt}{S_T}.
\end{aligned}
$$

Hence, for $s < T$

$$
D_s \widehat{\theta}_T = \frac{X_T D_s X_T - \widehat{\theta}_T D_s S_T}{S_T}.
$$

Thus

$$
\begin{aligned}
& \alpha_H \int_0^T \int_0^t D_s(\widehat{\theta}_T \Sigma_t)(t-s)^{2H-2} ds dt \\
= \; & \alpha_H \frac{X_T}{S_T} \int_0^T \int_0^t \Sigma_t D_s X_T (t-s)^{2H-2} ds dt - \alpha_H \frac{\widehat{\theta}_T}{S_T} \int_0^T \int_0^t \Sigma_t D_s S_T (t-s)^{2H-2} ds dt \\
& + \alpha_H \widehat{\theta}_T \int_0^T \int_0^t D_s \Sigma_t (t-s)^{2H-2} ds dt \\
:= \; & J_{1,T} - J_{2,T} + J_{3,T}
\end{aligned}
$$

We shall prove that for every $\varepsilon > 0$

$$
\frac{|J_{1,T}|}{T^\varepsilon} \longrightarrow 0, \qquad (25)
$$

$$
\frac{|J_{2,T}|}{T^\varepsilon} \longrightarrow 0, \qquad (26)
$$

$$
\frac{J_{3,T}}{T} \longrightarrow \frac{\theta+\rho}{\rho-\theta} H\Gamma(2H)[(-\rho)^{1-2H} - (-\theta)^{1-2H}] \qquad (27)
$$

almost surely as $T \to \infty$.
We first estimate $J_{1,T}$. Clearly, (14) implies

$$\begin{aligned} J_{1,T} &= \alpha_H \frac{X_T}{S_T} \int_0^T \int_0^t \Sigma_t D_s X_T (t-s)^{2H-2} dsdt \\ &= \frac{\alpha_H}{\rho-\theta} \frac{X_T}{S_T} \int_0^T \Sigma_t \int_0^t \left(\rho e^{-\rho(T-s)} - \theta e^{-\theta(T-s)}\right) (t-s)^{2H-2} dsdt \\ &= \frac{\alpha_H}{\rho-\theta} \frac{X_T}{S_T} \int_0^T \Sigma_t \int_0^t \left(\rho e^{-\rho(T-t+x)} - \theta e^{-\theta(T-t+x)}\right) x^{2H-2} dxdt. \end{aligned}$$

The last equality comes from making the change of variable $x = t - s$.
Hence

$$\frac{|J_{1,T}|}{T^{\varepsilon}} \leqslant c(H,\theta,\rho) \frac{|X_T|/T^{\varepsilon}}{S_T/T} \frac{\sup_{t\in[0,T]} |\Sigma_t|}{T^{\varepsilon}}.$$

Using (18), (21) and the point 5) of Lemma 21, the convergence (25) is obtained.
Next we estimate $J_{2,T}$. By (14) we have

$$\begin{aligned} J_{2,T} &= \alpha_H \frac{\widehat{\theta}_T}{S_T} \int_0^T \int_0^t \Sigma_t D_s S_T (t-s)^{2H-2} dsdt \\ &= 2\alpha_H \frac{\widehat{\theta}_T}{S_T} \int_0^T \int_0^t \Sigma_t \int_s^T X_u D_s X_u (t-s)^{2H-2} dudsdt \\ &= 2\alpha_H \frac{\widehat{\theta}_T}{(\rho-\theta)S_T} \int_0^T \int_0^t \Sigma_t \int_s^T X_u \left(\rho e^{-\rho(u-s)} - \theta e^{-\theta(u-s)}\right) (t-s)^{2H-2} dudsdt. \end{aligned}$$

Then

$$\begin{aligned} \frac{|J_{2,T}|}{T^{\varepsilon}} &\leqslant c(H,\theta,\rho) \frac{\left|\widehat{\theta}_T\right|}{S_T} \frac{\sup_{t\in[0,T]} |X_t| \sup_{t\in[0,T]} |\Sigma_t|}{T^{\varepsilon}} \int_0^T \int_0^t e^{-\min(\theta,\rho)(T-s)} (t-s)^{2H-2} dsdt \\ &\leqslant c(H,\theta,\rho) \frac{\left|\widehat{\theta}_T\right|}{S_T/T} \frac{\sup_{t\in[0,T]} |X_t| \sup_{t\in[0,T]} |\Sigma_t|}{T^{\varepsilon}} \\ &\longrightarrow 0 \end{aligned}$$

almost surely as $T \longrightarrow \infty$. The last convergence comes from (18), (21), Theorem 4 and the point 5) of Lemma 21. Thus, the convergence (26) is satisfied.
Finally, we estimate $J_{3,T}$. Using (21) and (14)

$$\begin{aligned} J_{3,T} &= \alpha_H \widehat{\theta}_T \int_0^T \int_0^t D_s \Sigma_t (t-s)^{2H-2} dsdt \\ &= \frac{\alpha_H \widehat{\theta}_T}{\rho-\theta} \int_0^T \int_0^t \left(e^{-\theta(t-s)} - e^{-\rho(t-s)}\right) (t-s)^{2H-2} dsdt. \end{aligned}$$

By l'Hôpital rule we obtain

$$\frac{J_{3,T}}{T} \longrightarrow \frac{\theta+\rho}{\rho-\theta} H\Gamma(2H)[\theta^{1-2H} - \rho^{1-2H}]$$

almost surely as $T \longrightarrow \infty$.
Using the above estimations (25), (26), (27) together with (24), Lemma 2, Theorem 4, the point 5) of Lemma 21 and Lemma 3 the desired result is then obtained. ■

Theorem 7 *We have the almost sure convergence*

$$\widehat{\rho}_T \longrightarrow \rho^*$$

as $T \to \infty$, where

$$\rho^* = \frac{\theta\rho(\theta+\rho)\eta^{\Sigma}}{\eta^X + (\theta+\rho)^2\eta^{\Sigma}}.$$

Proof. The proof is a straightforward consequence of Lemma 5 and Lemma 6. ■

Our approach to prove the asymptotic normality for both estimators $\widehat{\theta}_T$ and $\widehat{\rho}_T$ looks first at the normal convergence of the T-indexed second-chaos sequence based on the kernel which appears in the representation (15) of X. Thereafter, thanks to elementary stochastic calculus in the second chaos, these double stochastic integrals will be identified in an expression for the leading terms in $\widehat{\theta}_T - \theta^*$ in the proof of Theorem 8. A similar technique, plus the use of the chain rule of Young integrals and their relation to Skorohod integrals, is used to find again that the leading terms in $\widehat{\rho}_T - \rho^*$ are also linear combinations of the same double integrals; the analysis of the lower-order terms are less evident than for $\widehat{\theta}_T - \theta^*$; the proof of Theorem 8 records all the details.

We are ready to prove the asymptotic normality of $\left(\widehat{\theta}_T, \widehat{\rho}_T\right)$.

Theorem 8 *Assume that $H \in (\frac{1}{2}, \frac{3}{4})$. Then*

$$\sqrt{T}\left(\widehat{\theta}_T - \theta^*, \widehat{\rho}_T - \rho^*\right) \xrightarrow{law} \mathcal{N}(0, {}^t P\ \Gamma\ P)$$

where the matrices Γ and P are defined respectively in (A-5) and (35).

Proof. We express $\widehat{\theta}_T - \theta^*$ and $\widehat{\rho}_T - \rho^*$ as linear combinations of the double stochastic integrals identified in the previous theorem, plus lower-order terms.
The case of $\widehat{\theta}_T - \theta^*$ is rather straightforward. It follows from (16) that

$$\widehat{\theta}_T - \theta^* = \frac{\rho\theta\int_0^T X_t\Sigma_t dt - \int_0^T X_t\delta B_t}{S_T}.$$

Since

$$\int_0^T X_t\Sigma_t dt = \frac{1}{2}\Sigma_T^2$$

and

$$\begin{aligned}\frac{1}{\sqrt{T}}\int_0^T X_t\delta B_t^H &= \frac{1}{(\rho-\theta)\sqrt{T}}\int_0^T\int_0^t (\rho e^{-\rho(t-s)} - \theta e^{-\theta(t-s)})\delta B_s^H\delta B_t^H \\ &= \frac{1}{(\rho-\theta)\sqrt{T}}\left(\rho I_2(f_T^{\rho}) - \theta I_2(f_T^{\theta})\right)\end{aligned}$$

we can write

$$\sqrt{T}\left(\widehat{\theta}_T - \theta^*\right) = \frac{\frac{1}{(\rho-\theta)\sqrt{T}}\left(\theta I_2(f_T^{\theta}) - \rho I_2(f_T^{\rho})\right)}{S_T/T} + R_T^{\theta}. \tag{28}$$

where

$$R_T^{\theta} := \frac{\rho\theta}{2}\frac{\Sigma_T^2/\sqrt{T}}{S_T/T} \longrightarrow 0 \tag{29}$$

almost surely as $T \longrightarrow \infty$.
For $\widehat{\rho}_t - \rho^*$, the situation is significantly more complex. We have for every $0 \leqslant t \leqslant T$

$$\begin{aligned}\widehat{V}_t &= X_t + \widehat{\theta}_T \Sigma_t = V_t + (\widehat{\theta}_T - \theta)\Sigma_t = V_t + (\widehat{\theta}_T - \theta^*)\Sigma_t + \rho\Sigma_t \\ &= V_t - \frac{\rho}{\theta}(X_t - V_t) - \frac{1}{\theta}(\widehat{\theta}_T - \theta^*)(X_t - V_t) \\ &= \frac{\theta^*}{\theta} V_t - \frac{\rho}{\theta} X_t - \frac{1}{\theta}(\widehat{\theta}_T - \theta^*)(X_t - V_t)\end{aligned}$$

which leads to

$$\widehat{L}_T = \int_0^T \widehat{V}_t^2 dt = I_T + (\widehat{\theta}_T - \theta^*)(J_T + (\widehat{\theta}_T - \theta^*)K_T)$$

where

$$\begin{aligned}I_T &= \frac{1}{\theta^2}(\rho^2 S_T + (\theta^*)^2 L_T - 2\theta^* \rho P_T), \\ J_T &= \frac{1}{\theta^2}(2\rho S_T + 2\theta^* L_T - 2(\theta + 2\rho)P_T), \\ K_T &= \frac{1}{\theta^2}(S_T + L_T - 2P_T).\end{aligned}$$

Thus,

$$\widehat{L}_T(\widehat{\rho}_T - \rho^*) = I_T^V + (\widehat{\theta}_T - \theta^*)(J_T^V + (\widehat{\theta}_T - \theta^*)K_T^V)$$

where

$$\begin{aligned}I_T^V &= -\int_0^T \widehat{V}_t \delta \widehat{V}_t - \rho^* I_T = -\frac{\widehat{V}_T^2}{2} + \alpha_H \int_0^T \int_0^t D_s \widehat{V}_t (t-s)^{2H-2} ds dt - \rho^* I_T, \\ J_T^V &= -\rho^* J_T, \text{ and } K_T^V = -\rho^* K_T.\end{aligned}$$

On the other hand, using the formula (A-1) we obtain

$$\begin{cases} S_T = -\frac{X_T^2}{2\theta^*} + \frac{1}{\theta^*}\int_0^T X_s dB_s^H - \frac{\rho\theta}{2\theta^*}\Sigma_T^2, \\ P_T = -\frac{X_T V_T}{\theta^*} + \frac{1}{\theta^*}\int_0^T X_s dB_s^H + \frac{V_T^2}{2\theta^*}, \\ L_T = -\frac{V_T^2}{2\rho} + \frac{1}{\rho}\int_0^T V_s dB_s^H. \end{cases}$$

Furthermore, using the relation between Young and Skorohod integrals,

$$\begin{cases} S_T = -\frac{X_T^2}{2\theta^*} + \frac{1}{\theta^*}\int_0^T X_s \delta B_s^H + \frac{\alpha_H}{\theta^*}\int_0^T\int_0^t D_s X_t (t-s)^{2H-2} ds dt - \frac{\rho\theta}{2\theta^*}\Sigma_T^2, \\ P_T = -\frac{X_T V_T}{\theta^*} + \frac{1}{\theta^*}\int_0^T X_s \delta B_s^H + \frac{\alpha_H}{\theta^*}\int_0^T\int_0^t D_s X_t (t-s)^{2H-2} ds dt + \frac{V_T^2}{2\theta^*}, \\ L_T = -\frac{V_T^2}{2\rho} + \frac{1}{\rho}\int_0^T V_s \delta B_s^H + \frac{\alpha_H}{\rho}\int_0^T\int_0^t D_s V_t (t-s)^{2H-2} ds dt. \end{cases} \tag{30}$$

Setting

$$\lambda_T := \alpha_H \int_0^T \int_0^t e^{-(t-s)}(t-s)^{2H-2} ds dt$$

we can write

$$\begin{aligned}
\lambda_T^X := &\quad \alpha_H \int_0^T \int_0^t D_s X_t (t-s)^{2H-2} ds dt = \frac{1}{\rho - \theta}\left(\rho^{2-2H} - \theta^{2-2H}\right)\lambda_T \\
\lambda_T^V := &\quad \alpha_H \int_0^T \int_0^t D_s V_t (t-s)^{2H-2} ds dt = \rho^{1-2H}\lambda_T \\
\lambda_T^\Sigma := &\quad \alpha_H \int_0^T \int_0^t D_s \Sigma_t (t-s)^{2H-2} ds dt = \frac{1}{\theta}(\lambda_T^V - \lambda_T^X) = \frac{1}{\rho - \theta}\left(\theta^{1-2H} - \rho^{1-2H}\right)\lambda_T \\
\lambda_T^{\widehat{V}} := &\quad \alpha_H \int_0^T \int_0^t D_s \widehat{V}_t (t-s)^{2H-2} ds dt = \lambda_T^X + \widehat{\theta}_T \lambda_T^\Sigma + J_{1,T} - J_{2,T} \\
= &\quad \lambda_T^X + \theta^* \lambda_T^\Sigma + (\widehat{\theta}_T - \theta^*)\lambda_T^\Sigma + J_{1,T} - J_{2,T}.
\end{aligned}$$

The last equality comes from the fact that $D_s \widehat{V}_t = D_s X_t + \widehat{\theta}_T D_s \Sigma_t + \Sigma_t D_s \widehat{\theta}_T$.
Since

$$\lambda_T^X + \theta^* \lambda_T^\Sigma = \frac{\rho^*}{\theta^2}\left(-\frac{\rho^2}{\theta^*}\lambda_T^X - \frac{(\theta^*)^2}{\rho}\lambda_T^V + 2\rho\lambda_T^X\right)$$

we can write

$$\begin{aligned}
I_T^V &= -\frac{\widehat{V}_T^2}{2} + \lambda_T^{\widehat{V}} - \rho^* I_T \\
&= (\widehat{\theta}_T - \theta^*)\lambda_T^\Sigma - \frac{\rho^*}{\theta^2}\left[(-2\rho + \frac{\rho^2}{\theta^*})\int_0^T X_s \delta B_s^H + \frac{(\theta^*)^2}{\rho}\int_0^T V_s \delta B_s^H\right] + R_T
\end{aligned}$$

where

$$R_T = \frac{-\widehat{V}_T^2}{2} + J_{1,T} - J_{2,T} - \frac{\rho^*}{\theta^2}\left[\rho^2(-\frac{X_T^2}{2\theta^*} - \frac{\rho\theta}{2\theta^*}\Sigma_T^2 - (\theta^*)^2\frac{V_T^2}{2\rho} - 2\theta^*\rho(\frac{-X_T V_T}{\theta^*} + \frac{V_T^2}{2\theta^*})\right].$$

Combining previous estimations we obtain

$$\widehat{L}_T(\widehat{\rho}_T - \rho^*) = c_T^\rho I_2(f_T^\rho) + c_T^\theta I_2(f_T^\theta) + \frac{R_T^\theta}{\sqrt{T}}\left(\lambda_T^\Sigma + J_T^V + (\widehat{\theta}_T - \theta^*)K_T^V\right) + R_T$$

where

$$\begin{aligned}
c_T^\rho &= \frac{-\rho\lambda_T^\Sigma}{(\rho-\theta)S_T} - \frac{\rho\rho^*(-2\rho + \frac{\rho^2}{\theta^*})}{\theta^2(\rho-\theta)} - \frac{\rho^*(\theta^*)^2}{\rho\theta^2} - \frac{\rho J_T^V}{(\rho-\theta)S_T} - \frac{\rho(\widehat{\theta}_T - \theta^*)K_T^V}{(\rho-\theta)S_T} \\
&\longrightarrow \; c^\rho = \frac{-\rho\lambda^\Sigma}{(\rho-\theta)\eta^X} - \frac{\rho\rho^*(-2\rho + \frac{\rho^2}{\theta^*})}{\theta^2(\rho-\theta)} - \frac{\rho^*(\theta^*)^2}{\rho\theta^2} - \frac{\rho\lambda^J}{(\rho-\theta)\eta^X} \qquad (31)
\end{aligned}$$

almost surely as $T \longrightarrow \infty$, and

$$\begin{aligned}
c_T^\theta &= \frac{\theta\lambda_T^\Sigma}{(\rho-\theta)S_T} + \frac{\theta\rho^*(-2\rho + \frac{\rho^2}{\theta^*})}{\theta^2(\rho-\theta)} - \frac{\rho^*(\theta^*)^2}{\rho\theta^2} + \frac{\theta J_T^V}{(\rho-\theta)S_T} + \frac{\theta(\widehat{\theta}_T - \theta^*)K_T^V}{(\rho-\theta)S_T} \\
&\longrightarrow \; c^\theta = \frac{\theta\lambda^\Sigma}{(\rho-\theta)\eta^X} + \frac{\theta\rho^*(-2\rho + \frac{\rho^2}{\theta^*})}{\theta^2(\rho-\theta)} - \frac{\rho^*(\theta^*)^2}{\rho\theta^2} + \frac{\theta\lambda^J}{(\rho-\theta)\eta^X} \qquad (32)
\end{aligned}$$

almost surely as $T \longrightarrow \infty$. These last two convergences come from the fact that

$$\lambda_T^{\Sigma}/T \longrightarrow \lambda^{\Sigma} = \frac{2H\Gamma(2H)}{\rho-\theta}\left[\theta^{1-2H} - \rho^{1-2H}\right]$$

and

$$J_T^V/T \longrightarrow \lambda^J = \frac{2}{\theta^2}\left[\theta^*\eta^X - H\Gamma(2H)\theta^*\rho^{-2H}\right]$$

almost surely as $T \longrightarrow \infty$, because $\lambda_T/T \longrightarrow 2H\Gamma(2H)$ as $T \longrightarrow \infty$.
Thus

$$\sqrt{T}(\widehat{\rho}_T - \rho^*) = \frac{\frac{1}{\sqrt{T}}\left(c_T^{\rho} I_2(f_T^{\rho}) + c_T^{\theta} I_2(f_T^{\theta})\right)}{\frac{\widehat{L}_T}{T}} + R_T^{\rho} \tag{33}$$

where as $T \longrightarrow \infty$

$$R_T^{\rho} = R_T^{\theta}\frac{\left(\lambda_T^{\Sigma} + J_T^V + (\widehat{\theta}_T - \theta^*)K_T^V\right)}{\widehat{L}_T} + \frac{R_T/\sqrt{T}}{\frac{\widehat{L}_T}{T}} \longrightarrow 0 \tag{34}$$

almost surely.
Finally, with the expressions (28) and (33) on hand, and the almost-sure negligibility of their corresponding lower-order terms as proved in (29) and (34), we get

$$\sqrt{T}\left(\widehat{\theta}_T - \theta^*, \widehat{\rho}_T - \rho^*\right) = \frac{1}{\sqrt{T}}\left(I_2(f_T^{\theta}), I_2(f_T^{\rho})\right)\begin{pmatrix} \frac{\theta}{\rho-\theta}\frac{T}{S_T} & c_T^{\theta}\frac{T}{\widehat{L}_T} \\ \frac{\rho}{\theta-\rho}\frac{T}{S_T} & c_T^{\rho}\frac{T}{\widehat{L}_T}\end{pmatrix} + \left(R_T^{\theta}, R_T^{\rho}\right)$$

where as $T \longrightarrow \infty$

$$\begin{pmatrix} \frac{\theta}{\rho-\theta}\frac{T}{S_T} & c_T^{\theta}\frac{T}{\widehat{L}_T} \\ \frac{\rho}{\theta-\rho}\frac{T}{S_T} & c_T^{\rho}\frac{T}{\widehat{L}_T}\end{pmatrix} \longrightarrow P := \begin{pmatrix} \frac{\theta}{\rho-\theta}\frac{1}{\eta^X} & \frac{c^{\theta}}{\eta^{\widehat{L}}} \\ \frac{\rho}{\theta-\rho}\frac{1}{\eta^X} & \frac{c^{\rho}}{\eta^{\widehat{L}}}\end{pmatrix} \tag{35}$$

almost surely. Now, applying Slutsky's lemma and Theorem 22 combined with the above convergences, the proof is complete. ■

4 Discrete observation

Assume that the process X is observed equidistantly in time with the step size Δ_n: $t_i = i\Delta_n, i = 0, \ldots, n$, and $T_n = n\Delta_n$ denotes the length of the 'observation window'. The goal of this section is to construct two estimators $\check{\theta}_n$ and $\check{\rho}_n$ of θ and ρ respectively based on the sampling data $X_{t_i}, i = 0, \ldots, n$, and study their strong consistency and asymptotic normality. We also want to define estimators in such a way that consistency and normality results proved in Section 3 for the continuous-data estimators $\left(\widehat{\theta}_T, \widehat{\rho}_T\right)$ can be used to good effect in the discrete case. The basic strategy for this is therefore to look for ways of discretizing the MLE studied in Section 3. It turns out that the most efficient way of implementing this strategy is to define several intermediate estimators, starting with ones where only the denominators in $\left(\widehat{\theta}_T, \widehat{\rho}_T\right)$ are discretized, and then using an algebraic asymptotic interpretation of the numerators to avoid a direct discretization of the corresponding Young or Skorohod integrals. This method allows a rather direct use of the asymptotic normality Theorem 8 in Section 3, while for the strong consistency results, some of the almost-sure convergences proved in

Section 3 are used directly, and additional ones are newly established early on in this section. See Section 1.2 for other details about the heuristics which explain the choices made below in this Section.

For any given process Z, define

$$Q_n(Z) := \frac{1}{n}\sum_{i=1}^{n}(Z_{t_{i-1}})^2$$

The following well-known direct consequence of the Borel-Cantelli Lemma (see e.g. [18]), will allows us to turn convergence rates in the p-th mean into pathwise convergence rates. This is particularly efficient when working with sequences in Wiener chaos.

Lemma 9 *Let $\gamma > 0$ and $p_0 \in \mathbb{N}$. Moreover let $(Z_n)_{n\in\mathbb{N}}$ be a sequence of random variables. If for every $p \geqslant p_0$ there exists a constant $c_p > 0$ such that for all $n \in \mathbb{N}$,*

$$(E|Z_n|^p)^{1/p} \leqslant c_p \cdot n^{-\gamma},$$

then for all $\varepsilon > 0$ there exists a random variable η_ε such that

$$|Z_n| \leqslant \eta_\varepsilon \cdot n^{-\gamma+\varepsilon} \quad \textit{almost surely}$$

for all $n \in \mathbb{N}$. Moreover, $E|\eta_\varepsilon|^p < \infty$ for all $p \geqslant 1$.

As before we assume that $\Delta_n = t_{k+1} - t_k$ is a function of n only. Of some importance, particularly for the purpose of proving normal convergence theorems, is the case $n^{-\alpha}$ with a given $\alpha \in \mathbb{R}$. The case $\alpha > 0$ implies that the observation frequency must increase even as the horizon itself also increases. The case of $\alpha = 0$ is of special importance because it corresponds to a setup where the observation frequency is fixed ($\Delta_n = 1$, no in-fill asymptotics, only increasing horizon), which may be desirable in some applications. We will see that for some almost-sure convergence results, we may even take a time step Δ_n which grows with n. In other words, this allows for very sparse observations. We will also see that most almost-sure results are valid for the entire range $H \in (\frac{1}{2}, 1)$, while normal convergence results require $H \in (\frac{1}{2}, \frac{3}{4})$. We begin by recording and proving some important technical estimates.

Lemma 10 *Define $\delta_n(X) := \sqrt{T_n}\left(Q_n(X) - \frac{S_{T_n}}{T_n}\right)$. Then*

$$E\left[\delta_n^2(X)\right] \leqslant c(H,\theta,\rho)\min\left(n\Delta_n^{2H+1}, \frac{1}{n\Delta_n} + \Delta_n^{H+1} + \Delta_n^{4H-3}\sum_{j=1}^{n} j^{4H-4}\right). \tag{36}$$

In particular, if $\Delta_n \leqslant n^\alpha$ for some $\alpha \in (-\infty, 1/H)$, then

$$Q_n(X) \longrightarrow \eta^X \tag{37}$$

almost surely as $n \to \infty$.

Proof. The points 3) and 6) of Lemma 21 lead to

$$\begin{aligned} E\left[\delta_n^2(X)\right] &= \frac{1}{T_n}\sum_{i,j=1}^{n}\int_{t_{i-1}}^{t_i}\int_{t_{j-1}}^{t_j} E\left[\left(X_t^2 - X_{t_{i-1}}^2\right)\left(X_s^2 - X_{t_{j-1}}^2\right)\right] dsdt \\ &\leqslant c(H,\theta,\rho)n\Delta_n^{2H+1}. \end{aligned}$$

On the other hand, we have

$$
\begin{aligned}
& E\left[\delta_n^2(X)\right] \\
= & \frac{1}{T_n}\sum_{i,j=1}^{n}\int_{t_{i-1}}^{t_i}\int_{t_{j-1}}^{t_j} E\left(X_t^2 - X_{t_{i-1}}^2\right) E\left(X_s^2 - X_{t_{j-1}}^2\right) dsdt \\
& + \frac{1}{T_n}\sum_{i,j=1}^{n}\int_{t_{i-1}}^{t_i}\int_{t_{j-1}}^{t_j} E\left[\left(X_t - X_{t_{i-1}}\right)\left(X_s - X_{t_{j-1}}\right)\right] E\left[\left(X_t + X_{t_{i-1}}\right)\left(X_s + X_{t_{j-1}}\right)\right] dsdt \\
& + \frac{1}{T_n}\sum_{i,j=1}^{n}\int_{t_{i-1}}^{t_i}\int_{t_{j-1}}^{t_j} E\left[\left(X_t - X_{t_{i-1}}\right)\left(X_s + X_{t_{j-1}}\right)\right] E\left[\left(X_t + X_{t_{i-1}}\right)\left(X_s - X_{t_{j-1}}\right)\right] dsdt \\
:= & \frac{1}{T_n}\sum_{i,j=1}^{n}\left(D_1(i,j) + D_2(i,j) + D_3(i,j)\right).
\end{aligned}
$$

By using the points 2), 4) and 6) of Lemma 21 we obtain

$$
\begin{aligned}
\frac{1}{T_n}\sum_{i,j=1}^{n} D_1(i,j) &= \frac{1}{T_n}\left[\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} E\left(X_t^2 - X_{t_{i-1}}^2\right) dt\right]^2 \\
&\leqslant \frac{c(H,\theta,\rho)}{T_n}\left[\Delta_n \sum_{i=1}^{n} e^{-t_{i-1}/2}\right]^2 \\
&\leqslant \frac{c(H,\theta,\rho)}{T_n}\left[\frac{\Delta_n}{1 - e^{-\Delta_n/2}}\right]^2,
\end{aligned}
$$

$$
\begin{aligned}
& \frac{1}{T_n}\sum_{i=1}^{n}\left(D_2(i,i) + D_3(i,i)\right) \\
= & \frac{1}{T_n}\sum_{i=1}^{n}\left(\int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{t_i}\left[2(E(X_tX_s))^2 + 2(E(X_{t_{i-1}}^2))^2 - 4(E(X_{t_{i-1}}X_s))^2\right] dsdt\right) \\
\leqslant & \ c(H,\theta,\rho)\Delta_n^{H+1},
\end{aligned}
$$

and

$$
\begin{aligned}
& \frac{1}{T_n}\sum_{i\neq j=1}^{n}\left(D_2(i,j) + D_3(i,j)\right) \\
= & \frac{2}{T_n}\sum_{i\neq j=1}^{n}\left(\int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{t_i}\left[(E(X_tX_s))^2 - (E(X_{t_{i-1}}X_s))^2 - (E(X_{t_{i-1}}X_t))^2 + (E(X_{t_{i-1}}X_{t_{j-1}}))^2\right] dsdt\right) \\
\leqslant & \frac{c(H,\theta,\rho)}{T_n}\sum_{i<j=1}^{n}\Delta_n^{4H-2}|j-i-1|^{4H-4} \\
\leqslant & \ c(H,\theta,\rho)\Delta_n^{4H-3}\sum_{j=1}^{n} j^{4H-4}.
\end{aligned}
$$

Thus (36) is obtained. Now, using (13), Lemma 9 and (18) we will be able to assert the convergence (37), and thus the entire lemma, as soon as we can show that the right-hand side of (36) divided by T_n converges to 0 as fast as some negative power of n. Thus we only need to show that there exists $\varepsilon > 0$, such that as $n \to \infty$

$$q_n := \min\left(\Delta_n^{2H}, \frac{1}{(n\Delta_n)^2} + \frac{\Delta_n^H}{n} + \frac{1}{n}\Delta_n^{4H-4}\sum_{j=1}^{n} j^{4H-4}\right) \leqslant n^{-\varepsilon}.$$

Let us concentrate first on the second part of the minimum defining q_n. This is the sum of the three terms $(n\Delta_n)^{-2}$, Δ_n^H/n, and $n^{-1}\Delta_n^{4H-4}\sum_{j=1}^n j^{4H-4}$. The first of these three terms will tend to 0 like a negative power of n as soon as there exists $\varepsilon_1 > 0$ such that $\Delta_n \geqslant n^{-1+\varepsilon_1}$. The second term will tend to 0 like a negative power of n as soon as there exists $\varepsilon_2 > 0$ such that $\Delta_n \leqslant n^{1/H-\varepsilon_2}$. For the third term, we must separate the case $H < 3/4$ from the case $H \geqslant 3/4$. When $H < 3/4$, the series $\sum_{j=1}^n j^{4H-4}$ is bounded, so the last term in the second part of the min in q_n will tend to 0 like a negative power of n as soon as there exists $\varepsilon_3 > 0$ such that $\Delta_n \geqslant n^{-1/(4-4H)+\varepsilon_3}$. When $H > 3/4$, the series is bounded above by a constant times n^{4H-3}, yielding a contribution of $(\Delta_n/n)^{4H-4}$; so the last term in the second part of the min in q_n will tend to 0 like a negative power of n as soon as there exists $\varepsilon_4 > 0$ such that $\Delta_n \geqslant n^{-1+\varepsilon_4}$. The case $H = 3/4$ is done in the same fashion, with the same conclusion as when $H > 3/4$. Thus we have proved that for each fixed n, if there exist $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$ such that

$$\max\left(n^{-1+\varepsilon_1}, n^{-\frac{1}{4-4H}+\varepsilon_3}\right) \leqslant \Delta_n \leqslant n^{1/H-\varepsilon_2} \tag{38}$$

then for some $\varepsilon > 0$,

$$q_n \leqslant n^{-\varepsilon}.$$

On the other hand notice that for every $H \in (1/2, 1)$, there exist $\varepsilon_1, \varepsilon_3 > 0$ such that

$$\max\left(n^{-1+\varepsilon_1}, n^{-\frac{1}{4-4H}+\varepsilon_3}\right) \leqslant n^{-1/2}. \tag{39}$$

Thus for each fixed n, if we have

$$\Delta_n \leqslant n^{-1/2}, \tag{40}$$

using the first part of the min in the definition of q_n, we get

$$q_n \leqslant n^{-\varepsilon}$$

with $\varepsilon = H$. To conclude, by (39), for each fixed n, we are either in the case (38) or (40), so that $q_n \leqslant n^{-\varepsilon}$ in all cases as soon as $\Delta_n \leqslant n^{1/H-\varepsilon_2}$ for some $\varepsilon_2 > 0$. The proof of the lemma is complete. ∎

Lemma 11 *Define* $\delta_n(\Sigma) := \sqrt{T_n}\left(Q_n(\Sigma) - \frac{1}{T_n}\int_0^{T_n}\Sigma_t^2 dt\right)$. *Then*

$$E\left[\delta_n^2(\Sigma)\right] \leqslant c(H,\theta,\rho)\min\left(n\Delta_n^{2H+1}, \frac{1}{n\Delta_n} + \Delta_n^{H+1} + \Delta_n^{4H-3}\sum_{j=1}^{n} j^{4H-4}\right). \tag{41}$$

In particular, if $\Delta_n \leqslant n^\alpha$ *for some* $\alpha \in (-\infty, 1/H)$, *then*

$$Q_n(\Sigma) \longrightarrow \eta^\Sigma \tag{42}$$

almost surely as $n \to \infty$.
On the other hand if $n^{1+\alpha}\Delta_n^{H+1} \to 0$ for some $\alpha > 0$,

$$\left|Q_n(\widehat{\Sigma}) - Q_n(\Sigma)\right| \longrightarrow 0 \tag{43}$$

almost surely as $n \to \infty$, where

$$\widehat{\Sigma}_{t_i}^n = \Delta_n \sum_{j=1}^{i} X_{t_{j-1}}.$$

In addition, if $n^3\Delta_n^{2H+3} \to 0$,

$$\sqrt{T_n}\left|Q_n(\widehat{\Sigma}) - Q_n(\Sigma)\right| \longrightarrow 0 \tag{44}$$

in $L^2(\Omega)$ as $n \to \infty$.

Proof. By using same arguments as in the proof of Lemma 10, (21) and (19), we obtain (41) and (42).
Now, we prove the convergence (43). We can write

$$Q_n(\Sigma) - Q_n(\widehat{\Sigma}) \;=\; \frac{-1}{n}\sum_{i=1}^{n}\left(\Sigma_{t_{i-1}} - \widehat{\Sigma}_{t_{i-1}}\right)^2 + \frac{2}{n}\sum_{i=1}^{n}\Sigma_{t_{i-1}}\left(\Sigma_{t_{i-1}} - \widehat{\Sigma}_{t_{i-1}}\right).$$

Using the point 6) of Lemma 21

$$\begin{aligned}
E\left(\left(\Sigma_{t_{i-1}} - \widehat{\Sigma}_{t_{i-1}}\right)^2\right) &= \sum_{j=1}^{i-1}\sum_{k=1}^{i-1}\int_{t_{j-1}}^{t_j}\int_{t_{k-1}}^{t_k} E[(X_s - X_{t_{j-1}})(X_r - X_{t_{k-1}})]drds \\
&\leqslant c(H,\theta,\rho)\left(\sum_{j=1}^{i-1}\int_{t_{j-1}}^{t_j}|s - t_{j-1}|^H ds\right)^2 \\
&\leqslant c(H,\theta,\rho)\left(n\Delta_n^{H+1}\right)^2
\end{aligned}$$

Then, by Hölder inequality and the point 3) of Lemma 21 we obtain for every $p \geqslant 1$

$$\left(E\left[\left|Q_n(\Sigma) - Q_n(\widehat{\Sigma})\right|^p\right]\right)^{1/p} \leqslant c(H,\theta,\rho)\left[n^2\Delta_n^{2H+2} + n\Delta_n^{H+1}\right].$$

Thus, by (13), Lemma 9 and that fact that $n^{1+\alpha}\Delta_n^{H+1} \to 0$ for some $\alpha > 0$ the convergence (43) is obtained.
Furthermore, it is also easy to see that the convergence (44) is satisfied. ■

4.1 Auxiliary estimators $\tilde{\theta}$ and $\tilde{\rho}$

The first step in constructing a discrete-observation-based estimator for which the asymptotics of $\left(\widehat{\theta}_T, \widehat{\rho}_T\right)$ studied in Section 3 can be helpful, is to consider the following two auxiliary estimators $\widetilde{\theta}_n$ and $\widetilde{\rho}_n$ of θ^* and ρ^* respectively, by leaving the numerators in $\left(\widehat{\theta}_T, \widehat{\rho}_T\right)$ alone, and discretizing the denominators:

$$\widetilde{\theta}_n = -\frac{\frac{1}{T_n}\int_0^{T_n} X_t\delta X_t}{Q_n(X)}$$

and

$$\widetilde{\rho}_n(\Sigma) = -\frac{\frac{1}{T_n}\int_0^{T_n} \widehat{V}_t^n \delta \widehat{V}_t^n}{Q_n(X) + (\widetilde{\theta}_n)^2 Q_n(\Sigma)},$$

where

$$\widehat{V}_t^n = X_t + \widehat{\theta}_n \Sigma_t, \quad 0 \leqslant t \leqslant T_n. \tag{45}$$

and we recall that $Q_n(Z)$ is a notation for the Riemann-sum rectangle approximation $\frac{1}{n}\sum_{i=1}^{n}(Z_{t_{i-1}})^2$. We also consider the version of $\widetilde{\rho}_n(\Sigma)$ based only on discrete observations of Σ:

$$\widetilde{\rho}_n(\widehat{\Sigma}) = -\frac{\frac{1}{T_n}\int_0^{T_n} \widehat{V}_t^n \delta \widehat{V}_t^n}{Q_n(X) + (\widetilde{\theta}_n)^2 Q_n(\widehat{\Sigma})}.$$

Combining Lemma 3 and the almost-sure convergence (37) we deduce the strong consistency of $\widetilde{\theta}_n$.

Theorem 12 *Assume $H \in (1/2, 1)$. If $\Delta_n \leqslant n^{\alpha}$ for some $\alpha \in (-\infty, 1/H)$, then*

$$\widetilde{\theta}_n \longrightarrow \theta^*$$

almost surely as $n \to \infty$.

By Lemmas 6 and 11 it is easy also to deduce the strong consistency of $\widetilde{\rho}_n(\Sigma)$ and $\widetilde{\rho}_n(\widehat{\Sigma})$.

Theorem 13 *Assume $H \in (1/2, 1)$. If $\Delta_n \leqslant n^{\alpha}$ for some $\alpha \in (-\infty, 1/H)$, then*

$$\widetilde{\rho}_n(\Sigma) \longrightarrow \rho^*$$

almost surely as $n \to \infty$.
In addition, if $n^{1+\alpha}\Delta_n^{H+1} \to 0$ for some $\alpha > 0$,

$$\widetilde{\rho}_n(\widehat{\Sigma}) \longrightarrow \rho^*$$

almost surely as $n \to \infty$.

To establish the asymptotic normality of $\left(\widetilde{\theta}_n, \widetilde{\rho}_n\right)$, we can write

$$\sqrt{T_n}\left(\widetilde{\theta}_n - \theta^*\right) = \frac{\frac{S_{T_n}}{T_n}}{Q_n}\sqrt{T_n}(\widehat{\theta}_{T_n} - \theta^*) + \frac{\theta^*\sqrt{T_n}(\frac{S_{T_n}}{T_n} - Q_n(X))}{Q_n(X)}.$$

Similarly,

$$\sqrt{T_n}\left(\widetilde{\rho}_n - \rho^*\right) = \frac{\frac{\widehat{L}_{T_n}}{T_n}}{\widehat{Q}_n}\sqrt{T_n}(\widehat{\rho}_{T_n} - \rho^*) + \frac{\rho^*\sqrt{T_n}(\frac{\widehat{L}_{T_n}}{T_n} - \widehat{Q}_n)}{\widehat{Q}_n}.$$

Theorem 8 provides the convergence of the last summands in each of the two lines above. Combining this with the convergences we obtained in Lemmas 10 and 11, we obtain the following result.

Theorem 14 *Let $H \in (\frac{1}{2}, \frac{3}{4})$ and $n\Delta_n^{H+1} \to 0$. Then*

$$\sqrt{T_n}\left(\widetilde{\theta}_n - \theta^*, \widetilde{\rho}_n(\Sigma) - \rho^*\right) \overset{law}{\longrightarrow} \mathcal{N}\left(0, {}^t P\ \Gamma\ P\right)$$

In addition if $n^3\Delta_n^{2H+3} \to 0$,

$$\sqrt{T_n}\left(\widetilde{\theta}_n - \theta^*, \widetilde{\rho}_n(\widehat{\Sigma}) - \rho^*\right) \overset{law}{\longrightarrow} \mathcal{N}\left(0, {}^t P\ \Gamma\ P\right)$$

where P the matrix defined in (35).

4.2 X and Σ are observed

The problem with the auxiliary estimators $\widetilde{\theta}_n$ and $\widetilde{\rho}_n$ is that they still contain Skorohod integrals. In order to devise a further scheme that allows us to evaluate them, at least approximately, using discrete data only, we begin by using the discrete observations of X and Σ, and recalling that, from Lemmas 3 and 6, we have

$$\begin{aligned} \frac{1}{T}\int_0^T X_t \delta X_t &\longrightarrow -\rho\theta(\rho+\theta)\eta^X, \\ \frac{1}{T}\int_0^T \widehat{V}_t \delta \widehat{V}_t &\longrightarrow -\rho\theta(\rho+\theta)\eta^\Sigma \end{aligned}$$

where η^X and η^Σ, which are also functions of H, θ, ρ, are given in Lemma 2. Since these limits depend on the parameters we are trying to estimate, one strategy is to rewrite the strong consistency results of Theorems 12 and 13 for $\widetilde{\theta}_n$ and $\widetilde{\rho}_n(\Sigma)$ as implicit definitions of new estimators, where the numerators in the definitions of $\widetilde{\theta}_n$ and $\widetilde{\rho}_n(\Sigma)$ are replaced by their limits recalled above, and each instance of θ and ρ therein are replaced by the new estimator we are trying to define. The same substitution must be done with the expressions θ^* and ρ^*, since these are the limits of $\widetilde{\theta}_n$ and $\widetilde{\rho}_n$. In other words we consider only that the denominators in $\widetilde{\theta}_n$ and $\widetilde{\rho}_n$ contain data, and replace all other instances of (θ, ρ) in the limits in Theorems 12 and 13 by the pair of estimators we are trying to define. After some minor manipulations, this leads to the following definition of a new pair of estimators $\left(\check{\theta}_n, \check{\rho}_n\right)$ as solution of the system of the following two equations, if it exists:

$$\left\{ \begin{array}{l} \check{\theta}_n + \check{\rho}_n = \frac{H\Gamma(2H)[(\check{\rho}_n)^{2-2H} - (\check{\theta}_n)^{2-2H}]}{(\check{\rho}_n - \check{\theta}_n)Q_n(X)} \\ \frac{\left(\check{\theta}_n\right)^2 - (\check{\rho}_n)^2}{\left[(\check{\theta}_n)^{2-2H} - (\check{\rho}_n)^{2-2H} + \left(\check{\rho}_n + \check{\theta}_n\right)^2\left((\check{\rho}_n)^{-2H} - (\check{\theta}_n)^{-2H}\right)\right]} = \frac{H\Gamma(2H)}{Q_n(X) + (\check{\theta}_n + \check{\rho}_n)^2 Q_n(\Sigma)} \end{array} \right. . \tag{46}$$

We emphasize that the above is an implicit definition of $\left(\check{\theta}_n, \check{\rho}_n\right)$. It is also rather opaque. The system (46) can be simplified slightly using more elementary manipulations. We find that the definition of $\left(\check{\theta}_n, \check{\rho}_n\right)$ is equivalent to the following:

$$F\left(\check{\theta}_n, \check{\rho}_n\right) = (Q_n(X), Q_n(\Sigma))$$

where F is a positive function of the variables (x, y) in $(0, +\infty)^2$ defined by: for every $(x, y) \in (0, +\infty)^2$

$$F(x, y) = H\Gamma(2H) \times \left\{ \begin{array}{ll} \frac{1}{y^2 - x^2}\left(y^{2-2H} - x^{2-2H}, x^{-2H} - y^{-2H}\right) & \text{if } x \neq y \\ \left((1-H)x^{-2H}, Hx^{-2H-2}\right) & \text{if } x = y. \end{array} \right. \tag{47}$$

Interestingly, this shows that a good candidate for the discrete version of the least-squares estimator of (θ,ρ) is none other than a type of generalized method of moments estimator obtained via Lemma 2 after discretizing the expressions $S_T(X) := T^{-1}\int_0^T X_s^2 ds$ and $S_T(\Sigma) := T^{-1}\int_0^T \Sigma_s^2 ds$. We now consider the question whether System (46) has a unique solution $\left(\check{\theta}_n^2, \check{\rho}_n^2\right)$, and how this may imply strong consistency for these estimators.
Since for every $(x,y)\in(0,+\infty)^2$ with $x\neq y$

$$J_F(x,y) = \Gamma(2H+1)\begin{pmatrix} \frac{(1-H)x^{1-2H}\left(x^2-y^2\right)-x\left(x^{2-2H}-y^{2-2H}\right)}{(x^2-y^2)^2} & \frac{(1-H)y^{1-2H}\left(y^2-x^2\right)-y\left(y^{2-2H}-x^{2-2H}\right)}{(x^2-y^2)^2} \\ \frac{Hx^{-2H-1}\left(x^2-y^2\right)+x\left(x^{-2H}-y^{-2H}\right)}{(x^2-y^2)^2} & \frac{Hy^{-2H-1}\left(y^2-x^2\right)+y\left(y^{-2H}-x^{-2H}\right)}{(x^2-y^2)^2} \end{pmatrix}$$

the determinant of $J_F(x,y)$ is non-zero on in $(0,+\infty)^2$. So, F is a diffeomorphism in $(0,+\infty)^2$ and its inverse G has a Jacobian

$$J_G(a,b) = \frac{\Gamma(2H+1)}{\det J_F(x,y)}\begin{pmatrix} \frac{Hy^{-2H-1}\left(y^2-x^2\right)+y\left(y^{-2H}-x^{-2H}\right)}{(x^2-y^2)^2} & -\frac{(1-H)y^{1-2H}\left(y^2-x^2\right)-y\left(y^{2-2H}-x^{2-2H}\right)}{(x^2-y^2)^2} \\ -\frac{Hx^{-2H-1}\left(x^2-y^2\right)+x\left(x^{-2H}-y^{-2H}\right)}{(x^2-y^2)^2} & \frac{(1-H)x^{1-2H}\left(x^2-y^2\right)-x\left(x^{2-2H}-y^{2-2H}\right)}{(x^2-y^2)^2} \end{pmatrix};$$

where $(x,y) = G(a,b)$.
Hence, (37) and (42) lead to

$$\left(\check{\theta}_n, \check{\rho}_n\right) = G\left(Q_n(X), Q_n(\Sigma)\right) \longrightarrow G\left(\eta^X, \eta^\Sigma\right) = (\theta,\rho)$$

almost surely as $n\to\infty$ as soon as $\Delta_n \leqslant n^\alpha$ for some $\alpha\in(-\infty,1/H)$. Summarizing, we have proved the following.

Theorem 15 *Let $H\in(1/2,1)$ and assume that $\Delta_n \leqslant n^\alpha$ for some $\alpha\in(-\infty,1/H)$. Then, as $n\longrightarrow\infty$*

$$\left(\check{\theta}_n, \check{\rho}_n\right) \longrightarrow (\theta,\rho)$$

almost surely.

We may now prove a normal convergence result for $\left(\check{\theta}_n, \check{\rho}_n\right)$ based on Theorem 14. Note that the second part of Theorem 14 is not needed here because we rely on fully observed Σ in this section.

Theorem 16 *Suppose that $H\in(\frac{1}{2},\frac{3}{4})$ and $n\Delta_n^{H+1}\to 0$. Then*

$$\sqrt{T_n}\left(\check{\theta}_n-\theta, \check{\rho}_n-\rho\right) \xrightarrow{law} \mathcal{N}(0, {}^tM\ {}^tP\ \Gamma\ P\ M)$$

where the matrices Γ, P and M are defined respectively in (A-5), (35) and (50).

Proof. We have

$$\widetilde{\theta}_n = -\frac{\frac{1}{T_n}\int_0^{T_n} X_t \delta X_t}{Q_n(X)} := \frac{J_n^\theta(X)}{Q_n(X)}$$

and

$$\widetilde{\rho}_n(\Sigma) = -\frac{\frac{1}{T_n}\int_0^{T_n} \widehat{V}_t^n \delta \widehat{V}_t^n}{Q_n(X)+(\widetilde{\theta}_n)^2 Q_n(\Sigma)} := \frac{J_n^\rho(\widehat{V})}{Q_n(X)+(\widetilde{\theta}_n)^2 Q_n(\Sigma)}.$$

Then, we can write

$$\begin{aligned}
& \sqrt{T_n}\left(\check{\theta}_n-\theta, \check{\rho}_n-\rho)\right) \\
= & \sqrt{T_n}\left(G\left(\frac{J_n^\theta(X)}{\widetilde{\theta}_n},(\widetilde{\theta}_n)^{-2}\left(\frac{J_n^\rho(\widehat{V})}{\widetilde{\rho}_n(\Sigma)}-\frac{J_n^\theta(X)}{\widetilde{\theta}_n}\right)\right)-(\theta, \rho)\right) \\
= & \sqrt{T_n}\left(GoL\left(J_n^\theta(X), J_n^\rho(\widehat{V}), \widetilde{\theta}_n, \widetilde{\rho}_n(\Sigma)\right)-(\theta, \rho)\right)
\end{aligned}$$

where $L(r,s,u,v)=\left(\frac{r}{u},\frac{s}{u^2v}-\frac{r}{u^3}\right)$.
On the other hand for any $\varepsilon\in(0,1)$

$$\frac{1}{T}\int_0^T\int_0^t r^{2H-2}e^{-r}drdt=\Gamma(2H-1)+o(\frac{1}{T^\varepsilon})$$

because

$$\begin{aligned}
\frac{1}{T^{1-\varepsilon}}\int_0^T\int_t^\infty r^{2H-2}e^{-r}drdt & \leqslant \frac{1}{2T^{1-\varepsilon}}(1-e^{-T/2})\int_0^\infty r^{2H-2}e^{-r/2}dr \\
& \rightarrow 0.
\end{aligned}$$

Combining this together with (23) and the point 5) of Lemma 21 we can write

$$J_n^\theta(X)=(\rho+\theta)\eta^X+o(\frac{1}{\sqrt{T_n}}) \tag{48}$$

where $o(\frac{1}{\sqrt{T_n}})$ denotes a random variable such that $\sqrt{T_n}o(\frac{1}{\sqrt{T_n}})$ converges to zero almost surely as $T_n\rightarrow\infty$.
Similar argument leads to

$$J_n^\rho(\widehat{V})=\rho\theta(\rho+\theta)\eta^\Sigma+o(\frac{1}{\sqrt{T_n}}). \tag{49}$$

Since

$$GoL\left((\rho+\theta)\eta^X,\rho\theta(\rho+\theta)\eta^\Sigma,\theta^*,\rho^*\right)=(\theta,\rho)$$

we can write

$$\begin{aligned}
& \sqrt{T_n}\left(\check{\theta}_n-\theta, \check{\rho}_n-\rho)\right) \\
= & \sqrt{T_n}\left(GoL\left(J_n^\theta(X), J_n^\rho(\widehat{V}), \widetilde{\theta}_n, \widetilde{\rho}_n(\Sigma)\right)-GoL\left((\rho+\theta)\eta^X,\rho\theta(\rho+\theta)\eta^\Sigma,\theta^*,\rho^*\right)\right) \\
= & \sqrt{T_n}\Big[GoL\left(J_n^\theta(X), J_n^\rho(\widehat{V}), \widetilde{\theta}_n, \widetilde{\rho}_n(\Sigma)\right)-GoL\left((\rho+\theta)\eta^X,\rho\theta(\rho+\theta)\eta^\Sigma,\widetilde{\theta}_n, \widetilde{\rho}_n(\Sigma)\right) \\
& +GoL\left((\rho+\theta)\eta^X,-\rho\theta(\rho+\theta)\eta^\Sigma,\widetilde{\theta}_n, \widetilde{\rho}_n(\Sigma)\right)-GoL\left((\rho+\theta)\eta^X,\rho\theta(\rho+\theta)\eta^\Sigma,\theta^*,\rho^*\right)\Big] \\
:= & s_n+r_n.
\end{aligned}$$

From (48) and (49) we obtain $s_n\longrightarrow 0$ almost surely as $n\rightarrow\infty$.
On the other hand, by Taylor's formula

$$r_n=\sqrt{T_n}\left(\widetilde{\theta}_n-\theta^*,\widetilde{\rho}_n(\Sigma)-\rho^*\right)M+d_n$$

where

$$M = \begin{pmatrix} \frac{\partial h_1}{\partial u}(\theta^*, \rho^*) & \frac{\partial h_2}{\partial u}(\theta^*, \rho^*) \\ \frac{\partial h_1}{\partial v}(\theta^*, \rho^*) & \frac{\partial h_2}{\partial v}(\theta^*, \rho^*) \end{pmatrix} \tag{50}$$

with

$$h(u,v) = (h_1, h_2)(u,v) \quad = \quad GoL\left((\rho+\theta)\eta^X, \rho\theta(\rho+\theta)\eta^\Sigma, u, v\right).$$

We can write

$$h(u,v) \quad = \quad (G_1, G_2)\, og\,(u,v)$$

where

$$g(u,v) = (g_1, g_2)(u,v) = L\left((\rho+\theta)\eta^X, \rho\theta(\rho+\theta)\eta^\Sigma, u, v\right).$$

Moreover for $i = 1, 2$

$$\frac{\partial h_i}{\partial u}(u,v) = \frac{\partial G_i}{\partial a}(g(u,v))\frac{\partial g_1}{\partial u}(u,v) + \frac{\partial G_i}{\partial b}(g(u,v))\frac{\partial g_2}{\partial u}(u,v)$$

and

$$\frac{\partial h_i}{\partial v}(u,v) = \frac{\partial G_i}{\partial a}(g(u,v))\frac{\partial g_1}{\partial v}(u,v) + \frac{\partial G_i}{\partial b}(g(u,v))\frac{\partial g_2}{\partial v}(u,v).$$

On the other hand, d_n converges in distribution to zero, because

$$\|d_n\| \leqslant c(H, \theta, \rho)\sqrt{T_n}\|(\widetilde{\rho}_n(\Sigma) - \theta^*, \widetilde{\theta}_n - \rho^*)\|^2.$$

It is elementary that if for any $\omega \in \Omega$ there exists $n_0(\omega) \in \mathbb{N}$ such that $X_n(\omega) = Y_n(\omega)$ for all $n \geqslant n_0(\omega)$ and $X_n \xrightarrow{law} 0$ as $n \to \infty$, then $Y_n \xrightarrow{law} 0$ as $n \to \infty$.
Combining this with Theorem 14 the proof is completed. ■

4.3 X is observed

In the previous section, we encountered theorems in which X and Σ are both assumed to be fully observed in discrete time. Since Σ is the time-antiderivative of X, such an assumption corresponds, for instance, to the physical situation where X is the velocity of a particle, and Σ is its position.

In this section, we abandon such a framework, and assume instead that only X is observed in discrete time. Thus we consider the following pair of estimators $\left(\breve{\theta}_n, \breve{\rho}_n\right)$:

$$\left(\breve{\theta}_n, \breve{\rho}_n\right) = G\left(Q_n(X), Q_n(\widehat{\Sigma})\right)$$

where the deterministic explicit function G was identified in the previous section as the inverse of the function F given in (47). Equivalently, $\left(\breve{\theta}_n, \breve{\rho}_n\right)$ is the solution of the system (46), or its equivalent form (47), with Σ replaced by the process $\widehat{\Sigma}$, which relies only on observations of X. Using same arguments as in Section 4.2 and Lemma 11, but relying now on the second part of Theorem 13 (hence the stronger condition on Δ_n for the strong consistency result) and the second part of Theorem 14 (hence the stronger condition on Δ_n for the convergence in law result), we conclude the following.

Theorem 17 *If $n^{1+\alpha}\Delta_n^{H+1} \to 0$ for some $\alpha > 0$,*

$$\left(\breve{\theta}_n, \breve{\rho}_n\right) \longrightarrow (\theta, \rho)$$

almost surely as $n \to \infty$.

Theorem 18 *Let $H \in (\frac{1}{2}, \frac{3}{4})$. If $n^3\Delta_n^{2H+3} \to 0$, then, as $n \to \infty$*

$$\sqrt{T_n}\left(\breve{\theta}_n - \theta, \breve{\rho}_n - \rho\right) \xrightarrow{law} \mathcal{N}(0,{}^t M\ {}^tP\ \Gamma\ P\ M)$$

where the matrices Γ, P and M are defined respectively in (A-5), (35) and (50).

4.4 X and V are observed

When both X and V are observed, the estimator of θ^* based on continuous data is $\widehat{\theta}_T$ given in (9) but the estimator of ρ becomes the usual full-observation estimator of an fBm-driven Ornstein-Uhlenbeck process as in [14], i.e.

$$\overline{\rho}_T = -\frac{\int_0^T V_t \delta V_t}{\int_0^T V_t^2 dt}.$$

Following similar arguments as in the beginning of Section 4.2, the natural candidate for the estimator based on discrete data of X and V is the pair $\left(\underline{\theta}_n, \underline{\rho}_n\right)$ defined as the solution of the following system:

$$\begin{cases} \underline{\theta}_n + \underline{\rho}_n = \dfrac{\frac{H\Gamma(2H)}{(\underline{\rho}_n - \underline{\theta}_n)}[(\underline{\rho}_n)^{2-2H} - (\underline{\theta}_n)^{2-2H}]}{Q_n(X)} \\ \underline{\rho}_n = \left(\frac{H\Gamma(2H)}{Q_n(V)}\right)^{\frac{1}{2H}}. \end{cases}$$

We see that $\underline{\rho}_n$ is defined explicitly autonomously via the discrete-data-based statistic $Q_n(V)$. With $\underline{\rho}_n$ now known, elementary manipulations yield that $\underline{\theta}_n$ is precisely the solution of the following simple equation

$$(\underline{\theta}_n)^{2-2H} - \left(\frac{Q_n(X)}{H\Gamma(2H)}\right)(\underline{\theta}_n)^2 = \left(\underline{\rho}_n\right)^{2-2H} - \left(\frac{Q_n(X)}{H\Gamma(2H)}\right)\left(\underline{\rho}_n\right)^2.$$

Define

$$\overline{F}(x,y) = H\Gamma(2H) \times \begin{cases} \left(\frac{y^{2-2H} - x^{2-2H}}{y^2 - x^2}, y^{-2H}\right) & \text{if } x \neq y \\ \left((1-H)x^{-2H}, x^{-2H}\right) & \text{if } x = y. \end{cases}$$

Its Jacobian is given, for every $(x,y) \in (0,+\infty)^2$ such that $x \neq y$, by

$$J_{\overline{F}}(x,y) = \Gamma(2H+1)\begin{pmatrix} \frac{(1-H)x^{1-2H}\left(x^2-y^2\right) - x\left(x^{2-2H} - y^{2-2H}\right)}{(x^2-y^2)^2} & \frac{(1-H)y^{1-2H}\left(y^2-x^2\right) - y\left(y^{2-2H} - x^{2-2H}\right)}{(x^2-y^2)^2} \\ 0 & -y^{-2H-1} \end{pmatrix}.$$

Thus the Jacobian of is inverse $\overline{G}$ is as follows

$$J_{\overline{G}}(a,b) = \frac{\Gamma(2H+1)}{\det J_{\overline{F}}(x,y)}\begin{pmatrix} -y^{-2H-1} & -\frac{(1-H)y^{1-2H}\left(y^2-x^2\right) - y\left(y^{2-2H} - x^{2-2H}\right)}{(x^2-y^2)^2} \\ 0 & \frac{(1-H)x^{1-2H}\left(x^2-y^2\right) - x\left(x^{2-2H} - y^{2-2H}\right)}{(x^2-y^2)^2} \end{pmatrix}; (x,y) = \overline{G}(a,b).$$

Using same arguments as in Section 4.2 we obtain

Theorem 19 *Assume that $\Delta_n \leqslant n^{\alpha}$ for some $\alpha \in (-\infty, 1/H)$. Then, as $n \longrightarrow \infty$*

$$\left(\overline{\theta}_n, \overline{\rho}_n\right) \longrightarrow (\theta, \rho)$$

almost surely.

Theorem 20 *Suppose that $H \in (\frac{1}{2}, \frac{3}{4})$ and $n\Delta_n^{H+1} \to 0$. Then, as $n \to \infty$*

$$\sqrt{T_n}\left(\overline{\theta}_n - \theta, \overline{\rho}_n - \rho\right) \xrightarrow{law} \mathcal{N}(0, {}^tQ\ {}^tP\ \Gamma\ P\ Q)$$

where Γ and P are defined respectively in (A-5) and (35), and where

$$Q = \begin{pmatrix} \frac{\partial f_1}{\partial u}(\theta^*, \rho) & \frac{\partial f_2}{\partial u}(\theta^*, \rho) \\ \frac{\partial f_1}{\partial v}(\theta^*, \rho) & \frac{\partial f_2}{\partial v}(\theta^*, \rho) \end{pmatrix}$$

such that

$$(f_1, f_2)(u, v) = \overline{G} o l\,(u, v)$$

with

$$l(u, v) = \overline{L}\left((\rho + \theta)\eta^X, \rho^{1-2H}, u, v\right)$$

and

$$\overline{L}(r, s, u, v) := \left(\frac{r}{u}, \frac{s}{v}\right).$$

5 Appendix

In this appendix, we present some calculations used in the paper.

Fix $T > 0$. Let $f, g : [0, T] \longrightarrow \mathbb{R}$ be Hölder continuous functions of orders $\alpha \in (0, 1)$ and $\beta \in (0, 1)$ respectively with $\alpha + \beta > 1$. Young [29] proved that the Riemann-Stieltjes integral (so-called Young integral) $\int_0^T f_s dg_s$ exists. Moreover, if $\alpha = \beta \in (\frac{1}{2}, 1)$ and $\phi : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a function of class $\mathcal{C}^1$, the integrals $\int_0^{\cdot} \frac{\partial \phi}{\partial f}(f_u, g_u) df_u$ and $\int_0^{\cdot} \frac{\partial \phi}{\partial g}(f_u, g_u) dg_u$ exist in the Young sense and the following chain rule holds:

$$\phi(f_t, g_t) = \phi(f_0, g_0) + \int_0^t \frac{\partial \phi}{\partial f}(f_u, g_u) df_u + \int_0^t \frac{\partial \phi}{\partial g}(f_u, g_u) dg_u, \quad 0 \leqslant t \leqslant T. \tag{A-1}$$

As a consequence, if $H \in (\frac{1}{2}, 1)$ and $(u_t,\ t \in [0, T])$ is a process with Hölder paths of order $\alpha \in (1 - H, 1)$, the integral $\int_0^T u_s dB_s^H$ is well-defined as a Young integral. Suppose moreover that for any $t \in [0, T]$, $u_t \in D^{1,2}$, and

$$P\left(\int_0^T \int_0^T |D_s u_t||t - s|^{2H-2} ds dt < \infty\right) = 1.$$

Then, by [1], $u \in Dom\delta$ and for every $t \in [0, T]$,

$$\int_0^t u_s dB_s^H = \int_0^t u_s \delta B_s^H + H(2H - 1) \int_0^t \int_0^t D_s u_r |s - r|^{2H-2} dr ds. \tag{A-2}$$

In particular, when φ is a non-random Hölder continuous function of order $\alpha \in (1-H, 1)$, we obtain

$$\int_0^T \varphi_s dB_s^H = \int_0^T \varphi_s \delta B_s^H = B^H(\varphi). \tag{A-3}$$

In addition, for all $\varphi,\ \psi \in |\mathcal{H}|$,

$$E\left(\int_0^T \varphi_s dB_s^H \int_0^T \psi_s dB_s^H\right) = H(2H-1)\int_0^T \int_0^T \varphi(u)\psi(v)|u-v|^{2H-2} du dv.$$

Lemma 21 *Let $m, m' > 0$ and let X^m be the process defined in (15). Then,*

1) $\lambda(m,m') := H(2H-1)\int_0^\infty \int_0^\infty e^{-ms}e^{-m'r}|s-r|^{2H-2} drds = \frac{H\Gamma(2H)}{m+m'}\left(m^{1-2H} + m'^{1-2H}\right)$,

2) $0 \leqslant \lambda(m,m') - E(X_t^m X_t^{m'}) \leqslant c(H,m,m')e^{-t/2}$,

3) $\sup_{t\geqslant 0} E[|X_t^m|^p] \leqslant c(H,m,p) < \infty$,

4) $0 \leqslant E(X_t^m X_s^{m'}) \leqslant c(H,m,m')|t-s|^{2H-2}$,

5) *For every* $\varepsilon > 0$, $\frac{X_T^m}{T^\varepsilon} \to 0$ *almost surely as* $T \to \infty$,

6) $E(|X_t^m - X_s^m|^p) \leqslant c(H,m,p)|t-s|^{pH}$.

Proof. To prove equality 1), we just write

$$\begin{aligned}
\lambda(m,m') &= H(2H-1)\int_0^\infty \int_0^\infty e^{-ms}e^{-m'r}|s-r|^{2H-2} drds \\
&= H(2H-1)\int_0^\infty dse^{-ms}\int_0^s dre^{-m'r}|s-r|^{2H-2} \\
&\quad + H(2H-1)\int_0^\infty dse^{-ms}\int_s^\infty dre^{-m'r}|s-r|^{2H-2} \\
&= \frac{H\Gamma(2H)}{m+m'}\left(m^{1-2H} + m'^{1-2H}\right).
\end{aligned}$$

For the point 2) see [15]. For 3) and 6) we refer to [16], and for 4) and 5) see [14, Lemma 5.2 and Lemma 5.4] ■

Theorem 22 *Let $H \in (\frac{1}{2}, \frac{3}{4})$. Define for $m > 0$*

$$f_T^m(u,v) := \frac{1}{2}e^{-m|u-v|}\mathbb{1}_{\{[0,T]\}}^{\otimes 2}(u,v).$$

Then, as $T \longrightarrow \infty$,

$$\frac{1}{\sqrt{T}}\left(I_2(f_T^\theta), I_2(f_T^\rho)\right) \xrightarrow{law} \mathcal{N}(0,\Gamma) \tag{A-4}$$

where Γ is a symmetric nonnegative definite matrix which has the following explicit expression

$$\Gamma = \eta_H \begin{pmatrix} l_1 & l_3 \\ l_3 & l_2 \end{pmatrix} \tag{A-5}$$

where $l_1 = \theta^{1-4H}$, $l_2 = \rho^{1-4H}$, $l_3 = \frac{2\rho\theta}{(4H-1)(\rho^2-\theta^2)}\left[\theta^{1-4H} - \rho^{1-4H}\right]$ *and*

$$\eta_H = H^2(4H-1)\left[\Gamma(2H)^2 + \frac{\Gamma(2H)\Gamma(3-4H)\Gamma(4H-1)}{\Gamma(2-2H)}\right].$$

Proof. Notice that for (A-4) to hold it suffices that prove that for every $a, b \in \mathbb{R}$,

$$G_T := aI_2(f_T^\theta) + bI_2(f_T^\rho)$$

converges in law to $\mathcal{N}\left(0, (a,b)\Gamma^t(a,b)\right)$ as $T \longrightarrow \infty$.
Fix $a, b \in \mathbb{R}$. Since G_T is a multiple integral, by the isometry property of double stochastic integral I_2, we get the variance of G_T as follows

$$EG_T^2 = \frac{\alpha_H^2}{2}\left(a^2\frac{I_T^1}{T} + b^2\frac{I_T^2}{T} + 2ab\frac{I_T^3}{T}\right),$$

where

$$I_T^1 = \int_{[0,T]^4} e^{-\theta|t-s|}e^{-\theta|u-v|}|t-u|^{2H-2}|s-v|^{2H-2}dtdsdudv,$$

$$I_T^2 = \int_{[0,T]^4} e^{-\rho|t-s|}e^{-\rho|u-v|}|t-u|^{2H-2}|s-v|^{2H-2}dtdsdudv,$$

$$I_T^3 = \int_{[0,T]^4} e^{-\rho|t-s|}e^{-\theta|u-v|}|t-u|^{2H-2}|s-v|^{2H-2}dtdsdudv.$$

Using the same argument as in the proof of [14, Theorem 3.4], we have as $T \to \infty$

$$lim_{T\to\infty}\frac{\alpha_H^2}{2}\frac{I_T^1}{T} = \eta_H l_1$$

and

$$lim_{T\to\infty}\frac{\alpha_H^2}{2}\frac{I_T^2}{T} = \eta_H l_2.$$

Now, let us estimate I_T^3. We have

$$\begin{aligned}\frac{dI_T^3}{dT} &= 2\left[\int_{[0,T]^3} e^{-\rho(T-s)}e^{-\theta|u-v|}(T-u)^{2H-2}|s-v|^{2H-2}dsdudv\right]\\ &\quad +2\left[\int_{[0,T]^3} e^{-\rho|t-s|}e^{-\theta(T-v)}(T-t)^{2H-2}|s-v|^{2H-2}dsdudv\right]\\ &:= A_T(\rho,\theta) + A_T(\theta,\rho).\end{aligned}$$

Making the change of variables $T-s=x$, $T-u=y$ and $T-v=z$

$$A_T(\rho,\theta) = 2\int_{[0,T]^3} e^{-\rho x}e^{-\theta|y-z|}y^{2H-2}|z-x|^{2H-2}dxdydz.$$

This implies

$$lim_{T\to\infty}A_T(\rho,\theta) = A_\infty(\rho,\theta) = 2\int_{[0,\infty]^3} e^{-\rho x}e^{-\theta|y-z|}y^{2H-2}|z-x|^{2H-2}dxdydz.$$

Making the change of variables $z-x=w$, we obtain

$$\begin{aligned}A_\infty(\rho,\theta) &= 2\left[\int_{[0,\infty)^2}\int_{-x}^{\infty} e^{-\rho x}e^{-\theta|y-w-x|}y^{2H-2}|w|^{2H-2}dwdxdy\right]\\ &= 2\left[\int_{[0,\infty)^2}\int_{-x}^{y-x} e^{-\rho x}e^{-\theta(y-w-x)}y^{2H-2}|w|^{2H-2}dwdxdy\right]\\ &\quad +2\left[\int_{(0,\infty)^2}\int_{y-x}^{\infty} e^{-\rho x}e^{\theta(y-w-x)}y^{2H-2}|w|^{2H-2}dwdxdy\right].\end{aligned}$$

Integrating in x we get

$$\begin{aligned}A_\infty(\rho,\theta) &= \frac{2}{(-\rho+\theta)}\left[\int_0^\infty\int_{-\infty}^{+\infty}(e^{(-\rho+\theta)(y-w)}-e^{(-\rho+\theta)[(-w)\vee 0]})1_{[(y-w)-((-w)\vee 0)]_+}\right.\\ &\quad \left.\times e^{-\theta(y-w)}y^{2H-2}|w|^{2H-2}\quad dwdy\right]\\ &\quad -\frac{2}{(-\rho-\theta)}\left[\int_0^\infty\int_{-\infty}^{+\infty}e^{(-\theta-\rho)[(y-w)\vee 0]}e^{\theta(y-w)}y^{2H-2}|w|^{2H-2}dwdy\right]\\ &= \frac{2}{\theta-\rho}A^1(\rho,\theta)+\frac{2}{\rho+\theta}A^2(\rho,\theta).\end{aligned}$$

Furthermore

$$\begin{aligned}A^1(\rho,\theta) &= \int_0^\infty\int_0^\infty(e^{-\rho(y+w)}-e^{-\rho w}e^{-\theta y})y^{2H-2}w^{2H-2}dwdy\\ &\quad +\int_0^\infty\int_0^y(e^{-\rho(y-w)}-e^{-\theta(y-w)})y^{2H-2}w^{2H-2}dwdy\\ &= \left(\int_0^\infty y^{2H-2}e^{-\rho y}dy\right)^2-\left(\int_0^\infty y^{2H-2}e^{-\theta y}dy\right)\left(\int_0^\infty w^{2H-2}e^{-\rho w}dw\right)\\ &\quad +\int_0^\infty\int_w^\infty(e^{-\rho(y-w)}-e^{-\theta(y-w)})y^{2H-2}w^{2H-2}dydw\\ &= \Gamma(2H-1)^2\rho^{1-2H}\left[\rho^{1-2H}-\theta^{1-2H}\right]\\ &\quad +\int_0^\infty\int_0^\infty(e^{-\rho x}-e^{-\theta x})(x+w)^{2H-2}w^{2H-2}dxdw.\end{aligned}$$

Using $(x+w)^{2H-2}=\frac{1}{\Gamma(2-2H)}\int_0^\infty \xi^{1-2H}e^{-\xi(w+x)}d\xi$, the term A^1 becomes

$$\begin{aligned}A^1(\rho,\theta) &= \Gamma(2H-1)^2\rho^{1-2H}\left[\rho^{1-2H}-\theta^{1-2H}\right]\\ &\quad +\frac{\Gamma(2H-1)}{\Gamma(2-2H)}\int_0^\infty\int_0^\infty \xi^{2-4H}e^{-\xi x}(e^{-\rho x}-e^{-\theta x})(d\xi dx\\ &= \Gamma(2H-1)^2\rho^{1-2H}\left[\rho^{1-2H}-\theta^{1-2H}\right]\\ &\quad +\frac{\Gamma(2H-1)\Gamma(3-4H)\Gamma(4H-2)}{\Gamma(2-2H)}\left[\rho^{2-4H}-\theta^{2-4H}\right].\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
A^2(\rho,\theta) &= \int_0^\infty\int_0^\infty e^{-\rho(y+w)}y^{2H-2}w^{2H-2}dwdy \\
&\quad+\int_0^\infty\int_0^y e^{-\rho(y-w)}y^{2H-2}w^{2H-2}dwdy \\
&\quad+\int_0^\infty\int_y^\infty e^{\theta(y-w)}y^{2H-2}w^{2H-2}dwdy \\
&= \rho^{2-4H}\Gamma(2H-1)^2 \\
&\quad+\frac{\Gamma(2H-1)\Gamma(3-4H)\Gamma(4H-2)}{\Gamma(2-2H)}\left[\rho^{2-4H}+\theta^{2-4H}\right].
\end{aligned}$$

Thus, $A_\infty(\theta,\rho)$ is also obtained.
Consequently

$$\begin{aligned}
lim_{T\to\infty}\alpha_H^2\frac{I_T^3}{T} &= \alpha_H^2(A_\infty(\rho,\theta)+A_\infty(\theta,\rho)) \\
&= 2\eta_H l_3.
\end{aligned}$$

Finally, combining the above convegences we deduce that as $T\to\infty$

$$EG_T^2\longrightarrow(a,b)\ \Gamma\ {}^t(a,b).$$

On the other hand

$$\begin{aligned}
D_sG_T &= \frac{1}{\sqrt{T}}\left(\int_0^s\left(ae^{-\theta(s-t)}+be^{-\rho(s-t)}\right)\delta B_t+\int_s^T\left(ae^{-\theta(t-s)}+be^{-\rho(t-s)}\right)\delta B_t\right) \\
:= &\ \frac{1}{\sqrt{T}}\left(X_s^{a,b}+Y_{s,T}^{a,b}\right).
\end{aligned}$$

Hence

$$\begin{aligned}
\|DG_T\|_{\mathcal{H}}^2 &= \frac{\alpha_H}{T}\int_0^T\int_0^T\left(X_s^{a,b}+Y_{s,T}^{a,b}\right)\left(X_r^{a,b}+Y_{r,T}^{a,b}\right)|r-s|^{2H-2}dsdr \\
&= \frac{\alpha_H}{T}\int_0^T\int_0^T\left(X_s^{a,b}X_r^{a,b}+2X_r^{a,b}Y_{s,T}^{a,b}+Y_{s,T}^{a,b}Y_{r,T}^{a,b}\right)|r-s|^{2H-2}dsdr \\
:= &\ \frac{\alpha_H}{T}\left(A_T^{a,b}+B_T^{a,b}+C_T^{a,b}\right).
\end{aligned}$$

Since $X_s^{a,b}$ belongs to the first Wiener chaos of B^H,

$$E\left(\left|A_T^{a,b}-EA_T^{a,b}\right|^2\right)=2\int_{[0,T]^4}E(X_s^{a,b}X_r^{a,b})E(X_u^{a,b}X_v^{a,b})|u-r|^{2H-2}|v-s|^{2H-2}dsdrdudv.$$

Using similar arguments as in [14, Lemma 5.4 of web-only Appendix],

$$\begin{aligned}
E\left(\left|A_T^{a,b}-EA_T^{a,b}\right|^2\right) &\leqslant c(H,\theta,\rho)\int_{[0,T]^4}|s-r|^{2H-2}|v-u|^{2H-2}|u-r|^{2H-2}|v-s|^{2H-2}dsdrdudv \\
&\leqslant \frac{c(H,\theta,\rho)}{T^{4-8H}}\int_{[0,1]^4}|s-r|^{2H-2}|v-u|^{2H-2}|u-r|^{2H-2}|v-s|^{2H-2}dsdrdudv.
\end{aligned}$$

Using the same argument for $B_T^{a,b}$ and $C_T^{a,b}$ we conclude that

$$\begin{aligned}
& E\left(\left|\|DG_T\|_{\mathcal{H}}^2 - E\|DG_T\|_{\mathcal{H}}^2\right|^2\right) \\
\leqslant\ & \frac{c(H,\theta,\rho)}{T^{6-8H}} \int_{[0,1]^4} |s-r|^{2H-2}|v-u|^{2H-2}|u-r|^{2H-2}|v-s|^{2H-2} dsdrdudv \\
\longrightarrow\ & 0
\end{aligned}$$

as $T \longrightarrow \infty$, because $H < \frac{3}{4}$. This completes the proof of Theorem 22. ■

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