# Almost Sure Exponential Behavior of a Directed Polymer in a Fractional Brownian Environment

Frederi G. Viens*, Tao Zhang†

**Abstract**

This paper studies the asymptotic behavior of a one-dimensional directed polymer in a random medium. The latter is represented by a Gaussian field $B_H$ on $\mathbf{R}_+ \times \mathbf{R}$ with fractional Brownian behavior in time (*Hurst* parameter $H$) and arbitrary function-valued behavior in space. The partition function of such a polymer is

$$u(t) = \mathbf{E}_b \left[ \exp \int_0^t B_H(dr, b_r) \right].$$

Here $b$ is a continuous-time nearest neighbor random walk on $\mathbf{Z}$ with fixed intensity $2\kappa$, defined on a complete probability space $\mathbf{P}_b$ independent of $B_H$. The spatial covariance structure of $B_H$ is assumed to be homogeneous and periodic with period $2\pi$. For $H < \frac{1}{2}$, we prove existence and positivity of the Lyapunov exponent defined as the almost sure limit $\lim_{t\to\infty} t^{-1} \log u(t)$. For $H > \frac{1}{2}$, we prove that the upper and lower almost sure limits $\limsup_{t\to\infty} t^{-2H} \log u(t)$ and $\liminf_{t\to\infty} (t^{-2H} \log t) \log u(t)$ are non-trivial in the sense that they are bounded respectively above and below by finite, strictly positive constants. Thus, as $H$ passes through $1/2$, the exponential behavior of $u(t)$ changes abruptly. This can be considered as a phase transition phenomenon. Novel tools used in this paper include sub-Gaussian concentration theory via the Malliavin calculus, detailed analyses of the long-range memory of fractional Brownian motion, and an almost-superadditivity property.

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## 1 Introduction

### 1.1 The model

This article is concerned with a one-dimensional directed polymer in a fractional Brownian-type random environment in $\mathbf{R}$. Such a model can be described as follows. Initially, in the absence of any random medium, the polymer itself is modeled as a standard random walk $b = \{b_t : t \geq 0\}$, defined on a complete filtered probability space $\left(\Omega_b, \mathcal{F}^b, \left(\mathcal{F}_t^b\right)_{t\geq 0}, \left(\mathbf{P}_b^x\right)_{x\in\mathbf{R}}\right)$, where $\mathbf{P}_b^x$ stands for the law of the simple (nearest-neighbor) symmetric random walk on $\mathbf{Z}$ indexed by $t \in \mathbf{R}_+$, starting

*Dept. Statistics, Purdue University, W. Lafayette, IN 47907; viens@purdue.edu; **corresponding author**

†Dept. Mathematics, Purdue University, W. Lafayette, IN 47907; tzhang@math.purdue.edu

from the initial condition $x$. The corresponding expected value is denoted by $\mathbf{E}_b^x$, or simply by $\mathbf{E}_b$ when $x=0$.

The random environment is represented by a Gaussian field $B_H$ indexed on $\mathbf{R}_+ \times \mathbf{R}$, defined on another independent complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Denote by $\mathbf{E}$ the expected value with respect to $\mathbf{P}$. The covariance structure of $B_H$ is given by

$$\mathbf{E}\left[B_H(t,x)\, B_H(s,y)\right] = R_H(s,t)\, Q(x-y)\,, \tag{1}$$

where

$$R_H(s,t) = 2H \int_0^{s\wedge t} (t-r)^{H-\frac{1}{2}} (s-r)^{H-\frac{1}{2}}\, dr,$$

and $Q : \mathbf{R} \to \mathbf{R}$ is a given homogeneous covariance function satisfying some regularity conditions that will be specified later on. In particular, for every fixed $t \in \mathbf{R}$, the process $x \to t^{-H} B_H(t,x)$ is a homogeneous centered Gaussian field on $\mathbf{R}$ with covariance function $Q$. For fixed $x \in \mathbf{R}$, the process $t \to [Q(0)]^{-1/2} B_H(t,x)$ is a so-called Riemann–Liouville fractional Brownian motion with Hurst parameter $H$. We refer the reader to the Appendix (Section 7) for properties of this process, particularly Subsection 7.1 for definitions, and Subsection 7.3 for relations to the standard fractional Brownian motion. Henceforth we refer to the Riemann-Liouville fBm simply as fractional Brownian motion (fBm) with Hurst parameter $H$. The two versions of fBm have very similar properties: see [3], [4], [20], or Subsection 7.3. The reason for using the Riemann-Liouville version of fBm as opposed to standard fBm is to simplify some calculations; our results hold identically in the standard fBm case, but the calculation are denser, and we decided to avoid presenting these for the sake of clarity.

Once $b$ and $B_H$ are defined, we can define the polymer measure in the following way: for any $t>0$, the energy of a given *path* (or *configuration*) $b$ on $[0,t]$ is given via the Hamiltonian

$$H_t^x(b) = -\int_0^t B_H(dr, b_r + x)\,.$$

The completely rigorous meaning for this integral can be found in the next section. Notice that for any fixed path $b$, $H_t^x(b)$ is a centered Gaussian random variable. Based on this Hamiltonian, for any fixed $x \in \mathbf{R}$ and a given constant $\beta$ (interpreted as the inverse temperature of the system) we can define our random polymer measure as the Gibbs measure

$$d\hat{\mathbf{P}}_t^x(b) = \frac{e^{-\beta H_t^x}(b)}{u(t,x)}\, d\mathbf{P}_b^x(b)\,, \tag{2}$$

with

$$u(t,x) = \mathbf{E}_b^x\left[e^{-\beta H_t^x(b)}\right]. \tag{3}$$

The function $u(t,x)$ is referred to as the *partition function*. It obviously ensures that the polymer measure is a probability measure. It plays an important role in understanding the entire measure. This Gibbs measure, and its partition function, are random, as they depend on the randomness of $B_H$. In the nomenclature of mathematical physics, statements about the law of the configuration $b$ formulated using averages with respect to $\mathbf{P}$ (with respect to the randomness of $B_H$) are *annealed* statements, while statements formulated almost surely with respect to $\mathbf{P}$ are *quenched* statements. In this article, we are concerned primarily with quenched results, and more specifically with the almost-sure exponential rate of growth for large time of the partition function $u$ given in (3). To dispel any possible confusion, we note here that the phrase "almost surely" systematically denotes statements that hold with $\mathbf{P}$-probability 1.

## 1.2 The problem, and related works

When the Hurst parameter $H$ in the model is equal to $1/2$, the polymer's random environment is Brownian in time: its time-differential is white noise. This type of model has been studied extensively. [2] and [8] established the links between martingale theory and directed polymers in Brownian random environment, and over the last few years, several papers have studied different types of polymer models: the case of random walks in discrete potential is treated in [10], the case of Gaussian random walks in [19], [24], and the case of Brownian polymer in a Poisson potential is considered in [15]. Recently, [7] studied the wandering (superdiffusive) exponent for the continuous space Brownian polymer in a Brownian environment; its partition function was studied extensively in the recent works [18] and [26], while further work for small temperature is being investigated in [9].

The model $u$ in (3) is also, up to a time reversal, equal to the so-called *stochastic Anderson* model, which is the solution of a linear multiplicative stochastic heat equation driven by the random environment $B^H$ as its potential. The time-white noise case $H = 1/2$ has been a highly popular model for quite some time, introduced by the Russian mathematical physics school as a non-trivial basic model for more complex problems (see the review paper [21]). Its large-time asymptotics were first studied in discrete space $\mathbf{Z}^d$ in [12]. Properties of these discrete and continuous-space models were further investigated in a number of articles since then. We refer to the sharpest results know to date in continuous space in [18], and references therein.

When $H$ is any number in $(0, 1)$ other than $1/2$, the time-covariance structure of the random environment becomes non-trivial: instead of independent increments, we have long-range dependence (medium or long memory, in the language of time series, when $H \in (0, 1/2)$ and $H \in (1/2, 1)$ respectively) due to the fractional Brownian behavior. The resulting polymer model is more complicated. To the best of our knowledge no work has been devoted to it. One reason which is typically quoted for such lack of study is that fBm is neither a martingale nor a Markov process, making the standard artillery of probabilistic tools inapplicable. However, in modeling terms, the case of independent time-increments ($H = 1/2$) can only be considered as idealized. Real data typically exhibits correlations. This is becoming increasingly clear in such areas as financial econometrics and communications networks, where medium and long memory data seem to be the norm. These cases, which contrast sharply with the case of independent increments, are thus a good place to start investigating correlations for polymers and Anderson models. One point deserves clarification: the issue of spatial correlations has already been well understood (see [18]); our emphasis here is to introduce time correlations for the first time.

In this article we study the almost-sure large-time exponential behavior of the random Gibbs measure $u$'s partition function when $H \neq 1/2$. Because our main thrust is to show that the difficulties inherent in the random medium's fBm behavior can be overcome, we consider a situation which is otherwise relatively simple, while still using an infinite-dimensional noise term, to obtain non-trivial results, and in particular ones which do not coincide with the case $H = 1/2$; in particular we will prove that a clear phase transition occurs as $H$ passes through the value $1/2$ (see detailed description of results in the next subsection). We assume the inverse temperature $\beta = 1$ and the continuous-time nearest-neighbor random walk $b$ on $\mathbf{Z}$ has intensity $2\kappa$, where for some results ($H < 1/2$) the diffusion constant $\kappa$ will need to be small. Moreover, we require in our model that the homogeneous covariance function $Q$ in (1) be periodic with period $2\pi$. This implies that our model is then identical to one in which $b$ is the continuous-time nearest-neighbor random walk with unit step size, restricted to the unit circle, where the point on the unit circle is identified with its angle.

This model, as described above, has the interesting feature that, since $2\pi \notin \mathbf{Q}$, the random walk will visit infinitely many points on the unit circle. In this situation, the smoothness of $Q$ will play a visible role in some of our results, despite the fact our polymer steps only discretely in space: some of our proofs essentially require that $Q$ be twice differentiable, which is equivalent to requiring $B^H$ to be spatially differentiable almost surely (Assumption 1 on page 10).

All of our results also hold if we modify the random walk $b$ by changing its step size to a rational fraction of $2\pi$; in that case, it visits only finitely many points on the circle, making it unnecessary to define the medium's spatial covariance $Q$ on more than this finite set; the smoothness assumption on $Q$ can be achieved automatically by interpolation outside of this finite set. We do not comment on this point further.

The periodicity of $Q$ was chosen to ensure that the polymer stays in effect in a bounded domain. The size of this domain does not play a role in our results; they would remain true in the case of a circle with arbitrarily large radius, and are easily extended to this case; we do not comment on this point further herein.

According to (3), since the covariance function $Q$ is homogeneous, it follows that for every $x \in \mathbf{Z}$, $u(t,x)$ is identical to $u(t,0)$ in distribution. Because of this fact we will only need to consider the partition function

$$u(t) := u(t,0) = \mathbf{E}_b \left[ \exp \int_0^t B_H(dr, b_r) \right]. \tag{4}$$

Our object is to study the existence of the almost-sure limit of $\frac{1}{t} \log u(t)$ when $t \to \infty$ and $t \in \mathbf{N}$. We restrict $t$ to being an integer in order to apply Borel-Cantelli-type arguments easily. The proper notation for limits as $t$ tends to infinity is thus $\lim_{t\to\infty, t\in\mathbf{N}}$. In many cases, we will omit the notation $t \in \mathbf{N}$, writing only $\lim_{t\to\infty}$.

When the limit of $\frac{1}{t} \log u(t)$ exists and is finite, we will show it is positive. When the limit is infinite, we will investigate the proper scale needed to recuperate a finite positive limit instead. The former situation relates to $H < 1/2$, and is quantitatively similar to the case $H = 1/2$ which has been studied extensively in the aforementioned references, although the proofs require new concepts and tools. The latter case is when $H > 1/2$, and provides us with entirely new quantitative behaviors, including a clear phase transition when $H$ passes through the value $1/2$.

### 1.3 Structure of the article and summary of results

After some preliminaries and tools presented in Sections 2 and 3, we begin our study by looking at properties of the expectation of $\log u(t)$, denoted by $U(t)$. Section 4 shows that under the assumption that $\frac{\partial}{\partial x} B_H(t,x)$ exists almost surely for any fixed $t$ and $x$, $U(t)$ is almost superadditive when $H \in (0,1)$, a property defined and studied in that section. When $H = \frac{1}{2}$, this property of almost superadditivity becomes the property of superadditivity, which had been studied recently in [18] and [26].

Section 5 studies the case of $H < \frac{1}{2}$. In Subsection 5.1 it is shown that $U(t)$ grows at most linearly, that is, $\left\{t^{-1} U(t)\right\}_{t\in\mathbf{N}}$ is bounded. This property, together with the almost superadditivity, gives the existence, finiteness, and nonnegativity of $\lim_{t\to\infty} t^{-1} U(t)$. Subsection 5.3 connects $\log u(t)$ and $U(t)$ via a concentration theory, which implies that

$$\lim_{t\to\infty} \left( \frac{1}{t} \log u(t) - \frac{1}{t} U(t) \right) = 0, \quad \text{a.s.} \tag{5}$$

Combining all of these results we obtain that under spatial homogeneity of $B_H$,

$$\lambda := \lim_{t\to\infty} \frac{1}{t} \log u(t)$$

exists almost surely and $\lambda$ is a deterministic, finite, non-negative real number. This is called the almost sure Lyapunov exponent of $u$. In Subsection 5.2, positivity of $\lambda$ is obtained when $H \in (H_0, 1/2]$ and $\kappa \le \kappa_0$, where $H_0, \kappa_0$ are values depending only on $Q$, and assuming that $Q$ is not identically constant (for instance, the case of discrete spatial white noise, i.e. $\{B^H(\cdot, x)\}_x$ IID, is covered, since it is $Q(0) > 0$ and $Q(x) = 0$ for all $x \neq 0$).

Section 6 deals with the case of $H > \frac{1}{2}$. In this case, $\{t^{-1}U(t)\}_{t\in\mathbf{N}}$ is unbounded, which indicates that $t^{-1}\log u(t)$ blows up as well. Therefore we try to find a deterministic function $L(t)$ such that $\lim_{t\to\infty} L(t)^{-1} \log u(t)$ exists almost surely. If such a function $L$ can be found so that this limit is finite and non-zero, we call this $L$ the *exponential rate function* of $u(t)$. Subsection 6.1 gives the concentration result (5) which also holds when $H > \frac{1}{2}$, but for slightly different reasons than when $H < \frac{1}{2}$. Subsection 6.2 shows that $\{t^{-2H}U(t)\}_{t\in\mathbf{N}}$ is bounded. This, plus the concentration result, gives that

$$\limsup_{t\to\infty} \frac{1}{t^{2H}} \log u(t) \le \lambda^*, \quad \text{a.s.}$$

for some deterministic, finite, positive real number $\lambda^*$. In Subsection 6.3, we perform a detailed analysis of the Hamiltonian's covariance structure, and combine it with time discretization techniques similar to those used in [14] and [29] to get the lower bound on $L(t)$. We obtain that

$$\liminf_{t\to\infty} \left(\frac{t^{2H}}{\log t}\right)^{-1} \log u(t) \ge \lambda_*, \quad \text{a.s.}$$

for some deterministic, finite, positive real number $\lambda_*$. In particular we get lower and upper bounds on $L(t)$, if it exists, when $H > 1/2$:

$$\frac{t^{2H}}{\log t} \le L(t) \le t^{2H}.$$

We can summarize these results as follows. Let

$$\alpha := \lim_{t\to\infty} \frac{\log\log u(t)}{\log t}.$$

There exist non-random constants $H_0 \in (0, 1/2)$ and $\kappa_0 > 0$ such that

1) when $H \in (H_0, \frac{1}{2}]$ and $\kappa \in (0, \kappa_0]$, $\alpha = 1$;

2) when $H \in (1/2, 1)$ for all $\kappa > 0$, $\alpha = 2H$.

It is notable that when $H$ passes through $\frac{1}{2}$ there is a phase transition for the order of the exponential rate. When $H \le 1/2$, the partition function has a Lyapunov exponent, just like in the case $H = 1/2$, i.e. $\log u(t)$ is almost surely asymptotically linear; when $H > 1/2$, the Lyapunov exponent is infinite, and the correct rate of increase of $\log u(t)$ seems to be closest to $t^{2H}$.

## 2 Preliminary calculations

In this section we give the precise meaning of the partition function in (4). Since the covariance function $Q$ is homogeneous and periodic with period of $2\pi$, we have a random Fourier series representation for $B_H(t,x)$: there exists $\{q_k\}_{k\in\mathbf{Z}}$ a sequence of non-negative real numbers such that $q_{-k}=q_k$ and

$$Q(x)=\sum_{k=-\infty}^{\infty} q_k e^{ikx},$$

and the Gaussian field $B_H$ can be written as a random Fourier series

$$B_H(t,x)=\sum_{k=-\infty}^{\infty}\sqrt{q_k}\; e^{ikx}B_{H,k}(t)$$

where all $B_{H,k}$'s, $k\in\mathbf{Z}_+$, are *i.i.d.* complex-valued fBm's with a common Hurst parameter $H$, and $B_{H,-k}=\overline{B_{H,k}}$. This last condition ensures that $B_H(t,x)$ is real-valued. The integral in (4) can hence be written as

$$\int_0^t B_H(dr,b_r)=\sum_{k=-\infty}^{\infty}\sqrt{q_k}\int_0^t e^{ikb_r}B_{H,k}(dr).$$

The Wiener integral with respect to fBm is discussed in Subsections 7.4 and 7.5 in the Appendix. It follows that there exist *i.i.d.* standard complex-valued Wiener processes $W_k$, $k\geq 0$ such that, with $K_H^*$ the standard transfer operator for our fBm (see its definition (54) in Subsection 7.4 of the Appendix), for a fixed nearest neighbor path $b$,

$$\int_0^t B_H(dr,b_r)=\sum_{k=-\infty}^{\infty}\sqrt{q_k}\int_0^t\left[K_H^* e^{ikb_\cdot}\right](t,r)\,W_k(dr)$$

provided $W_{-k}=\overline{W_k}$. Therefore, the partition function $u(t)$ can be expressed using random Fourier series of Wiener integrals with respect to standard Wiener processes, as

$$u(t)=\mathbf{E}_b\left[\exp\left\{\sum_{k=-\infty}^{\infty}\sqrt{q_k}\int_0^t\left[K_H^* e^{ikb_\cdot}\right](t,r)\,W_k(dr)\right\}\right]. \tag{6}$$

Measurability and integrability of the expression inside the expectation $\mathbf{E}_b$ jointly in $(b,\omega)\in\Omega_b\times\Omega$ is a standard issue that can easily be resolved by $L^2$ approximations: see for instance [14]; we do not comment on this further.

Since $b$ is a continuous-time nearest-neighbor random walk, we can look at the partition function $u(t)$ from another viewpoint to get a discrete representation, by decomposing the average $\mathbf{E}_b$ over the jump times and jump positions of $b$. If a trajectory $b$ is fixed, then between two jump times $t_j$, $t_{j+1}$ of $r\to b_r$, the value of that path is fixed, say at $x_j$, and we see that $\int_{t_j}^{t_{j+1}}B_H(dr,b_r)$ is just the increment $B_H(t_{j+1},x_j)-B_H(t_j,x_j)$. Formula (4) hence becomes

$$u(t)=\mathbf{E}_b\left[\exp X\left(\tilde{t},\tilde{x}\right)\right]$$

if we write $\tilde{t}=(0=t_0<t_1<t_2<\cdots<t_{N_t}<t_{N_t+1}=t)$ and $\tilde{x}=(0=x_0,x_1,x_2,\ldots,\;x_{N_t})$ for the successive times and locations of the jumps of the path $r\to b_r$ and set

$$X\left(\tilde{t},\tilde{x}\right)=\sum_{j=0}^{N_t}\left[B_H(t_{j+1},x_j)-B_H(t_j,x_j)\right]. \tag{7}$$

The number $N_t$ of jumps of the path $r \to b_r$ before time $t$, defines a Poisson process with intensity $2\kappa$ so that:

$$p(t,m) := \mathbf{P}_b(N_t = m) = e^{-2\kappa t}\frac{(2\kappa t)^m}{m!}$$

and given the value of $N_t$, the jump times $t_j$ are uniformly distributed between 0 and $t$, and the location of the jumps, which are independent of the times of jumps, are all equally distributed, that is, each nearest-neighbor path $x_1, x_2, \ldots, x_{N_t}$ has equal probability $2^{-N_t}$. Consequently, the expectation giving the value of $u(t)$ can be written in the discrete form

$$u(t) = \sum_{m=0}^{\infty} p(t,m) \sum_{\tilde{x}\in\mathcal{P}_m} \frac{1}{2^m} \int_{\tilde{t}\in\mathcal{S}(t,m)} \frac{m!}{t^m} e^{X_m(\tilde{t},\tilde{x})} dt_1 \cdots dt_m \tag{8}$$

Here $\mathcal{P}_m$ denotes the set of all nearest neighbor paths of length $m$, while $\mathcal{S}(t,m)$ is the simplex $\left\{\tilde{t} : 0 < t_1 < t_2 < \cdots < t_m < t\right\}$, so that $\frac{m!}{t^m}\, dt_1 \cdots dt_m$ is indeed the uniform distribution on $\mathcal{S}(t,m)$ and $X_m\left(\tilde{t},\tilde{x}\right)$ is the $X\left(\tilde{t},\tilde{x}\right)$ defined in (7) with $N_t = m$.

Furthermore, for each fixed $m$, $X_m\left(\tilde{t},\tilde{x}\right)$ can be written as a random Fourier series:

$$X_m\left(\tilde{t},\tilde{x}\right) := \sum_{k=-\infty}^{\infty} \sqrt{q_k} \sum_{j=0}^{m} \int_{t_j}^{t_{j+1}} f_j^{m,\tilde{t},\tilde{x}}(r)\, W_k(dr) \tag{9}$$

with each $f_j^{m,\tilde{t},\tilde{x}}(r)$, $j = 0, 1, \ldots, m$, being defined by

$$f_j^{m,\tilde{t},\tilde{x}}(r) = \sqrt{2H}\left[e^{ikx_j}(t_{j+1}-r)^{H-\frac{1}{2}} + \sum_{\ell=j+1}^{m} e^{ikx_\ell}\left[(t_{\ell+1}-r)^{H-\frac{1}{2}} - (t_\ell - r)^{H-\frac{1}{2}}\right]\right]. \tag{10}$$

To prove this formula, notice first that each increment in (7) has the random Fourier series representation

$$B_H(t_{j+1},x_j) - B_H(t_j,x_j) = \sum_{k=-\infty}^{\infty} \sqrt{q_k}\, e^{ikx_j}\left[B_{H,k}(t_{j+1}) - B_{H,k}(t_j)\right],$$

From the Wiener integral representation of fBm (52) (in Appendix 7) we know that

$$\begin{aligned}
&B_{H,k}(t_{j+1}) - B_{H,k}(t_j)\\
&= \int_0^{t_{j+1}} \sqrt{2H}\,(t_{j+1}-r)^{H-\frac{1}{2}}\, W_k(dr) - \int_0^{t_j} \sqrt{2H}\,(t_j-r)^{H-\frac{1}{2}}\, W_k(dr).
\end{aligned}$$

Now for each fixed $k$, it follows that

$$\begin{aligned}
&\sum_{j=0}^{m} e^{ikx_j}\left[B_{H,k}(t_{j+1}) - B_{H,k}(t_j)\right]\\
&= \sum_{j=0}^{m} e^{ikx_j}\left[\int_0^{t_{j+1}} \sqrt{2H}\,(t_{j+1}-r)^{H-\frac{1}{2}}\, W_k(dr) - \int_0^{t_j} \sqrt{2H}\,(t_j-r)^{H-\frac{1}{2}}\, W_k(dr)\right]\\
&= \sum_{j=0}^{m} \int_{t_j}^{t_{j+1}} e^{ikx_j}\sqrt{2H}\,(t_{j+1}-r)^{H-\frac{1}{2}}\, W_k(dr)\\
&+ \sum_{j=0}^{m}\sum_{\ell=0}^{j-1} \int_{t_\ell}^{t_{\ell+1}} e^{ikx_j}\sqrt{2H}\left[(t_{j+1}-r)^{H-\frac{1}{2}} - (t_j-r)^{H-\frac{1}{2}}\right] W_k(dr).
\end{aligned}$$

After exchanging the order of summation over $j$ and $\ell$ and then summing over all $k$'s, we obtain (9) and (10). These formula have the merit of decomposing $X_m\left(\tilde{t},\tilde{x}\right)$ into a series of independent noise terms. They also make the time dependence structure appear explicitly in $X_m\left(\tilde{t},\tilde{x}\right)$, since the variance of each independent term is an explicit function of all the jump times of $b$, in contrast to what would hold when $H=1/2$, where the $j$th independent term depends only on the $j$th jump time interval of $b$.

# 3 Two Tools

In this section we introduce two useful tools which will serve in the following sections. The first one (see [1]) is the Dudley entropy upper bound often known as Dudley-Fernique theorem, for expected suprema of Gaussian fields. Let $\{Y_t\}_{t\in T}$ be a separable Gaussian field on an arbitrary index set $T$, endowed with the canonical metric

$$\delta\left(t,s\right)=\sqrt{\mathbf{E}\left[\left(Y_t-Y_s\right)^2\right]}.$$

**Theorem 3.1 (Dudley-Fernique)** *There exists a universal constant $K_{univ}>0$ such that*

$$\mathbf{E}\left[\sup_{t\in T}Y_t\right]\leq K_{univ}\int_0^{\infty}\sqrt{N\left(\varepsilon\right)}\,d\varepsilon \tag{11}$$

*where $N\left(\varepsilon\right)$ is the* metric entropy *of $\left(T,\delta\right)$, i.e. the smallest number of balls of radius $\varepsilon$ in the canonical metric $\delta$ required to cover the set $T$.*

The second tool is concerned with Malliavin derivatives. Let $M$ be a white-noise measure indexed on $\mathbf{R}_+\times\mathbf{R}$, on a complete probability space $\left(\Omega,\mathcal{F},\mathbf{P}\right)$ where $\mathcal{F}$ is the $\sigma$-field generated by $M$, endowing $\mathbf{R}_+$ with the interpretation of a positive time axis. More precisely, $M$ is a $\sigma$-additive Gaussian random measure in the sense of $L^2\left(\Omega\right)$, defined by saying that for any Borel sets $A\in\mathcal{B}\left(\mathbf{R}_+\right)$ and $B\in\mathcal{B}\left(\mathbf{R}\right)$, $M\left(A\times B\right)$ is Gaussian random variable $\mathcal{N}\left(0,\left|A\right|\mu\left(B\right)\right)$ where $\left|\cdot\right|$ is the Lebesgue measure and $\mu$ is a $\sigma$-finite measure on $\mathbf{R}$, and moreover if $A\times B\cap A'\times B'=\emptyset$ then $M\left(A\times B\right)$ and $M\left(A'\times B'\right)$ are independent. The filtration generated by $M$ is the sequence $\{\mathcal{F}_t\}_{t\geq 0}$ defined by setting $\mathcal{F}_t$ to be the $\sigma$-field generated by all random variables $M\left(\left[0,s\right]\times B\right)$ where $s\leq t$ and $B\in\mathcal{B}\left(\mathbf{R}\right)$. For a random variable $F$ in the space $L^2\left(\Omega,\mathcal{F},\mathbf{P}\right)$ of all square-integrable $\mathcal{F}_\infty$-random variables, its Malliavin derivative $DF$ with respect to $M$, if it exists, is a random field on $\mathbf{R}_+\times\mathbf{R}$ in accordance with the usual definitions from the theory of abstract Wiener spaces. The domain of Malliavin derivative $D$ is defined as $\mathbf{D}^{1,2}$ (meaning that $DF\in L^2\left(\mathbf{R}_+\times\mathbf{R}\times\Omega\right)$). One may consult Chapter 1 in [22] for details. For our purpose, it is sufficient to notice the following two important properties of the operator $D$.

1. Let $f$ be a non-random function in $L^2\left(\mathbf{R}_+\times\mathbf{R},ds\times\mu\left(dx\right)\right)$ and define

   $$F=\int_{\mathbf{R}_+\times\mathbf{R}}f\left(s,x\right)M\left(ds,dx\right).$$

   Let $g$ be a function in $C^1\left(\mathbf{R}\right)$ and $g'$ the usual derivative of $g$. The random variable $G=g\left(F\right)$ has the Malliavin derivative given by

   $$D_{s,x}G=g'\left(F\right)f\left(s,x\right)$$

   provided that $g\left(F\right),g'\left(F\right)\in L^2\left(\Omega,\mathcal{F},\mathbf{P}\right)$. In particular, $D_{s,x}F=f\left(s,x\right)$.

2. If $G$ has a Malliavin derivative and $G$ is $\mathcal{F}_t$-measurable for some $t \geq 0$, then for all $x \in \mathbf{R}$ and all $s > t$ we have $D_{s,x} G = 0$.

Notice that if $G \in \mathbf{D}^{1,2}$, then $DG \in L^2(\Omega \times \mathbf{R}_+ \times \mathbf{R}, \mathbf{P} \times ds \times \mu(dx))$, and we see immediately that $G$ described in the first property above is indeed in $\mathbf{D}^{1,2}$.

The following result estimates the centered moments of a random variable by using its Malliavin derivative. We refer to a convenient place to find its statement and proof.

**Lemma 3.2 (Lemma 10 in [18])** *Let $G$ be a centered random variable in the space $L^2(\Omega, \mathcal{F}, \mathbf{P})$. Assume $G \in \mathbf{D}^{1,2}$ and $G$ is $\mathcal{F}_t$–measurable. Then for every integer $p \geq 0$, there exists a constant $C_p$ which depends only on $p$ such that*

$$\mathbf{E}\left[G^{2p}\right] \leq C_p \left\{ \mathbf{E}\left[ \int_{\mathbf{R}} \mu(dx) \int_0^t \left(\mathbf{E}\left[D_{s,x} G \,|\mathcal{F}_s\right]\right)^2 ds \right] \right\}^p . \tag{12}$$

**Remark 3.3** *In particular, if $M$ is indexed on $\mathbf{R}_+ \times \mathbf{Z}$ and $\mu$ is the uniform unit mass measure, i.e., $\mu(k) = 1$ for all $k \in \mathbf{Z}$, then (12) becomes*

$$\mathbf{E}\left[G^{2p}\right] \leq C_p \left\{ \mathbf{E}\left[ \sum_{k=-\infty}^{\infty} \int_0^t \left(\mathbf{E}\left[D_{s,k} G \,|\mathcal{F}_s\right]\right)^2 ds \right] \right\}^p . \tag{13}$$

# 4 Almost Superadditivity

As denoted in Section 2, $U(t) = \mathbf{E}[\log u(t)]$ is the expectation of $\log u(t)$. In the case of $H = \frac{1}{2}$, where $B_H(\cdot, x)$ is Brownian motion for every fixed $x$, Rovira and Tindel [26] in the homogeneous case, and more generally Florescu and Viens [18] show that $\{U(t)\}_{t \in \mathbf{N}}$ is superadditive. However, when $H \neq \frac{1}{2}$, this property does not hold; instead, it turns out that the sequence $\{U(t)\}_{t \in \mathbf{N}}$ is a so-called *almost superadditive sequence*. In this section, we establish this important property and its basic consequences.

## 4.1 Almost Superadditivity and Convergence

We first give an important result about almost superadditive sequences of numbers.

**Definition 4.1** *If $\{f(n)\}_{n \in \mathbf{N}}$ is a sequence of real numbers, and $\{\epsilon(n)\}_{n \in \mathbf{N}}$ a sequence of non-negative numbers, such that*

$$f(m+n) \geq f(m) + f(n) - \epsilon(m+n)$$

*for any $m, n \in \mathbf{N}$, then we say $\{f(n)\}_{n \in \mathbf{N}}$ is an almost superadditive sequence relative to $\{\epsilon(n)\}_{n \in \mathbf{N}}$.*

**Remark 4.2** *In the above definition, if $\epsilon(n) = 0$ identically, then $\{f(n)\}_{n \in \mathbf{N}}$ is a superadditive sequence in the usual sense.*

It is well known that if $\{f(n)\}_{n \in \mathbf{N}}$ is a superadditive sequence, then the sequence $\{f(n)/n\}_{n \in \mathbf{N}}$ either converges to its supremum (if it is finite) or diverges properly to $+\infty$. For almost superadditive sequences, convergence to a supremum does not hold in general, but we do have the following analogous result.

**Theorem 4.3** *Let $\{f(n)\}_{n\in\mathbf{N}}$ be an almost superadditive sequence relative to $\{\epsilon(n)\}_{n\in\mathbf{N}}$ and furthermore, assume that*

$$i)\ \lim_{n\to\infty}\frac{\epsilon(n)}{n}=0;\quad ii)\ \sum_{n=1}^{\infty}\frac{\epsilon(2^n)}{2^n}<\infty.$$

*(1) If $\sup_n \frac{f(n)}{n}<\infty$, then $\lim_{n\to\infty}\frac{f(n)}{n}$ exists;*

*(2) If $\sup_n \frac{f(n)}{n}=\infty$, then $\left\{\frac{f(n)}{n}\right\}$ diverges properly to $\infty$.*

The proof of this theorem is provided in the Appendix, Subsection 7.6, for completeness.

## 4.2 Almost Superadditivity of $U(t)$

It is trivial to see that $U(t)\geq 0$ for all $t$. Indeed this can be shown by using Jensen's inequality and Fubini theorem:

$$U(t)=\mathbf{E}[\log u(t)]=\mathbf{E}\left[\log\left(\mathbf{E}_b\left[\exp\int_0^t B_H(dr,b_r)\right]\right)\right]\geq \mathbf{E}_b\mathbf{E}\left[\int_0^t B_H(dr,b_r)\right]=0.$$

For $H\neq 1/2$, we need some spatial regularity of $B_H(t,x)$:

**Assumption 1 (Spatial Regularity)** *For any fixed $t$ and $x$, $\frac{\partial B_H}{\partial x}(t,x)$ exists almost surely.*

**Remark 4.4** *Assumption 1 is equivalent to $\sum_{k=-\infty}^{\infty}k^2 q_k<\infty$. For convenience we denote*

$$Q_1:=\left(\sum_{k=-\infty}^{\infty}k^2 q_k\right)^{\frac{1}{2}}.$$

Now we give the main result of this section:

**Theorem 4.5** *Under Assumption 1, for each $H\in(0,1)$, there exists a positive constant $\tilde{C}_{Q,H}$, depending only on $Q$ and $H$, such that*

$$U(t+s)\geq U(t)+U(s)-\tilde{C}_{Q,H}(s\vee t)^H. \tag{14}$$

As a prelude to the proof, we recall the probability measure defined by the polymer's law, with a slightly more explicit notation than in (2) to emphasize the fact that its randomness depends on that of $B_H$.

**Definition 4.6** *$\hat{\mathbf{P}}_{b,B_H,t}$ is a random probability measure on the same space as $\mathbf{P}_b$ such that*

$$\hat{\mathbf{P}}_{b,B_H,t}[A]=\mathbf{E}_b\left[\frac{\exp\int_0^t B_H(dr,b_r)}{u(t)}\mathbf{1}_A\right]. \tag{15}$$

*Denote by $\hat{\mathbf{E}}_{b,B_H,t}$ the expected value with respect to $\hat{\mathbf{P}}_{b,B_H,t}$.*

**Proof. [Proof of Theorem 4.5]**

*Step 1. Setup and Strategy.* Let $s, t$ be fixed. Without loss of generality, assume $t \leq s$. Using the probability measure $\hat{\mathbf{P}}_{b,B_H,t}$ in (15), we have

$$\begin{aligned} &\log u(t+s) - \log u(t) \\ =&\log \mathbf{E}_b \left[ \frac{\exp \int_0^t B_H(dr, b_r)}{u(t)} \mathbf{E}_b \left\{ \exp \int_t^{t+s} B_H(dr, b_r - b_t + b_t) \middle| b_t \right\} \right] \\ =&\log \hat{\mathbf{E}}_{b,B_H,t} \left[ \mathbf{E}_b \left\{ \exp \int_t^{t+s} B_H(dr, b_r - b_t + y) \middle| b_t = y \right\} \right]. \end{aligned} \tag{16}$$

When $y$ is fixed, from the representation of the Wiener integral it follows that

$$\begin{aligned} &\int_t^{t+s} B_H(dr, b_r - b_t + y) \\ &= \sum_{k=-\infty}^{\infty} \sqrt{q_k} \int_0^{t+s} \left[ K_H^* e^{ik(b_\cdot - b_t + y)} \right] (t+s, r) W_k(dr) \\ &- \sum_{k=-\infty}^{\infty} \sqrt{q_k} \int_0^{t} \left[ K_H^* e^{ik(b_\cdot - b_t + y)} \right] (t, r) W_k(dr) \\ &= \sum_{k=-\infty}^{\infty} \sqrt{q_k} \int_t^{t+s} \left[ K_H^* e^{ik(b_\cdot - b_t + y)} \right] (t+s, r) W_k(dr) \\ &+ \sum_{k=-\infty}^{\infty} \sqrt{q_k} \int_0^{t} \left\{ \left[ K_H^* e^{ik(b_\cdot - b_t + y)} \right] (t+s, r) - \left[ K_H^* e^{ik(b_\cdot - b_t + y)} \right] (t, r) \right\} W_k(dr) \\ &=: Y_1(y) + Y_2(y). \end{aligned} \tag{17}$$

We will investigate the properties of $Y_1(y)$ and $Y_2(y)$. Let $b'$ be the process defined by $b'_r = b_{r+t} - b_t$. It is clear that $b'$ is independent of $b_t$, and identically distributed. The term involving $Y_1$ is similar to $u(s)$, modulo a shift by $t$ into the future for both $b$ and $W$; using the stationarity of the increments of both $b$ and $W$ in time, we will see that the term $U(s)$ can be made to appear using $Y_1$, by injecting (17) into (16). The price to pay for this involves $Y_2$; since $Y_1$ and $Y_2$ are independent under $\mathbf{P}$, it will be sufficient to study $Y_2$ to find this price, which will yield the theorem's "almost" correction. It is also useful to note that $Y_1(y)$ and $Y_2(y)$ are functions of $b'$, and are therefore independent of the path $b$ up to time $t$.

*Step 2. Calculating* $Y_1$. To calculate $Y_1(y)$, we notice that for $r \in [t, t+s]$, if $r' = r - t$, then

$$\begin{aligned} &\left[ K_H^* e^{ik(b_\cdot - b_t + y)} \right] (t+s, r) \\ &= K_H(t+s, r) e^{ik(b_r - b_t + y)} + \int_r^{t+s} \left( e^{ik(b_\tau - b_t + y)} - e^{ik(b_r - b_t + y)} \right) \frac{\partial K_H}{\partial \tau}(\tau, r) \, d\tau \\ &= K_H(s, r') e^{ik\left(b_{r'+t} - b_t + y\right)} + \int_{r'}^{s} \left( e^{ik\left(b_{\tau'+t} - b_t + y\right)} - e^{ik\left(b_{r'+t} - b_t + y\right)} \right) \frac{\partial K_H}{\partial \tau'}(\tau', r') \, d\tau' \\ &= \left[ K_H^* e^{ik(b_{\cdot+t} - b_t + y)} \right] (s, r') \\ &= \left[ K_H^* e^{ik(b'_\cdot + y)} \right] (s, r'). \end{aligned}$$

Define the shifted potential $\theta_t W$ by $\theta_t W(t', x) = W(t + t', x)$ for all $x \in \mathbf{R}$ and $t' \geq 0$. Then we get

$$Y_1(y) = \sum_{k=-\infty}^{\infty} \sqrt{q_k} \int_0^s \left[K_H^* e^{ik(b'_{\cdot}+y)}\right](s, r)\, \theta_t W_k(dr).$$

As we said, $\theta_t W$ has the same distribution as $W$ and $b'$ has the same distribution as $b$. Thus, $\mathbf{E}_{b'}\left[e^{Y_1(y)}\right]$ has the same distribution as $u(s, y)$, and hence as $u(s)$ because the random field $u(s, x)$ is spatially homogeneous.

*Step 3. Estimating* $Y_2$. We now calculate $Y_2(y)$. For $r \in [0, t)$, we can express

$$\begin{aligned}
&\left[K_H^* e^{ik(b_{\cdot}-b_t+y)}\right](t+s, r) - \left[K_H^* e^{ik(b_{\cdot}-b_t+y)}\right](t, r)\\
&= K_H(t+s, r)\, e^{ik(b_r-b_t+y)} + \int_r^{t+s} \left(e^{ik(b_\tau-b_t+y)} - e^{ik(b_r-b_t+y)}\right) \frac{\partial K_H}{\partial \tau}(\tau, r)\, d\tau\\
&- K_H(t, r)\, e^{ik(b_r-b_t+y)} - \int_r^{t} \left(e^{ik(b_\tau-b_t+y)} - e^{ik(b_r-b_t+y)}\right) \frac{\partial K_H}{\partial \tau}(\tau, r)\, d\tau\\
&= K_H(t+s, r)\, e^{ik(b_r-b_t+y)} - K_H(t, r)\, e^{ik(b_r-b_t+y)}\\
&+ \int_t^{t+s} \left(e^{ik(b_\tau-b_t+y)} - e^{ik(b_r-b_t+y)}\right) \frac{\partial K_H}{\partial \tau}(\tau, r)\, d\tau\\
&= \int_t^{t+s} e^{ik(b_\tau-b_t+y)} \frac{\partial K_H}{\partial \tau}(\tau, r)\, d\tau\\
&= \int_t^{t+s} e^{ik\left(b'_{\tau-t}+y\right)} \frac{\partial K_H}{\partial \tau}(\tau, r)\, d\tau.
\end{aligned}$$

Therefore we have

$$Y_2(y) = \sum_{k=-\infty}^{\infty} \sqrt{q_k} \int_0^t \left(\int_t^{t+s} e^{ik\left(b'_{\tau-t}+y\right)} \frac{\partial K_H}{\partial \tau}(\tau, r)\, d\tau\right) W_k(dr).$$

For each fixed path $b$, $\{Y_2(y) : y \in [0, 2\pi]\}$ is a Gaussian field indexed by $[0, 2\pi]$, and is identical to the same field indexed by all of $\mathbf{R}$ because $2\pi$ is the period of the covariance function $Q$. The canonical metric $\delta(y_1, y_2)$ of $Y_2$ is defined by the formula

$$\begin{aligned}
\delta^2(y_1, y_2) &:= \mathbf{E}\left[(Y_2(y_1) - Y_2(y_2))^2\right]\\
&= \sum_{k=-\infty}^{\infty} q_k \int_0^t \left|\int_t^{t+s} \left(e^{ik\left(b'_{\tau-t}+y_1\right)} - e^{ik\left(b'_{\tau-t}+y_2\right)}\right) \frac{\partial K_H}{\partial \tau}(\tau, r)\, d\tau\right|^2 dr \qquad (18)
\end{aligned}$$

We estimate $\delta$ as follows:

$$\begin{aligned}
\delta^2(y_1, y_2) &\leq |y_1 - y_2|^2 \left(\sum_{k=-\infty}^{\infty} k^2 q_k\right) \int_0^t |K_H(t+s-r) - K_H(t-r)|^2\, dr\\
&= |y_1 - y_2|^2 Q_1^2 \int_0^t 2H \left|(t+s-r)^{H-\frac{1}{2}} - (t-r)^{H-\frac{1}{2}}\right|^2 dr\\
&= |y_1 - y_2|^2 Q_1^2 s^{2H} \int_0^{\frac{t}{s}} 2H \left|r^{H-\frac{1}{2}} - (1+r)^{H-\frac{1}{2}}\right|^2 dr\\
&\leq |y_1 - y_2|^2 Q_1^2 s^{2H} \int_0^1 2H \left|r^{H-\frac{1}{2}} - (1+r)^{H-\frac{1}{2}}\right|^2 dr. \qquad (19)
\end{aligned}$$

Therefore we obtain an upper bound on $\delta$

$$\delta\left(y_1, y_2\right) \leq Q_1 L_H s^H \left|y_1 - y_2\right| \tag{20}$$

with

$$L_H := \left(\int_0^1 2H \left|r^{H-\frac{1}{2}} - (1+r)^{H-\frac{1}{2}}\right|^2 dr\right)^{\frac{1}{2}}. \tag{21}$$

Applying the Dudley-Fernique Theorem 3.1, we can estimate a lower bound on $Y_2\left(y\right)$. Let $N\left(\varepsilon\right)$ be the minimum number of $\varepsilon$–balls covering the interval $[0, 2\pi]$ (in the metric $\delta$). According to the above estimate (20) on $\delta$, we can construct an $\varepsilon$–net $V_\varepsilon$ which covers the interval $[0, 2\pi]$ and the number of elements in $V_\varepsilon$ is no more than $\frac{2\pi Q_1 L_H s^H}{\varepsilon} + 1$. Also when $\varepsilon \geq 2\pi Q_1 L_H s^H =: \varepsilon_{\max}$ it is trivial that $N\left(\varepsilon\right) = 1$. We get

$$\begin{aligned}
\mathbf{E}\left[\sup_{y\in[0,2\pi]} \left\{-Y_2\left(y\right)\right\}\right] &\leq K_{univ} \int_0^{\varepsilon_{\max}} \left|\log \frac{2\pi Q_1 L_H s^H}{\varepsilon}\right|^{\frac{1}{2}} d\varepsilon \\
&= 2\pi K_{univ} Q_1 L_H s^H \int_0^\infty 2r^2 e^{-r^2} dr \\
&= \sqrt{\pi^3} K_{univ} Q_1 L_H s^H.
\end{aligned}$$

Here the inequality comes from (11) and the fact that $-Y_2\left(y\right)$ has the same distribution as $Y_2\left(y\right)$; $K_{univ}$ is a positive universal constant. It follows that

$$\mathbf{E}\left[\inf_{y\in[0,2\pi]} Y_2\left(y\right)\right] = -\mathbf{E}\left[\sup_{y\in[0,2\pi]} \left\{-Y_2\left(y\right)\right\}\right] \geq -\sqrt{\pi^3} K_{univ} Q_1 L_H s^H. \tag{22}$$

This is one point where having $b$ be limited to a compact set is crucial. If $b$ were allowed to wander in all of $\mathbf{R}$, the above expectations would be infinite.

*Step 4. Putting the estimates together.* Now go back to (16). As noted before, since $b'$ is independent of $b_t$, thus $Y_1\left(y\right)$ and $Y_2\left(y\right)$ are also independent of $b_t$. Therefore

$$\begin{aligned}
&\mathbf{E}_b\left[\exp\left(Y_1\left(y\right) + \inf_{z\in[0,2\pi]} Y_2\left(z\right)\right)\middle| b_t = y\right] \\
&= \mathbf{E}_{b'}\left[\exp\left(Y_1\left(y\right) + \inf_{z\in[0,2\pi]} Y_2\left(z\right)\right)\right]\Big|_{y=b_t},
\end{aligned}$$

where the notation $\mathbf{E}_{b'}\left[\cdots\right]|_{y=b_t}$ means that first one takes the expectation with respect to $b'$ with $y$ fixed, and then one replaces $y$ by $b_t$. It follows that, to evaluate the last quantity in (16), we can write

$$\begin{aligned}
&\hat{\mathbf{E}}_{b,B_H,t}\left[\mathbf{E}_b\left\{\exp\int_t^{t+s} B_H\left(dr, b_r - b_t + y\right)\middle| b_t = y\right\}\right] \\
=&\hat{\mathbf{E}}_{b,B_H,t}\left[\mathbf{E}_b\left\{\exp\left\{Y_1\left(y\right) + Y_2\left(y\right)\right\}\middle| b_t = y\right\}\right] \\
\geq&\hat{\mathbf{E}}_{b,B_H,t}\left[\mathbf{E}_b\left\{\exp\left\{Y_1\left(y\right) + \inf_{z\in[0,2\pi]} Y_2\left(z\right)\right\}\middle| b_t = y\right\}\right] \\
=&\hat{\mathbf{E}}_{b,B_H,t}\left[\mathbf{E}_{b'}\left[\exp\left\{Y_1\left(y\right) + \inf_{z\in[0,2\pi]} Y_2\left(z\right)\right\}\right]\Big|_{y=b_t}\right] \\
=&\hat{\mathbf{E}}_{b,B_H,t}\left[\mathbf{E}_{b'}\left[e^{Y_1\left(b_t\right)} \exp\left\{\inf_{z\in[0,2\pi]} Y_2\left(z\right)\right\}\right]\right].
\end{aligned} \tag{23}$$

In the iterated expectation notation in the last line above, and below, a shorthand notation is introduced: we replaced $Y_1(y)$ in conjunction with the notation $|_{y=b_t}$, by the more compact notation $Y_1(b_t)$; still, first $b_t$ is considered as a constant, while $\mathbf{E}_{b'}$ is taken, and then it is replaced by $b_t$ before the second expectation is taken.

We introduce another random probability measure $\tilde{\mathbf{P}}_{b,B_H,t}$ on the same space as $\mathbf{P}_b$ by

$$\tilde{\mathbf{P}}_{b,B_H,t}[A] = \hat{\mathbf{E}}_{b,B_H,t}\left[\mathbf{E}_{b'}\left[\frac{e^{Y_1(b_t)}}{\hat{\mathbf{E}}_{b,B_H,t}\mathbf{E}_{b'}\left[e^{Y_1(b_t)}\right]}\mathbf{1}_A\right]\right],$$

and denote by $\tilde{\mathbf{E}}_{b,B_H,t}$ the corresponding expected value; thence we reexpress the last line in (23) as

$$\begin{aligned}&\hat{\mathbf{E}}_{b,B_H,t}\left[\mathbf{E}_{b'}\left[e^{Y_1(b_t)}\exp\left\{\inf_{z\in[0,2\pi]}Y_2(z)\right\}\right]\right]\\ &=\left(\hat{\mathbf{E}}_{b,B_H,t}\left[\mathbf{E}_{b'}\left[e^{Y_1(b_t)}\right]\right]\right)\tilde{\mathbf{E}}_{b,B_H,t}\left[\exp\left\{\inf_{z\in[0,2\pi]}Y_2(z)\right\}\right]. \qquad (24)\end{aligned}$$

Taking (16), (23) and (24) together yields the following estimation:

$$\begin{aligned}&\log u(t+s)-\log u(t)\\ &=\log\hat{\mathbf{E}}_{b,B_H,t}\left[\mathbf{E}_b\left\{\exp\int_t^{t+s}B_H(dr,b_r-b_t+y)\bigg|\,b_t=y\right\}\right]\\ &\geq\log\hat{\mathbf{E}}_{b,B_H,t}\mathbf{E}_{b'}\left[e^{Y_1(b_t)}\exp\left\{\inf_{z\in[0,2\pi]}Y_2(z)\right\}\right]\\ &=\log\hat{\mathbf{E}}_{b,B_H,t}\mathbf{E}_{b'}\left[e^{Y_1(b_t)}\right]+\log\tilde{\mathbf{E}}_{b,B_H,t}\left[\exp\left\{\inf_{z\in[0,2\pi]}Y_2(z)\right\}\right]\\ &\geq\hat{\mathbf{E}}_{b,B_H,t}\left[\log\mathbf{E}_{b'}\left[e^{Y_1(b_t)}\right]\right]+\tilde{\mathbf{E}}_{b,B_H,t}\left[\inf_{z\in[0,2\pi]}Y_2(z)\right].\end{aligned}$$

Taking the expectation with respect to $\mathbf{P}$, we get

$$\begin{aligned}&U(t+s)-U(t)\\ &\geq\mathbf{E}\left[\hat{\mathbf{E}}_{b,B_H,t}\left[\log\mathbf{E}_{b'}\left[e^{Y_1(b_t)}\right]\right]+\tilde{\mathbf{E}}_{b,B_H,t}\left[\inf_{z\in[0,2\pi]}Y_2(z)\right]\right]\\ &=\hat{\mathbf{E}}_{b,B_H,t}\left[\mathbf{E}\left[\log\mathbf{E}_{b'}\left[e^{Y_1(b_t)}\right]\right]\right]+\tilde{\mathbf{E}}_{b,B_H,t}\mathbf{E}\left[\inf_{z\in[0,2\pi]}Y_2(z)\right].\end{aligned}$$

As indicated before, $\mathbf{E}_{b'}\left[e^{Y_1(y)}\right]$ has the same distribution as $u(s)$. Therefore

$$\hat{\mathbf{E}}_{b,B_H,t}\left[\mathbf{E}\left[\log\mathbf{E}_{b'}\left[e^{Y_1(b_t)}\right]\right]\right]=\hat{\mathbf{E}}_{b,B_H,t}[U(s)]=U(s).$$

Combining this with the lower bound in (22) we get

$$U(t+s)-U(t)\geq U(s)-\sqrt{\pi^3}K_{univ}Q_1L_Hs^H.$$

The proof is therefore completed by setting

$$\tilde{C}_{Q,H}=\sqrt{\pi^3}K_{univ}Q_1L_H, \qquad (25)$$

where the three constants $K_{univ}$, $Q_1$, and $L_H$ are given respectively in Theorem 3.1, Assumption 1, and equation (21). ∎

**Remark 4.7** *It follows from Definition 4.1 and Theorem 4.5 that* $\{U(t)\}_{t\in\mathbf{N}}$ *is an almost superadditive sequence relative to* $\{\epsilon(t)\}$ *where*

$$\epsilon(t) = \tilde{C}_{Q,H} t^H. \tag{26}$$

**Remark 4.8** *From (21), when* $H \to \frac{1}{2}$, *we can see that* $L_H \to 0$ *and therefore* $\tilde{C}_{Q,H} \to 0$. *That* $\tilde{C}_{Q,H} = 0$ *for* $H = 1/2$ *coincides with the fact that* $\{U(t)\}$ *is a superadditive sequence when* $H = 1/2$, *where the polymer is in a Brownian environment.*

For $\epsilon(t)$ given in (26), it is not hard to see that $\lim_{t\to\infty} t^{-1}\epsilon(t) = 0$; and furthermore,

$$\sum_{n=1}^{\infty} \frac{\epsilon(2^n)}{2^n} = \tilde{C}_{Q,H} \sum_{n=1}^{\infty} 2^{n(H-1)} = \tilde{C}_{Q,H} \left( \frac{2^{H-1}}{1-2^{H-1}} \right) = \frac{\tilde{C}_{Q,H}}{2^{1-H}-1} < \infty.$$

Therefore, by Theorem 4.3, if $\sup_{t\in\mathbf{N}} t^{-1}U(t) < \infty$, then $\lim_{t\to\infty} t^{-1}U(t)$ exists and is finite; otherwise, $\lim_{t\to\infty} t^{-1}U(t)$ diverges properly to $\infty$.

We will see that when $H < \frac{1}{2}$, $\left\{t^{-1}U(t)\right\}_{t\in\mathbf{N}}$ is in fact bounded and thus converges, while when $H > \frac{1}{2}$, the story is quite different: $\left\{t^{-1}U(t)\right\}_{t\in\mathbf{N}}$ diverges to $\infty$ properly. We now study each case separately.

# 5 Exponential Behavior when $H < \frac{1}{2}$

In this section, we study the exponential behavior of $u(t)$ when $H < \frac{1}{2}$. We prove that under certain conditions, the Lyapunov exponent of $u(t)$ exists almost surely and is strictly positive.

**Assumption 2**

$$C^*_{Q,H} := \frac{\sqrt{Q(0) - Q(2)}}{(H+1)\sqrt{\pi}} - \frac{\sqrt{\pi^3} K_{univ} Q_1 L_H}{2^{1-H}-1} > 0. \tag{27}$$

**Remark 5.1** $\sqrt{\pi^3} K_{univ} Q_1 L_H$ *is the constant* $\tilde{C}_{Q,H}$ *in Theorem 4.5. Recall that* $K_{univ}$, $Q_1$, *and* $L_H$ *are given respectively in Theorem 3.1, Assumption 1, and equation (21).*

**Remark 5.2** *As* $H \to 1/2$, *(21) shows that* $L_H \to 0$, *which then means that there exists some* $H_0 < 1/2$ *such that when* $H > H_0$, *Assumption 2 is equivalent to the nondegeneracy assumption*

$$Q(0) > Q(2). \tag{28}$$

**Remark 5.3** *Condition (28), which can easily be weakened to* $Q(0) > Q(x)$ *for some* $x \neq 0$, *simply means that the random field* $B^H(t,x)$ *is not identically constant in* $x$. *Condition (28) is thus satisfied for all non-trivial potentials* $B^H$, *including the case of spatial white noise (*$B^H(t,x)$ *and* $B^H(s,y)$ *independent for all* $x \neq y$, *i.e.* $Q(0) > 0$ *and* $Q(x) = 0$ *for all* $x \neq 0$*).*

The main result established in this section is the following.

**Theorem 5.4** *Let* $H \in \left(0, \frac{1}{2}\right)$ *and assume Assumptions 1 and 2 are satisfied. Then there exists* $\kappa_0 > 0$ *such that for any* $\kappa < \kappa_0$,

$$\lambda := \lim_{t\to\infty, t\in\mathbf{N}} \frac{1}{t} \log u(t)$$

*exists almost surely, is deterministic, finite, and positive.*

### 5.1 Sublinear Growth of $U(t)$

Recall from the preliminary notation and calculations in Section 2 that we have a summation expression for $u(t)$ given by

$$u(t) = \sum_{m=0}^{\infty} p(t,m) \sum_{\tilde{x} \in \mathcal{P}_m} \frac{1}{2^m} \int_{\tilde{t} \in \mathcal{S}(t,m)} \frac{m!}{t^m} e^{X_m(\tilde{t},\tilde{x})} dt_1 \cdots dt_m, \tag{29}$$

where

$$X_m\left(\tilde{t},\tilde{x}\right) := \sum_{k=-\infty}^{\infty} \sqrt{q_k} \sum_{j=0}^{m} \int_{t_j}^{t_{j+1}} f_j^{m,\tilde{t},\tilde{x}}(s)\, W_k(ds)$$

with each $f_j^{m,\tilde{t},\tilde{x}}(s)$, $j = 0, 1, \ldots, m$, being defined in (10).

**Lemma 5.5** *For each fixed $m$ and each fixed path $b = \left(\tilde{t},\tilde{x}\right)$, the variance of $X_m\left(\tilde{t},\tilde{x}\right)$ is bounded from above by*

$$\mathbf{E}\left[X_m^2\left(\tilde{t},\tilde{x}\right)\right] \le 4Q(0)\,(m+1)^{1-2H}\, t^{2H}. \tag{30}$$

**Proof.** For each fixed $m$, $\tilde{t}$ and $\tilde{x}$, $X_m\left(\tilde{t},\tilde{x}\right)$ is a centered Gaussian random variable with variance

$$\mathbf{E}\left[X_m^2\left(\tilde{t},\tilde{x}\right)\right] = \sum_{k=-\infty}^{\infty} q_k \sum_{j=0}^{m} \int_{t_j}^{t_{j+1}} \left|f_j^{m,\tilde{t},\tilde{x}}(s)\right|^2 ds.$$

We can estimate the bound for each $f_j^{m,\tilde{t},\tilde{x}}(s)$ by

$$\begin{aligned}
\left|f_j^{m,\tilde{t},\tilde{x}}(s)\right| &= \sqrt{2H}\left| e^{ikx_j}\left(t_{j+1}-s\right)^{H-\frac{1}{2}} + \sum_{\ell=j+1}^{m} e^{ikx_\ell}\left[\left(t_{\ell+1}-s\right)^{H-\frac{1}{2}} - \left(t_\ell - s\right)^{H-\frac{1}{2}}\right]\right| \\
&\le \sqrt{2H}\left\{\left(t_{j+1}-s\right)^{H-\frac{1}{2}} + \sum_{\ell=j+1}^{m}\left[\left(t_\ell-s\right)^{H-\frac{1}{2}} - \left(t_{\ell+1}-s\right)^{H-\frac{1}{2}}\right]\right\} \\
&= \sqrt{2H}\left[2\left(t_{j+1}-s\right)^{H-\frac{1}{2}} - (t-s)^{H-\frac{1}{2}}\right] \\
&\le 2\sqrt{2H}\left(t_{j+1}-s\right)^{H-\frac{1}{2}}.
\end{aligned} \tag{31}$$

Therefore one obtains

$$\begin{aligned}
\mathbf{E}\left[X_m^2\left(\tilde{t},\tilde{x}\right)\right] &\le 4\sum_{k=-\infty}^{\infty} q_k \sum_{j=0}^{m} \int_{t_j}^{t_{j+1}} 2H\left(t_{j+1}-s\right)^{2H-1} ds \\
&= 4Q(0)\sum_{j=0}^{m}\left(t_{j+1}-t_j\right)^{2H}.
\end{aligned}$$

Since $\sum_{j=0}^{m}\left(t_{j+1}-t_j\right) = t$ and $H < \frac{1}{2}$, we get that $\sum_{j=0}^{m}\left(t_{j+1}-t_j\right)^{2H}$ attains its maximum when $t_{j+1}-t_j = \frac{t}{m+1}$ for $j = 0, 1, \ldots, m$. This translates as the conclusion of the lemma. ■

Thanks to this lemma, we are able to prove that the growth rate of $U(t)$ is at most linear.

**Theorem 5.6** *When $H \in \left(0, \frac{1}{2}\right)$, there exists a constant $C_{Q,H,\kappa}$, depending only on $Q$, $H$ and $\kappa$, such that*

$$\frac{U(t)}{t} \leq C_{Q,H,\kappa}, \quad t \in \mathbf{N}.$$

**Proof.** In this proof, $c_1$, $c_2$, *etc.* denote universal constants unless indicated otherwise.

First note that Jensen's inequality yields

$$U(t) = \mathbf{E}[\log u(t)] \leq \log \mathbf{E}[u(t)]. \tag{32}$$

It follows from (29) that

$$\mathbf{E}[u(t)] = \sum_{m=0}^{\infty} p(t,m) \sum_{\tilde{x} \in \mathcal{P}_m} \frac{1}{2^m} \int_{\tilde{t} \in \mathcal{S}(t,m)} \frac{m!}{t^m} \mathbf{E}\left[e^{X_m(\tilde{t},\tilde{x})}\right] dt_1 \cdots dt_m.$$

Since $Y$ is a centered Gaussian random variable, we get

$$\mathbf{E}\left[e^{X_m(\tilde{t},\tilde{x})}\right] = \exp\left\{\frac{1}{2}\mathbf{E}\left[X_m^2(\tilde{t},\tilde{x})\right]\right\} \leq \exp\left\{2Q(0)(m+1)^{1-2H} t^{2H}\right\}$$

for each $m$, thanks to Lemma 5.5. It therefore follows that

$$\mathbf{E}[u(t)] \leq \sum_{m=0}^{\infty} p(t,m) \exp\left\{2Q(0)(m+1)^{1-2H} t^{2H}\right\}. \tag{33}$$

Here $p(t,m)$ is the probability that a Poisson process with parameter $2\kappa$ has exactly $m$ jumps before time $t$. It is known that $p(t,m)$ attains its maximum when $m \sim \alpha t$ for some constant $\alpha > 0$ and decays exponentially when $m$ is large. Therefore it is natural that the summation of all terms up to $m = \alpha t$ will contribute the principal part of the above summation of series.

We now split the summation in (33) into two parts, at the point $m = \lfloor \alpha t \rfloor$, where $\alpha$ is a positive constant depending only on $Q$, $H$ and $\kappa$, which will be determined later; that is,

$$\begin{aligned}
&\sum_{m=0}^{\infty} p(t,m) \exp\left\{2Q(0)(m+1)^{1-2H} t^{2H}\right\} \\
&= \left(\sum_{m=0}^{\lfloor \alpha t \rfloor - 1} + \sum_{m=\lfloor \alpha t \rfloor}^{\infty}\right) p(t,m) \exp\left\{2Q(0)(m+1)^{1-2H} t^{2H}\right\}.
\end{aligned}$$

Here $\lfloor x \rfloor$ denotes the greatest integer less than or equal to $x$.

The first part in the summation can be controlled by

$$\begin{aligned}
&\sum_{m=0}^{\lfloor \alpha t \rfloor - 1} p(t,m) \exp\left\{2Q(0)(m+1)^{1-2H} t^{2H}\right\} \\
&\leq \exp\left\{2Q(0)(\alpha t)^{1-2H} t^{2H}\right\} = \exp\left\{2Q(0)\alpha^{1-2H} t\right\}.
\end{aligned}$$

As for the second part in the summation, let us denote by $J_m$ each term in it, for each integer $m \geq \lfloor \alpha t \rfloor$. Since $p(t,m) = e^{-2\kappa t}\frac{(2\kappa t)^m}{m!}$, we have

$$J_m = e^{-2\kappa t}\frac{(2\kappa t)^m}{m!} \exp\left\{2Q(0)(m+1)^{1-2H} t^{2H}\right\}.$$

From Stirling's formula $m! \simeq \sqrt{2\pi m + o(m)}\; e^{-m}m^m$ when $m$ is large, we can control $J_m$ from above by

$$J_m \leq \frac{c_1 (2\kappa t)^m e^{-2\kappa t}}{\sqrt{2\pi m}\; e^{-m}m^m} \exp\left\{2Q(0)(m+1)^{1-2H} t^{2H}\right\}.$$

If $\alpha$ is chosen such that $2Q(0) \leq \alpha^{2H}$, then

$$\begin{aligned} J_m &\leq \frac{c_1 (2\kappa t)^m e^{-2\kappa t}}{\sqrt{2\pi m}\; e^{-m}m^m} \exp\left\{(\alpha t)^{2H}(m+1)^{1-2H}\right\} \\ &\leq \frac{c_1 (2\kappa t)^m e^{-2\kappa t}}{\sqrt{2\pi m}\; e^{-m}m^m}\; e^{m+1} \end{aligned}$$

since $m+1 \geq \lfloor \alpha t \rfloor + 1 > \alpha t$. Hence

$$J_m \leq \frac{c_1 e^{-2\kappa t+1}}{\sqrt{2\pi}} \left(\frac{2\kappa t e^2}{m}\right)^m \leq \frac{c_1 e^{-2\kappa t+1}}{\sqrt{2\pi}} \left(\frac{2\kappa t e^2}{\alpha t - 1}\right)^m.$$

If $\alpha$ is further chosen such that $\frac{2\kappa t e^2}{\alpha t - 1} < \frac{1}{2}$, then

$$\sum_{m=\lfloor \alpha t \rfloor}^{\infty} J_m < \sum_{m=\lfloor \alpha t \rfloor}^{\infty} \frac{c_1 e^{-2\kappa t+1}}{\sqrt{2\pi}} \left(\frac{1}{2}\right)^m < \frac{c_1 e}{\sqrt{2\pi}}.$$

Therefore by taking

$$\alpha = \alpha_{Q,H,\kappa} = \max\left((2Q(0))^{\frac{1}{2H}}, 4\kappa e^2 + 1\right), \tag{34}$$

we obtain

$$\mathbf{E}[u(t)] \leq \exp\left\{2Q(0)(\alpha_{Q,H,\kappa})^{1-2H} t\right\} + \frac{c_1 e}{\sqrt{2\pi}}. \tag{35}$$

It follows from (32) that

$$U(t) \leq \log \mathbf{E}[u(t)] \leq c_2 Q(0)(\alpha_{Q,H,\kappa})^{1-2H} t.$$

Let

$$C_{Q,H,\kappa} = c_2 Q(0)(\alpha_{Q,H,\kappa})^{1-2H},$$

then the proof is finished. ■

## 5.2 Positivity

We proved in the last Subsection that $\sup_{t\in\mathbf{N}} t^{-1}U(t) < \infty$. Since $\{U(t)\}_{t\in\mathbf{N}}$ is also an almost superadditive sequence as a consequence of Theorem 4.5, Theorem 4.3 implies that

$$\lambda := \lim_{t\to\infty} \frac{U(t)}{t}$$

exists as long as Assumption 1 is satisfied. In that case $\lambda$ is nonnegative because $U(t) \geq 0$. Furthermore, if we can prove that $\lambda$ is strictly positive, then $U(t)$ will grow at a linear rate asymptotically; otherwise, $U(t)$ would grow slower than linearly. In this Subsection we therefore investigate the positivity of $\lambda$.

Let $\epsilon(t) = \sqrt{\pi^3} K_{univ} Q_1 L_H t^H$ and recall from Theorem 4.5 that we have $U(t+s) \geq U(t) + U(s) - \epsilon(t+s)$ for any $s, t \in \mathbf{N}$, under Assumption 1 .

Even though the limit of $U(t)/t$ is not attained increasingly with our non-superadditive sequence, we can still use the exact form of our error term $\epsilon(t+s)$ to quantify precisely the maximum distance that $\lambda$ might dip below $U(t)/t$ for any given $t$. Specifically we have the following

**Lemma 5.7** *Under the hypotheses and notation of Theorem 4.5 and its proof, we have the existence of $\lambda$ such that for any $t \geq 0$,*

$$\lambda = \lim_{t\to\infty} \frac{U(t)}{t} \geq \frac{1}{t}\left(U(t) - \frac{\sqrt{\pi^3} K_{univ} Q_1 L_H t^H}{2^{1-H}-1}\right).$$

**Proof.** The existence of $\lambda$ was established in the discussion above. Denote $\tilde{U}(t) = \frac{U(t)}{t}$ and $\tilde{\epsilon}(t) = \frac{\epsilon(t)}{t}$. Now, let $t$ be fixed; then

$$\tilde{U}(2t) \geq \tilde{U}(t) - \tilde{\epsilon}(2t),$$

follows from almost superadditivity. We obtain that, by induction,

$$\tilde{U}(2^n t) \geq \tilde{U}(t) - \sum_{i=1}^{n} \tilde{\epsilon}\left(2^i t\right).$$

Letting $n \to \infty$, we obtain

$$\lambda \geq \tilde{U}(t) - \sum_{i=1}^{\infty} \tilde{\epsilon}\left(2^i t\right).$$

Furthermore, we calculate

$$\sum_{i=1}^{\infty} \tilde{\epsilon}\left(2^i t\right) = \sqrt{\pi^3} K_{univ} Q_1 L_H t^{H-1} \sum_{i=1}^{\infty} 2^{i(H-1)} = \frac{\sqrt{\pi^3} K_{univ} Q_1 L_H t^{H-1}}{2^{1-H}-1}.$$

The result of the lemma now follows. ■

A lower bound estimate on $U(t)$ is in need here. Even though the next lemma is not particularly difficult to establish, it does represent the specific technical reason we are able to prove positivity for our polymer Lyapunov exponent.

**Lemma 5.8** *For any $t$, it holds that*

$$U(t) \geq -2\kappa t + \log(2\kappa t) - \log 2 + \left(\frac{\sqrt{Q(0) - Q(2)}}{(H+1)\sqrt{\pi}}\right) t^H.$$

**Proof.** In order to obtain a lower bound on $U(t)$, we need consider no more than the path $b$ with exactly one jump before time $t$. Then, ignoring all other terms in (8),

$$U(t) \geq \mathbf{E}\left[\log\left(p(t,1) \int_0^t \frac{ds}{t} \frac{e^{X_1(s,+1)} + e^{X_1(s,-1)}}{2}\right)\right],$$

where

$$X_1(s,j) = \int_0^s B_H(dr,0) + \int_s^t B_H(dr,j), \quad j = \pm 1.$$

Since $a+b \geq \max(a,b)$ when $a,b \geq 0$, we therefore have

$$\begin{aligned} U(t) &\geq \mathbf{E}\left[\log\left(p(t,1)\int_0^t \frac{ds}{t}\frac{\max\left(e^{X_1(s,+1)},e^{X_1(s,-1)}\right)}{2}\right)\right] \\ &= \mathbf{E}\left[\log\left(\frac{p(t,1)}{2}\int_0^t \frac{ds}{t}\exp\left\{\max\left(X_1(s,+1),X_1(s,-1)\right)\right\}\right)\right]. \end{aligned}$$

It follows from Jensen's inequality that

$$\begin{aligned} U(t) &\geq \log p(t,1) - \log 2 + \int_0^t \frac{ds}{t}\mathbf{E}\left[\max\left(X_1(s,+1),X_1(s,-1)\right)\right] \\ &= -2\kappa t + \log(2\kappa t) - \log 2 + \int_0^t \frac{ds}{t}\mathbf{E}\left[\max\left(X_1(s,+1),X_1(s,-1)\right)\right]. \end{aligned}$$

Note that $\max(a,b) = (a+b+|a-b|)/2$. Since $X_1(s,1)$ and $X_1(s,-1)$ are centered Gaussian random variables, we have

$$\mathbf{E}\left[\max\left(X_1(s,1),X_1(s,-1)\right)\right] = \frac{1}{2}\mathbf{E}\left[\left|X_1(s,1)-X_1(s,-1)\right|\right] = \frac{\sigma(s)}{\sqrt{2\pi}}$$

with

$$\begin{aligned} \sigma(s) &= \sqrt{Var\left[X_1(s,1)-X_1(s,-1)\right]} \\ &= \sqrt{2\left[Q(0)-Q(2)\right]\mathbf{E}\left[\left(B_H(t)-B_H(s)\right)^2\right]} \\ &\geq \sqrt{2\left[Q(0)-Q(2)\right]}\,(t-s)^H. \end{aligned}$$

For the last inequality, see Section 7.3 in the Appendix. Therefore, we get the conclusion of the lemma. ■

Our positivity result can now be easily established.

**Theorem 5.9** *Let $H \in \left(0,\frac{1}{2}\right)$ and assume Assumptions 1 and 2 are satisfied. Then there exists $\kappa_0 > 0$ such that for any $\kappa < \kappa_0$,*

$$\lambda := \lim_{t\to\infty, t\in\mathbf{N}} \frac{U(t)}{t}$$

*exists, is finite, and positive.*

**Proof.** As proved before, $\lambda$ exists, is finite, and nonnegative under Assumption 1. Furthermore, in Lemma 5.7, we have shown that for any $t$,

$$\lambda \geq \frac{1}{t}\left(U(t) - \frac{\sqrt{\pi^3}K_{univ}Q_1L_Ht^H}{2^{1-H}-1}\right).$$

We therefore get

$$\lambda \geq \frac{1}{t}\left[-2\kappa t+\log(2\kappa t)-\log 2+\left(\frac{\sqrt{Q(0)-Q(2)}}{(H+1)\sqrt{\pi}}-\frac{\sqrt{\pi^3}K_{univ}Q_1L_H}{2^{1-H}-1}\right)t^H\right]$$

by virtue of Lemma 5.8. Since Assumption 2 is also satisfied, we have

$$\lambda \geq \frac{1}{t}\left(-2\kappa t+\log(2\kappa t)-\log 2+C^*_{Q,H}t^H\right)$$

with $C^*_{Q,H}>0$.

Now assume $\kappa<\frac{1}{4}$ and choose $t=t_\kappa=\lfloor\frac{1}{2\kappa}\rfloor$, then

$$1\geq 2\kappa t_\kappa>1-2\kappa>\frac{1}{2}.$$

Therefore

$$\lambda \geq \frac{1}{t_\kappa}\left(-1-2\log 2+C^*_{Q,H}t_\kappa^H\right).$$

Denote $C^0_{Q,H}=\left(\frac{1+2\log 2}{C^*_{Q,H}}\right)^{\frac{1}{H}}$ and let $\kappa_0=\min\left(\frac{1}{4},\frac{1}{2\left(1+C^0_{Q,H}\right)}\right)$, then for any $\kappa<\kappa_0$, we have $t_\kappa>C^0_{Q,H}$ and consequently,

$$\lambda \geq \frac{1}{t_\kappa}\left(-1-2\log 2+C^*_{Q,H}t_\kappa^H\right)>0.$$

■

**Remark 5.10** *When all the assumptions in Theorem 5.9 are satisfied, there further exists $\kappa_1>0$ such that for any $\kappa<\kappa_1$,*

$$\lambda \geq \frac{1}{t_\kappa}\left(-1-2\log 2+C^*_{Q,H}t_\kappa^H\right)\geq\frac{1}{2}C^*_{Q,H}t_\kappa^{H-1}\geq\frac{1}{2}C^*_{Q,H}(2\kappa)^{1-H}.$$

*This gives a lower bound on how fast $\lambda$ may decrease as the diffusion constant $\kappa$ goes to 0. We see that the largest lower bounds are obtained for $H$ close to 1/2, i.e. for $B_H$ that is more regular in time. This contrasts sharply with results such as in [18], where the dependence of a similar $\lambda$ on $\kappa$ shows a larger lower bound when $H$ is smallest, but for a random medium with fractional Brownian behavior in space, not time. Memory length in space and in time seem to have opposite effects. This question will be investigated further in a separate article.*

### 5.3 Concentration Theory

So far we have studied the properties of $U(t)$. In this Subsection, we will show an important relation between the asymptotic behavior of $U(t)$ and that of $u(t)$.

**Theorem 5.11** *When $H\in\left(0,\frac{1}{2}\right)$, it holds almost surely that*

$$\lim_{t\to\infty,t\in\mathbf{N}}\left(\frac{1}{t}\log u(t)-\frac{1}{t}U(t)\right)=0.$$

We will prove this theorem by applying the Malliavin derivative Lemma 3.2 to $t^{-1}\log u(t)$. Since in the expression (29) for $u(t)$, each $X_m\left(\tilde{t},\tilde{x}\right)$ has a random Fourier series representation, we must use the discrete version (13) of this lemma. It is therefore necessary to find a bound on the Malliavin derivative $D_{s,k}\log u(t)$, for $s\geq 0$ and $k\in\mathbf{Z}$.

**Lemma 5.12** *There exists a positive constant $C_{Q,H,\kappa}$, depending only on $Q$, $H$ and $\kappa$, such that*

$$\mathbf{E}\left[\sum_{k=-\infty}^{\infty}\int_0^t\left(\mathbf{E}\left[D_{s,k}\log u(t)\middle|\mathcal{F}_s\right]\right)^2ds\right]\leq C_{Q,H,\kappa}t.$$

**Proof.** *Step 1. Setup.* Since

$$u(t)=\sum_{m=0}^{\infty}p(t,m)\sum_{\tilde{x}\in\mathcal{P}_m}\frac{1}{2^m}\int_{\tilde{t}\in\mathcal{S}(t,m)}\frac{m!}{t^m}e^{X_m(\tilde{t},\tilde{x})}dt_1\cdots dt_m,$$

we have

$$\begin{aligned}D_{s,k}\log u(t)&=\frac{1}{u(t)}\sum_{m=0}^{\infty}p(t,m)\sum_{\tilde{x}\in\mathcal{P}_m}\frac{1}{2^m}\int_{\tilde{t}\in\mathcal{S}(t,m)}\frac{m!}{t^m}e^{X_m(\tilde{t},\tilde{x})}D_{s,k}X_m\left(\tilde{t},\tilde{x}\right)dt_1\cdots dt_m\\&=:\hat{\mathbf{E}}_{b,B_H,t}\left[Y\right],\end{aligned}$$

where $\hat{\mathbf{E}}_{b,B_H,t}$ is the expectation under the polymer measure, i.e. the probability measure $\hat{\mathbf{P}}_{b,B_H,t}$ which we defined in (15). Here $Y$ is a random variable defined as follows: for fixed $m$ and fixed $\tilde{t}\in\mathcal{S}(t,m)$ and $\tilde{x}\in\mathcal{P}_m$, conditionally on the event that $N_t=m$ and the jump times and positions of $b$ are given by $\tilde{t}$ and $\tilde{x}$, $Y=D_{s,k}X_m\left(\tilde{t},\tilde{x}\right)$.

From Jensen's inequality for the probability measure $\hat{\mathbf{P}}_{b,B_H,t}$ it follows that

$$\begin{aligned}&\left(D_{s,k}\log u(t)\right)^2\\\leq&\hat{\mathbf{E}}_{b,B_H,t}\left[Y^2\right]\\=&\frac{1}{u(t)}\sum_{m=0}^{\infty}p(t,m)\sum_{\tilde{x}\in\mathcal{P}_m}\frac{1}{2^m}\int_{\tilde{t}\in\mathcal{S}(t,m)}\frac{m!}{t^m}e^{X_m(\tilde{t},\tilde{x})}\left|D_{s,k}X_m\left(\tilde{t},\tilde{x}\right)\right|^2dt_1\cdots dt_m.\end{aligned}\tag{36}$$

*Step 2. The Malliavin derivative of* $\log u(t)$. Note that for fixed $m$,

$$X_m\left(\tilde{t},\tilde{x}\right):=\sum_{k=-\infty}^{\infty}\sqrt{q_k}\sum_{j=0}^{m}\int_{t_j}^{t_{j+1}}f_j^{m,\tilde{t},\tilde{x}}(s)\,W_k(ds)$$

with each $f_j^{m,\tilde{t},\tilde{x}}(s)$, $j=0,1,\ldots,m$, is defined in (10). Therefore we can calculate $D_{s,k}X_m$ and use the estimate (31) to get

$$\begin{aligned}\left|D_{s,k}X_m\left(\tilde{t},\tilde{x}\right)\right|&=\sqrt{q_k}\left|\sum_{j=0}^{m}f_j^{m,\tilde{t},\tilde{x}}(s)\,\mathbf{1}_{[t_j,t_{j+1}]}(s)\right|\\&\leq 2\sqrt{q_k}\sum_{j=0}^{m}\sqrt{2H}\left(t_{j+1}-s\right)^{H-\frac{1}{2}}\mathbf{1}_{[t_j,t_{j+1}]}(s).\end{aligned}\tag{37}$$

Denote

$$g_m\left(\tilde{t},s\right)=\sum_{j=0}^{m}\sqrt{2H}\left(t_{j+1}-s\right)^{H-\frac{1}{2}}\mathbf{1}_{[t_j,t_{j+1}]}\left(s\right).$$

Applying result (37) to inequality (36) we get

$$\begin{aligned}&\left(D_{s,k}\log u\left(t\right)\right)^2\\ \leq&4q_k\left(\frac{1}{u\left(t\right)}\sum_{m=0}^{\infty}p\left(t,m\right)\sum_{\tilde{x}\in\mathcal{P}_m}\frac{1}{2^m}\int_{\tilde{t}\in\mathcal{S}(t,m)}\frac{m!}{t^m}e^{X_m\left(\tilde{t},\tilde{x}\right)}\left|g_m\left(\tilde{t},s\right)\right|^2dt_1\cdots dt_m\right).\end{aligned}$$

*Step 3. The $L^2$ norm of $\log u\left(t\right)$'s Malliavin derivative.* From the result of the previous step we get

$$\begin{aligned}&\int_0^t\mathbf{E}\left[\left(D_{s,k}\log u\left(t\right)\right)^2\right]ds\\ &\leq 4q_k\mathbf{E}\left[\frac{1}{u\left(t\right)}\sum_{m=0}^{\infty}p\left(t,m\right)\sum_{\tilde{x}\in\mathcal{P}_m}\frac{1}{2^m}\int_{\tilde{t}\in\mathcal{S}(t,m)}\frac{m!}{t^m}e^{X_m\left(\tilde{t},\tilde{x}\right)}\left(\int_0^t\left|g_m\left(\tilde{t},s\right)\right|^2ds\right)dt_1\cdots dt_m\right].\end{aligned}$$

Since

$$\int_0^t\left|g_m\left(\tilde{t},s\right)\right|^2ds=\sum_{j=0}^{m}\int_{t_j}^{t_{j+1}}2H\left(t_{j+1}-s\right)^{2H-1}ds=\sum_{j=0}^{m}\left(t_{j+1}-t_j\right)^{2H},$$

we can control it by

$$\int_0^t\left|g_m\left(\tilde{t},s\right)\right|^2ds\leq\left(m+1\right)^{1-2H}t^{2H}$$

because $H<\frac{1}{2}$ and $\sum_{j=0}^{m}\left(t_{j+1}-t_j\right)=t$. One then obtains that

$$\begin{aligned}&\int_0^t\mathbf{E}\left[\left(D_{s,k}\log u\left(t\right)\right)^2\right]ds\\ &\leq 4q_k\mathbf{E}\left[\frac{t^{2H}}{u\left(t\right)}\sum_{m=0}^{\infty}p\left(t,m\right)\sum_{\tilde{x}\in\mathcal{P}_m}\frac{1}{2^m}\int_{\tilde{t}\in\mathcal{S}(t,m)}\frac{m!}{t^m}e^{X_m\left(\tilde{t},\tilde{x}\right)}\left(m+1\right)^{1-2H}dt_1\cdots dt_m\right].\end{aligned}$$

And we finally obtain

$$\begin{aligned}&\mathbf{E}\left[\sum_{k=-\infty}^{\infty}\int_0^t\left(\mathbf{E}\left\{D_{s,k}\log u\left(t\right)\middle|\mathcal{F}_s\right\}\right)^2ds\right]\\ &\leq\mathbf{E}\left[\sum_{k=-\infty}^{\infty}\int_0^t\mathbf{E}\left\{\left(D_{s,k}\log u\left(t\right)\right)^2\middle|\mathcal{F}_s\right\}ds\right]\\ &=\sum_{k=-\infty}^{\infty}\int_0^t\mathbf{E}\left[\left(D_{s,k}\log u\left(t\right)\right)^2\right]ds\\ &\leq 4Q\left(0\right)\mathbf{E}\left[\frac{t^{2H}}{u\left(t\right)}\sum_{m=0}^{\infty}p\left(t,m\right)\sum_{\tilde{x}\in\mathcal{P}_m}\frac{1}{2^m}\int_{\tilde{t}\in\mathcal{S}(t,m)}\frac{m!}{t^m}e^{X_m\left(\tilde{t},\tilde{x}\right)}\left(m+1\right)^{1-2H}dt_1\cdots dt_m\right].\end{aligned}$$

*Step 4. Dealing with the Poisson law.* To find an upper bound on the expectation in the last line above, we can use the same technique as in the proof of Theorem 5.6. We split the summation in the expectation into two parts at the point $m = \lfloor \alpha t \rfloor$:

$$\begin{aligned}&\sum_{m=0}^{\infty} p(t,m) \sum_{\tilde{x}\in\mathcal{P}_m} \frac{1}{2^m} \int_{\tilde{t}\in\mathcal{S}(t,m)} \frac{m!}{t^m} e^{X_m(\tilde{t},\tilde{x})} (m+1)^{1-2H}\, dt_1\cdots dt_m \\ &= \left(\sum_{m=0}^{\lfloor \alpha t\rfloor -1} + \sum_{m=\lfloor \alpha t\rfloor}^{\infty}\right) p(t,m) \sum_{\tilde{x}\in\mathcal{P}_m} \frac{1}{2^m} \int_{\tilde{t}\in\mathcal{S}(t,m)} \frac{m!}{t^m} e^{X_m(\tilde{t},\tilde{x})} (m+1)^{1-2H}\ \ dt_1\cdots dt_m.\end{aligned}$$

Here $\alpha$ is a constant depending only on $Q$, $H$ and $\kappa$; we will choose it later.

The first part of the expectation of the summation above is then bounded from above as

$$\begin{aligned}I &:= \mathbf{E}\left[\frac{t^{2H}}{u(t)} \sum_{m=0}^{\lfloor \alpha t\rfloor -1} p(t,m) \sum_{\tilde{x}\in\mathcal{P}_m} \frac{1}{2^m} \int_{\tilde{t}\in\mathcal{S}(t,m)} \frac{m!}{t^m} e^{X_m(\tilde{t},\tilde{x})} (m+1)^{1-2H}\, dt_1\cdots dt_m\right] \\ &\le \mathbf{E}\left[\frac{t^{2H}}{u(t)} (\alpha t)^{1-2H} \sum_{m=0}^{\lfloor \alpha t\rfloor -1} p(t,m) \sum_{\tilde{x}\in\mathcal{P}_m} \frac{1}{2^m} \int_{\tilde{t}\in\mathcal{S}(t,m)} \frac{m!}{t^m} e^{X_m(\tilde{t},\tilde{x})} dt_1\cdots dt_m\right] \\ &\le \alpha^{1-2H} t.\end{aligned}$$

Meanwhile, we can control the second part of the expected summation as

$$\begin{aligned}J &:= \mathbf{E}\left[\frac{t^{2H}}{u(t)} \sum_{m=\lfloor \alpha t\rfloor}^{\infty} p(t,m) \sum_{\tilde{x}\in\mathcal{P}_m} \frac{1}{2^m} \int_{\tilde{t}\in\mathcal{S}(t,m)} \frac{m!}{t^m} e^{X_m(\tilde{t},\tilde{x})} (m+1)^{1-2H}\, dt_1\cdots dt_m\right] \\ &= t^{2H} \sum_{m=\lfloor \alpha t\rfloor}^{\infty} p(t,m) \sum_{\tilde{x}\in\mathcal{P}_m} \frac{1}{2^m} \int_{\tilde{t}\in\mathcal{S}(t,m)} \frac{m!}{t^m} \mathbf{E}\left[\frac{e^{X_m(\tilde{t},\tilde{x})}}{u(t)}\right] (m+1)^{1-2H}\, dt_1\cdots dt_m \\ &\le t^{2H} \sum_{m=\lfloor \alpha t\rfloor}^{\infty} p(t,m) \sum_{\tilde{x}\in\mathcal{P}_m} \frac{1}{2^m} \int_{\tilde{t}\in\mathcal{S}(t,m)} \frac{m!}{t^m} (m+1)^{1-2H} \\ &\qquad\qquad \times \left(\mathbf{E}\left[e^{2X_m(\tilde{t},\tilde{x})}\right]\right)^{\frac{1}{2}} \left(\mathbf{E}\left[(u(t))^{-2}\right]\right)^{\frac{1}{2}} dt_1\cdots dt_m.\end{aligned}$$

We first have

$$\mathbf{E}\left[e^{2X_m(\tilde{t},\tilde{x})}\right] = \exp\left\{2\mathbf{E}\left[X_m^2(\tilde{t},\tilde{x})\right]\right\} \le \exp\left\{8Q(0)\, t^{2H} (m+1)^{1-2H}\right\},$$

thank to result (30). Notice that function $\phi(x) = x^{-2}$ is convex on $\mathbf{R}_+\backslash\{0\}$. Therefore we also have

$$\begin{aligned}\mathbf{E}\left[(u(t))^{-2}\right] &= \mathbf{E}\left[\left(\mathbf{E}_b\left[\exp\left\{\int_0^t B_H(ds,b_s)\right\}\right]\right)^{-2}\right] \\ &\le \mathbf{E}\mathbf{E}_b\left[\exp\left\{-2\int_0^t B_H(ds,b_s)\right\}\right].\end{aligned}$$

Let $\tilde{B}_H$ be a fractional Brownian motion with spatial covariance $\tilde{Q} = 4Q$, then for any fixed path $b$, $-2\int_0^t B_H(ds, b_s)$ has the same distribution as $\int_0^t \tilde{B}_H(ds, b_s)$. Applying the result (35), with $\alpha_{4Q,H,\kappa}$ as defined in (34),

$$\mathbf{E}\mathbf{E}_b\left[\exp\left\{-2\int_0^t B_H(ds, b_s)\right\}\right] \le \exp\left\{8Q(0)\left(\alpha_{4Q,H,\kappa}\right)^{1-2H} t\right\} + \frac{c_1 e}{\sqrt{2\pi}}$$
$$\le \exp\left\{8c_2 Q(0)\left(\alpha_{4Q,H,\kappa}\right)^{1-2H} t\right\}.$$

Therefore,

$$\begin{aligned} J &\le t^{2H} \sum_{m=\lfloor \alpha t \rfloor}^{\infty} p(t,m) \sum_{\tilde{x}\in\mathcal{P}_m} \frac{1}{2^m} \int_{\tilde{t}\in\mathcal{S}(t,m)} \frac{m!}{t^m}(m+1)^{1-2H} dt_1 \cdots dt_m \\ &\qquad \times \exp\left\{4Q(0)\, t^{2H}(m+1)^{1-2H} + 4c_2 Q(0)\left(\alpha_{4Q,H,\kappa}\right)^{1-2H} t\right\} \\ &\le t^{2H} \sum_{m=\lfloor \alpha t \rfloor}^{\infty} \frac{c_1 (2\kappa t)^m e^{-2\kappa t}}{\sqrt{2\pi m}\; e^{-m} m^m} (m+1)^{1-2H} \\ &\qquad \times \exp\left\{4Q(0)\, t^{2H}(m+1)^{1-2H} + 4c_2 Q(0)\left(\alpha_{4Q,H,\kappa}\right)^{1-2H} t\right\} \end{aligned}$$

where Stirling's formula is used.

*Step 5. Optimizing over $\alpha$.* If $\alpha$ is chosen such that $4Q(0) \le \alpha^{2H}$ and $4c_2 Q(0)\left(\alpha_{4Q,H,\kappa}\right)^{1-2H} \le \alpha$, then

$$\begin{aligned} J &\le t^{2H} \sum_{m=\lfloor \alpha t \rfloor}^{\infty} \frac{c_1 (2\kappa t)^m e^{-2\kappa t}}{\sqrt{2\pi m}\; e^{-m} m^m} (m+1)^{1-2H} e^{2(m+1)} \\ &\le t^{2H} \sum_{m=\lfloor \alpha t \rfloor}^{\infty} \frac{c_1 e^{-2\kappa t+2}}{\sqrt{2\pi}} \left(\frac{2\kappa t e^4}{\alpha t - 1}\right)^m. \end{aligned}$$

Let $\alpha$ be further chosen such that $\frac{2\kappa t e^4}{\alpha t-1} < \frac{1}{2}$. Then

$$J \le t^{2H} \sum_{m=\lfloor \alpha t \rfloor}^{\infty} \frac{c_1 e^{-2\kappa t+2}}{\sqrt{2\pi}} \left(\frac{1}{2}\right)^m \le \left(\frac{c_1 e^2}{\sqrt{2\pi}}\right) t^{2H}.$$

By taking

$$\alpha = \tilde{\alpha}_{Q,H,\kappa} := \max\left(\left(4Q(0)\right)^{\frac{1}{2H}}, 4c_2 Q(0)\left(\alpha_{4Q,H,\kappa}\right)^{1-2H}, 4\kappa e^4 + 1\right),$$

we finally obtain

$$\begin{aligned} &\mathbf{E}\left[\sum_{k=-\infty}^{\infty} \int_0^t \left(\mathbf{E}\left[D_{s,k}\log u(t) \middle| \mathcal{F}_s\right]\right)^2 ds\right] \\ &\le 4Q(0)(I+J) \\ &\le 4Q(0)\left(\left(\tilde{\alpha}_{Q,H,\kappa}\right)^{1-2H} t + \frac{c_1 e^2}{\sqrt{2\pi}}\, t^{2H}\right) \le C_{Q,H,\kappa} t. \end{aligned}$$

Here the constant is given by

$$C_{Q,H,\kappa} = 4Q(0)\left((\tilde{\alpha}_{Q,H,\kappa})^{1-2H} + \frac{c_1 e^2}{\sqrt{2\pi}}\right).$$

■

Now we can complete the proof of Theorem 5.11.

**Proof. [Proof of Theorem 5.11]**

We apply Lemma 3.2 to $G = t^{-1}\log u(t)$. For every $p \geq 1$

$$\begin{aligned}\mathbf{E}\left[(G-\mathbf{E}[G])^{2p}\right] &\leq C_p\left\{\frac{1}{t^2}\mathbf{E}\left[\sum_{k=-\infty}^{\infty}\int_0^t (\mathbf{E}[D_{s,k}\log u(t)|\,\mathcal{F}_s])^2\, ds\right]\right\}^p \\ &\leq C_p\left(\frac{C_{Q,H,\kappa}}{t}\right)^p =: \frac{C_{Q,H,\kappa,p}}{t^p}.\end{aligned}$$

Here the last inequality comes from Lemma 5.12. Therefore by Chebyshev's inequality, for fixed integer $t$, for any constant $C(t) > 0$,

$$\mathbf{P}\left[\left|\frac{1}{t}\log u(t) - \frac{1}{t}U(t)\right| > C(t)\right] \leq \frac{C_{Q,H,\kappa,p}}{t^p\,[C(t)]^{2p}}.$$

Let $t^p[C(t)]^{2p} = t^\beta$, that is, $C(t) = t^{-(p-\beta)/(2p)}$. By choosing $\beta > 1$ but close to 1 enough, we only need to require $p > 1$ to guarantee that $\lim_{t\to\infty} C(t) = 0$, and then we are able to apply the Borel-Cantelli lemma and obtain that, almost surely,

$$\lim_{t\to\infty}\left(\frac{1}{t}\log u(t) - \frac{1}{t}U(t)\right) = 0.$$

■

The main result of this section is now trivial.

**Proof. [Proof of Theorem 5.4]** Theorem 5.4 is a consequence of combining Theorems 4.5, 5.6, 5.9 and 5.11 together. ■

# 6 Exponential Behavior when $H > \frac{1}{2}$

## 6.1 Concentration Theory

When $H > \frac{1}{2}$, we still have the concentration result just as in the case of $H < \frac{1}{2}$.

**Theorem 6.1** *When $H \in (1/2, 1)$, it holds almost surely that*

$$\lim_{t\to\infty, t\in\mathbf{N}}\left(\frac{1}{t}\log u(t) - \frac{1}{t}U(t)\right) = 0.$$

This theorem extends Theorem 5.11 to all $H \in (0,1)$. However, the Lyapunov exponent of $u(t)$ actually blows up in this case because

$$\lim_{t\to\infty}\frac{1}{t}U(t) = +\infty, \quad \text{a.s.}$$

This fact is not straightforward: it is implied by the lower bound result in Subsection 6.3 (Theorem 6.7) below. Because of this we seek instead the almost-sure existence of $\lim_{t\to\infty} \frac{1}{L(t)} \log u(t)$ for some deterministic function $L(t)$ which grows faster than linearly. Independent of the existence of such a function $L(t)$, we have the following immediate corollary to Theorem 6.1.

**Corollary 6.2** *For any deterministic function $L(t)$ which grows faster than $t$, by which we mean that $\lim_{t\to\infty} t^{-1}L(t) = +\infty$, it follows immediately that almost surely,*

$$\liminf_{t\to\infty, t\in\mathbf{N}} \frac{1}{L(t)} \log u(t) = \liminf_{t\to\infty, t\in\mathbf{N}} \frac{1}{L(t)} U(t), \quad a.s.$$

*and*

$$\limsup_{t\to\infty, t\in\mathbf{N}} \frac{1}{L(t)} \log u(t) = \limsup_{t\to\infty, t\in\mathbf{N}} \frac{1}{L(t)} U(t), \quad a.s..$$

Notice that both a.s.-$\liminf_{t\to\infty} \frac{1}{L(t)} \log u(t)$ and a.s.-$\limsup_{t\to\infty} \frac{1}{L(t)} \log u(t)$ are deterministic real numbers since $U(t)$ is deterministic.

**Definition 6.3** *We call a deterministic function $L(t)$ the exponential rate function of $u(t)$ if the following non-trivial limits hold almost surely simultaneously:*

$$\liminf_{t\to\infty, t\in\mathbf{N}} \frac{1}{L(t)} \log u(t) > 0 \tag{38}$$

$$\limsup_{t\to\infty, t\in\mathbf{N}} \frac{1}{L(t)} \log u(t) < \infty. \tag{39}$$

*If the above two limits coincide, the common value $\lambda^*$ is called the Lyapunov exponent with respect to the exponential rate function $L(t)$. If only (38) is known to be satisfied, we say $L(t)$ is a proper lower bound on the exponential rate function of $u(t)$; on the other hand, if only (39) is known to be satisfied, we say $L(t)$ is a proper upper bound on the exponential rate function of $u(t)$.*

**Remark 6.4** *When $H > \frac{1}{2}$, the lower bound result of Subsection 6.3 (Theorem 6.7) implies that with $L(t)$ any lower bound on the exponential rate function of $u(t)$, we must have $\lim_{t\to\infty} L(t)/t = \infty$.*

**Proof. [Proof of Theorem 6.1]**

The proof applies Lemma 3.2 to $G = t^{-1} \log u(t)$, just as we had done in the case of $H < 1/2$. The calculations here, however, are not the same; it turns out they are simpler. Recall that we have the random Fourier series representation of $u(t)$ as (6),

$$u(t) = \mathbf{E}_b \left[ \exp\left\{ \sum_{k=-\infty}^{\infty} \sqrt{q_k} \int_0^t \left[ K_H^* e^{ikb\cdot} \right](t,s) W_k(ds) \right\} \right].$$

When $H > \frac{1}{2}$, the operator $K_H^*$ is more regular than when $H < 1/2$. Its action on $e^{ikb\cdot}$ can be written as

$$\left[ K_H^* e^{ikb\cdot} \right](t,s) = \int_s^t \sqrt{2H} \left( H - \frac{1}{2} \right) e^{ikb_r} (r-s)^{H-\frac{3}{2}} dr.$$

Therefore we have

$$u(t) = \mathbf{E}_b \left[ \exp \left\{ \sum_{k=-\infty}^{\infty} \sqrt{q_k} \int_0^t W_k(ds) \left[ \int_s^t \sqrt{2H} \left( H - \frac{1}{2} \right) e^{ikb_r} (r-s)^{H-\frac{3}{2}} dr \right] \right\} \right]. \tag{40}$$

We calculate that, for each $k$,

$$\begin{aligned} & D_{s,k}G \\ &= \frac{1}{t} \frac{1}{u(t)} \mathbf{E}_b \left[ \sqrt{q_k} \left( \int_s^t \sqrt{2H} \left( H - \frac{1}{2} \right) e^{ikb_r} (r-s)^{H-\frac{3}{2}} dr \right) \exp \left\{ \int_0^t B_H(ds, b_s) \right\} \right] \\ &= \frac{1}{t} \hat{\mathbf{E}}_{b,B_H,t} \left[ \sqrt{q_k} \int_s^t \sqrt{2H} \left( H - \frac{1}{2} \right) e^{ikb_r} (r-s)^{H-\frac{3}{2}} dr \right] \end{aligned}$$

when $s \leq t$; and $D_{s,k}G = 0$ when $s > t$. Here, $\hat{\mathbf{E}}_{b,B_H,t}$ is the expectation under the random measure $\hat{\mathbf{P}}_{b,B_H,t}$ defined in (15). Since $H > \frac{1}{2}$, we can control $D_{s,k}G$ by

$$\begin{aligned} |D_{s,k}G| &\leq \frac{1}{t} \hat{\mathbf{E}}_{b,B_H,t} \left[ \sqrt{q_k} \int_s^t \sqrt{2H} \left( H - \frac{1}{2} \right) (r-s)^{H-\frac{3}{2}} dr \right] \\ &= \frac{\sqrt{2Hq_k}\,(t-s)^{H-\frac{1}{2}}}{t}. \end{aligned}$$

Applying Lemma 3.2 for every $p \geq 1$, we get

$$\begin{aligned} \mathbf{E}\left[ (G - \mathbf{E}[G])^{2p} \right] &\leq C_p \left\{ \mathbf{E} \left[ \sum_{k=-\infty}^{\infty} \int_0^t \left( \mathbf{E}\{ D_{s,k}G \,|\mathcal{F}_s \} \right)^2 ds \right] \right\}^p \\ &\leq C_p \left\{ \frac{1}{t^2} \sum_{k=-\infty}^{\infty} q_k \int_0^t 2H (t-s)^{2H-1} ds \right\}^p \\ &= \frac{C_p [Q(0)]^p}{t^{p(2-2H)}} =: \frac{C_{Q,p}}{t^{p(2-2H)}}. \end{aligned}$$

What we have just proved is that for any $t \in \mathbf{N}$,

$$\mathbf{E}\left[ \left( \frac{1}{t} \log u(t) - \frac{1}{t} U(t) \right)^{2p} \right] \leq \frac{C_{Q,p}}{t^{p(2-2H)}}.$$

The remainder of the proof, using Chebyshev's inequality and the Borel-Cantelli lemma, is now identical to the same arguments in the proof of Theorem 5.11 with the exception that one uses $C(t) = t^{[\beta - p(2-2H)]/(2p)}$ and that we thus only need to require $p > \frac{1}{2-2H}$. ■

## 6.2 Exponential Rate Function: Upper Bound

One straightforward fact about $U(t)$ is that $t^{-2H}U(t)$ is bounded. To see this, we consider the formula (40) in the proof of the last theorem. Let $Y$ be defined by $u(t) = \mathbf{E}_b[\exp Y(t)]$, i.e.

$$Y(t) := \sum_{k=-\infty}^{\infty} \sqrt{q_k} \int_0^t W_k(ds) \left[ \int_s^t \sqrt{2H} \left( H - \frac{1}{2} \right) e^{ikb_r} (r-s)^{H-\frac{3}{2}} dr \right].$$

Then for each fixed path $b$, $Y(t)$ is a centered Gaussian random variable and its variance is

$$\begin{aligned}
\mathbf{E}\left[Y^2(t)\right] &= \sum_{k=-\infty}^{\infty} q_k \int_0^t \left| \int_s^t \sqrt{2H}\left(H-\frac{1}{2}\right) e^{ikb_r}(r-s)^{H-\frac{3}{2}}\, dr \right|^2 ds \\
&= \sum_{k=-\infty}^{\infty} q_k \int_0^t \left| \int_s^t \sqrt{2H}\left(H-\frac{1}{2}\right) (r-s)^{H-\frac{3}{2}}\, dr \right|^2 ds \\
&= Q(0)\, t^{2H}.
\end{aligned}$$

Then with Jensen's inequality and Fubini theorem we get

$$U(t) = \mathbf{E}\left[\log u(t)\right] \le \log \mathbf{E}_b \mathbf{E}\left[e^{Y(t)}\right] \le \frac{1}{2} Q(0)\, t^{2H}.$$

Therefore we have $t^{-2H} U(t) \le Q(0)/2$. This fact plus the concentration result of Corollary 6.2, yield the following.

**Theorem 6.5** *When $H \in \left(\frac{1}{2}, 1\right)$, there exits a deterministic, finite real number $\lambda^* \le \frac{Q(0)}{2}$ such that*

$$\limsup_{t\to\infty, t\in\mathbf{N}} \frac{1}{t^{2H}} \log u(t) \le \lambda^*, \quad a.s.$$

**Remark 6.6** *We cannot say the function $L_+(t) := t^{2H}$ is the exponential rate function of $u(t)$ because it is not clear whether $\liminf_{t\to\infty} \frac{1}{t^{2H}} U(t)$ is strictly positive or not. But in either case $L_+(t)$ is a proper upper bound on the exponential rate function.*

## 6.3 Exponential Rate Function: Lower Bound

The proof of our main result in this subsection is technical because it deals very specifically with the long-term correlation structure of the increments of fBm. Thanks to our concentration result, the basic problem we need to tackle can be summarized by seeking to maximize $X_m\left(\tilde{t}, \tilde{x}\right)$ over all possible paths $\left(\tilde{t}, \tilde{x}\right)$ of length $m$, assuming these paths are allowed to depend on the randomness of $B_H$, and then finding that maximum's expectation. Even when $H = 1/2$, it is not known what the form of such a maximizing path is. A technique proposed early on in the case $H = 1/2$ in [12], before it was known that concentration inequalities make it useful to evaluate $\mathbf{E}\left[\sup_{\tilde{t},\tilde{x}} X_m\left(\tilde{t}, \tilde{x}\right)\right]$ for any lower bound estimation, was to restrict $\tilde{t}$ to jump at fairly regularly spaced intervals, and then to choose each increment of $\tilde{x}$ in such a way as to maximize the corresponding term in the analogue of $X_m\left(\tilde{t}, \tilde{x}\right)$, in a step-by-step way. This work was greatly facilitated by the independence of the increments in $X_m\left(\tilde{t}, \tilde{x}\right)$. We have decided to combine this original strategy with the concentration result, in our case. To deal with the dependence of our increments, we single out a representative term among the series that forms each increment, and base our random maximization of the corresponding increment of $\tilde{x}$ on it. This means that we then have to investigate the effect of our choice for $\tilde{x}$ on all the other terms of the said series; this is where a detailed understanding of how the correlations play out is necessary.

**Theorem 6.7** *Assume $Q(0) > Q(2)$. When $H \in \left(\frac{1}{2}, 1\right)$, there exits a deterministic, positive real number $\lambda_*$ such that*

$$\liminf_{t\to\infty, t\in\mathbf{N}} \left(\frac{t^{2H}}{\log t}\right)^{-1} \log u(t) \ge \lambda_*, \quad a.s.$$

**Proof.** *Step 1. Initial strategy and remarks.* Define the filtration notation

$$\mathcal{F}^{W}_{(u,v]} := \sigma \left\{ W(r,x) - W(u,x) : u < r \leq v, x \in \mathbf{Z} \right\}, \quad \mathcal{F}^{W}_{u} := \mathcal{F}^{W}_{[0,u]}.$$

We write

$$\begin{aligned} B_H\left([u,v],x\right) &:= B_H(v,x) - B_H(u,x) \\ &= \int_u^v K_H(v,r)\, W(dr,x) + \int_0^u \left(K_H(v,r) - K_H(u,r)\right) W(dr,x) \\ &=: I^H(u,v;x) + J^H(u,v;x), \end{aligned}$$

where $K_H(u,r) = \sqrt{2H}\,(u-r)^{H-\frac{1}{2}}$. This is a way to single out the "innovations" part of the increment $B_H([u,v],x)$, denoted by $I^H(u,v;x)$ above, which is measurable with respect to $\mathcal{F}^W_{(u,v]}$; therefore it is independent of the other part of the increment, denoted above by $J^H(u,v;x)$, because the latter is measurable with respect to $\mathcal{F}^W_u$. We will perform the maximization mentioned above on the innovations parts only, and then investigate its effects on the other parts.

Our goal is to maximize $X_m\left(\tilde{t},\tilde{x}\right)$ over possible random ($B_H$-dependent) choices of $\tilde{x} \in \mathcal{P}_m$, where, with fixed $m$ and fixed simplex of jump times $\tilde{t} \in \mathcal{S}(t,m)$,

$$X_m\left(\tilde{t},\tilde{x}\right) = \sum_{j=0}^{m} B_H\left([t_j,t_{j+1}],x_j\right).$$

In fact, we will only look at calculating the expectation $\mathbf{E}\left[X_m\left(\tilde{t},\tilde{x}^*\right)\right]$ for a specific choice of $\tilde{x}^*$; this will of course yield a lower bound on a maximized $\mathbf{E}\left[\sup_{\tilde{x}} X_m\left(\tilde{t},\tilde{x}\right)\right]$. We introduce the shorthand notation $I^H(t_j,t_{j+1};x_j) =: I_j(x_j)$, and $J^H(t_j,t_{j+1};x_j) =: J_j(x_j)$. Thus

$$X_m\left(\tilde{t},\tilde{x}\right) = \sum_{j=0}^{m} I_j(x_j) + \sum_{j=1}^{m} J_j(x_j).$$

We choose the specific $\tilde{x}^*$ by maximizing the terms $I_j(x_j)$ step by step. This maximization is not sufficient to obtain our final theorem if we then ignore the role of $J_j(x_j)$, but it turns out that the choices made to maximize $I_j(x_j)$ are also beneficial to making $J_j(x_j)$ large. This unexpected bonus only works because of the positivity of increment correlations when $H > \frac{1}{2}$. When $H < \frac{1}{2}$, none of these arguments are needed because the correct exponential scale is $L(t) = t$, which works in conjunction with our almost superadditivity result.

*Step 2. The maximizing path $\tilde{x}^*$.* First we define $x_0^* = 0$ (we have no choice there) and for all $j \geq 0$, assuming we have defined $x_j^*$ as measurable with respect to $\mathcal{F}^W_{t_{j+1}}$, let $x^*_{j+1}$ be the value among the pair $\left\{x_j^*+1, x_j^*-1\right\}$ which maximizes $\max\left\{I_{j+1}\left(x_j^*+1\right); I_{j+1}\left(x_j^*-1\right)\right\}$.

We claim the fact that $W$ is spatially homogeneous implies that we can write $x^*_{j+1} = x^*_j + \varepsilon^*_{j+1}$ where $\varepsilon^*_{j+1}$ is measurable with respect to $\mathcal{F}^W_{(t_{j+1},t_{j+2}]}$, and thus is independent of $\mathcal{F}^W_{t_{j+1}}$, and therefore of $x^*_j$. Indeed, first, by definition, $\varepsilon^*_{j+1} = \arg\max\left\{I_{j+1}\left(x^*_j+z\right) : z \in \{-1,+1\}\right\}$.Consider now the two-dimensional random vector $\left(I_{j+1}\left(x_j^*+1\right); I_{j+1}\left(x_j^*-1\right)\right)$. We claim that this vector is jointly Gaussian and independent of the random variable $x_j^*$. Let $f,g$ be two test functions. Since the random field $\{I_{j+1}(z), z \in \mathbf{Z}\}$ depends only on $\mathcal{F}^W_{(t_{j+1},t_{j+2}]}$, we have, by conditioning on $x^*_j$ which is independent of $\mathcal{F}^W_{(t_{j+1},t_{j+2}]}$,

$$\begin{aligned}
&\mathbf{E}\left[f\left(I_{j+1}\left(x^*_j+1\right)\right)g\left(I_{j+1}\left(x^*_j-1\right)\right)\right]\\
&=\mathbf{E}\left[\mathbf{E}\left[f\left(I_{j+1}\left(x^*_j+1\right)\right)g\left(I_{j+1}\left(x^*_j-1\right)\right)|\mathcal{F}^W_{t_{j+1}}\right]\right]\\
&=\mathbf{E}\left[\mathbf{E}\left[f\left(I_{j+1}(z+1)\right)g\left(I_{j+1}(z-1)\right)\right]|_{z=x^*_j}\right]\\
&=\mathbf{E}\left[f\left(I_{j+1}(+1)\right)g\left(I_{j+1}(-1)\right)\right],
\end{aligned}$$

where in the last equality we used the spatial homogeneity of $W$. Since the pair $(I_{j+1}(+1); I_{j+1}(-1))$ is jointly Gaussian, this proves that the pair $\left(I_{j+1}\left(x^*_j+1\right); I_{j+1}\left(x^*_j-1\right)\right)$ is also jointly Gaussian with the same law. It also proves that $\varepsilon^*_{j+1}$ has the same law as $\arg\max\{I_{j+1}(z) : z \in \{-1,+1\}\}$. On the other hand, the conditional part of the calculation above can be repeated as

$$\begin{aligned}
&\mathbf{E}\left[f\left(I_{j+1}\left(x^*_j+1\right)\right)g\left(I_{j+1}\left(x^*_j-1\right)\right)|\mathcal{F}^W_{t_{j+1}}\right]\\
&=\mathbf{E}\left[f\left(I_{j+1}(z+1)\right)g\left(I_{j+1}(z-1)\right)\right]|_{z=x^*_j}\\
&=\mathbf{E}\left[f\left(I_{j+1}(+1)\right)g\left(I_{j+1}(-1)\right)\right];
\end{aligned}$$

this proves that $\left(I_{j+1}\left(x^*_j+1\right); I_{j+1}\left(x^*_j-1\right)\right)$ is independent of $\mathcal{F}^W_{t_{j+1}}$, and thus so is $\varepsilon^*_{j+1}$.

Summarizing this step, we have proved that with the sequence $x^*$ defined by

$$x^*_{j+1} = x^*_j + \varepsilon^*_{j+1}$$

where $x^*_0 = 0$ and

$$\varepsilon^*_{j+1} = \arg\max\left\{I_{j+1}\left(x^*_j+z\right) : z \in \{-1,+1\}\right\},$$

then $\varepsilon^*_{j+1}$ is measurable with respect to $\mathcal{F}^W_{(t_{j+1},t_{j+2}]}$, is independent of $\mathcal{F}^W_{t_{j+1}}$, and has the same distribution as $\arg\max\{I_{j+1}(z) : z \in \{-1,+1\}\}$.

*Step 3. A special expression for the non-innovation terms.* A first analysis of the $J$ terms evaluated at $\tilde{x}^*$, using again $W$'s spatial homogeneity, reveals a very interesting property. For any fixed $j$, we can further decompose such terms as follows:

$$\begin{aligned}
J_j\left(x^*_j\right) &= \sum_{k=0}^{j-1}\int_{t_k}^{t_{k+1}}\left(K_H(t_{j+1},r)-K_H(t_j,r)\right)W\left(dr,x^*_j\right)\\
&=\sum_{k=0}^{j-1}\int_{t_k}^{t_{k+1}}\left(K_H(t_{j+1},r)-K_H(t_j,r)\right)W\left(dr,x^*_k+\sum_{\ell=k+1}^{j}\varepsilon^*_\ell\right).
\end{aligned}$$

Note that to evaluate $\mathbf{E}\left[X_m\left(\tilde{t},\tilde{x}^*\right)\right]$, the contribution of the term $J_j\left(x^*_j\right)$ is simply its expectation. We claim that we have

$$\begin{aligned}
&\mathbf{E}\left[J_j\left(x^*_j\right)\right]\\
&=\sum_{k=0}^{j-1}\mathbf{E}\left[\int_{t_k}^{t_{k+1}}\left(K_H(t_{j+1},r)-K_H(t_j,r)\right)W\left(dr,x^*_k+\sum_{\ell=k+1}^{j}\varepsilon^*_\ell\right)\right] &(41)\\
&=\sum_{k=0}^{j-1}\mathbf{E}\left[\int_{t_k}^{t_{k+1}}\left(K_H(t_{j+1},r)-K_H(t_j,r)\right)W\left(dr,x^*_k\right)\right]. &(42)
\end{aligned}$$

The elimination of the terms $\sum_{\ell=k+1}^{j}\varepsilon_{\ell}^{*}$ when going from (41) to (42) above holds because of the following facts (similar to the argument in Step 2). As noted in Step 2, $\sum_{\ell=k+1}^{j}\varepsilon_{\ell}^{*}$ is measurable with respect to $\vee_{\ell=k+1}^{j}\mathcal{F}_{(t_\ell,t_{\ell+1}]}^{W}$, and thus is independent of $\mathcal{F}_{t_{k+1}}^{W}$. To calculate the expectation $\mathbf{E}$ in (41), we can calculate first the expectation $\mathbf{E}$ conditioned on the value of sum $\sum_{\ell=k+1}^{j}\varepsilon_{\ell}^{*}$. This value then vanishes because of homogeneity of $W$ in space. To be precise, we have the following expression for each term in the sum over $k$ in (41):

$$\begin{aligned}
&\mathbf{E}\left[\int_{t_k}^{t_{k+1}}\left(K_H\left(t_{j+1},r\right)-K_H\left(t_j,r\right)\right)W\left(dr,x_k^{*}+\sum_{\ell=k+1}^{j}\varepsilon_{\ell}^{*}\right)\right]\\
&=\mathbf{E}\left[\mathbf{E}\left\{\int_{t_k}^{t_{k+1}}\left(K_H\left(t_{j+1},r\right)-K_H\left(t_j,r\right)\right)W\left(dr,x_k^{*}+c\right)\middle|\sum_{\ell=k+1}^{j}\varepsilon_{\ell}^{*}=c\right\}\Bigg|_{c=\sum_{\ell=k+1}^{j}\varepsilon_{\ell}^{*}}\right]\\
&=\mathbf{E}\left[\mathbf{E}\left\{\int_{t_k}^{t_{k+1}}\left(K_H\left(t_{j+1},r\right)-K_H\left(t_j,r\right)\right)W\left(dr,x_k^{*}\right)\middle|\sum_{\ell=k+1}^{j}\varepsilon_{\ell}^{*}=c\right\}\right]\\
&=\mathbf{E}\left[\int_{t_k}^{t_{k+1}}\left(K_H\left(t_{j+1},r\right)-K_H\left(t_j,r\right)\right)W\left(dr,x_k^{*}\right)\right];
\end{aligned}$$

(42) is thus proved.

*Step 4. Evaluating the maximized expectation.* Introducing the notation

$$J_{j,k}\left(x\right)=\int_{t_k}^{t_{k+1}}\left(K_H\left(t_{j+1},r\right)-K_H\left(t_j,r\right)\right)W\left(dr,x\right),$$

we thus only need to calculate

$$\mathbf{E}\left[X_m\left(\tilde{t},\tilde{x}^{*}\right)\right]=\sum_{j=0}^{m}\mathbf{E}\left[I_j\left(x_j^{*}\right)\right]+\sum_{j=0}^{m}\sum_{k=0}^{j-1}\mathbf{E}\left[J_{j,k}\left(x_k^{*}\right)\right].$$

*Step 4.1. Simplifying the dependence on $x^{*}$.* For any fixed $x$, $J_{j,k}\left(x\right)\in\mathcal{F}_{(t_k,t_{k+1}]}^{W}$, and $J_{j,k}$ is homogeneous in $x$, so that since $x_k^{*}$ can be decomposed as the sum $x_{k-1}^{*}+\varepsilon_k^{*}$ where $\varepsilon_k^{*}\in\mathcal{F}_{(t_k,t_{k+1}]}$ and $x_{k-1}^{*}$ is independent of $\mathcal{F}_{(t_k,t_{k+1}]}^{W}$, using the same argument as in the previous two steps, the expectation $\mathbf{E}\left[J_{j,k}\left(x_k^{*}\right)\right]$ is actually equal to $\mathbf{E}\left[J_{j,k}\left(\varepsilon_k^{*}\right)\right]$. In fact, by the same token, in this last formula, $\varepsilon_k^{*}$ can simply be understood (has the same distribution) as the value $+1$ or $-1$ which maximizes $\max\left\{I_k\left(z\right):z\in\{-1,+1\}\right\}$. This is of course the same argument used in Step 2 to prove that $\varepsilon_j^{*}$ has the same distribution as

$$\varepsilon_j^{*}=\arg\max\left\{I_j\left(z\right):z\in\{-1,+1\}\right\},$$

and we will abusively use the same notation $\varepsilon_j^{*}$ for both of these, because we will only need to refer to their common law. Again, this coincidence of laws, homogeneity, and independence, imply that $\mathbf{E}\left[I_j\left(x_j^{*}\right)\right]$ can also be written as $\mathbf{E}\left[I_j\left(\varepsilon_j^{*}\right)\right]=\mathbf{E}\left[\max\left\{I_j\left(z\right):z\in\{-1,+1\}\right\}\right]$. Thus our task is only to evaluate

$$\mathbf{E}\left[X_m\left(\tilde{t},\tilde{x}^{*}\right)\right]=\sum_{j=0}^{m}\mathbf{E}\left[I_j\left(\varepsilon_j^{*}\right)\right]+\sum_{j=0}^{m}\sum_{k=0}^{j-1}\mathbf{E}\left[J_{j,k}\left(\varepsilon_k^{*}\right)\right].$$

*Step 4.2. Covariance structure evaluation.* In order to perform such an evaluation, Step 4.1 proves that we only need to investigate, for each fixed $j \le m$, and $k \le j-1$, the covariance structure of the 4-dimensional Gaussian vector $\left(Z_k^+, Z_k^-, Z_{j,k}^+, Z_{j,k}^-\right)$ where $Z_k^+ = I_k(+1)$, $Z_k^- = I_k(-1)$, $Z_{j,k}^+ = J_{j,k}(+1)$, and $Z_{j,k}^- = J_{j,k}(-1)$. Indeed, for example, we have

$$\mathbf{E}\left[I_j\left(\varepsilon_j^*\right)\right] = \mathbf{E}\left[Z_j^{\varepsilon_j^*}\right] = \mathbf{E}\left[Z_j^{\arg\max\left\{Z_j^{\varepsilon}:\varepsilon=\pm1\right\}}\right]$$

and

$$\mathbf{E}\left[J_{j,k}\left(\varepsilon_k^*\right)\right] = \mathbf{E}\left[Z_{j,k}^{\varepsilon_k^*}\right] = \mathbf{E}\left[Z_{j,k}^{\arg\max\left\{Z_k^{\varepsilon}:\varepsilon=\pm1\right\}}\right],$$

where we abusively confuse the notation $Z^+$ with $Z^{+1}$, and $Z^-$ with $Z^{-1}$.

We note that $\mathbf{E}\left[\left(Z_k^+\right)^2\right] = \mathbf{E}\left[\left(Z_k^-\right)^2\right]$ and $\mathbf{E}\left[Z_k^+ Z_{j,k}^+\right] = \mathbf{E}\left[Z_k^- Z_{j,k}^-\right]$, since the pair $\left(Z_k^+, Z_{j,k}^+\right)$ has the same distribution as $\left(Z_k^-, Z_{j,k}^-\right)$ by homogeneity. We then first calculate the covariance $\sigma_{j,k} = \mathbf{E}\left[Z_k^+ Z_{j,k}^+\right] = \mathbf{E}\left[Z_k^- Z_{j,k}^-\right]$ and get

$$\begin{aligned}\sigma_{j,k} &= \mathbf{E}\left[Z_k^+ Z_{j,k}^+\right] \\ &= Q(0)\int_{t_k}^{t_{k+1}} K_H(t_{k+1}, r)\left(K_H(t_{j+1}, r) - K_H(t_j, r)\right) dr \\ &= Q(0)\int_{t_k}^{t_{k+1}} 2H\left(t_{k+1} - r\right)^{H-\frac{1}{2}}\left(\left(t_{j+1} - r\right)^{H-\frac{1}{2}} - \left(t_j - r\right)^{H-\frac{1}{2}}\right) dr. \qquad (43)\end{aligned}$$

On the other hand we can calculate trivially that

$$\begin{aligned}\sigma_k^2 = \mathbf{E}\left[\left(Z_k^+\right)^2\right] = \mathbf{E}\left[\left(Z_k^-\right)^2\right] &= Q(0)\int_{t_k}^{t_{k+1}} 2H\left(t_{k+1} - r\right)^{2H-1} dr \\ &= Q(0)\left(t_{k+1} - t_k\right)^{2H}. \qquad (44)\end{aligned}$$

Let

$$\alpha_{j,k} = \frac{\sigma_{j,k}}{\sigma_k^2}, \qquad (45)$$

which is obviously positive. Then we can represent $Z_{j,k}^+$ using a centered Gaussian random variable $Y_{j,k}^+$ which is independent of $Z_k^+$ , as follows,

$$Z_{j,k}^+ = \alpha_{j,k} Z_k^+ + Y_{j,k}^+.$$

We can do the same for $Z_{j,k}^-$ and get

$$Z_{j,k}^- = \alpha_{j,k} Z_k^- + Y_{j,k}^-. \qquad (46)$$

In order to express the correlation between the '+' r.v.'s and the '−' r.v.'s, it is sufficient to note that if we have two Gaussian random variables $Z^+$ and $Z^-$ identically expressed using increments from $W(\cdot, +1)$ and $W(\cdot, -1)$ respectively, because of the tensor-product structure of

$W$, we immediately get that there exists a random variable $\bar{Z}$ independent of $Z^+$ and distributed identically to $Z^+$, such that

$$Z^- = \frac{Q(2)}{Q(0)} Z^+ + \frac{R(2)}{Q(0)} \bar{Z},$$

where $R(2) = \sqrt{Q^2(0) - Q^2(2)}$. Therefore we can rewrite (46) as

$$Z_{j,k}^- = \alpha_{j,k} \left( \frac{Q(2)}{Q(0)} Z_k^+ + \frac{R(2)}{Q(0)} \bar{Z}_k \right) + Y_{j,k}^-,$$

where $\bar{Z}_k$ and $Z_k^+$ are independent and identically distributed.

We are now in a position to prove the non-obvious fact that $Y_{j,k}^+$, which is independent of $Z_k^+$ by definition, is also independent of $Z_k^-$. Indeed,

$$\begin{aligned}
\mathbf{E}\left[Y_{j,k}^+ Z_k^-\right] &= \frac{1}{Q(0)} \mathbf{E}\left[Y_{j,k}^+ \left(Q(2) Z_k^+ + R(2) \bar{Z}_k\right)\right] \\
&= 0 + \frac{R(2)}{Q(0)} \mathbf{E}\left[\left(Z_{j,k}^+ - \alpha_{j,k} Z_k^+\right) \bar{Z}_k\right] \\
&= 0 + \mathbf{E}\left[Z_{j,k}^+ \left(Z_k^- - \frac{Q(2)}{Q(0)} Z_k^+\right)\right] - 0 \\
&= \mathbf{E}\left[Z_{j,k}^+ Z_k^-\right] - \frac{Q(2)}{Q(0)} \mathbf{E}\left[Z_{j,k}^+ Z_k^+\right] \\
&= 0.
\end{aligned}$$

The last equality comes from the tensor-product structure of $W$'s distribution, again. Independence follows from the jointly Gaussian property of $\left(Z_k^+, Z_k^-, Z_{j,k}^+, Z_{j,k}^-\right)$. Similarly, $Y_{j,k}^-$ is independent of both $Z_k^-$ and $Z_k^+$.

From these independence properties, since $\varepsilon_k^*$ depends by definition only on $Z_k^+$ and $Z_k^-$, we conclude that $\varepsilon_k^*$ is independent of $Y_{j,k}^+$ and of $Y_{j,k}^-$. Putting these facts together, we obtain the following calculation:

$$\begin{aligned}
\mathbf{E}\left[J_{j,k}(\varepsilon_k^*)\right] &= \mathbf{E}\left[Z_{j,k}^{\varepsilon_k^*}\right] \\
&= \mathbf{E}\left[\mathbf{1}_{\varepsilon_k^*=+1}\left(\alpha_{j,k} Z_k^+ + Y_{j,k}^+\right)\right] + \mathbf{E}\left[\mathbf{1}_{\varepsilon_k^*=-1}\left(\alpha_{j,k} Z_k^- + Y_{j,k}^-\right)\right] \\
&= \alpha_{j,k} \mathbf{E}\left[\mathbf{1}_{\varepsilon_k^*=+1} Z_k^+\right] + \alpha_{j,k} \mathbf{E}\left[\mathbf{1}_{\varepsilon_k^*=-1} Z_k^-\right] \\
&= \alpha_{j,k} \mathbf{E}\left[Z_k^{\varepsilon_k^*}\right]. \qquad (47)
\end{aligned}$$

*Step 4.3. Calculation of the entire maximized expectation.* We can now evaluate $\mathbf{E}\left[X_m\left(\tilde{t},\tilde{x}^*\right)\right]$. Applying (47) we first get

$$\begin{aligned}
\mathbf{E}\left[X_m\left(\tilde{t},\tilde{x}^*\right)\right] &= \sum_{j=0}^{m}\mathbf{E}\left[I_j\left(\varepsilon_j^*\right)\right] + \sum_{j=0}^{m}\sum_{k=0}^{j-1}\mathbf{E}\left[J_{j,k}\left(\varepsilon_k^*\right)\right] \\
&= \sum_{j=0}^{m}\mathbf{E}\left[Z_j^{\varepsilon_j^*}\right] + \sum_{j=0}^{m}\sum_{k=0}^{j-1}\alpha_{j,k}\mathbf{E}\left[Z_k^{\varepsilon_k^*}\right] \\
&\geq \sum_{k=0}^{m-1}\mathbf{E}\left[Z_k^{\varepsilon_k^*}\right] + \sum_{k=0}^{m-1}\sum_{j=k+1}^{m}\alpha_{j,k}\mathbf{E}\left[Z_k^{\varepsilon_k^*}\right] \\
&= \sum_{k=0}^{m-1}\mathbf{E}\left[Z_k^{\varepsilon_k^*}\right]\left(1+\sum_{j=k+1}^{m}\alpha_{j,k}\right). \qquad (48)
\end{aligned}$$

The inequality used above is a minor modification, where some small positive terms are thrown out in order to get a more tractable expression; it is not of any fundamental importance. Now we notice that for each $k \leq m$,

$$\mathbf{E}\left[Z_k^{\varepsilon_k^*}\right] = \mathbf{E}\left[\max\left(Z_k^+,Z_k^-\right)\right] = \frac{1}{\sqrt{2\pi}}\sqrt{\mathbf{E}\left[\left(Z_k^+-Z_k^-\right)^2\right]} = \sigma_k\sqrt{\frac{1}{\pi}\left(1-\frac{Q(2)}{Q(0)}\right)};$$

and meanwhile, (43) and (45) yield

$$\begin{aligned}
1+\sum_{j=k+1}^{m}\alpha_{j,k} &= \frac{1}{\sigma_k^2}\left(\sigma_k^2+\sum_{j=k+1}^{m}\sigma_{j,k}\right) \\
&= \frac{Q(0)}{\sigma_k^2}\int_{t_k}^{t_{k+1}} 2H\left(t_{k+1}-r\right)^{H-\frac{1}{2}}\left(t-r\right)^{H-\frac{1}{2}}dr.
\end{aligned}$$

Therefore (48) becomes

$$\begin{aligned}
&\mathbf{E}\left[X_m\left(\tilde{t},\tilde{x}^*\right)\right] \\
&\geq Q(0)\sqrt{\frac{1}{\pi}\left(1-\frac{Q(2)}{Q(0)}\right)}\left(\sum_{k=0}^{m-1}\frac{1}{\sigma_k}\int_{t_k}^{t_{k+1}} 2H\left(t_{k+1}-r\right)^{H-\frac{1}{2}}\left(t-r\right)^{H-\frac{1}{2}}dr\right) \\
&= 2H\sqrt{\frac{Q(0)-Q(2)}{\pi}}\left(\sum_{k=0}^{m-1}\left(t_{k+1}-t_k\right)^{-H}\int_{t_k}^{t_{k+1}}\left(t_{k+1}-r\right)^{H-\frac{1}{2}}\left(t-r\right)^{H-\frac{1}{2}}dr\right)
\end{aligned}$$

where we used (44). We can manipulate the integral in each term in the above sum further with a change of variable, and get

$$\begin{aligned}
&\int_{t_k}^{t_{k+1}}\left(t_{k+1}-r\right)^{H-\frac{1}{2}}\left(t-r\right)^{H-\frac{1}{2}}dr \\
&= \left(t_{k+1}-t_k\right)^{2H}\int_0^1 r^{H-\frac{1}{2}}\left(\frac{t-t_{k+1}}{t_{k+1}-t_k}+r\right)^{H-\frac{1}{2}}dr \\
&\geq \left(H+\frac{1}{2}\right)^{-1}\left(t_{k+1}-t_k\right)^{H+\frac{1}{2}}\left(t-t_{k+1}\right)^{H-\frac{1}{2}}.
\end{aligned}$$

The final result for this step is therefore

$$\mathbf{E}\left[X_m\left(\tilde{t},\tilde{x}^*\right)\right] \geq \frac{2H}{H+\frac{1}{2}}\sqrt{\frac{Q(0)-Q(2)}{\pi}}\left(\sum_{k=0}^{m-1}\left(t_{k+1}-t_k\right)^{\frac{1}{2}}\left(t-t_{k+1}\right)^{H-\frac{1}{2}}\right).$$

*Step 5. Restricting the time intervals.* It is clear at this point that the lower bound on any maximization of $\mathbf{E}\left[X_m\left(\tilde{t},\tilde{x}\right)\right]$ will be affected greatly by the time intervals between jumps. We now define $V_a(t,m)$ to be a subset of $\mathcal{S}(t,m)$, consisting of all $\tilde{t}$ such that the jump time $t_j$'s are quite evenly distributed: let

$$V_a(t,m) = \left\{\tilde{t}\in\mathcal{S}(t,m) : \frac{jt}{m}-\frac{at}{m}\leq t_j\leq\frac{jt}{m},\quad j=1,2,\ldots,m\right\}$$

with $0<a<1$. For any $\tilde{t}\in V_a(t,m)$, $m>1$, we have from the conclusion of Step 4

$$\begin{aligned}
\mathbf{E}\left[X_m\left(\tilde{t},\tilde{x}^*\right)\right] &\geq \frac{2H}{H+\frac{1}{2}}\sqrt{\frac{Q(0)-Q(2)}{\pi}}\sqrt{\frac{(1-a)t}{m}}\left(\sum_{k=0}^{m-1}\left(t-\frac{(k+1)t}{m}\right)^{H-\frac{1}{2}}\right)\\
&= \frac{2H}{H+\frac{1}{2}}\sqrt{\frac{(Q(0)-Q(2))(1-a)}{\pi}}\left(t^H m^{-H}\sum_{k=1}^{m}(m-k)^{H-\frac{1}{2}}\right)\\
&\geq \frac{2H}{H+\frac{1}{2}}\sqrt{\frac{(Q(0)-Q(2))(1-a)}{\pi}}\left(t^H m^{-H}\int_1^m (m-r)^{H-\frac{1}{2}}\,dr\right)\\
&\geq \frac{H}{\left(H+\frac{1}{2}\right)^2}\sqrt{\frac{(Q(0)-Q(2))(1-a)}{\pi}}\left(t^H\sqrt{m}\right). \qquad (49)
\end{aligned}$$

The last inequality above gives a lower bound on $\mathbf{E}\left[X_m\left(\tilde{t},\tilde{x}^*\right)\right]$ which is uniform on the specific set $V_a(t,m)$. We can calculate the probability of the set $V_a(t,m)$ under $\mathbf{P}_b$ given $N_t=m$:

$$\begin{aligned}
p_a &= \mathbf{P}_b\left[V_a(t,m)\,|N_t=m\right]\\
&= \int_{\tilde{t}\in V_a(t,m)}\frac{m!}{t^m}dt_1\cdots dt_m\\
&= \frac{m!}{t^m}\left(\int_{\frac{t}{m}-\frac{at}{m}}^{\frac{t}{m}}dt_1\right)\left(\int_{\frac{2t}{m}-\frac{at}{m}}^{\frac{2t}{m}}dt_2\right)\cdots\left(\int_{t-\frac{at}{m}}^{t}dt_m\right) = \frac{a^m m!}{m^m}.
\end{aligned}$$

*Step 6. Lower bound on $U$.* We may now obtain a lower bound on $U(t)$ by using the lower bound (49) on the set $V_a(t,m)$: we only need to throw out all paths such that $\tilde{t}$ is not in $V_a(t,m)$, and keep only the single trajectory $\tilde{x}^*$. We do this for a single value of $m$, discarding all other choices. Define the constant $C_{Q,H}=H\left(H+\frac{1}{2}\right)^{-2}\pi^{-1/2}\sqrt{Q(0)-Q(2)}$. We thus have

$$\begin{aligned}
U(t) &\geq \mathbf{E}\left[\log\left[p(t,m)\sum_{\tilde{x}\in\mathcal{P}_m}\frac{1}{2^m}\int_{\tilde{t}\in\mathcal{S}(t,m)}e^{X_m\left(\tilde{t},\tilde{x}\right)}\frac{m!}{t^m}dt_1\cdots dt_m\right]\right]\\
&\geq \mathbf{E}\left[\log\left[\frac{1}{2^m}p(t,m)\,p_a\int_{\tilde{t}\in V_a(t,m)}e^{X_m\left(\tilde{t},\tilde{x}^*\right)}\frac{m!}{p_a t^m}dt_1\cdots dt_m\right]\right]\\
&\geq \log\left[\frac{1}{2^m}p(t,m)\,p_a\right]+C_{Q,H}\sqrt{1-a}\;t^H\sqrt{m}.
\end{aligned}$$

The last inequality is true because $\frac{m!}{p_a t^m} dt_1 \cdots dt_m$ is indeed a probability measure on the set $V_a(t,m)$, and therefore we can use Jensen's inequality and Fubini theorem to obtain

$$\begin{aligned}\mathbf{E}\left[\log \int_{\tilde{t}\in V_a(t,m)} e^{X_m(\tilde{t},\tilde{x^*})} \frac{m!}{p_a t^m} dt_1 \cdots dt_m\right] &\geq \mathbf{E}\left[\int_{\tilde{t}\in V_a(t,m)} X_m\left(\tilde{t},\tilde{x^*}\right) \frac{m!}{p_a t^m} dt_1 \cdots dt_m\right] \\ &\geq C_{Q,H}\sqrt{1-a}\ t^H\sqrt{m}\end{aligned}$$

while adopting the inequality (49).

Since $p(t,m) = e^{-2\kappa t}\frac{(2\kappa t)^m}{m!}$ and $p_a = \frac{a^m m!}{m^m}$, we therefore have

$$U(t) \geq -2\kappa t + m(\log(a\kappa) + \log t - \log m) + C_{Q,H}\sqrt{1-a}\ t^H\sqrt{m}. \tag{50}$$

*Step 7. Choosing an optimal m; conclusion.* Now choose $a = \frac{3}{4}$ (this choice is somewhat arbitrary) and choose $m = \left\lceil \frac{\alpha^2 t^{2H}}{(\log t)^2} \right\rceil$ for some $\alpha$. This implies that $m\log m \leq \frac{1}{4}C_{Q,H}t^H\sqrt{m}$ for large $t$. By calculating

$$\frac{\alpha t^H}{\log t}(2\log\alpha + 2H\log t - 2\log\log t) \leq \frac{1}{4}C_{Q,H}t^H,$$

we are in a position to choose an optimal $\alpha = \alpha^*$: it suffices to choose $\alpha^* > \frac{C_{Q,H}}{8H}$ but close to $\frac{C_{Q,H}}{8H}$. It follows that with $m$ being chosen this way, we have $2\kappa t \ll t^H\sqrt{m}$ and $m\left(\log\left(\frac{3}{4}\kappa\right) + \log t\right) \asymp t^H\sqrt{m}$ for large $t$. Therefore we obtain a lower bound on $U(t)$ for large $t$:

$$U(t) \geq \frac{1}{8}C_{Q,H}t^H\left(\frac{\alpha^* t^H}{\log t}\right) = \left(\frac{\alpha^* C_{Q,H}}{8}\right)\frac{t^{2H}}{\log t}. \tag{51}$$

Now invoking the concentration result of Corollary 6.2, we have proved the theorem. ■

**Remark 6.8** *This theorem validates the fact we indicated at the beginning of this section, that is*

$$\lim_{t\to\infty}\frac{1}{t}\log u(t) = +\infty, \quad a.s.$$

**Remark 6.9** *We cannot say the function* $L_-(t) := t^{2H}/\log t$ *is the exponential rate function of* $u(t)$ *since we do not know whether* $\limsup_{t\to\infty}\left(t^{2H}/\log t\right)^{-1}U(t)$ *is finite or not. But in either case* $L_-(t)$ *is a proper lower bound on the exponential rate function.*

**Remark 6.10** *The lower bound given in (51) is sharp in the sense that its order cannot be improved in the context of our arguments.*

Indeed, we see in (50) there are two negative terms: $-2\kappa t$, $-m\log m$; and three positive terms: $m\log(a\kappa)$, $m\log t$, $C_{Q,H}\sqrt{1-a}\ t^H\sqrt{m}$. When $m \gg t^{2H}/(\log t)^2$, it is obvious that $m\log m \gg m\log t \gg m$ and $m\log m \gg t^H\sqrt{m}$ for large $t$. Then the sum of all these negative and positive terms gives us a negative lower bound of $U(t)$, which is of no help since we know $U(t) \geq 0$. When $m \ll t^{2H}/(\log t)^2$, it is also easy to see that we get either a negative lower bound or a lower bound smaller than $t^{2H}/\log t$. Therefore the sharp estimation on the lower bound of $U(t)$ will be obtained when $m \asymp t^{2H}/(\log t)^2$, just as we did above. For such $m$, we see that

$$m\log m \asymp m\log t \asymp t^H\sqrt{m} \asymp \frac{t^{2H}}{\log t}$$

for large $t$. This shows that the lower bound in (51) cannot be improved with our methods.

### 6.4 Discussion

We realize that there is a gap between the lower bound and upper bound on the exponential rate function of $u(t)$. Unfortunately we have not found a way to make them coincide at this point. However, the discrepancy here is quite small, in the sense that for any positive $\varepsilon$ close to 0,

$$\lim_{t\to\infty}\frac{1}{t^{2H-\varepsilon}}\log u(t)=\infty,\quad \text{a.s.}$$

Therefore, if $L(t)$ is of the power form $t^{\beta}$, $\beta$ can only be $2H$.

Recall that in Subsection 6.3, to estimate the lower bound of $U(t)$, we construct an "optimal" jump path $\tilde{x}^*$ for fixed jump number $m$ and jump time $\tilde{t}$, which maximizes the innovations part of the increments. This choice also turns out to be beneficial for the non-innovations part of the increments. Then we obtain formula (49), a lower bound on $\mathbf{E}\left[X_m\left(\tilde{t},\tilde{x}^*\right)\right]$ uniform over $\tilde{t}$ in the specific set $V_a(t,m)$, a subset of $\mathcal{S}(t,m)$, with approximately evenly spaced jump times. It is worth pointing out that even if (49) held uniformly on the set of all jump time sequences, $\mathcal{S}(t,m)$, our estimation of the lower bound of $U(t)$, (51), would not be improved. The reason is that if (49) holds for all $\tilde{t}\in\mathcal{S}(t,m)$, we have

$$\begin{aligned}
U(t) &\geq \mathbf{E}\left[\log\left[p(t,m)\sum_{\tilde{x}\in\mathcal{P}_m}\frac{1}{2^m}\int_{\tilde{t}\in\mathcal{S}(t,m)}e^{X_m(\tilde{t},\tilde{x})}\frac{m!}{t^m}dt_1\cdots dt_m\right]\right]\\
&\geq \mathbf{E}\left[\log\left[\frac{1}{2^m}p(t,m)\int_{\tilde{t}\in\mathcal{S}(t,m)}e^{X_m(\tilde{t},\tilde{x}^*)}\frac{m!}{t^m}dt_1\cdots dt_m\right]\right]\\
&\geq \log\left[\frac{1}{2^m}p(t,m)\right]+C_{Q,H}t^H\sqrt{m}.
\end{aligned}$$

Since $p(t,m)=e^{-2\kappa t}\frac{(2\kappa t)^m}{m!}$, using Stirling's formula $m!=\sqrt{2\pi m+o(m)}\,e^{-m}m^m$, we still only get

$$U(t)\geq -2\kappa t+m\left(\log(c\kappa)+\log t-\log m\right)+C_{Q,H}t^H\sqrt{m}.$$

In other words, considering only those $\tilde{t}\in V_a(t,m)$ is not a restriction.

There are two possible ways of getting a sharper lower bound estimation on $U(t)$. 1) Instead of taking only one term in the expression of $u(t)$, take many terms, that is, consider many $m$'s simultaneously. But for too many different $m$'s, the various behaviors of $X_m\left(\tilde{t},\tilde{x}^*\right)$ might deviate from each other, and putting them together without using Jensen's inequality would entail a difficult calculation involving their correlations. We may need other tools. 2) Use another construction of the "optimal" path $\tilde{x}^*$ which can yield larger increments. However, in such a construction we may lose the independence of the $\varepsilon_j$'s, which will make the covariance calculation extremely hard. It may be that a better place to look for improving our work for $H>1/2$ is in the upper bound. However, one first needs some intuition as to which of the two rate functions $t^{2H}$ and $t^{2H}\log^{-1}t$ is closest to the truth, which is an entirely open issue at this stage.

## 7 Appendix: Riemann–Liouville Fractional Brownian Motion

In this section, we list some results about the Riemann–Liouville fractional Brownian motion, while the very last subsection is the proof of our analytic almost-superadditive Theorem 4.3.

### 7.1 RL-fBm: definition

**Definition 7.1** *Let $(\Omega, \mathbf{P}; W)$ be a canonical Wiener space. A Riemann–Liouville fractional Brownian motion (RL–fBm) $B_H$ with respect to $(\Omega, \mathbf{P}; W)$ is a centered Gaussian process on $\mathbf{R}_+$, defined as a Wiener integral against the Wiener process $W$: for any $t \geq 0$,*

$$B_H(t) := \int_0^t \sqrt{2H}\,(t-u)^{H-\frac{1}{2}}\,W(du) \tag{52}$$

*where $H \in (0,1)$ is called Hurst parameter. We also denote the integrand in (52) by*

$$K_H(t,s) = K_H(t-s) := \sqrt{2H}\,(t-s)^{H-\frac{1}{2}}, \quad s \leq t.$$

**Remark 7.2** *When $H = \frac{1}{2}$, $B_H$ is a standard Brownian motion.*

**Remark 7.3** *Unlike in the first 6 sections of this paper, we use the notation $B_H$ here to denote a scalar RL-fBm, not an infinite-dimensional one with spatial covariance $Q$.*

The covariance structure of $B_H$ is given by

$$R_H(s,t) := \mathbf{E}\left[B_H(s)\,B_H(t)\right] = 2H\int_0^{s\wedge t}(t-r)^{H-\frac{1}{2}}(s-r)^{H-\frac{1}{2}}\,dr. \tag{53}$$

Specifically, when $t > s$, it turns out that $R_H(s,t) = \frac{2H}{H+\frac{1}{2}}\,t^H s^H \left(\frac{s}{t}\right)^{\frac{1}{2}}\ {}_2F_1\left(\frac{1}{2}-H, 1, H+\frac{1}{2}, \frac{s}{t}\right)$ ,where ${}_2F_1$ is the hypergeometric function. In the next section, we find some much more tractable estimations of the covariance structure of $B_H$.

### 7.2 Increments of $B_H$

Define the so-called squared canonical metric of $B_H$ as the variance of its increments, i.e. $\delta^2(s,t) := \mathbf{E}\left[(B_H(t) - B_H(s))^2\right]$. It is crucial to provide sharp bounds on $\delta^2(s,t)$.

**Proposition 7.4** *For all $H \in (0,1)$, with $C_H = 2H\int_0^\infty \left((1+y)^{H-1/2} - y^{H-1/2}\right)^2 dy$,*

$$(t-s)^{2H} \leq \delta^2(s,t) \leq C_H\,(t-s)^{2H}.$$

**Proof.** We have

$$\begin{aligned}
\delta^2(s,t) &= \mathbf{E}\left[(B_H(t) - B_H(s))^2\right] \\
&= \mathbf{E}\left[\left(\int_0^s \left[K_H(t-r) - K_H(s-r)\right] W(dr) + \int_s^t K_H(t-r)\,W(dr)\right)^2\right] \\
&= \int_0^s \left[K_H(t-r) - K_H(s-r)\right]^2 dr + \int_s^t \left[K_H(t-r)\right]^2 dr \\
&= \int_0^s \left[K_H(t-r) - K_H(s-r)\right]^2 dr + (t-s)^{2H}.
\end{aligned}$$

The lower bound of the proposition follows immediately. For the upper estimate, it suffices to write the additional term above as follows:

$$\begin{aligned}\int_0^s [K_H(t-r) - K_H(s-r)]^2\, dr &= \int_0^s \left[(t-r)^{H-1/2} - (s-r)^{H-1/2}\right]^2 dr \\ &= (t-s)^{2H} \int_0^{s/(t-s)} \left((1+y)^{H-1/2} - y^{H-1/2}\right) \\ &\leq C\,(t-s)^{2H}.\end{aligned}$$

■

## 7.3 Relation to Standard Fractional Brownian Motion

The standard fractional Brownian motion $B_H^f$ can be written as a Wiener integral against $W$ as

$$B_H^f(t) = \int_{-\infty}^0 \left[(t-u)^{H-\frac{1}{2}} - u^{H-\frac{1}{2}}\right] W(du) + \int_0^t (t-u)^{H-\frac{1}{2}}\, W(du).$$

This is called the moving average representation of $B_H^f$. We notice here that $B_H^f(t)$ is the sum of the RL-fBm and of a random process that is differentiable. Another formula, similar to (52), also holds for standard fBm, with, instead of the quantity $\sqrt{2H}\,(t-u)^{H-\frac{1}{2}}$, a more complicated kernel $K(t,u)$; this is often called the kernel representation of fBm. See [23] for details. It is well-known that $B_H^f$ has stationary increments, that is, for any $t, h \in \mathbf{R}_+$,

$$\mathbf{E}\left[\left(B_H^f(t+h) - B_H^f(t)\right)^2\right] = h^{2H}.$$

The RL–fBm $B_H$ fails to have such a property. However, we can calculate by how much the increments of $B_H$ fail to be stationary. Indeed, as shown in the previous Subsection, there exist constants $c_H, C_H > 0$ such that for all $s, t, h \in \mathbf{R}_+$,

$$c_H \mathbf{E}\left[\left(B_H^f(t+h) - B_H^f(t)\right)^2\right] \leq \mathbf{E}\left[(B_H(t+h) - B_H(t))^2\right] \leq C_H \mathbf{E}\left[\left(B_H^f(t+h) - B_H^f(t)\right)^2\right].$$

This shows that the covariance structure of $B_H$ is commensurate with that of $B_H^f$. Therefore $B_H$ shares the same regularity properties as $B_H^f$, and several other crucial properties; we list the most important ones here:

(i) $B_H(0) = 0$;

(ii) $B_H$ is not a martingale; $B_H$ is not a Markov process;

(iii) $B_H$ is adapted to a Brownian filtration;

(iv) Almost every path of $B_H$ is $\alpha$–Hölder continuous whenever $\alpha \in (0, H)$; more precisely, $f(r) = r^H \sqrt{\log r^{-1}}$ is, up to a constant, an almost-sure uniform modulus of continuity for $B_H$;

(v) $B_H$ is self–similar with parameter $H$: for any constant $a > 0$, the law of $\{B_H(at) : t \in \mathbf{R}_+\}$ and the law of $\left\{a^H B_H(t) : t \in \mathbf{R}_+\right\}$ are identical.

**Remark 7.5** *Property (iv) indicates that $B_H$ is less (resp. more) regular than Brownian motion when $H \in \left(0, \frac{1}{2}\right)$ (resp. $H \in \left(\frac{1}{2}, 1\right)$); the standard fractional Brownian motion is the only continuous stochastic process with finite variance that is self-similar with parameter $H$ and has stationary increments.*

## 7.4 Wiener Integral with respect to $B_H$

Just as is done with regular Brownian motion, we can give a proper definition of the Wiener integral with respect to $B_H$.

Let $B_H$ be a RL–fBm with respect to the Wiener space $(\Omega, \mathbf{P}; W)$. Let $\varphi$ be a deterministic measurable function on $\mathbf{R}_+$. For any fixed $T > 0$, we define the operator $K_H^*$ on $\varphi$ by

$$[K_H^* \varphi](T, s) := K_H(T, s)\varphi(s) + \int_s^T (\varphi(r) - \varphi(s)) \frac{\partial K_H}{\partial r}(r, s)\, dr \tag{54}$$

if it exists. When $H > 1/2$, the actual domain of this operator can be extended beyond bonafide functions, but for simplicity, and because we will not need to consider the action of $K^*$ on non-functional elements, we define the functional domain of $K^*$ as the set of functions $\varphi$ such that $[K_H^* \varphi](T, \cdot) \in L^2([0, T])$. We denote this space of functions by $|\mathcal{H}|$, and denote

$$\|\varphi\|_{|\mathcal{H}|} = \|K_H^* \varphi\|_{L^2([0,T])} = \int_0^T \left| K_H(T, s)\varphi(s) + \int_s^T (\varphi(r) - \varphi(s)) \frac{\partial K_H}{\partial r}(r, s)\, dr \right|^2 ds. \tag{55}$$

This $|\mathcal{H}|$ is the so-called *canonical functional Hilbert space* of $B_H$ on $[0, T]$. When $H > 1/2$, we note that $|\mathcal{H}|$ contains all continuous functions: indeed, the singularity at $r = s$ for $\frac{\partial K_H}{\partial r}$ in the Riemann integral in the definition (54) has power $H - 3/2 > -1$. On the other hand, when $H < 1/2$, this singularity is not integrable, and one can only guarantee existence of the said Riemann integral if $\varphi$ is itself sufficiently regular: we then see that $|\mathcal{H}|$ contains all functions which are $\alpha$-Hölder-continuous with $\alpha > 1/2 - H$. The reader may have already noticed that usage of the operator $K^*$ made in this article is for functions $\varphi$ which are differentiable, so that they are in $|\mathcal{H}|$ for any $H$. By integration by parts, it is easy to see that when $H > \frac{1}{2}$, exploiting its regularity, the operator $K_H^*$ in (54) can be rewritten as

$$[K_H^* \varphi](T, s) = \int_s^T \varphi(r) \frac{\partial K_H}{\partial r}(r, s)\, dr.$$

For any function $\varphi$ in $|\mathcal{H}|$ we define the Wiener integral of $\varphi$ with respect to $B_H$ on $[0, T]$ as the centered Gaussian random variable given by

$$\int_0^T \varphi(r)\, B_H(dr) = \int_0^T [K_H^* \varphi](T, r)\, W(dr). \tag{56}$$

## 7.5 Complex-Valued Processes

**Definition 7.6** *We say $B_H$ is a complex valued RL–fBm on $\mathbf{R}_+$, with Hurst parameter $H$, if $B_H = \frac{1}{\sqrt{2}}(B_{H,1} + iB_{H,2})$, where $B_{H,1}$ and $B_{H,2}$ are independent (real) RL–fBm's on $\mathbf{R}_+$, with the same Hurst parameter $H$. The covariance structure of $B_H$ is then given by*

$$R_H(s, t) := \mathbf{E}\left[B_H(s)\overline{B_H(t)}\right] = 2H \int_0^{s \wedge t} (t - r)^{H - \frac{1}{2}} (s - r)^{H - \frac{1}{2}}\, dr.$$

The definition of the Wiener integral can now be extended to complex-valued RL–fBm. We first extend the Hilbert space $|\mathcal{H}|$ to a larger space consisting of all complex valued deterministic functions $\varphi = \varphi_1 + i\varphi_2$ with $\varphi_1, \varphi_2 \in |\mathcal{H}|$. Without ambiguity, we still denote the new space by

$|\mathcal{H}|$. It is notable that the operator $K_H^*$ defined in (54) can be extended to an operator on $|\mathcal{H}|$, denoted by $K_H^*$ again, given by

$$[K_H^* \varphi](T, \cdot) := [K_H^* \varphi_1](T, \cdot) + i [K_H^* \varphi_2](T, \cdot).$$

The norm given in formula (55) is valid as well.

**Definition 7.7** *Let $B_H = \frac{1}{\sqrt{2}}(B_{H,1} + iB_{H,2})$ be a complex-valued RL–fBm. For any $\varphi \in |\mathcal{H}|$, say $\varphi = \varphi_1 + i\varphi_2$, we define the Wiener integral of $\varphi$ with respect to $B_H$ on $[0, T]$ by*

$$\begin{aligned} \int_0^T \varphi(r) B_H(dr) := &\left( \int_0^T \varphi_1(r) B_{H,1}(dr) - \int_0^T \varphi_2(r) B_{H,2}(dr) \right) \\ &+ i \left( \int_0^T \varphi_1(r) B_{H,2}(dr) + \int_0^T \varphi_2(r) B_{H,1}(dr) \right). \end{aligned}$$

Let $W_1$ and $W_2$ be two independent Wiener processes such that

$$B_{H,j}(t) = \int_0^t K_H(t, s) W_j(dr), \quad j = 1, 2;$$

and let $W = \frac{1}{\sqrt{2}}(W_1 + iW_2)$ which is a complex-valued Wiener process, then the above Wiener integral can be written in exactly the same form as (56).

### 7.6 Proof of Theorem 4.3

Derriennic and Harchem [17] proved the first part while considering a general case where $\{f(n)\}_{n \in \mathbf{N}}$ is an almost superadditive sequence of integrable functions. We follow their techniques here.

Let $\tilde{f}(n) = \frac{f(n)}{n}$ and $\tilde{\epsilon}(n) = \frac{\epsilon(n)}{n}$. Assume $\sup_n \tilde{f}(n) < \infty$. We first prove $\left\{\tilde{f}(2^n)\right\}$ converges. From the almost superadditivity we have

$$f\left(2^{n+i}\right) \geq 2f\left(2^{n+i-1}\right) - \epsilon\left(2^{n+i}\right),$$

which is equivalent to

$$\tilde{f}\left(2^{n+i}\right) \geq \tilde{f}\left(2^{n+i-1}\right) - \tilde{\epsilon}\left(2^{n+i}\right);$$

by induction we therefore get

$$\tilde{f}\left(2^{n+k}\right) \geq \tilde{f}(2^n) - \sum_{i=1}^{k} \tilde{\epsilon}\left(2^{n+i}\right).$$

As $k \uparrow \infty$ ($n$ fixed), it follows that

$$\liminf_{k \to \infty} \tilde{f}\left(2^k\right) = \liminf_{k \to \infty} \tilde{f}\left(2^{n+k}\right) \geq \tilde{f}(2^n) - \sum_{i=1}^{\infty} \tilde{\epsilon}\left(2^{n+i}\right);$$

and then letting $n \uparrow \infty$, we get

$$\liminf_{k \to \infty} \tilde{f}\left(2^k\right) \geq \limsup_{n \to \infty} \tilde{f}(2^n)$$

since $\sum_{n=1}^{\infty} \tilde{\epsilon}(2^n) < \infty$. Therefore $\lim_{n\to\infty} \tilde{f}(2^n)$ exists and is finite; we denote this limit as $f^*$.

We claim that $\{f(n)\}_{n\in\mathbf{N}}$ is bounded from below. Indeed, since $\left\{\tilde{f}(2^n)\right\}$ converges, it is bounded from below; meanwhile, $\tilde{\epsilon}(n)$ is bounded from above since $\lim_{n\to\infty} \tilde{\epsilon}(n) = 0$. Let $M$ be a positive number such that $\tilde{f}(2^n) \geq -M$ and $\tilde{\epsilon}(n) \leq M$ for all $n \in \mathbf{N}$. Then

$$\begin{aligned} f\left(2^i + 2^j\right) &\geq f\left(2^i\right) + f\left(2^j\right) - \epsilon\left(2^i + 2^j\right) \\ &= 2^i \tilde{f}\left(2^i\right) + 2^j \tilde{f}\left(2^j\right) - \left(2^i + 2^j\right) \tilde{\epsilon}\left(2^i + 2^j\right) \\ &\geq -2M\left(2^i + 2^j\right). \end{aligned}$$

Any integer $n$ can be decomposed as a sum of powers of 2, say $n = \sum_{i=1}^{\ell} 2^{k_i}$. Thus we get, by induction,

$$f(n) \geq -2M \sum_{i=1}^{\ell} 2^{k_i} = -2Mn.$$

Together with the assumption $\sup_n \tilde{f}(n) < \infty$ we see $\left\{f^* - \tilde{f}(n)\right\}$ is bounded. Let

$$l^+ := \limsup_{n\to\infty} \left(f^* - \tilde{f}(n)\right)^+;$$

here $x^+$ is the positive part of $x$, i.e., $x^+ = x \vee 0$. Obviously $l^+$ is finite. There exist two sequences $\{t_j\}$ and $\{n_j\}$ such that $l^+ = \lim_{j\to\infty} \left(f^* - \tilde{f}(t_j)\right)^+$ and $2^{n_j} \leq t_j \leq 2^{n_j+1}$. Noting that $\frac{1}{2} \leq \frac{2^{n_j}}{t_j} \leq 1$, we may assume $\lim_{j\to\infty} \frac{2^{n_j}}{t_j} = \alpha$ with $\frac{1}{2} \leq \alpha \leq 1$; otherwise, we only need take a subsequence. Since

$$f(t_j) \geq f(2^{n_j}) + f(t_j - 2^{n_j}) - \epsilon(t_j),$$

we obtain

$$\left(f^* - \tilde{f}(t_j)\right)^+ \leq \frac{2^{n_j}}{t_j} \left(f^* - \tilde{f}(2^{n_j})\right)^+ + \left(1 - \frac{2^{n_j}}{t_j}\right) \left(f^* - \tilde{f}(t_j - 2^{n_j})\right)^+ + \tilde{\epsilon}(t_j).$$

Letting $j \to \infty$ we then get $l^+ \leq (1-\alpha) l^+$ and hence $l^+ = 0$.

On the other hand, let

$$l^- := \limsup_{n\to\infty} \left(f^* - \tilde{f}(n)\right)^-;$$

$l^-$ is finite therefore. Here $x^-$ is the negative part of $x$, *i.e.*, $x^- = (-x) \vee 0$. We can construct two sequences $\{s_j\}$ and $\{m_j\}$ such that $l^- = \lim_{j\to\infty} \left(f^* - \tilde{f}(s_j)\right)^-$, $2^{m_j-1} \leq s_j \leq 2^{m_j}$ and $\lim_{j\to\infty} \frac{s_j}{2^{m_j}} = \beta$ with $\frac{1}{2} \leq \beta \leq 1$. Since

$$f(2^{m_j}) \geq f(s_j) + f(2^{m_j} - s_j) - \epsilon(2^{m_j}),$$

we obtain

$$\frac{s_j}{2^{m_j}} \left(f^* - \tilde{f}(s_j)\right)^- \leq \left(f^* - \tilde{f}(2^{m_j})\right)^- + \left(1 - \frac{s_j}{2^{m_j}}\right) \left(f^* - \tilde{f}(2^{m_j} - s_j)\right)^+ + \tilde{\epsilon}(2^{m_j}).$$

Letting $j \to \infty$ we then get $\beta l^- \leq (1-\beta) l^+ = 0$ and hence $l^- = 0$.

Note that

$$\limsup_{n\to\infty}\left|f^* - \tilde{f}(n)\right| \le l^+ + l^- = 0.$$

Therefore we have proved that

$$\lim_{n\to\infty}\tilde{f}(n) = f^*.$$

Now, let us assume $\sup_n \frac{f(n)}{n} = \infty$. We can still get the inequality

$$\liminf_{n\to\infty}\tilde{f}(2^n) \ge \limsup_{n\to\infty}\tilde{f}(2^n).$$

This is to say $\left\{\tilde{f}(2^n)\right\}$ either converges or diverges properly to $\infty$. However, if $\left\{\tilde{f}(2^n)\right\}$ converges, say, $f^* = \lim_{n\to\infty}\tilde{f}(2^n)$ is finite, then following the above arguments we will get $l^+ = l^- = 0$. (Note that whether $l^-$ is finite is not crucial in the arguments.) This implies $\{f(n)\}_{n\in\mathbf{N}}$ indeed converges, hence is bounded, which is a contradiction. Therefore, $\left\{\tilde{f}(2^n)\right\}$ diverges properly to $\infty$.

The claim that $\{f(n)\}_{n\in\mathbf{N}}$ is bounded from below is thus true. Let $l_- := \liminf_{n\to\infty}\tilde{f}(n)$, finite or infinite, then $l_- > -\infty$. Let $\{t_j\}$ and $\{n_j\}$ be two sequences such that $l_- = \lim_{j\to\infty}\tilde{f}(t_j)$, $2^{n_j} \le t_j \le 2^{n_j+1}$ and $\lim_{j\to\infty}\frac{2^{n_j}}{t_j} = \alpha$ with $\frac{1}{2} \le \alpha \le 1$. Since

$$\tilde{f}(t_j) \ge \frac{2^{n_j}}{t_j}\tilde{f}(2^{n_j}) + \left(1 - \frac{2^{n_j}}{t_j}\right)\tilde{f}(t_j - 2^{n_j}) - \tilde{\epsilon}(t_j),$$

we obtain

$$\begin{aligned}\lim_{j\to\infty}\tilde{f}(t_j) &\ge \alpha\lim_{j\to\infty}\tilde{f}(2^{n_j}) + (1-\alpha)\limsup_{j\to\infty}\tilde{f}(t_j - 2^{n_j})\\ &\ge \alpha\lim_{j\to\infty}\tilde{f}(2^{n_j}) + (1-\alpha)\liminf_{n\to\infty}\tilde{f}(n).\end{aligned}$$

This implies $l_- = \infty$, i.e., $\{f(n)\}_{n\in\mathbf{N}}$ diverges properly to $\infty$.

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