Sharp estimation of the almost-sure Lyapunov exponent for the Anderson model in continuous space

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Abstract

In this article we study the exponential behavior of the continuous stochastic Anderson model, i.e. the solution of the stochastic partial differential equation $u(t,x) = 1 + \int_0^t \kappa \Delta_x u(s,x)\, ds + \int_0^t W(ds,x)\, u(s,x)$, when the spatial parameter x is continuous, specifically $x \in \mathbf{R}$, and W is a Gaussian field on $\mathbf{R}_+ \times \mathbf{R}$ that is Brownian in time, but whose spatial distribution is widely unrestricted. We give a partial existence result of the Lyapunov exponent defined as $\lim_{t\to\infty} t^{-1} \log u(t,x)$. Furthermore, we find upper and lower bounds for $\limsup_{t\to\infty} t^{-1} \log u(t,x)$ and $\liminf_{t\to\infty} t^{-1} \log u(t,x)$ respectively, as functions of the diffusion constant κ which depend on the regularity of W in x. Our bounds are sharper, work for a wider range of regularity scales, and are significantly easier to prove than all previously known results. When the uniform modulus of continuity of the process W is in the logarithmic scale, our bounds are optimal.

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1 Preliminaries

1.1 Introduction

This article studies the almost-sure large-time exponential behavior of the so-called *(stochastic parabolic) Anderson model* in $\mathbf{R}$, i.e., the solution of the following stochastic parabolic partial differential equation with linear multiplicative potential: for all $x \in \mathbf{R}$ and all $t \geq 0$,

$$u(t,x) = 1 + \int_0^t \kappa \frac{\partial^2 u}{\partial x^2}(s,x)\, ds + \int_0^t W(ds,x)\, u(s,x)\,. \tag{1}$$

Here the *potential* W is a centered Gaussian field on $\mathbf{R}_+ \times \mathbf{R}$ that is Brownian in the time parameter t when the space parameter x is fixed, and has an arbitrary covariance structure in the space parameter x. All previous work on this Anderson model with space-time-dependent potential, whether in continuous or discrete space, has concentrated on the case where W is homogeneous in space, e.g. [3], [4], [5], [6], [7], [8], [10], [16], [17]. Our article makes no such assumption, asking only for milder regularity and non-degeneracy, and proving results that strengthen all existing ones, using simpler, more efficient proofs. This introduction contains a detailed narrative explaining the nature and significance of this article's qualitative and quantitative results; a casual reader will find, in Section 1.1.3, a short guide to extracting precise statements of all results.

1.1.1 Qualitative outline and significance of results

This article connects the regularity properties of W with the quantitative behavior of u, a direction which was never achieved precisely before, with potentially important consequences for the physical systems connected to the Anderson model. We refer to [7] and [11] for specific physical motivations in astrophysics, hydrodynamics, and other fields. In general modeling terms, our ultimate regularity result says that if the potential's spatial modulus of continuity is known with some precision, one obtains sharp bounds on the exponential rate of increase of u in time, and conversely under certain circumstances (Corollary 29). One can then argue that if a given rate of increase is observable for u, which is typically the case for physical systems modeled by u as time evolutions in a random potential W, then the regularity properties of the random medium W can be estimated with excellent precision. This can be achieved with little or no need for statistical inference, which is particularly useful in the many situations where W is typically not directly observable.

Section 2 deals with general non-quantitative results on the existence of a Lyapunov exponent (see (2)). Sections 3 and 4 provide lower and upper bounds respectively on this exponential behavior. Section 5 investigates the exact quantitative meaning of these bounds for some specific scales. The remainder of this introduction gives a detailed account of our results, indicating which tools are employed, and comparing our results and techniques to those used in the above-cited references.

Under some very mild boundedness and non-degeneracy conditions on the spatial covariance structure of W, we prove that the so-called *almost sure Lyapunov exponent* λ defined by

$$\lambda = \lim_{t\to\infty} \frac{1}{t} \log u(t,x), \tag{2}$$

if it exists, is non-random, uniformly positive, and typically does not depend on x. We also give a clear deterministic criterion under which existence of λ holds. This is achieved in Section 2. Our proof techniques are sharp in the sense that, in the spatially homogeneous case, existence of λ follows immediately, at least when t tends to infinity along an arithmetic sequence. They are also more efficient than those used in the works cited above. Our non-degeneracy hypothesis is used to prove a crucial positivity result in Section 2.4 despite the lack of any super- or sub-additivity property (such properties are crucial in previous works). Inspired by a new idea in [15] in the context of directed polymers, we pioneer a use of the Malliavin calculus for establishing existence of λ. In our situation, the Gaussian property of W appears to be far from necessary. Beyond being a measure of our methods' efficiency, this is an indication that many known almost sure results for the Anderson model may hold with non-Gaussian potentials. We will investigate this last idea in another article.

Regardless of whether the Lyapunov exponent exists, we are able to find upper and lower bounds for $\limsup_{t\to\infty} t^{-1} \log u(t,x)$, and $\liminf_{t\to\infty} t^{-1} \log u(t,x)$ respectively, as functions of the diffusion constant κ when κ is small enough, in Sections 3 and 4. The methods employed are very simple compared with the existing upper- and lower-bound proofs in the above-cited references in continuous space; they are no more complex than the proofs in the simpler case of $x \in \mathbf{Z}$. Our techniques are also more powerful since they do not require homogeneity. We borrowed a crucial idea based on Gaussian supremum estimation techniques originally introduced in the present article's second-named author's own work [6], and further sharpened in [4], which is to consider the expected value of the supremum over all Feynman-Kac paths of the potential integrated along each path. Both of these references are in discrete space $x \in \mathbf{Z}^d$ with homogeneous potential. In contrast to [4] and [6], however, we work in continuous space, and our probabilistic estimations herein draw heavily on the new ideas of Section 2.

One can compare our bounds with those previously obtained, when the potential W is assumed to be essentially H-Hölder-continuous in space, but not H'-Hölder-continuous for any $H' > H$. In this situation, the lower bound, derived in Section 3, is of the order $\kappa^{H/(H+1)}$. The only previously known result in continuous space, computed in the same Hölder scale for W homogeneous, can be found in [17]: a lower bound of order $\kappa^{H/(H+1)} / \log^{(1-H)/(1+H)}(1/\kappa)$. Moreover the techniques used in [17] are excruciatingly complicated, and are limited to the said Hölder scale only. Our new result improves the one in [17] slightly, extends beyond the Hölder scale, applies to the non-homogenous case, and the proofs are comparatively much simpler.

In Section 5, we show that our lower bound is in fact optimal when the spatial regularity of W is in a logarithmic scale: we find that $\liminf_{t\to\infty} t^{-1} \log u(t,x)$ is bounded below by a constant multiple of $\left(\log \kappa^{-1}\right)^{-\beta}$ when W admits the function $\left(\log \kappa^{-1}\right)^{-\beta/2+1/2}$ as an almost-sure uniform modulus of continuity on any interval in $\mathbf{R}$. This is precisely the same value, up to a constant, as the upper bound on $\limsup_{t\to\infty} t^{-1} \log u(t,x)$ which we obtain in Section 4. Even in this very irregular scale, our upper bound is an improvement over the known result in continuous space, which was obtained for homogeneous W in [7], namely the order $(\log \kappa^{-1})^{-1}$. Our upper bound derivation, constructed for the nonhomogeneous case, is again simpler than the proof in [7]. In the Hölder scale, we obtain an even better result, namely an upper bound of the order $\kappa^{H/(3H+1)}$. We can see that for small H, the difference between our upper and lower bounds become negligible, which is consistent with the fact that our bounds are sharp in the logarithmic scale, since that scale can be understood as living within the case $H = 0$.

The authors of [17] used a spatial discretization technique first introduced in [7], improving the original error estimate of [7] significantly. In our proof of the lower bound, we do not need to use any discretization. We only rely on a discretization for the upper bound proof, and then again essentially only in its original form as given in [7], i.e. without resorting to the exceedingly delicate analytic arguments of the improved error estimate in [17].

1.1.2 Quantitative conclusions and heuristics

The global quantitative conclusion we can draw from our estimations is that, regardless of any spatial homogeneity, the almost-sure exponential rate of increase of $u(t,x)$ in large time (and, if it exists, the Lyapunov exponent) is closely related to the local spatial regularity of W: if W is precisely H-Hölder-continuous in space, the rate is sandwiched between $\kappa^{H/(H+1)}$ and $\kappa^{H/(3H+1)}$; if W is more irregular yet, specifically if $(\log \kappa^{-1})^{-\gamma}$ is a sharp almost-sure uniform modulus of continuity for W in space, then the rate is, up to a constant, equal to $(\log \kappa^{-1})^{-2\gamma-1}$, and a converse of this result in the logarithmic scale holds.

In view of the (small) gap between our upper and lower bounds, it is difficult to give an intuitive idea, at least in the Hölder scale, of why such results should even hold; if such an idea were discernable, we would be in a position to formulate a conjecture as to what the true Lyapunov exponent should be in all cases. In the logarithmic scale, however, things are a little more clear, when one compares discrete and continuous models. In the discrete case, one can consult [4] for a simple heuristic, based on the Gaussian property of the increments of W and on the Poisson law for the sequence of jump times of the Feynman-Kac paths, to see why the Lyapunov exponent should be of order $\log^{-1}(\kappa^{-1})$.

In one interpretation, to draw a link between discrete and continuous space cases, one can say that the discrete case falls within the continuous framework: since the discrete-space W can be considered as discontinuous on $\mathbf{R}$ at the points of $\mathbf{Z}$, and since there is a great deal of independence of the increments of W in space (the hypothesis is that $\{W(\cdot,x) : x \in \mathbf{Z}\}$ are independent), it is natural to find a Lyapunov exponent which corresponds to a case where W features no continuity: this is precisely what can be observed in the logarithmic scale, where a discontinuous case, corresponding in the notation of Corollary 27 to the case $\beta = 1$, yields exactly the discrete case result (compare in particular with Corollary 28). That the Lyapunov exponent gets smaller when β increases from 1 can be explained as follows. What makes the Lyapunov exponent non-zero is the Feynman-Kac path b's abilities to seek out zones where W is large. In discrete space, one is stuck with a specific discretization step $\varepsilon = 1$, which boasts a fixed amount of independence at any scale, helping in b's search for high levels of W; in continuous space, the higher the regularity, the smaller one may take the discretization step ε (for example, in the power scale $\varepsilon = \kappa^{1/(6H+2)}$, in the log scale

$\varepsilon \leq \sqrt{\kappa} \log^{3\beta/2}(1/\kappa)$, see Section 5), and thus the more dependence of W's spatial increments one is able to exploit to restrict b's search.

Another, less optimistic, interpretation, points to the fact that the analogy between discrete and continuous space may only be taken so far. Indeed, the discrete model is one which should be called "space-time-white-noise" since W is independent at every site of $\mathbf{Z}$. In continuous space, between the limit of continuous fields W (e.g. case $\beta > 1$) and the case of space-time white noise, there is an entire scale of regularity of W: in fractional derivatives terminology, an entire 1/2-derivative must be taken, in a Schwartz distribution sense, to go from the case $\beta = 1$ to the case of white noise. In this sense, there should be much more space between the bounds of Corollary 27 and any result for space-time white noise in continuous space. This would contradict any analogy between discrete and continuous Anderson models driven by space-time white noise. A way out of this "paradox" may come from the fact that, unlike the continuous Anderson model we study in this article, driven by a potential that is a bonafide function in space, the same model driven by space-time white-noise does not seem to have a Feynman-Kac representation, or even a proper physical meaning. Alternately the paradox may point to a real physical difference between discrete and continuous space models in all cases.

1.1.3 A casual reader's guide to our results' precise statements

The casual reader can skip Section 1.2 except for definitions (3), (4), and the Feynman-Kac formula (8). After having taken into consideration the first two conditions **(E)** and **(E')** in Section 1.3, the reader will find the main existence result – relating $U(t,x) := \mathbf{E} \log u(t,x)$ and a possible existence of λ – in the statements of Theorem 2 and Proposition 3, both at the beginning of Section 2.1; the remainder of that section, and any further references to the quantities $\lambda_-(x)$ and $\lambda_+(x)$ can be ignored on a first reading. The crux of the proof of the almost-sure existence Theorem 2 is established in Proposition 11 (Section 2.3). Of fundamental quantitative importance for the entire paper is the quantity $U_*(t) := \inf_x U(t,x)$, whose super-additivity is studied in Proposition 5 (Section 2.2), and for which the basic positivity result $\sup_t U_*(t)/t > 0$ is given in Proposition 12 (Section 2.4). All the above results are of a qualitative nature.

Our quantitative results are best appreciated in the two examples of the Hölder and logarithmic regularity scales of Section 5: after having read Conditions **(H)** and **(L)** therein, with notation relative to Conditions **(E'-)** and **(E")** of Section 1.3, the reader will appreciate the first three corollaries of Section 5 (Corollaries 25, 26, 27). Precise, more general upper and lower bound results, i.e. not restricted to any given regularity scale, are given respectively in Sections 3 and 4, in Theorems 14 and 23, still under hypotheses defined in Section 1.3. The last two results of the article, Corollaries 28 and 29 in Section 5, show to what extent the Lyapunov exponent and the potential's modulus of continuity are intertwined.

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1.2 The structure of W and the Feynman-Kac Formula

We define W specifically as follows: it is a separable centered Gaussian field on $\mathbf{R}_{+} \times \mathbf{R}$, defined under some probability space $(\Omega, \mathcal{F}, \mathbf{P})$, such that for all s, t in $\mathbf{R}_{+}$ and all x, y in $\mathbf{R}$:

$$\mathbf{E}\left[W(t,x)\, W(s,y)\right] = \min(s,t)\, Q(x,y), \tag{3}$$

where $\mathbf{E}$ is the mathematical expectation with respect to $\mathbf{P}$, and where Q is a bonafide covariance function for a real-valued separable Gaussian process on $\mathbf{R}$.[1]

We define the *spatial canonical metric* δ of W, by $\delta^{2}(x,y) = \mathbf{E}\left[(W(1,x) - W(1,y))^{2}\right]$. Then, we trivially have,

$$\delta^{2}(x,y) = Q(x,x) + Q(y,y) - 2Q(x,y). \tag{4}$$

Among the various conditions on Q and/or δ which we will use in this article, we mention here that a convenient condition leading to our lower bound result, which was referred to above as a non-H-Hölder continuity condition, is of the type $\delta(x,y) \geq c\,|x-y|^{H}$ for $|x-y|$ small. For our upper bound result there is a condition of similar type: $\delta(x,y) \leq C\,|x-y|^{H}$. However we will see, particularly in Section 5, that we are not restricted to this Hölder scale, and that we can choose other, more irregular functions $f(|x-y|)$ than $|x-y|^{H}$.

It is occasionally convenient to represent Q as follows: we can assume that there exists some positive sigma-finite measure μ on some measurable set Λ, and some measurable function f in $L^{2}(\mathbf{R} \times \Lambda; dx \times d\mu)$ such that

$$Q(x,x') = \int_{\Lambda} f(x,y)\, \overline{f(x',y)}\, \mu(dy), \tag{5}$$

where the bar denotes complex conjugation. Information on this representation can be found in P. Major's text [13]. To fix ideas, we assume that $\Lambda = \mathbf{R}$, and indeed all useful examples can be found in this case. As a classical example, $f(x,y) = e^{ixy}$ and μ is symmetric and of mass one if and only if W real-valued and spatially homogeneous.

When the Anderson model equation (1) is understood in the so-called Stratonovich sense (this sense is used in all works mentioned in the Introduction), it is known that, with b representing a Wiener process started at 0 with variance κ defined on some other probability space $(\Omega_{b}, \mathcal{F}_{b}, \mathbf{P}_{b})$ which is not related to W, we have for fixed t and x, the so-called Stochastic Feynman-Kac formula

$$u(t,x) = \mathbf{E}_{b}\left[\exp\left(\int_{0}^{t} W(ds, b_{t} - b_{s} + x)\right)\right], \tag{6}$$

[1] Many of the results in this article are valid if we replace $\mathbf{R}$ by $\mathbf{R}^{d}$ for some integer $d > 1$, with essentially identical proofs; for the sake of notational simplicity, we work with $d = 1$ throughout. Our Positivity and Lower Bound sections are non-trivial to extend to higher dimensions: in fact, the transience properties of b in higher dimensions make it less than clear that any generalization of our techniques is possible. The extension of the Upper Bound section to $d > 1$ would presumably be feasible, but the price to pay are a certain number of technical difficulties due to the dependence of the jump times and the discrete-time path in our discretization method, as were encountered in [7].

as long as the regularity of W in the space parameter is sufficient to allow the right-hand side of (6) to actually make sense from a measure-theoretic standpoint. When W is almost-surely uniformly continuous in space, this formula can easily be established, by referring to the proof in [7], or a different proof in [16], although these two articles make the additional unnecessary assumption that W is Hölder-continuous in space in order to justify formula (6). We do not know of another published proof of formula (6) under mere uniform continuity of W, yet we do not elaborate further on this issue, since it is only tangential to our purpose.

For the reader who is simply curious as to whether the stochastic integral in formula (6), and its unusual notation, is well-defined, it is convenient to assume that the spectral measure μ defining Q actually has a density: $d\mu/dy = q(y)$. Then, it is sufficient to recall that W can be represented using a white-noise (independently and homogeneously scattered Gaussian) measure M on $\mathbf{R}_+ \times \mathbf{R}$, by the formula

$$W(t,x) = \int_{\mathbf{R}_+ \times \mathbf{R}} \mathbf{1}_{[0,t]}(s) \, M(ds,dy) \sqrt{q(y)} f(x,y), \tag{7}$$

so that we can take the following formula as a definition:

$$\int_0^t W(ds, b_t - b_s + x) := \int_{[0,t]\times\mathbf{R}} M(ds,dy) \sqrt{q(y)} f(b_t - b_s + x, y).$$

This is a so-called Wiener-Itô integral with respect to the white noise measure M associated with W. Also, note that $\sqrt{q}$ can be absorbed into f. If Q does not have a density q, one can still define the stochastic integral in (6) by referring to the spectral measure of W itself. For more details and a more general representation, consult [13].

We can already see that when t, x are fixed, we have the following non-time-reversed Feynman-Kac formula:

$$u(t,x) = \mathbf{E}_b \left[\exp\left(\int_0^t W(ds, x + b_s) \right) \right], \tag{8}$$

where the equality holds in distribution under $\mathbf{P}$. This formula holds by the independence and stationarity of the increments of W in time, by reversing time in the stochastic integral in (6). It is crucial to note that formula (8) does NOT hold for two values $u(t,x)$ and $u(t',x)$ simultaneously. When trying to estimate the law of such a pair, one must revert to the formula with time reversal. In this article, the formula (8) is only used to exploit the law of $u(t,x)$ for fixed t, x, i.e. the marginal distributions of the stochastic process $u(\cdot, x)$.

As a last aid in understanding the structure of the Feynman-Kac formula, we mention that for any fixed continuous path b and any fixed x in $\mathbf{R}$, the process $X_\cdot^b(x)$ defined by

$$X_t^b(x) := \int_0^t W(ds, x + b_s)$$

is in fact a $\left\{\mathcal{F}_t^W\right\}_{t\geq 0}$-Martingale, where $\left\{\mathcal{F}_t^W\right\}_t$ is the filtration induced by W, and that one can easily calculate the joint variation

$$< X_\cdot^b(x), X_\cdot^{b'}(x) > (t) = \int_0^t Q(x + b_s, x + b'_s) ds.$$

The techniques used in this article do not need to refer to this specific Martingale property. We are currently investigating directions where this property may be useful for tackling open problems related to our current results (see end of Section 2.1).

1.3 Gaussian models - Assumptions

In this section we provide *all* the assumptions on W needed for our proofs. At the beginning of each section we will specify the particular hypotheses needed. In order to make sure that the Feynman-Kac formula holds, we will take almost-sure continuity of W in space as a standing assumption in this entire article (see Remark 1 and paragraph following).

Define

$$\Delta(\varepsilon) := \inf\{\delta(x,y) : |x-y| \geq \varepsilon\},$$
$$\delta(\varepsilon) := \sup\{\delta(x,y) : |x-y| \leq \varepsilon\}$$

Note that these are increasing functions for $\varepsilon \geq 0$. Consider the following conditions on the spatial distribution of W.

(E) [Boundedness of the variances]

$$\sup\{Q(x,x) : x \in \mathbf{R}\} < \infty$$

(E') [Nondegeneracy]. There exists $\varepsilon_0 > 0$ such that

$$c_0 := |\Delta(\varepsilon_0)|^2 > 0.$$

(E'-) [Specific nondegeneracy]. There exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$,

$$\Delta(\varepsilon) > 0, \quad \lim_{\varepsilon \to 0^+} \Delta(\varepsilon) = 0.$$

(E") [Specific regularity]. The following limit holds:

$$\lim_{\varepsilon \to 0^+} \delta(\varepsilon) = 0.$$

Here follow an analysis of these conditions, including remarks on equivalent formulations and significance.

- The boundedness condition **(E)** forces the generic magnitude of W not to get too big. The nondegeneracy condition **(E')** essentially only implies that W is not flat in space. These are very weak conditions on Q. In particular, one can see that the homogeneous case is a small subset of those random fields satisfying **(E)** and **(E')**. These conditions are all that is needed to derive all our basic existence and positivity results. Condition **(E')** is equivalent to the existence of $\varepsilon_0 > 0$ such that $\delta(x,y) \geq \sqrt{c_0}$ for $|x-y| \geq \varepsilon_0$.

- The uniform nondegeneracy hypothesis **(E')** has to be strengthened to **(E'-)** for certain proofs in the Lower Bound and Upper Bound sections, when more quantitative arguments are used. This condition is equivalent to the existence of a positive increasing function $\bar{\Delta}$ such that $\delta(x,y) \geq \bar{\Delta}(\varepsilon)$ for all $|x-y| \geq \varepsilon$.

- The generic upper bound found in 1998 in [7] was of the order $\log^{-1}(\kappa^{-1})$, an order which coincides with the discrete space bounds, according to [6] and [4]. In order to have a better upper bound in continuous space, it is necessary to assume something like an upper-bound analogue of Condition **(E')**. This idea was already conjectured in [17], in their Hölder-scale condition (H), although no attempt was made there to formulate an upper bound. Here, with Condition **(E"),** not only are we able to prove an upper bound, but unlike in [17], spatial homogeneity will not be needed, and we are not restricted to the Hölder scale. Condition **(E")** is equivalent to the existence of a positive increasing function $\bar{\delta}$ with $\lim_{0+} \bar{\delta} = 0$ and $\delta(x,y) \leq \bar{\delta}(\varepsilon)$ for all $|x-y| \leq \varepsilon$.

Remark 1 *Recall that we have as a standing assumption that W is almost-surely continuous. The theory of Gaussian regularity can be used to argue that we should then have, in the context of Condition **(E")**,* $\delta(r) = o\left(\left(\log\left(r^{-1}\right)\right)^{-1/2}\right)$.

- More specifically, under Condition **(E")** on δ, the Gaussian regularity theory implies that $\delta(r)\left(\log\left(r^{-1}\right)\right)^{1/2}$ is an almost-sure modulus of continuity for W in space. For example if $\delta(r) \leq r^H$, **(E")** implies that W is γ-Hölder-continuous in x for all $\gamma < H$. This is the situation studied in [17]. Condition **(E")** is not restricted to the Hölder scale since it encompasses the following logarithmic scale of regularity, studied in detail in [18] and in [14], given by

$$\delta(r) = \left(\log\left(r^{-1}\right)\right)^{-\beta}$$

  for any $\beta > 0$: it is simple to see that continuity of W (i.e. the above Remark 1) only imposes the restriction $\beta > 1$. This is a consequence of the so-called Dudley-Fernique theory for Gaussian regularity [see details in [18] for example] which implies that if $\beta \leq 1$, the corresponding W is not uniformly continuous, in fact is unbounded, on any interval in space. In this highly irregular situation, it is not clear to us that there is any way to prove the Feynman-Kac formula; if there were a rigorous interpretation of the formula, for example by using approximations, we conjecture that it would be impossible to manipulate the formula to derive upper- and lower-bound results such as those we obtain herein. Thus the condition $\beta > 1$ in our logarithmic scale seems to be necessary for any development using our techniques.

2 Existence of the almost-sure Lyapunov exponent

2.1 Introduction

In this section we study the existence of

$$\lambda = \lim_{t\to\infty} \frac{\log u(t,x)}{t},$$

for a general Gaussian field W. Of paramount importance is the quantity

$$U(t,x) = \mathbf{E}\left[\log u(t,x)\right]. \tag{9}$$

The goal of this entire section is to establish the following result.

Theorem 2 *Assume **(E)** and **(E')** are satisfied. With U defined in (9), if*

$$\lambda(x_0) := \lim_{\substack{t\to\infty \\ t\in\mathbf{N}}} \frac{1}{t} U(t,x_0) \tag{10}$$

exists for a fixed $x_0 \in \mathbf{R}$, then the Lyapunov exponent for the solution of the stochastic parabolic Anderson model (1) *restricted to integers, i.e.* $\lim_{t\to\infty, t\in\mathbf{N}} t^{-1} \log u(t,x_0)$, *exists* for that x_0 *almost surely, is finite, positive, and non-random. Specifically for any x_0 satisfying (10), almost surely*

$$\lim_{\substack{t\to\infty \\ t\in\mathbf{N}}} \frac{1}{t} \log u(t,x_0) = \lambda(x_0) = \lim_{\substack{t\to\infty \\ t\in\mathbf{N}}} \frac{1}{t} U(t,x_0).$$

Under some circumstances, one can prove that $\lambda(x_0)$ is constant almost everywhere, e.g as in the case of compact space (see Remark and 4 and Proposition 8), or in the next proposition, which is obtained for free.

Proposition 3 (The Homogeneous case) *Under the hypothesis of Theorem 2, if in addition the Gaussian field W is spatially homogeneous, then the Lyapunov exponent of the Anderson model restricted to integers*

$$\lambda(x) := \lim_{\substack{t\to\infty \\ t\in\mathbf{N}}} \frac{1}{t} U(t,x) = \lim_{\substack{t\to\infty \\ t\in\mathbf{N}}} \frac{1}{t} \log u(t,x)$$

exists almost surely, is finite, positive, non-random, and does not depend on x; i.e. there exists $\lambda_* > 0$ such that $\lambda(x) = \lambda_*$ for all x.

In this paragraph, we give an overview of the structure and results of the Section. The first step is to study the expected value $U(t,x)$. Without a strong hypothesis such as homogeneity, it may not be possible to control this quantity in large time. However, we show that the study of the function $\inf_x U(t,x)$ can be fruitful. A property of superadditivity is proven for this function which implies the existence of the limit

$$\lambda_* = \lim_{t\to\infty} t^{-1} \inf_x U(t,x).$$

Then, we connect this limit with $t^{-1}U(t,x)$. If the limit of $t^{-1}U(t,x)$ exists when $t \to \infty$ for a particular x then this limit is at least as large as λ_*. Subsection 2.4 is devoted to the proof of positivity of λ_* under the non-degeneracy condition **(E')**, which then immediately implies the uniform positivity of $\lambda_-(x) := \liminf_{t\to\infty} t^{-1}U(t,x)$. Although Subsection 2.4 is relegated to the end of the present Section 2 for purposes of readability, the proof of $\lambda_* > 0$ is entirely self-contained, within Subsection 2.4. A study of $\lambda_-(x)$, without the use of the nondegeneracy condition **(E')**, is also given (Proposition 8), which includes a partial result of constancy of $\lambda_-(x)$.

Remark 4 *We leave it to the reader to rephrase the statements of Theorem 2 and Proposition 3 without the assumption* ***(E')****: one only needs to replace "positivity" by "non-negativity". It is also useful, and trivial, to see that if $\lambda_-(x)$ is constant, then the Lyapunov exponents identified in Theorem 2 and Proposition 3 coincide with this constant at any point x. See Proposition 8 for non-homogeneous cases where this constancy holds.*

The final step in proving the almost sure statement in Theorem 2 and Proposition 3 is to make the connection between

$$\lim_{t\to\infty} U(t,x)/t = \lim_{t\to\infty} \mathbf{E}\left[t^{-1}\log u(t,x)\right]$$

and $\lim_{t\to\infty} t^{-1}\log u(t,x)$ for those values of $x \in \mathbf{R}$ for which the first limit exists. To this end we estimate the Malliavin derivative of $\log u(t,x)$ efficiently, and use non-Gaussian concentration inequalities in order to derive an almost-sure result (Proposition 11): we obtain that $[U(t,x) - \log u(t,x)]/t$ converges to 0 almost surely; this holds regardless of the behavior of $U(t,x)$, but it is only when U is asymptotically linear that an almost-sure Lyapunov exponent can be deduced.

Arguably, Proposition 11 can be valuable even if $U(t,x)$ is not asymptotically linear in t. More precisely we can reformulate the proposition as

$$u(t,x) = \exp\left(U(t,x) + o(t)\right)$$

where it is known from Section 2.4 that $U(t,x)$ is the dominant term. We also prove herein that $U(t,x) \le t\sup_x Q(x,x)$, which means that U does not grow faster than linearly. Thus Proposition 11 gives a deterministic function around which the almost-sure exponential rate of change of u concentrates, even if it is not asymptotically constant. Such a situation occurs when $\lambda_+(x) := \limsup_{t\to\infty} U(t,x)/t > \lambda_-(x) := \limsup_{t\to\infty} U(t,x)/t$. One then has an exponential rate of increase $t^{-1}\log u(t,x)$ which, almost surely, oscillates between the values $\lambda_+(x)$ and $\lambda_-(x)$. Identifying examples of this situation is an open problem. Presumably, one should be able to find such an example if the potential W is highly inhomogeneous in space (e.g. such that $Var[W(1,x)]$ achieves at least two very different levels).

Our final existence result is expressed as a limit of the continuous time process $t^{-1}\log u(t,x)$ along a fixed sequence of times. We use the sequence of positive integer times in this article, although other sequences can be considered successfully. The majority of previous papers on the almost-sure existence of Anderson models' Lyapunov exponents also work with the sequence of integers, but often ignore the fact that this does not prove existence of the limit of $t^{-1}\log u(t,x)$ when t is allowed to tend to infinity along arbitrary sequences of times. The articles [7] and [16] do consider results along all possible sequences simultaneously, but do not prove any existence results, and thus fall short of addressing the real problem. We are well aware of this problem in our present work as well.

To give some insight as to why this is a much harder problem than many may believe, note that one would need to show, for example, that $[\log u(t,x) - \log u(n,x)]/n$ converges to 0 as $n \to \infty$ for all $t \in [n-1, n]$. One needs only to attempt writing down Ito's formula for the difference $\log u(t,x) - \log u(n,x)$ to see that what appears to make the estimation so arduous is precisely the time-reversal in the Feynman-Kac formula (6). We suspect that if W is sufficiently regular in space, the result may be true, but all our attempts have failed so far, even in the homogeneous case.

2.2 Convergence in the mean

Let $U(t,x)$ be defined by (9) for all $x \in \mathbf{R}$ and all $t \in \mathbf{R}_+$. The problem of existence of the Lyapunov exponent when $x \in \mathbf{R}$ has never been solved, and in the case when W is not homogeneous, the question of existence has not been answered even for $x \in \mathbf{Z}$. One way to understand why the non-homogeneous case is more difficult resides in the fact that the superadditive or subadditive properties do not hold in general for $U(\cdot, x)$. However, consider the quantity

$$U_*(t) := \inf_{x\in\mathbf{R}} U(t,x) = \inf_{x\in\mathbf{R}} \mathbf{E}[\log u(t,x)].$$

We have:

Proposition 5 *Under the hypothesis* ***(E)****, U_* is a superadditive function (i.e., $U_*(t+s) \geq U_*(t) + U_*(s)$ for all s,t in $\mathbf{R}_+$). The limit $\lambda_* := \lim_{t\to\infty} t^{-1}U_*(t)$ exists, is non-negative, is finite, and equals* $\sup_t U_*(t)/t$.

Proof. In this proof, we will make use of the following notation. For t fixed,

$$\mathbf{E}_b[F(b_\cdot - b_t + y)]|_{y=b_t}$$

where F depends on $b_\cdot$ only via the values b_r for $r \geq t$, represents the quantity

$$\mathbf{E}_b[F(b_\cdot - b_t + y)],$$

where, after the expectation is taken, the fixed value y is replaced with the random value b_t. By the independence of increments of b, the above quantities are of course equal to

$$\mathbf{E}_b\left[F(b_\cdot)|\mathcal{F}_t^b\right].$$

By the Feynman-Kac formula (8), and by conditioning inside $\mathbf{E}_b$ with respect to the filtration $\left\{\mathcal{F}_t^b\right\}_{t\geq 0}$ generated by b, and using the independence of increments of b, we have

$$
\begin{aligned}
U(t+s,x) &= \mathbf{E}\left[\log\left(\mathbf{E}_b\left\{e^{\int_0^{t+s} W(dr,x+b_r)}\right\}\right)\right] \\
&= \mathbf{E}\left[\log\left(\mathbf{E}_b\left\{e^{\int_0^{t} W(dr,x+b_r)}e^{\int_t^{t+s} W(dr,b_t+b_r-b_t+x)}\right\}\right)\right] \\
&= \mathbf{E}\left[\log\left(\mathbf{E}_b\left\{e^{\int_0^{t} W(dr,x+b_r)}\mathbf{E}_b\left[e^{\int_t^{t+s} W(dr,b_t+b_r-b_t+x)}\middle|\,\mathcal{F}_t^b\right]\right\}\right)\right] \\
&= \mathbf{E}\left[\log\left(\mathbf{E}_b\left\{e^{\int_0^{t} W(dr,x+b_r)}\ \mathbf{E}_b\left[e^{\int_t^{t+s} W(dr,b_r-b_t+y+x)}\right]\Big|_{y=b_t}\right\}\right)\right].
\end{aligned}
$$

Now define the shifted potential $\theta_t W$ by $\theta_t W(t',x) = W(t+t',x) - W(t,x)$ for all $x \in \mathbf{R}$ and $t' \geq 0$. Note that then, $\theta_t W$ has the same distribution as W, and that $r \mapsto b_r - b_t$ for t fixed and $r \geq t$ has the same distribution as $r \mapsto b_{r-t}$. Thus, we can rewrite things as:

$$
\begin{aligned}
&U(t+s,x) \\
&= \mathbf{E}\left[\log\left(\mathbf{E}_b\left[e^{\int_0^{t} W(dr,x+b_r)}\right]\mathbf{E}_b\left\{\frac{e^{\int_0^{t} W(dr,x+b_r)}}{u(t,x)}\mathbf{E}_b\left[e^{\int_0^{s} \theta_t W(dr',y+b_{r'}+x)}\right]\Big|_{y=b_t}\right\}\right)\right] \\
&= U(t,x) + \mathbf{E}\left[\log\left(\mathbf{E}_b\left\{\frac{e^{\int_0^{t} W(dr,b_r+x)}}{u(t,x)}\mathbf{E}_b\left[e^{\int_0^{s} \theta_t W(dr',b_{r'}+y+x)}\right]\Big|_{y=b_t}\right\}\right)\right].
\end{aligned}
$$

Definition 6 *We define a (random) measure $\mathbf{P}_{b,W,t,x}$ on the same space as $\mathbf{P}_b$ by the formula*

$$
\mathbf{P}_{b,W,t,x}[A] = \mathbf{E}_b\left[\frac{\exp\left(\int_0^t W(dr,b_r+x)\right)}{u(t,x)}\mathbf{1}_A\right]. \tag{11}
$$

Remark 7 *By the Feynman-Kac formula (8), we have $\mathbf{P}_{b,W,t,x}[\Omega] = 1$ so* (11) *clearly defines a probability measure.*

Now, using Jensen's inequality for the logarithm, we get

$$
\begin{aligned}
U(t+s,x) &= U(t,x) + \mathbf{E}\left[\log\left(\mathbf{E}_{b,W,t,x}\left\{\mathbf{E}_b\left[e^{\int_0^{s} \theta_t W(dr',b_{r'}+y+x)}\right]\Big|_{y=b_t}\right\}\right)\right] \\
&\geq U(t,x) + \mathbf{E}\left[\mathbf{E}_{b,W,t,x}\left\{\log \mathbf{E}_b\left[e^{\int_0^{s} \theta_t W(dr',b_{r'}+y+x)}\right]\Big|_{y=b_t}\right\}\right].
\end{aligned}
$$

It is important to note that we may not use Fubini's theorem here because $\mathbf{P}_{b,W,t,x}$ depends on the randomness in W. However, we can revert to the original notation, which allows us to use Fubini safely, and then exploit the fact that the terms involving W without the shift

θ_t are independent of those involving this shift, to obtain:

$$U(t+s,x) \geq U(t,x) + \mathbf{E}\left[\mathbf{E}_b\left\{\frac{e^{\int_0^t W(dr,b_r+x)}}{u(t,x)}\log \mathbf{E}_b\left[e^{\int_0^s \theta_t W(dr',b_{r'}+y+x)}\right]\Big|_{y=b_t}\right\}\right]$$
$$= U(t,x) + \mathbf{E}_b\left[\mathbf{E}\left\{\frac{e^{\int_0^t W(dr,b_r+x)}}{u(t,x)}\log \mathbf{E}_b\left[e^{\int_0^s \theta_t W(dr',b_{r'}+y+x)}\right]\Big|_{y=b_t}\right\}\right]$$
$$= U(t,x) + \mathbf{E}_b\left\{\mathbf{E}\left[\frac{e^{\int_0^t W(dr,b_r+x)}}{u(t,x)}\right]\mathbf{E}\left[\log \mathbf{E}_b\left[e^{\int_0^s \theta_t W(dr',b_{r'}+y+x)}\right]\right]\Big|_{y=b_t}\right\}.$$

Taking an infimum over all values of x yields

$$U_*(t+s) \geq U_*(t) + \inf_{x\in\mathbf{R}}\mathbf{E}_b\left\{\mathbf{E}\left[\frac{e^{\int_0^t W(dr,b_r+x)}}{u(t,x)}\right]\mathbf{E}\left[\log \mathbf{E}_b\left[e^{\int_0^s \theta_t W(dr',b_{r'}+y+x)}\right]\right]\Big|_{y=b_t}\right\}.$$

However, we can obtain a lower bound by taking an infimum over all possible values of $x+y$, after the expectation with respect to $\mathbf{P}$ is taken in the second term in the product above, but before replacing y by b_t. We then recognize the quantity $U_*(s)$ because $\mathbf{P}$ is θ_t-invariant. Then, using Fubini again we get

$$U_*(t+s)$$
$$\geq U_*(t) + \inf_{x\in\mathbf{R}}\mathbf{E}_b\left\{\mathbf{E}\left[\frac{e^{\int_0^t W(dr,b_r+x)}}{u(t,x)}\right]\inf_{x+y\in\mathbf{R}}\left\{\mathbf{E}\left[\log \mathbf{E}_b\left[e^{\int_0^s \theta_t W(dr',b_{r'}+y+x)}\right]\right]\right\}\Big|_{y=b_t}\right\}$$

$$= U_*(t) + \inf_{x\in\mathbf{R}}\mathbf{E}\left\{\mathbf{E}_b\left[\frac{e^{\int_0^t W(dr,b_r+x)}}{u(t,x)}U_*(s)\Big|_{y=b_t}\right]\right\}$$
$$= U_*(t) + U_*(s)\inf_{x\in\mathbf{R}}\mathbf{E}\left\{\mathbf{E}_b\left[\frac{\exp\left(\int_0^t W(dr,b_r+x)\right)}{u(t,x)}\right]\right\} = U_*(t) + U_*(s).$$

Here we used Remark 7 which ends the proof of the proposition's first statement. The remaining statements are nearly trivial. First, since U_* is super-additive, $U_*(t)/t$ has a limit when $t\to\infty$ which equals $\sup_t U_*(t)/t$. Thus we only need to show that $U_*(t)/t$ is bounded for all t. For b fixed, we calculate the variance of the centered Gaussian r.v. $\int_0^t W(ds,b_s+x)$ using (7) as

$$\mathbf{E}\left[\left(\int_0^t W(ds,b_s+x)\right)^2\right] = \int_0^t\int_{y\in\mathbf{R}}|f(b_s+x,y)|^2\mu(dy)\,ds$$
$$= \int_0^t Q(b_s+x,b_s+x)\,ds \leq t\sup_{z\in\mathbf{R}}Q(z,z).$$

Then, note that by Jensen's inequality and Fubini's lemma, and the hypothesis **(E)**,

$$
\begin{aligned}
U(t,x) &= \mathbf{E}\log\mathbf{E}_b\left(\exp\int_0^t W(ds,b_s+x)\right) \leq \log\mathbf{E}\mathbf{E}_b\left(\exp\int_0^t W(ds,b_s+x)\right) \\
&= \log\mathbf{E}_b\mathbf{E}\left(\exp\int_0^t W(ds,b_s+x)\right) = \log\mathbf{E}_b\exp\left(\frac{1}{2}\mathbf{E}\left[\left(\int_0^t W(ds,b_s+x)\right)^2\right]\right) \\
&\leq \log\mathbf{E}_b\exp\left(\frac{t}{2}\sup_{z\in\mathbf{R}}Q(z,z)\right) = \frac{t}{2}\sup_{z\in\mathbf{R}}Q(z,z).
\end{aligned}
$$

This proves that the limit of $U_*(t)/t$ is bounded above by $2^{-1}\sup_{z\in\mathbf{R}}Q(z,z)$. To prove that this limit is non-negative, we use Jensen's inequality by moving the logarithm inside $\mathbf{E}_b$, to get that:

$$
U(t,x) \geq \mathbf{E}\mathbf{E}_b\left(\log\exp\int_0^t W(ds,b_s+x)\right) = 0, \tag{12}
$$

finishing the proof of the proposition. ■

Proposition 5 is of crucial importance for the proof of the lower bound in Section 3. Moreover, in the homogeneous case this proposition enables us to conclude that the existence of the Lyapunov exponent holds, as is spelled out in Corollary 3, where the Lyapunov exponent is seen to be constant. The next proposition investigates the possible constancy of another notion of lower bound.

Proposition 8 *Define*

$$
\lambda_-(x) := \liminf_{t\to\infty}\frac{1}{t}U(t,x).
$$

*Assume Condition **(E)**. Let $\lambda_{\inf} := \inf_x \lambda_-(x)$. Then either the function $\lambda_-(x)$ is bounded away from its infimum on any finite interval, or $\lambda_-(x) = \lambda_{\inf}$ for Lebesgue-almost every x.*

The latter situation occurs when the SPDE (1) is defined for x in a compact smooth manifold.

Proof. To establish the first statement, we can write:

$$
\begin{aligned}
u(t,x) &= \mathbf{E}_b\left[e^{\int_0^1 W(ds,x+b_s)}e^{\int_1^t W(ds,x+b_s)}\right] \\
&= \mathbf{E}_b\left[\mathbf{E}_b\left[e^{\int_0^1 W(ds,x+b_s)}e^{\int_1^t W(ds,x+b_s)}\middle|\mathcal{F}_1^b\right]\right] \\
&= \mathbf{E}_b\left[e^{\int_0^1 W(ds,x+b_s)}\mathbf{E}_b\left[e^{\int_1^t W(ds,x+b_1+b_s-b_1)}\middle|\mathcal{F}_1^b\right]\right] \\
&= \mathbf{E}_b\left[e^{\int_0^1 W(ds,x+b_s)}\mathbf{E}_b\left[e^{\int_0^{t-1}\theta_1 W(dr,x+b_1+b_r)}\right]\right],
\end{aligned}
$$

where we used the independence of the increments of b on $[1, t-1]$ respectively $[0, 1]$. Now using Jensen's inequality, Fubini, and the identical distribution of W and $\theta_1 W$, we obtain:

$$\begin{aligned}
U(t,x) &= \mathbf{E}\left[\log \mathbf{E}_b\left[e^{\int_0^1 W(ds,x+b_s)}\mathbf{E}_b\left[e^{\int_0^{t-1}\theta_1 W(dr,x+b_1+b_r)}\right]\right]\right] \\
&\geq \mathbf{E}\left[\mathbf{E}_b\left[\int_0^1 W(ds, x+b_s) + \log \mathbf{E}_b\left[e^{\int_0^{t-1}\theta_1 W(dr,x+b_1+b_r)}\right]\right]\right] \\
&= \mathbf{E}_b\left[\mathbf{E}\left[W(ds, x+b_s)\right] + \mathbf{E}\left[\log \mathbf{E}_b\left[e^{\int_0^{t-1} W(dr,x+b_1+b_r)}\right]\right]\right] \\
&= \mathbf{E}_b\left[\mathbf{E}\left[\log u(t-1, x+b_1)\right]\right] \\
&= \mathbf{E}_b\left[U(t-1, x+b_1)\right].
\end{aligned}$$

Recall from (12) that for any x, $U(t,x)/t \geq 0$. Therefore, by Fatou's lemma we have, for each fixed x,

$$\liminf_{t\to\infty} \frac{1}{t} U(t,x) \geq \mathbf{E}_b\left[\liminf_{t\to\infty} \frac{1}{t} U(t-1, x+b_1)\right],$$

i.e., for all x,

$$\lambda_-(x) \geq \mathbf{E}_b\left[\lambda_-(x+b_1)\right] = \int \rho(dz)\,\lambda_-(x+z). \tag{13}$$

where $\rho(dz)$ is the Gaussian measure $(2\pi)^{-1/2}\,dz \exp(-z^2/2)$.

We proceed with a proof by contradiction. Let $\lambda_{\inf} = \inf_x \lambda_-(x)$. Assume λ_- is not a constant a.e. Therefore, since ρ and the Lebesgue measure are equivalent, we have

$$\rho(x : \lambda_-(x) > \lambda_{\inf}) > 0.$$

Hence by monotone convergence, There exists $\varepsilon > 0$ such that

$$\rho(x : \lambda_-(x) \geq \lambda_{\inf} + \varepsilon) > 0.$$

This means that there is a set $\mathcal{I}$ of positive Lebesgue measure such that for all $x \in \mathcal{I}$, $\lambda_-(x) \geq \lambda_{\inf} + \varepsilon$. By definition, even if $\lambda_{\inf}$ is not attained, there exists a sequence $(x_n)_n$ such that $\lambda_-(x_n)$ converges to $\lambda_{\inf}$. Now, using (13), we have for each n,

$$\begin{aligned}
\lambda_-(x_n) &\geq \int \rho(dz)\,\lambda_-(x_n+z) \\
&= \int_{\mathcal{I}-x_n} \rho(dz)\,\lambda_-(x_n+z) + \int_{\mathbf{R}\setminus\mathcal{I}-x_n} \rho(dz)\,\lambda_-(x_n+z) \\
&\geq (\varepsilon + \lambda_{\inf})\,\rho(\mathcal{I}-x_n) + \lambda_{\inf}\left(1-\rho(\mathcal{I}-x_n)\right) \\
&= \lambda_{\inf} + \varepsilon\rho(\mathcal{I}-x_n)
\end{aligned}$$

If we assume that the sequence $(x_n)_n$ has an accumulation point (non-infinite), then as $n \to \infty$, $\rho(\mathcal{I}-x_n)$ will tend $\rho(\mathcal{I})$, and since $\lambda_-(x_n)$ tends to $\lambda_{\inf}$, we obtain a contradiction.

If on the contrary we cannot assume that $(x_n)_n$ has an accumulation point, this implies that with

$$\varepsilon_M + \lambda_{\inf} := \inf \{\lambda_- (x) : x \in [-M, M]\},$$

$\varepsilon_M > 0$ for any M, which is the first alternative of the proposition.

To establish the third statement, we must reinterpret the law of b in the Feynman-Kac formula to be that of the Markov process whose generator is the Laplace-Beltrami operator on a smooth compact manifold. Since the Lebesgue measure is still absolutely continuous with respect to the law of b_1, the previous arguments still hold, and we can apply the second alternative of the proposition, since $(x_n)_n$ must have an accumulation point. ■

Remark 9 *The constancy of λ_- in Proposition 8 is not needed for any of the other results in this paper to hold. In this sense, this constancy property appears as a bonus in our results, for which homogeneity is not required.*

2.3 Almost-sure convergence

We begin with a lemma from stochastic analysis. The *filtration* of M is the family of sigma-fields $(\mathcal{F}_t)_{t\geq 0}$ defined by setting $\mathcal{F}_t$ to be the sigma-field generated by all the random variables $M([0,s] \times B)$ where $s \leq t$ and B is a Borel set in $\mathbf{R}$. For a random variable F in the space $L^2(\Omega, \mathcal{F}, \mathbf{P})$ generated by M, its Malliavin derivative DF with respect to M, when it exists, is a random field on $\mathbf{R}_+ \times \mathbf{R}$ in accordance with the usual definitions from the theory of abstract Wiener spaces. One may consult the corresponding chapter in [12] for a precise definition. For our purposes, it will be sufficient to note the following two important properties of D.

1. Let f be a non-random function in $L^2(\mathbf{R}_+ \times \mathbf{R}, ds \times \mu(dy))$. For any fixed $t \geq 0$, let $F = \iint_{\mathbf{R}_+\times\mathbf{R}} f(s,y) M(ds,dy)$.Let g be a function in $C^1(\mathbf{R})$, and let g' be the usual derivative of g. Then $G = g(F)$ has a Malliavin derivative given for all $s \geq 0$ and all $y \in \mathbf{R}$ by

$$D_{s,y}G = g'(F) f(s,y)$$

as long as $g'(F)$ is in $L^2(\Omega)$. Note in particular that $D_{s,y}F = f(s,y)$

2. If G has a Malliavin derivative and G is $\mathcal{F}_t$-measurable for some $t \geq 0$, then for all $y \in \mathbf{R}$ and all $s > t$ we have $D_{s,y}G = 0$.

It is informative to note that D is the only closable operator that satisfies all multi-dimensional analogues of the first condition above (g in $C^1(\mathbf{R}^d)$, d arbitrary). This fact will not be used herein. It is convenient to define the domain of the Malliavin derivative D as the so-called set $\mathbf{D}^{1,2}$. The book [12] can again be consulted for definitions and properties of this set, but here it is sufficient to say that when $G \in \mathbf{D}^{1,2}$, then $DG \in L^2(\Omega \times \mathbf{R}_+ \times \mathbf{R}, \mathbf{P}\times ds \times \mu(dy))$, and one immediately sees that the G described in the first property above is indeed in $\mathbf{D}^{1,2}$.

Lemma 10 *Let G be a centered random variable in $L^2(\Omega, \mathcal{F}, \mathbf{P})$. Assume $G \in \mathbf{D}^{1,2}$ and G is $\mathcal{F}_t$-measurable. Then for every integer $k \geq 0$, there exists a constant C_k that only depends on k such that*

$$\mathbf{E}\left[G^{2k}\right] \leq C_k \left\{ \mathbf{E}\left[\int_{y \in \mathbf{R}} \mu(dy) \int_0^t \left(\mathbf{E}\left[D_{s,y} G | \mathcal{F}_s\right]\right)^2 ds \right] \right\}^k .$$

Proof. We need the following version of the Clark-Ocone representation theorem (see [12]). Since $G \in \mathbf{D}^{1,2}$ and $\mathbf{E}G = 0$,

$$G = \iint_{\mathbf{R}_+ \times \mathbf{R}} \mathbf{E}\left[D_{s,y} G | \mathcal{F}_s\right] M(ds, dy).$$

Note that the stochastic integral above is of Itô type, since the integrand $\mathbf{E}\left[D_{s,y}G | \mathcal{F}_s\right]$ is adapted to the filtration $(\mathcal{F}_s)_{s \geq 0}$ of M. In particular, assume now, as we may by our hypothesis, that G is $\mathcal{F}_t$-measurable for some fixed $t \geq 0$. Then we can rewrite the above formula as $G = Y(t)$ where the stochastic process Y is defined for all $r \in [0, t]$ by

$$Y(r) = \iint_{[0,r] \times \mathbf{R}} \mathbf{E}\left[D_{s,y} G | \mathcal{F}_s\right] M(ds, dy).$$

Since the integral is of Itô type, with square-integrable integrand (using the hypothesis $G \in \mathbf{D}^{1,2}$), we see that Y is a martingale and that its quadratic variation is given by

$$\frac{\langle Y \rangle (ds)}{ds} = \int_R \left(\mathbf{E}\left[D_{s,y} G | \mathcal{F}_s\right]\right)^2 \mu(dy),$$

from which we immediately get

$$\mathbf{E}\left[\langle Y \rangle (r)\right] = \mathbf{E} \int_{\mathbf{R}} \int_0^s \left(\mathbf{E}\left[D_{s,y} G | \mathcal{F}_s\right]\right)^2 ds \mu(dy).$$

The Burkholder-Davis-Gundy inequality can be applied, yielding in particular the statement of the lemma when $r = t$. ■

We now apply this lemma to $G = t^{-1} \log u(t, x)$ where t and x are fixed. By property 2 above, we have $D_{s,y} G = 0$ for $s > t$. By property 1 above, the operator D is clearly linear, and operates only on the randomness of M, so that we may write, using the formula (8) for u,

$$\begin{aligned} D_{s,y} G &= \frac{1}{t} \frac{1}{u(t,x)} \mathbf{E}_b \left[D_{s,y} \exp\left(\iint_{[0,t] \times \mathbf{R}} f(b_s + x, y) M(ds, dy) \right) \right] \\ &= \frac{1}{t} \frac{1}{u(t,x)} \mathbf{E}_b \left[f(b_s + x, y) \exp\left(\iint_{[0,t] \times \mathbf{R}} f(b_s + x, y) M(ds, dy) \right) \right]. \end{aligned}$$

We rewrite this formula using the probability measure $\mathbf{P}_{b,W,t,x}$ defined by (11). We have

$$D_{s,y} G = \frac{1}{t} \mathbf{E}_{b,W,t,x}\left[f(b_s + x, y)\right]. \tag{14}$$

Now using the previous lemma for $k \geq 1$ coupled with several uses of Jensen's inequality and Fubini's lemma, plus hypothesis **(E)**, we get

$$
\begin{aligned}
\mathbf{E}\left[(G-\mathbf{E}[G])^{2k}\right] &\leq \frac{1}{t^{2k}} C_k \left\{\mathbf{E}\left[\int_{y\in\mathbf{R}} \mu(dy) \int_0^t \left(\mathbf{E}\left[\mathbf{E}_{b,W,t,x}\left[f(b_s+x,y)\right]|\mathcal{F}_s\right]\right)^2 ds\right]\right\}^k \\
&\leq \frac{1}{t^{2k}} C_k \left\{\mathbf{E}\left[\int_{y\in\mathbf{R}} \mu(dy) \int_0^t \mathbf{E}\left[\mathbf{E}_{b,W,t,x}\left[|f(b_s+x,y)|^2\right]|\mathcal{F}_s\right] ds\right]\right\}^k \\
&= \frac{1}{t^{2k}} C_k \left\{\mathbf{E}\left[\int_0^t \mathbf{E}\left[\mathbf{E}_{b,W,t,x}\left[\int_{y\in\mathbf{R}} \mu(dy) |f(b_s+x,y)|^2\right]|\mathcal{F}_s\right] ds\right]\right\}^k \\
&= \frac{1}{t^{2k}} C_k \left\{\mathbf{E}\left[\int_0^t \mathbf{E}\left[\mathbf{E}_{b,W,t,x}\left[Q(b_s+x,b_s+x)\right]|\mathcal{F}_s\right] ds\right]\right\}^k \qquad (15)\\
&\leq \frac{1}{t^k} C_k \sup_{x\in\mathbf{R}} Q(x,x)^k .
\end{aligned}
$$

In the remainder of this section and the next one, we assume that t can take only positive integer values. What we have just proved is that for any fixed $x \in \mathbf{R}$ and $t \in \mathbf{N}$, we have

$$
\mathbf{E}\left[\left(\frac{1}{t}\log u(t,x) - \frac{1}{t}U(t,x)\right)^{2k}\right] \leq t^{-k} C_{Q,k} \qquad (16)
$$

where $C_{Q,k}$ is a constant depending only on k and Q. Now by Chebyshev's inequality, for any constant $C(t)$,

$$
\mathbf{P}\left[\left|\frac{1}{t}\log u(t,x) - \frac{1}{t}U(t,x)\right| > C(t)\right] \leq \frac{C_{Q,k}}{t^k C(t)^{2k}}. \qquad (17)
$$

To be able to apply the Borel-Cantelli lemma, we may for example require that $t^k C(t)^{2k} = t^\beta$ where $\beta > 1$. This means $C(t) = t^{-(k-\beta)/2k}$, so that by choosing $\beta - 1 > 0$ and small enough, we only need to require that $k > 1$ to guarantee that $\lim_{t\to\infty} C(t) = 0$. In particular, we can state the following result.

Proposition 11 *Almost surely, for any fixed $x \in \mathbf{R}$,*

$$
\lim_{\substack{t\to\infty \\ t\in\mathbf{N}}} \left(\frac{1}{t}\log u(t,x) - \frac{1}{t}U(t,x)\right) = 0.
$$

Combining this with the result of the previous section, the proof of Theorem 2 is complete. To prove Proposition 3, it is sufficient to note that by homogeneity, $U(t,x)$ does not depend on x, so that $U(t,x) \equiv U_*(t)$ is superadditive, and thus $\lambda = \lambda_* = \lim_{t\to\infty} U(t,x)/t$ exists, so that Proposition 3 follows immediately from Theorem 2 and Proposition 5.

2.4 Positivity

The purpose of this section is only to establish the next proposition, whose proof does not depend on any of the results described above. The proposition provides a structure and a crucial ingredient for the proof Theorem 14.

Proposition 12 *In the notation of Proposition 5, if* ***(E)*** *and* ***(E')*** *hold,* $\lambda_* > 0$.

Proof. The main idea in this proof is to restrict the Feynman-Kac paths b to regions where the canonical metric δ^2 is bounded below, using Condition **(E')**. Throughout the proof, except for **Step 3** where the superadditive limiting property of U_* is used, our time $t > 0$ is fixed (large enough) and $x \in \mathbf{R}$ is fixed. We first choose a pair (x_0, y_0) such that $x_0 = x$ and $y_0 = x - \varepsilon_0$. By Condition **(E')** for all $x' \geq x_0$ and $y' \leq y_0$, we have $\delta^2(x', y') \geq c_0$. It will be notationally convenient to keep the identities of x_0 and x separate. We will also introduce the shorthand notation $b_s^x := x + b_s$. Here we continue to use a standard Brownian motion b starting from 0 under $\mathbf{P}_b$, so that b^x starts from x under $\mathbf{P}_b$.

Step 1. *Controlling the probabilities of* b^x *being outside of* $[x_0, y_0]$. Let

$$A_+ := \left\{ \inf_{s \in [t,2t]} b_s^x \geq x_0 \right\}, \text{ and } A_- := \left\{ \sup_{s \in [t,2t]} b_s^x \leq y_0 \right\}.$$

We begin with a simple result, whose proof we include for completeness. As stated, it refers to the law of standard Brownian motion started from $x = x_0$. If one prefers to use the Brownian motion started from x with variance κ (as is required when referring to the Feynman-Kac formula for equation (1)), one only needs to replace t by κt in the statement of the lemma below. This modification changes nothing to the usage of the lemma in the current proof of the proposition.

Lemma. *For any* $c > 2$*, there exists* t_0 *non-dependent on* x *such that if* $t \geq t_0$ *then,* $P_b[A_+] > (c\pi t)^{-1/2}$ *and* $P_b[A_-] > (c\pi t)^{-1/2}$.

Proof of the lemma. In this proof, as always, note that under $\mathbf{P}_b$, b is a standard Brownian motion started from 0. Then, with T_{-1} the first hitting time of -1 by b, and the Markov property at time t, we can write

$$\begin{aligned}
\mathbf{P}_b[A_+] &= \int_{x_0}^{\infty} \mathbf{P}_b[b_t + x \in dz] \, \mathbf{P}_b\left[\inf_{s \in [t,2t]} (b_s^x - b_t^x) + z \geq x_0 | b_t^x = z \right] \\
&= \int_{x_0}^{\infty} \mathbf{P}_b[b_t + x \in dz] \, \mathbf{P}_b\left[\inf_{s \in [0,t]} b_s \geq x_0 - z \right] \\
&\geq \int_{x_0+1}^{\infty} \mathbf{P}_b[b_t + x \in dz] \, \mathbf{P}_b\left[\inf_{s \in [0,t]} b_s \geq -1 \right] \\
&= \mathbf{P}_b\left[b_1 \geq (x_0 + 1 - x) / \sqrt{t} \right] \mathbf{P}_b[T_{-1} > t].
\end{aligned}$$

In the last expression, as $t \to \infty$, the first term converges to $1/2$ uniformly in x since $x_0 - x = 0$, while the second satisfies $\lim_{t\to\infty} \mathbf{P}_b [T_{-1} > t] \sqrt{t} = \sqrt{2/\pi}$. The result follows for A_+. The proof for A_- starts identically. We then arrive at the fact that

$$\mathbf{P}_b [A_-] \geq \mathbf{P}_b \left[b_1 \leq (y_0 - 1 - x) / \sqrt{t}\right] \mathbf{P}_b [T_1 > t] .$$

Since $y_0 - 1 - x = -1 - \varepsilon_0$, the first factor in the last expression converges to $1/2$ uniformly in x, while the second satisfies $\lim_{t\to\infty} \mathbf{P}_b [T_1 > t] \sqrt{t} = \sqrt{2/\pi}$. The lemma follows. ■

Continuing with the proof of the proposition, let $x_1 > x_0$ and let $y_1 < y_0$, and define

$$\begin{aligned}\tilde{A}_+ &= \{x_1 \geq b_s^x \geq x_0 : \forall s \in [t, 2t]\} , \\ \tilde{A}_- &= \{y_1 \leq b_s^x \leq y_0 : \forall s \in [t, 2t]\} .\end{aligned}$$

If x_1 (resp. y_1) tends to $+\infty$ (resp. $-\infty$), then $\mathbf{P}_b \left[\tilde{A}_+\right]$ tends to $\mathbf{P}_b [A_+]$ (resp. $\mathbf{P}_b \left[\tilde{A}_-\right]$ tends to $\mathbf{P}_b [A_-]$). Therefore, using the above lemma, for any fixed $t \geq t_0$, there exist fixed values of x_1 and y_1 (which may depend on t, x, ε_0), such that

$$\mathbf{P}_b \left[\tilde{A}_+\right] \geq \frac{1}{4\sqrt{t}} \text{ and } \mathbf{P}_b \left[\tilde{A}_-\right] \geq \frac{1}{4\sqrt{t}}.$$

Step 2. *Restricting b^x.* Let $X_b = X_b (2t) = \int_0^{2t} W (ds, b_s^x)$. We have

$$U (2t, x) = \mathbf{E} \left[\log \mathbf{E}_b \left[e^{X_b}\right]\right] \geq \mathbf{E} \left[\log \mathbf{E}_b \left[e^{X_b} \mathbf{1}_{\tilde{A}_+} + e^{X_b} \mathbf{1}_{\tilde{A}_-}\right]\right]$$

$$\begin{aligned}&\geq \mathbf{E} \left[\log \mathbf{E}_b \left[\max \left\{e^{X_b} \mathbf{1}_{\tilde{A}_+} ; e^{X_b} \mathbf{1}_{\tilde{A}_-}\right\}\right]\right] \\ &\geq \mathbf{E} \left[\max \left\{\log \mathbf{E}_b \left[e^{X_b} \mathbf{1}_{\tilde{A}_+}\right] ; \log \mathbf{E}_b \left[e^{X_b} \mathbf{1}_{\tilde{A}_-}\right]\right\}\right] \\ &= \mathbf{E} \left[\max \left\{\log \left(\mathbf{E}_b \left[e^{X_b} | \tilde{A}_+\right] \mathbf{P}_b \left[\tilde{A}_+\right]\right) ; \log \left(\mathbf{E}_b \left[e^{X_b} | \tilde{A}_-\right] \mathbf{P}_b \left[\tilde{A}_-\right]\right)\right\}\right] .\end{aligned}$$

By the result of the previous step, and using Jensen's inequality, we have for any $t \geq t_0$

$$\begin{aligned}U (2t, x) &\geq -\log \left(4\sqrt{t}\right) + \mathbf{E} \left[\max \left\{\mathbf{E}_b \left[X_b | \tilde{A}_+\right] ; \mathbf{E}_b \left[X_b | \tilde{A}_-\right]\right\}\right] \\ &= -\log 4 - (\log t) / 2 + \mathbf{E} \left[\max \left\{\tilde{Z}_+, \tilde{Z}_-\right\}\right] .\end{aligned}$$

Here we have introduced

$$\tilde{Z}_+ := \mathbf{E}_b \left[X_b | \tilde{A}_+\right] \quad \text{and} \quad \tilde{Z}_- := \mathbf{E}_b \left[X_b | \tilde{A}_-\right] ;$$

these form a pair of centered jointly Gaussian random variables. Indeed, they are both linear combinations of values of a single centered Gaussian field. This implies that the random variable $\tilde{Z}_{+}-\tilde{Z}_{-}$ is centered Gaussian. Now let $\sigma=\left(\mathbf{E}\left[\left(\tilde{Z}_{+}-\tilde{Z}_{-}\right)^{2}\right]\right)^{1/2}$. Then $\left(\tilde{Z}_{+}-\tilde{Z}_{-}\right)/\sigma$ is a standard normal random variable. Thus we can write $\mathbf{E}\left[\left|\tilde{Z}_{+}-\tilde{Z}_{-}\right|/\sigma\right]=\sqrt{2/\pi}$. This, plus the trivial fact that $\max\left(a,b\right)=\left(\left|a-b\right|+a+b\right)/2$, imply

$$\begin{aligned}\mathbf{E}\left[\max\left\{\tilde{Z}_{+},\tilde{Z}_{-}\right\}\right]&=2^{-1}\mathbf{E}\left[\left|\tilde{Z}_{+}-\tilde{Z}_{-}\right|\right]\\&=\left(2\pi\right)^{-1/2}\left(\mathbf{E}\left[\left(\tilde{Z}_{+}-\tilde{Z}_{-}\right)^{2}\right]\right)^{1/2}.\end{aligned}$$

We thus conclude

$$U\left(2t,x\right)\geq-\log 4-\left(\log t\right)/2+\left(2\pi\right)^{-1/2}\left(\mathbf{E}\left[\left(\tilde{Z}_{+}-\tilde{Z}_{-}\right)^{2}\right]\right)^{1/2}.$$

Step 3. *Statement of strategy.* By definition of U_{*}, and by the fact that $\lambda_{*}=\sup_{t}U_{*}\left(t\right)/t$, we have $\lambda_{*}\geq\inf_{x}U\left(2t,x\right)/(2t)$ for any fixed t. So we only need to identify a single value t such that $U\left(2t,x\right)$ is bounded below uniformly in x. From the result of the previous step, the proposition will thus be established if we can prove that $\log t=o\left(\mathbf{E}\left[\left(\tilde{Z}_{+}-\tilde{Z}_{-}\right)^{2}\right]\right)^{1/2}$, and that this holds uniformly in x. In fact, we will prove more, namely that with c_{0} the constant identified in Condition **(E')**, for some $t_{0}>0$, for any fixed $t\geq t_{0}$,

$$\mathbf{E}\left[\left(\tilde{Z}_{+}-\tilde{Z}_{-}\right)^{2}\right]\geq c_{0}t.$$

Step 4. *Calculating the variance of $\tilde{Z}_{+}-\tilde{Z}_{-}$.* We introduce a new, time-free, random field: let $\left\{\tilde{W}\left(x\right):x\in\mathbf{R}\right\}$ denote a centered Gaussian field satisfying $\mathbf{E}\left[\tilde{W}\left(x\right)\tilde{W}\left(y\right)\right]=Q\left(x,y\right)$. Also, for every fixed $s\in\left[0,2t\right]$, let $\tilde{Z}_{+}\left(s\right)=\mathbf{E}_{b}\left[\tilde{W}\left(b_{s}^{x}\right)|\tilde{A}_{+}\right]$ and $\tilde{Z}_{-}\left(s\right)=\mathbf{E}_{b}\left[\tilde{W}\left(b_{s}^{x}\right)|\tilde{A}_{-}\right]$. We now prove the following formula.

$$\mathbf{E}\left[\left(\tilde{Z}_{+}-\tilde{Z}_{-}\right)^{2}\right]=\int_{0}^{2t}\mathbf{E}\left[\left(\tilde{Z}_{+}\left(s\right)-\tilde{Z}_{-}\left(s\right)\right)^{2}\right]ds.$$

To begin with, writing squares of expected values $\left(\mathbf{E}_{b}\left[F\left(b^{x}\right)\right]\right)^{2}$ as expectations of products $\mathbf{E}_{b,b'}\left[F\left(b^{x}\right)F\left(b'^{x}\right)\right]$ where under the measure $\mathbf{P}_{b,b'}$, b^{x} and b'^{x} are independent copies of b^{x}, we have

$$\begin{aligned}\mathbf{E}\left[\left(\tilde{Z}_+-\tilde{Z}_-\right)^2\right] &= \mathbf{E}\left[\mathbf{E}_{b,b'}\left[\int_0^{2t} W\left(ds,b_s^x\right)\int_0^{2t} W\left(ds,b_s'^x\right)|\tilde{A}_+\left(b^x\right)\cap\tilde{A}_+\left(b'^x\right)\right]\right]\\ &+\mathbf{E}\left[\mathbf{E}_{b,b'}\left[\int_0^{2t} W\left(ds,b_s^x\right)\int_0^{2t} W\left(ds,b_s'^x\right)|\tilde{A}_-\left(b^x\right)\cap\tilde{A}_-\left(b'^x\right)\right]\right]\\ &-2\mathbf{E}\left[\mathbf{E}_{b,b'}\left[\int_0^{2t} W\left(ds,b_s^x\right)\int_0^{2t} W\left(ds,b_s'^x\right)|\tilde{A}_+\left(b^x\right)\cap\tilde{A}_-\left(b'^x\right)\right]\right]\\ &=\mathbf{E}_{b,b'}\left[\int_0^{2t} Q\left(b_{s,}^x b_s'^x\right)ds|\tilde{A}_+\left(b^x\right)\cap\tilde{A}_+\left(b'^x\right)\right]\\ &+\mathbf{E}_{b,b'}\left[\int_0^{2t} Q\left(b_{s,}^x b_s'^x\right)ds|\tilde{A}_-\left(b^x\right)\cap\tilde{A}_-\left(b'^x\right)\right]\\ &-2\mathbf{E}_{b,b'}\left[\int_0^{2t} Q\left(b_{s,}^x b_s'^x\right)ds|\tilde{A}_+\left(b^x\right)\cap\tilde{A}_-\left(b'^x\right)\right].\end{aligned}$$

Now using Fubini's theorem to bring the time integration outside, and using the formula $\mathbf{E}\left[\tilde{W}\left(x\right)\tilde{W}\left(y\right)\right]=Q\left(x,y\right)$, and another Fubini to bring this new $\mathbf{E}$ outside, we obtain

$$\begin{aligned}\mathbf{E}\left[\left(\tilde{Z}_+-\tilde{Z}_-\right)^2\right] &= \int_0^{2t}\mathbf{E}\mathbf{E}_{b,b'}\left[\tilde{W}\left(b_s^x\right)\tilde{W}\left(b_s'^x\right)|\tilde{A}_+\left(b^x\right)\cap\tilde{A}_+\left(b'^x\right)\right]ds\\ &+\int_0^{2t}\mathbf{E}\mathbf{E}_{b,b'}\left[\tilde{W}\left(b_s^x\right)\tilde{W}\left(b_s'^x\right)|\tilde{A}_-\left(b^x\right)\cap\tilde{A}_-\left(b'^x\right)\right]ds\\ &-2\int_0^{2t}\mathbf{E}\mathbf{E}_{b,b'}\left[\tilde{W}\left(b_s^x\right)\tilde{W}\left(b_s'^x\right)|\tilde{A}_+\left(b^x\right)\cap\tilde{A}_-\left(b'^x\right)\right]ds.\end{aligned}$$

Reintroducing squares and products of expectations with respect to $\mathbf{P}_b$ we obtain

$$\begin{aligned}\mathbf{E}\left[\left(\tilde{Z}_+-\tilde{Z}_-\right)^2\right] &= \int_0^{2t}\mathbf{E}\left[\left(\mathbf{E}_b\left[\tilde{W}\left(b_s^x\right)|\tilde{A}_+\right]\right)^2\right]ds+\int_0^{2t}\mathbf{E}\left[\left(\mathbf{E}_b\left[\tilde{W}\left(b_s^x\right)|\tilde{A}_-\right]\right)^2\right]ds\\ &-2\int_0^{2t}\mathbf{E}\left[\mathbf{E}_b\left[\tilde{W}\left(b_s^x\right)|\tilde{A}_+\right]\mathbf{E}_b\left[\tilde{W}\left(b_s^x\right)|\tilde{A}_-\right]\right]ds\\ &=\int_0^{2t}\mathbf{E}\left[\left(\mathbf{E}_b\left[\tilde{W}\left(b_s^x\right)|\tilde{A}_+\right]-\mathbf{E}_b\left[\tilde{W}\left(b_s^x\right)|\tilde{A}_-\right]\right)^2\right]ds\end{aligned}$$

which is what we set out to prove in this step.

Step 5. *Estimating the variance of* $\tilde{Z}_+-\tilde{Z}_-$. *Conclusion.* First we discard the entire first half of the expression for $\mathbf{E}\left[\left(\tilde{Z}_+-\tilde{Z}_-\right)^2\right]$ just obtained, for $s\in[0,t]$, yielding the lower bound

$$\mathbf{E}\left[\left(\tilde{Z}_+-\tilde{Z}_-\right)^2\right]\geq\int_t^{2t}\mathbf{E}\left[\left(\tilde{Z}_+\left(s\right)-\tilde{Z}_-\left(s\right)\right)^2\right]ds. \tag{18}$$

Our initial goal here is to show that $\tilde{Z}_+ - \tilde{Z}_-$ can be expressed as an increment of $\tilde{W}$ itself, albeit between random sites. By the assumption on the continuity of W, almost surely, $\tilde{W}$ defined in Step 4 is continuous on all of $\mathbf{R}$. Therefore the set $I_+ := \left\{\tilde{W}(x) : x \in [x_0, x_1]\right\}$ is a closed interval. Also, for all $s \in [t, 2t]$, under the conditioning $\tilde{A}_+$, we have $b_s^x \in [x_0, x_1]$. This implies that for each fixed s, $\tilde{Z}_+(s)$ is a convex combination of points in the interval I_+; indeed, to be specific, if we denote by $f_{s,+}(dy)$ the distribution of b_s^x given $\tilde{A}_+$, $f_{s,+}$ is supported by $[x_0, x_1]$, has total mass 1, and we have $\tilde{Z}_+(s) = \int_{x_0}^{x_1} f_{s,+}(dy)\,\tilde{W}(y)$. Therefore, since I_+ is convex, $\tilde{Z}_+(s) \in I_+$. This proves there exists a point $x_{s,+}^* \in [x_0, x_1]$ such that $\tilde{Z}_+(s) = \tilde{W}\left(x_{s,+}^*\right)$. Similarly, there exists a point $x_{s,-}^* \in [y_1, y_0]$ such that $\tilde{Z}_+(s) = \tilde{W}\left(x_{s,-}^*\right)$. Note that $x_{s,+}^*$ and $x_{s,-}^*$ are random; yet they are bounded as indicated, and conditional on $x_{s,+}^*$ and $x_{s,-}^*$, $\tilde{W}\left(x_{s,+}^*\right)$ and $\tilde{W}\left(x_{s,-}^*\right)$ are jointly Gaussian, with covariance given using the function Q.

Now we can write

$$\begin{aligned}
\mathbf{E}\left[\left(\tilde{Z}_+(s) - \tilde{Z}_-(s)\right)^2\right] &= \mathbf{E}\left[\left(\tilde{W}\left(x_{s,+}^*\right) - \tilde{W}\left(x_{s,-}^*\right)\right)^2\right] \\
&= \mathbf{E}\left[\mathbf{E}\left[\left(\tilde{W}\left(x_{s,+}^*\right) - \tilde{W}\left(x_{s,-}^*\right)\right)^2 \middle| x_{s,+}^*; x_{s,-}^*\right]\right] \\
&= \mathbf{E}\left[\delta\left(x_{s,+}^*; x_{s,-}^*\right)^2\right].
\end{aligned}$$

Since $x_{s,+}^*$ and $x_{s,-}^*$are supported by $[x_0, x_1]$ and $[y_1, y_0]$ respectively, we can use the lower bound on δ^2 given in Condition **(E')**, which, with the lower bound (18), yields

$$\mathbf{E}\left[\left(\tilde{Z}_+ - \tilde{Z}_-\right)^2\right] \geq \int_t^{2t} c_0 ds = c_0 t,$$

which, by Step 3, ends the proof of the proposition. ■

3 Lower Bound

With this section, we begin the quantitative analysis of the exponential behavior of u in large time. We note that existence of the Lyapunov exponent is not required for any of the results below.

Lemma 13 *With the notations in the previous Section let $\lambda_-(x) = \liminf_{t\to\infty} U(t,x)/t$. Then $\lambda_-(x)$ is a lower bound for the exponential behavior of the solution of the Stochastic parabolic Anderson PDE for x fixed, almost surely:*

$$\liminf_{\substack{t\to\infty \\ t\in\mathbf{N}}} \frac{\log u(t,x)}{t} \geq \lambda_-(x).$$

Proof. Let x be fixed and let $\beta < \lambda_-(x)$. We have, for $k > 1$,

$$\begin{aligned}\mathbf{P}\left[\frac{\log u(t,x)}{t} \leq \beta\right] &= \mathbf{P}\left[\log u(t,x) \leq \beta t\right] \\ &= \mathbf{P}\left[\log u(t,x) - U(t,x) \leq -U(t,x) + \beta t\right] \\ &\leq \frac{\mathbf{E}\left[\left|\log u(t,x) - U(t,x)\right|^{2k}\right]}{\left(U(t,x) - \beta t\right)^{2k}}.\end{aligned}$$

Using the calculations in Subsection 2.3, more specifically (16), we obtain

$$\mathbf{P}\left[\frac{\log u(t,x)}{t} \leq \beta\right] \leq \frac{t^k C_{Q,k}}{\left(U(t,x) - \beta t\right)^{2k}}. \tag{19}$$

There exists a t_0 so large that $\forall t \geq t_0$:

$$U(t,x) > \left(\lambda_-(x) - \frac{\lambda_-(x) - \beta}{2}\right) t.$$

Thus, (19) is continued by

$$\mathbf{P}\left[\frac{\log u(t,x)}{t} \leq \beta\right] \leq \frac{t^k C_{Q,k}}{\left(\left(\frac{\lambda_- - \beta}{2}\right) t\right)^{2k}} = C_{Q,k}\left(\frac{2}{\lambda_-(x) - \beta}\right)^{2k} t^{-k}. \tag{20}$$

With $k = 2$ we see that the probability is summable for $t \in \mathbf{N}$ since $\beta < \lambda_-(x)$. Therefore, we can apply the Borel-Cantelli lemma to assert that there exists an almost-surely finite integer $t_-(\omega)$ such that for every integer $t \geq t_-(\omega)$, $\log u(t,x)/t \geq \beta$. In conclusion, for any $\beta < \lambda_-(x)$ and any $x \in \mathbf{R}$, we have almost surely

$$\liminf_{\substack{t\to\infty \\ t\in\mathbf{N}}} \frac{\log u(t,x)}{t} \geq \beta.$$

Since β can be taken arbitrarily close to $\lambda_-(x)$, the result of the lemma follows. ■

We now have a clear method for finding lower bounds for the Lyapunov exponent: a lower bound for $\lambda_-(x)$ implies almost surely the same lower bound for $\liminf_{\substack{t\to\infty \\ t\in\mathbf{N}}} t^{-1} \log u(t,x)$, which is the starting point of the next theorem's proof.

Theorem 14 (Lower Bound for the Lyapunov Exponent) *There exists a universal constant c_u such that if $u(t,x)$ is the solution of the stochastic parabolic Anderson PDE* (1), *under Conditions **(E)** and **(E'-)**, we have for small κ, for fixed x, almost surely:*

$$\liminf_{\substack{t\to\infty \\ t\in\mathbf{N}}} \frac{\log u(t,x)}{t} \geq \frac{c_u}{\sqrt{32\pi}}\kappa\left(\Delta^{-1}\left(\frac{c_u}{\sqrt{t_0(\kappa)}}\right)\right)^{-2} = \frac{c_u}{t_0(\kappa)\sqrt{32\pi}},$$

where $t_0(\kappa)$ is the unique solution of the equation

$$c_u = \sqrt{t_0}\Delta\left(\sqrt{\kappa t_0}\right). \tag{21}$$

Proof. Step 1. *Strategy.* To prove this theorem, we need to recall some earlier facts found in Section 2. Since we trivially have $\lambda_-(x) \geq \lambda_*$ by definition, we will seek only to bound λ_* from below. By Proposition 5, we have that $\lambda_* \geq U_*(t)/t$ for any t. Therefore, we only need to identify a single time t_0 such that $U(t_0, x)/t_0$ exceeds the announced lower bound $c_u/(t_0\sqrt{32\pi})$ uniformly for all x.

Step 2. *Using prior setup with new scaling.* We will proceed similarly to the proof of Proposition 12. Now we should interpret b^x as having variance κ and starting point x, but in this proof it will be sufficient to use the notation b, as always, for our standard Brownian motion under $\mathbf{P}_b$, started from 0. Multiplying b by $\sqrt{\kappa}$ will then yield the right variance. We modify the definitions of $\tilde{A}_+$ and $\tilde{A}_-$. The new choices we make for these events are not symmetric, unlike in the proof of Proposition 12, which will result in a much larger lower bound than if we had kept the choices made in that proof. We take

$$\tilde{A}_+ := \left\{\sqrt{t}x_1 \geq \sqrt{\kappa}b_s \geq \sqrt{t}\,(x_0 - x) : \forall s \in [t, 2t]\right\},$$

and

$$\tilde{A}_- := \left\{\sqrt{t}y_1 \leq \sqrt{\kappa}b_s \leq \sqrt{t}\,(y_0 - x) : \forall s \in [t, 2t]\right\}.$$

Note that we have the following equalities in law under $\mathbf{P}_b$:

$$\tilde{A}_+ \overset{law}{=} \left\{x_1 \geq \sqrt{\kappa}b_{s'} \geq (x_0 - x) : \forall s' \in [1, 2]\right\},$$

$$\tilde{A}_- \overset{law}{=} \left\{y_1/\sqrt{\kappa} \leq b_{s'} \leq (y_0 - x)/\sqrt{\kappa} : \forall s' \in [1, 2]\right\}$$

which proves in particular that the probabilities $\mathbf{P}_b\left[\tilde{A}_+\right]$ and $\mathbf{P}_b\left[\tilde{A}_-\right]$ do not depend on t. By letting $x_0 = x$, $x_1 = \sqrt{\kappa}$, we get that $\mathbf{P}_b\left[\tilde{A}_+\right]$ does not depend on x or κ. To get the same effect on $\tilde{A}_-$, we may take $y_0 - x = -\sqrt{\kappa}$, and $y_1 = -2\sqrt{\kappa}$. In other words, there is a positive universal constant C_u such that $\log\left(\min\left\{\mathbf{P}_b\left[\tilde{A}_+\right]; \mathbf{P}_b\left[\tilde{A}_-\right]\right\}\right) = -C_u$. In any event, since $\tilde{A}_+$ and $\tilde{A}_-$ are disjoint, we still have from the proof of Proposition 12

$$U(2t, x) \geq -C_u + \mathbf{E}\left[\max\left\{\tilde{Z}_+, \tilde{Z}_-\right\}\right].$$

and

$$\mathbf{E}\left[\max\left\{\tilde{Z}_+, \tilde{Z}_-\right\}\right] \geq \frac{1}{\sqrt{2\pi}}\left(\int_t^{2t} \mathbf{E}\left[\delta\left(x^*_{s,+}; x^*_{s,-}\right)^2\right] ds\right)^{1/2},$$

where here the random variables $x^*_{s,+}$ and $x^*_{s,-}$ are bounded respectively below and above by $x_0\sqrt{t}$ and $y_0\sqrt{t}$. The other conclusion we can draw is that with these choices of x's and y's, we get $|x_0 - y_0| = \sqrt{\kappa}$.

Step 3. *Optimization of the parameters.* Using Condition **(E'-)**, we see that

$$\delta\left(x^*_{s,+};x^*_{s,-}\right)^2 \geq \Delta^2\left(\varepsilon\right)$$

where $\varepsilon = |x_0 - y_0|\sqrt{t} = \sqrt{\kappa t}$, as long as we can guarantee that $\sqrt{\kappa t}$ can be made small when κ is small. We would then have

$$U\left(2t,x\right) \geq -C_u + \sqrt{t/\left(2\pi\right)}\Delta\left(\sqrt{\kappa t}\right).$$

It is clear we need to choose $t = t_0 = t_0\left(\kappa\right)$ as a function of κ, and that an optimal choice, up to multiplicative universal constants, is one such that

$$2C_u\sqrt{2\pi} = \sqrt{t_0}\Delta\left(\sqrt{\kappa t_0}\right).$$

Since by Condition **(E'-)** Δ is a bijective function (near 0) with inverse Δ^{-1}, we would have to take

$$\kappa = \frac{1}{t_0}\left[\Delta^{-1}\left(\frac{2C_u\sqrt{2\pi}}{\sqrt{t_0}}\right)\right]^2.$$

This relation can of course be inverted to write t_0 as a function of κ, although the expression cannot be as explicit. We also have $\sqrt{\kappa t_0} = \Delta^{-1}\left(C_u\sqrt{2\pi/t_0}\right)$, and we see that since $\lim_{r\to 0}\Delta\left(r\right) = 0$, the same holds for Δ^{-1}, and therefore κt_0 is small as long as t_0 is large enough.

Step 4. *Checking t_0 can be made large enough when κ is small.* Since the above choice for t_0 implies

$$\frac{U\left(2t_0,x\right)}{2t_0} \geq \frac{C_u}{2t_0} = \frac{C_u}{2}\kappa\left(\Delta^{-1}\left(\frac{2C_u\sqrt{2\pi}}{\sqrt{t_0}}\right)\right)^{-2},$$

we will be able to conclude the proof of the theorem as long as we can justify that when κ is small, t_0 is large, since by Step 3, this would also imply that $\sqrt{\kappa t_0}$ is small, allowing the use of Condition **(E'-)**. Since W is assumed to be almost-surely continuous, the theory of Gaussian regularity (see for example [18]) implies that $\Delta\left(r\right) = o\left(\log^{-1/2}\left(1/r\right)\right)$. In particular, we can assume that for small r, $\Delta\left(r\right) < \log^{-1/2}\left(1/r\right)$. Equivalently, for small x, we have

$$\Delta^{-1}\left(x\right) > e^{-1/x^2}.$$

Combining this inequality with the expression for κ above yields

$$\kappa > \frac{1}{t_0}\exp\left(-t_0/(4\pi C_u)\right),$$

which indeed implies that if κ is small, t_0 will have to be large. Hence the claim that κt_0 can be made small enough is justified, and the proof of the theorem is complete. Note that the universal constant c_u in the statement of the theorem equals $2C_u\sqrt{2\pi}$. ■

Corollary 15 *By possibly adjusting the leading constant by a universal positive factor less than* 1*, the previous theorem holds even if the* $\liminf$ *is taken over all times* $t \in \mathbf{R}_+$ *(removing the subscript* $t \in \mathbf{N}$*).*

Proof. The technique used in [17] and in [7] to handle the infimum of $u(t,x)$ for all $t \in [n-1,n]$ where $n \in \mathbf{N}$, can be used here again with no additional difficulty. We omit these details since no new idea is required from those introduced in [17]. It is worth mentioning that this technique cannot be adapted to solving the open problem at the end of Section 2.1. This is because in [17], the authors only show a lower bound for the $\liminf$ of the quantity $n^{-1}\log\left(\inf\{u(t,x) : t \in [n-1,n]\}\right)$. The same idea of how to handle all $t \in [n-1,n]$ simultaneously is used in the upper bound context in [7]. But when putting the two together, a gap will always exist between lower and upper bounds. Thus the open problem at the end of Section 2.1 remains. ■

4 Upper Bound

For the upper bound we will use a discretization technique similar to those in [7] or [17], while making some necessary improvements. We will approximate the Brownian path in the Feynman-Kac formula (8) with a path that stays in $\varepsilon\mathbf{Z}$ where ε is a small positive number that will be chosen as a function of κ.

4.1 Notations and basic results

For any Brownian motion path in $\mathcal{C}$ the space of continuous functions, let $t_0 = 0$ and for $i = 1, 2, 3, \ldots$ let t_i be the first time after t_{i-1} that $b_t - b_{t_{i-1}}$ exits $[-\varepsilon, \varepsilon]$. We define the discretized path $\tilde{b}$ as the right-continuous path that jumps at each time t_i to the position $x_i := b_{t_i}$, and that is constant between jump times. For any time t, we define N_t as the number of jumps of $\tilde{b}$ up to time t. Denote by

$$S(t,n) = \left\{ \tilde{t} = (t_1, t_2, \ldots, t_n) \middle| \, 0 = t_0 \le t_1 \le t_2 \le \cdots \le t_n \le t \right\}$$

the simplex of all the possible sequences of n jump times, and by P_n the set of all possible visited sites $\tilde{x} = (x_0, x_1, \ldots, x_n)$. Under $\mathbf{P}_b$ the inter-jump times $T_i = t_i - t_{i-1}$ are independent and identically distributed and are independent of $\tilde{x}$. The sequence $\tilde{x}$ itself under $\mathbf{P}_b$ is a symmetric nearest-neighbor random walk on $\varepsilon\mathbf{Z}$ started at x. Here and throughout, x is fixed.

For $\tilde{b}$ the discretized version of b, using the convention $t_{N_t+1} := t$, we define:

$$X_{N_t}(\tilde{t}, \tilde{x}) := \int_0^t W(ds, \tilde{b}_s) = \sum_{i=0}^{N_t} \left(W(t_{i+1}, x_i) - W(t_i, x_i) \right),$$

$$\tilde{u}(t) := \tilde{u}(t,x) = \mathbf{E}_b \left[\exp(2X_{N_t}(\tilde{t}, \tilde{x})) \right].$$

Let $\lambda_+ = \limsup_{t\to\infty} t^{-1}\log u(t,x)$ and $\tilde{\lambda}_+ = \limsup_{t\to\infty} t^{-1}\log \tilde{u}(t)$. We may write almost surely using the Cauchy-Schwarz inequality:

$$u(t,0) \leq \mathbf{E}_b\left[e^{2X_t^b(x)-2X_{N_t}(\tilde{t},\tilde{x})}\right]^{1/2}\tilde{u}(t)^{1/2},$$

and thus,

$$\lambda_+ \leq \frac{1}{2}\left(\xi + \tilde{\lambda}_+\right), \tag{22}$$

with $\xi = \limsup_{t\to\infty}\log \mathbf{E}_b \exp(2X_t^b(x) - 2X_{N_t}(\tilde{t},\tilde{x}))$. This last quantity is the error committed by discretizing b, i.e. by replacing λ_+ by (a constant multiple of) $\tilde{\lambda}_+$. We seek an upper bound on both ξ and $\tilde{\lambda}_+$. We note here that both ξ and $\tilde{\lambda}_+$ are relative to the Gaussian field $2W$ rather than W.

4.2 Error estimate

We quote here a result that was originally in established in [7], and used subsequently in [17], more specifically Proposition 3 therein. We do not repeat the proof, noting instead that the measurability conditions that are required for a rigorous proof do not need to assume the Hölder-type conditions of [7] or [17], but that without continuity of W (see Remark 1 for an interpretation in the context of Condition **(E")**), we do not believe that the Feynman-Kac formula even holds, or that the following result can be established. We have almost-surely:

$$\xi \leq \limsup_{n\to\infty}\frac{1}{n}\log \mathbf{E}_b\left[\exp\left(K\sigma_{n,b}^2\right)\right], \tag{23}$$

where, $\sigma_{n,b}^2 = \sup_{t\in[n-1,n]}\mathbf{E}\left[\left(X_t^b(x) - X_{N_t}(\tilde{t},\tilde{x})\right)^2\right]$ and K a universal constant. We can estimate $\sigma_{n,b}^2$ using the assumption **(E")** as follows:

$$\begin{aligned}\sigma_{n,b}^2 &= \sup_{t\in[n-1,n]}\mathbf{E}\left[\left(X_t^b(x) - X_{N_t}(\tilde{t},\tilde{x})\right)^2\right]\\ &= \sup_{t\in[n-1,n]}\mathbf{E}\left[\left(\int_0^t W(ds,b_s) - \int_0^t W(ds,\tilde{b}_s)\right)^2\right]\\ &= \sup_{t\in[n-1,n]}\int_0^t \delta^2\left(b_s,\tilde{b}_s\right)ds \leq \int_0^n \delta^2\left(\left|b_s-\tilde{b}_s\right|\right)ds.\end{aligned}$$

Now using the fact that the two processes b and $\tilde{b}$ are never more than ε apart, we obtain:

$$\sigma_{n,b}^2 \leq n\delta^2(\varepsilon).$$

Finally, using this last estimate in (23) we find that the approximation error is bounded as

$$\xi \leq K\delta^2(\varepsilon) \tag{24}$$

4.3 Setup for use of Gaussian supremum estimates

We can write

$$\begin{aligned}\tilde{u}(t) &= E_b\left[e^{2X_{N_t}(\tilde{t},\tilde{x})}\right] \\ &= \sum_{n=1}^{\infty} \mathbf{E}_b\left[e^{2X_{N_t}(\tilde{t},\tilde{x})}\middle| N_t \in [t\alpha(n-1), t\alpha n]\right] \mathbf{P}_b\left[N_t \in [t\alpha(n-1), t\alpha n]\right]. \end{aligned} \tag{25}$$

At this point, let us notice that every discretization $\tilde{b}$ is characterized by the number of jumps up to time t, the times of those jumps, and the direction of the jumps. That is, every path $\tilde{b}$ is equivalent to its triplet $(N_t, \tilde{t}, \tilde{x})$. For any positive integer n and any $\alpha > 0$, let us then define

$$T_{n\alpha} = \left\{ \left(N_t, \tilde{t}, \tilde{x}\right) \middle| N_t \leq tn\alpha; \tilde{t} \in S(t, N_t); \tilde{x} \in P_{N_t} \right\}.$$

We will use the notations

$$X^*_{n,t,\alpha} = \sup_{T_{n\alpha}} 2X_{N_t}\left(\tilde{t}, \tilde{x}\right),$$

and

$$f(n\alpha, t) = \mathbf{E}\left[\sup_{T_{n\alpha}} 2X_{N_t}\left(\tilde{t}, \tilde{x}\right)\right].$$

In order to find an upper bound for the Lyapunov exponent we invoke two classical theorems from Gaussian processes theory that can be found in [1].

Theorem 16 (V. Sudakov - C. Borell) *Let T be a Polish space and $\{X_t\}_{t\in T}$ be a centered, separable, Gaussian field with $\sup_{t\in T} X_t < \infty$ a.s.. Then $\mathbf{E}(\sup_{t\in T} X_t) < \infty$ and for all $\lambda > 0$ we have:*

$$P\left(|\sup_{t\in T} X_t - \mathbf{E}\left(\sup_{t\in T} X_t\right)| > \lambda\right) \leq 2e^{-\frac{\lambda^2}{2\sigma_T^2}} \tag{26}$$

where $\sigma_T^2 = \sup_{t\in T} E(X_t^2)$.

For a separable Gaussian field $\{X_t\}_{t\in T}$ on T we use the following notation for its canonical metric on the space T:

$$\rho(s,t) = \sqrt{\mathbf{E}\left[(X_t - X_s)^2\right]}.$$

Theorem 17 (Dudley-Fernique) *There exists a universal constant $\mathbf{K} > 0$ such that:*

$$\mathbf{E}\left(\sup_{t\in T} X_t\right) \leq \mathbf{K}\int_0^{\infty} \sqrt{\log N(\eta)}\, d\eta \tag{27}$$

where $N(\eta)$ is the smallest number of balls of radius η in the metric ρ required to cover the space T.

Let us estimate the entropy function $N(\eta)$ for the field $X_{N_t}\left(\tilde{t},\tilde{x}\right)$ defined over $T_{n\alpha}$. Let $m\leq tn\alpha$, fixed. When $N_t=m$ is fixed, for $\tilde{t},\tilde{s}\in S(t,m)$ and $\tilde{x}\in P_m$ fixed, our metric is defined as:

$$d\left((m,\tilde{t},\tilde{x}),(m,\tilde{s},\tilde{x})\right)^2=\mathbf{E}\left(X_m\left(\tilde{t},\tilde{x}\right)-X_m\left(\tilde{s},\tilde{x}\right)\right)^2 \tag{28}$$

Remark 18 *The diameter of T under the metric d (the maximum distance of our metric) does not exceed $2\sqrt{t\max_x Q(x,x)}$*

This fact is trivial to see since $\mathbf{E}\left[\left(\int_0^t W(ds,x)\right)^2\right]\leq t\max_x Q(x,x)$. Define now $\bar{T}_m$ similarly to $T_{\alpha n}$, but with the number of jumps fixed, equal with m.

$$\bar{T}_m=\left\{\left(N_t,\tilde{t},\tilde{x}\right)\middle|\,N_t=m;\tilde{t}\in S(t,m);\tilde{x}\in P_m\right\}$$

We obviously have $T_{\alpha n}=\cup_{k\leq[\alpha nt]}\bar{T}_k$.

4.4 Gaussian estimations

Lemma 19 *For t, m and $\tilde{x}$ fixed, the canonical metric for the Gaussian field $\{X_m(\tilde{t},\tilde{x})\}_{\bar{T}_m}$ satisfies:*

$$d\left((m,\tilde{t},\tilde{x}),(m,\tilde{s},\tilde{x})\right)\leq\sqrt{4\sup_x Q(x,x)\sum_{i=0}^m|t_i-s_i|}.$$

Proof. This result is very similar to Lemma 2.1 in [6] for the homogeneous case. We have chosen to reprove it since the difference, in our non-homogeneous setting, is not that trivial. First note that by Cauchy-Schwarz,

$$Q(x,y)\leq\sqrt{Q\left(x,x\right)Q\left(y,y\right)}\leq\sup_x Q(x,x). \tag{29}$$

Using our metric, we can write:

$$\begin{aligned}d\left((m,\tilde{t},\tilde{x}),(m,\tilde{s},\tilde{x})\right)^2&=\sum_{i=0}^m Q(x_i,x_i)(t_{i+1}-t_i)+\sum_{i=0}^m Q(x_i,x_i)(s_{i+1}-s_i)\\&-2\mathbf{E}\sum_{i=0}^m\left(W(t_{i+1},x_i)-W(t_i,t_i)\right)\sum_{j=0}^m\left(W(s_{j+1},x_j)-W(s_j,t_j)\right)\\&\leq 2t\sup_x Q(x,x)-2\sum_{i=0}^m Q(x_i,x_i)\left|\Delta t_i\cap\Delta s_i\right|-2\sum_{i,j=0,i\neq j}^m Q(x_i,x_j)\left|\Delta t_i\cap\Delta s_j\right|.\end{aligned}$$

Using the notation $\Delta t_i=[t_i,t_{i+1}]$ and respectively $\Delta s_j=[s_j,s_{j+1}]$ we obviously have:

$$\sum_{i,j=0}^m\left|\Delta t_i\cap\Delta s_j\right|=\sum_{i=0}^m\left|\Delta t_i\cap\Delta s_i\right|+\sum_{i,j=0;i\neq j}^m\left|\Delta t_i\cap\Delta s_j\right|=t,$$

and now using (29), we obtain:

$$\sum_{i,j=0,i\neq j}^{m} Q(x_i,x_j)\left|\Delta t_i\cap\Delta s_j\right| \geq -\sup_x Q(x,x)\sum_{i,j=0,i\neq j}^{m}\left|\Delta t_i\cap\Delta s_j\right|$$
$$= -\sup_x Q(x,x)\left(t-\sum_{i=0}^{m}\left|\Delta t_i\cap\Delta s_i\right|\right).$$

Putting everything back together, we get:

$$d\left((m,\tilde{t},\tilde{x}),(m,\tilde{s},\tilde{x})\right)^2 \leq 4\sup_x Q(x,x)\left(t-\sum_{i=0}^{m}\left|\Delta t_i\cap\Delta s_i\right|\right),$$

and now the lemma follows since we have

$$t \leq \sum_{i=0}^{m}\left|\Delta t_i\cap\Delta s_i\right| + \sum_{i=0}^{m}\left|t_i-s_i\right|.$$

■

Now, in order to use the Dudley-Fernique Theorem 17, we will need to count how many balls of radius η are needed to cover $T_{n\alpha}$ in the metric d. To do that, first we will see how many balls are needed to cover $\bar{T}_m$, for each $m \leq t\alpha n$. Since $T_{n\alpha} = \cup_{m\leq t\alpha n}\bar{T}_m$, the number of balls required to cover $T_{n\alpha}$ is less than the sum of the number of balls required to cover each $\bar{T}_m$.

Suppose that we are working on $\bar{T}_m$. For any given sequence of jump times there are at most 2^m possible sequences $\tilde{x}\in P_m$. Thus, if we cover the simplex $S(t,m)$ with N balls we will need $2^m N$ total balls to cover $\bar{T}_m$. From Lemma 19 an upper bound for the metric (28) is: $d^2(\tilde{t},\tilde{s}) \leq 4Q(0)\sum_{i=1}^{m}|t_i-s_i|$. Using this upper bound, we now exhibit a lattice of times $\tilde{t}_{center}$ such that $d^2(\tilde{t},\tilde{t}_{center}) < \eta^2$, and then we will count how many points are in our lattice.

The next few paragraphs are similar to calculations performed originally in [7]. The reader familiar with those can skip directly to equation (30). Consider the partition of the interval $[0,t]$ by kt points with $k := 4\sup_x Q(x,x)m\eta^{-2}$: $\{0,\frac{1}{k},\frac{2}{k},\ldots,\frac{tk}{k}=t\}$. For any sequence in $S(t,m)$, say $t_1,t_2,\cdots,t_m$, for any $i=1,2,\cdots,m$, we can find a point $j(i)\,t/k$ in the partition such that $|t_i-\frac{jt}{k}|<\frac{1}{k}$. Consider $\tilde{t}_{center} := (j(i)\,t/k)_{i=1}^m$ for a fixed sequence $(j(i))_{i=1}^m$ of non-decreasing non-negative integers from 1 to tk, and consider the set of points $\tilde{t}$ satisfying $|t_i-\frac{j(i)t}{k}|<\frac{1}{k}$ for all $i=1,\cdots,m$. We obtain a cover of the entire set $S(t,m)$ by allowing $j(\cdot)$ to span all such sequences. Moreover, for each such $\tilde{t}_{center}$ we have

$$d^2(\tilde{t},\tilde{t}_{center}) \leq 4\sup_x Q(x,x)\sum_{i=1}^{n}\left|t_i-\frac{i}{k}\right| < 4\sup_x Q(x,x)\frac{m}{k} = \eta^2.$$

Hence, the balls centered at all the $\tilde{t}_{center}$'s thus constructed, with radius η in the metric d, cover the set $S(t,m)$. The total number of these balls is the number of nondecreasing sequences of length m with values in $\{1, 2, \cdots, kt\}$. This number is easily computed to be the binomial coefficient $\binom{kt+m-1}{m}$. We can also show, using Stirling's formula, that

$$\binom{kt+m-1}{m} \leq 3^m (kt+m)^m m^{-m}.$$

Now since the set $\bar{T}_m$ is the union of the sets $S(t,m) \times \{\tilde{x}\}$ where $\tilde{x}$ spans the set of all nearest-neighbor paths of length m, the total number $N_m(\eta)$ of balls needed to cover $\bar{T}_m$ is trivially bounded above as follows:

$$N_m(\eta) \leq 2^m 3^m (kt+m)^m m^{-m} = 6^m \left(\frac{4 \sup_x Q(x,x) t}{\eta^2} + 1 \right)^m,$$

where we used the definition of k.

Therefore, the following is again a trivial upper bound on the number $N(\eta)$ of η-balls required to cover $T_{n\alpha}$:

$$N(\eta) \leq \sum_{m=1}^{t\alpha n} N_m(\eta) \leq t\alpha n N_{t\alpha n}(\eta) \leq t\alpha n 6^{t\alpha n} \left(\frac{4 \sup_x Q(x,x) t}{\eta^2} + 1 \right)^{t\alpha n}.$$

For simplicity of notation, in the next few lines, we are going to use m instead of $t\alpha n$. We use the Dudley-Fernique Theorem 17 to obtain:

$$\begin{aligned}
\mathbf{E}\left(\sup_{(m,\tilde{t},\tilde{x}) \in T_{t\alpha n}} X_m(\tilde{t},\tilde{x}) \right) &\leq \mathbf{K} \int_0^{\eta_{max}} \sqrt{\log N(\eta)} d\eta \\
&\leq \mathbf{K} \int_0^{2\sqrt{t \sup_x Q(x,x)}} \sqrt{\log m 12^m \left(\frac{4 \sup_x Q(x,x) t}{\eta^2} \right)^m} \\
&\leq \mathbf{K} \int_0^{2\sqrt{t \sup_x Q(x,x)}} \sqrt{\log m + m \log 12 + m \log \frac{4 \sup_x Q(x,x) t}{\eta^2}} d\eta \\
&\leq \mathbf{K}\sqrt{m} \int_0^{2\sqrt{t \sup_x Q(x,x)}} \sqrt{1 + \log 12 + \log \frac{4 \sup_x Q(x,x) t}{\eta^2}} d\eta.
\end{aligned}$$

Now denote $c_1 = 1 + \log 12$, $c_2 = 4 \sup_x Q(x,x)$. We are going to make the change of variables: $\sqrt{c_1 + \log \frac{c_2 t}{\eta^2}} = z$. With this change of variables, the endpoints of integration become $+\infty$ and $\sqrt{c_1}$ and $d\eta = -\sqrt{c_2 t}\, 2z \exp(-z^2/2) \exp(c_1)$. Then, using the fact that the second moment of a standard normal is equal to 1, we have:

$$\begin{aligned}
\mathbf{E}\left(\sup_{(m,\tilde{t},\tilde{x}) \in T_{t\alpha n}} X_m(\tilde{t},\tilde{x}) \right) &\leq 2\mathbf{K}\sqrt{m}\sqrt{c_2 t} e^{c_1} \sqrt{2\pi} \int_{c_1}^{\infty} z^2 e^{-\frac{z^2}{2}} dz \\
&\leq K_Q \sqrt{mt}
\end{aligned}$$

for some constant K_Q depending only on Q (via the factor $\sup_x Q^{1/2}(x,x)$). Thus, substituting back $m = t\alpha n$, we find

$$f(n\alpha, t) = 2\mathbf{E}\left(\sup_{T_{n\alpha}} X_{N_t}(\tilde{t}, \tilde{x})\right) = \mathbf{E}\left(X^*_{n,t,\alpha}\right) \leq 2tK_Q\sqrt{n\alpha}. \tag{30}$$

To proceed further, we will need to prove a lower bound on $f(n\alpha, t)$ as well. We will need to use the strengthened non-degeneracy hypothesis **(E'-)**.

Lemma 20 *Under the hypothesis* ***(E'-)****, there exists a constant C_Q such that for any fixed $\varepsilon > 0$:*

$$f(n\alpha, t) \geq C_Q \Delta(\varepsilon) t\sqrt{n\alpha}.$$

Proof. Let us fix an $m \leq n\alpha t$. We will consider the function:

$$h(m,t) := \mathbf{E}\left[\sup_{\bar{T}_m} X_m(\tilde{t}, \tilde{x})\right],$$

with the same $\bar{T}_m$ defined earlier. We obviously have $h(m,t) \leq f(n\alpha t)$, for all $m \leq n\alpha t$. The idea is to maximize the increments defining X step by step. We will pick specific sequences $\tilde{t}$ and $\tilde{x}$ and the value of $\mathbf{E}X_m(\tilde{t}, \tilde{x}))$ will be a lower bound for the expected supremum over all the sequences. Let the times in the sequence $\tilde{t}$ be equally spaced i.e., $t_k = kt/m$, $k = 0, 1, \ldots, m$ and write $W(t_{k+1}, x) - W(t_k, x) = W_k(x)$. We let $x_0 = 0$ and choose x_k recursively as follows:

$$x_k = \begin{cases} x_{k-1} + \varepsilon & \text{if} \quad W_k(x_{k-1} + \varepsilon) \geq W_k(x_{k-1} - \varepsilon) \\ x_{k-1} - \varepsilon & \text{otherwise} \end{cases}.$$

Since this is just a particular sequence in $\bar{T}_m$, we obviously have:

$$h(m,t) \geq \sum_{k=0}^{m-1} \mathbf{E}\left[W_k(x_k)\right].$$

By the independence and scaling of the increments of W in time, and the spatial distribution of W, we immediately see that the values $W_k(x_k)$ are independent – which does not seem to be crucial here –, but more importantly that the distribution of $W_k(x_k)$ is equal to that of the random variable:

$$\sqrt{t/m}\max\left(Z_k, Z'_k\right)$$

where Z_k, Z'_k is a pair of centered Gaussian variables that are independent of x_{k-1} and satisfy $\mathbf{E}\left(Z_k^2\right) = Q\left(x_{k-1} - \varepsilon\right)$, $\mathbf{E}\left(Z_k'^2\right) = Q\left(x_{k-1} + \varepsilon\right)$, and $\mathbf{E}\left[\left(Z_k - Z'_k\right)^2\right] = \delta\left(x_{k-1} + \varepsilon, x_{k-1} - \varepsilon\right)$. Now we can use the Hypotheses **(E'-)**, coupled with the expected value of the maximum of two Gaussian random variables. Thus, for some constant $C_Q > 0$ depending only on Q, we have for all k:

$$\mathbf{E}\left[\max\left(Z_k, Z_k'\right)\right] \geq C_Q \Delta\left(\varepsilon\right).$$

This proves that for any fixed $\varepsilon > 0$,

$$h\left(t, m\right) \geq m C_Q \Delta\left(\varepsilon\right) \sqrt{\frac{t}{m}}.$$

We immediately obtain

$$f\left(n\alpha, t\right) \geq h\left(t, n\alpha t\right) \geq C_Q \Delta\left(\varepsilon\right) t \sqrt{n\alpha}.$$

■

Remark 21 *Strictly speaking, the full strength of Condition **(E'-)** is not necessary to prove this lemma. The only thing we need to assume is that for some increasing function $\bar{\Delta}$ that is positive except at 0, for all $|x-y|$ small, $\delta\left(x, y\right) \geq \bar{\Delta}\left(|x-y|\right)$.*

The previous lemma is only needed for the application of the Borel-Cantelli lemma which follows now. Since in the end we will choose the value ε depending only on κ, $C_Q\Delta\left(\varepsilon\right)$ appears as a fixed positive value that does not depend on n and t, and thus, it will be legitimate to apply the Borel-Cantelli lemma assuming $C_Q\Delta\left(\varepsilon\right)$ is a positive constant. Next, using the Sudakov-Borell inequality 16 and Lemma 20, we compute:

$$\begin{aligned}
\mathbf{P}\left[X^*_{n,t,\alpha} > 2f(n\alpha, t)\right] &= \mathbf{P}\left[X^*_{n,t,\alpha} - f(n\alpha, t) > f(n\alpha, t)\right] \\
&= \mathbf{P}\left[X^*_{n,t,\alpha} - \mathbf{E}\left(\sup_{T_{n\alpha}} 2X_{N_t}(\tilde{t}, \tilde{x})\right) > f(n\alpha, t)\right] \\
&\leq 2\exp\left(-\frac{1}{2t} f^2(n\alpha, t)\right) \leq 2e^{-\frac{1}{2}C_Q^2\Delta^2(\varepsilon)n\alpha t},
\end{aligned}$$

using (30) and the fact that $\sigma_T^2 = t$. To simplify notations, let us denote $c := C_Q\Delta\left(\varepsilon\right)$. We consider the probability:

$$\begin{aligned}
\mathbf{P}\left[A_{t,\alpha}\right] &:= \mathbf{P}\left[\exists n \geq 1 \text{ such that } X^*_{n,t,\alpha} > 2f(n\alpha, t)\right] \\
&\leq \sum_{n\geq 1} \mathbf{P}\left[X^*_{n,t,\alpha} > 2f(n\alpha, t)\right] \\
&\leq \sum_{n\geq 1} 2e^{-\frac{c^2 n\alpha t}{2}} \\
&= 2e^{-c^2\alpha t/2} \frac{1}{1 - e^{-c^2\alpha t/2}} \\
&\leq 4e^{-c^2\alpha t/2}.
\end{aligned}$$

Since $c > 0$, this probability is summable for $t \in \mathbf{N}$. We can then apply Borel-Cantelli lemma to obtain that there exists a $t_0(\omega) < \infty$ a.s. such that for all $t \geq t_0(\omega)$ $A^C_{t\alpha}$ is true. That means that there exists $t_0(\omega) < \infty$ a.s. such that for all $t \geq t_0(\omega)$:

$$\forall n \geq 1, \quad X^*_{n,t,\alpha} \leq 2f(n\alpha, t).$$

Now we are ready to continue (25). For t large enough $t \geq t_0(\omega)$, we have:

$$\tilde{u}(t) \leq \sum_{n=1}^{\infty} e^{4K_Q\sqrt{n\alpha}\,t}\mathbf{P}_b\left[N_t \in [t\alpha(n-1), t\alpha n]\right]. \tag{31}$$

4.5 Estimating the distribution of the number of jumps

According to the last inequality (31) above, we must find a sharp estimation of the distribution of the number of jumps up to time t. We first connect this distribution with the jump times themselves. More specifically, we denote the kth jump time by:

$$S_k := T_1 + T_2 + \cdots + T_k,$$

where T_i is the ith inter-jump time. We then have:

$$\{N_t \geq k\} = \{S_k \leq t\}.$$

We know that the inter-jump times T_i are independent and identically distributed random variables independent of the sequence $\tilde{x} = (x_0, x_1, \ldots, x_k)$, and T_1 is the first exit time of Brownian motion started at 0 with variance κ from the interval $[-\varepsilon, \varepsilon]$. To avoid confusion, we will denote $T_1^{(\kappa)}$ this first exit time for the Brownian motion with variance κ. A standard scaling argument easily connects its distribution with that of the first exit time $\tilde{T} = \tilde{T}_1$ of a standard Brownian motion from the interval $[-1, 1]$. If $T_1^{(\kappa)}, T_2^{(\kappa)}, \ldots, T_n^{(\kappa)}$ denote the times between crossings of the grid $\varepsilon\mathbf{Z}$ for a Brownian motion with variance κ then:

$$T_i^{(\kappa)} \overset{\text{Distribution}}{=} \frac{\varepsilon^2}{\kappa}\tilde{T}.$$

While the distribution of $\tilde{T}$ is known explicitly as a series, the corresponding formula is difficult to work with. To proceed with the estimation of the distribution of N_t, we will perform a specific estimation of $\tilde{T}$. Let m be a fixed integer ≥ 1. Then we have:

$$\begin{aligned}
\mathbf{P}_b\left[N_t \geq m\right] &= \mathbf{P}_b\left[S_m \leq t\right] \\
&= \mathbf{P}_b\left[T_1^{(\kappa)} + T_2^{(\kappa)} + \cdots + T_m^{(\kappa)} \leq t\right] \\
&= \mathbf{P}_b\left[\tilde{T}_1 + \tilde{T}_2 + \cdots + \tilde{T}_m \leq \frac{\kappa}{\varepsilon^2}t\right] \\
&\leq \sum_{\tilde{x} \in \tilde{P}_m} \mathbf{P}_b\left[\tilde{T}_1 + \tilde{T}_2 + \cdots + \tilde{T}_m \leq \frac{\kappa}{\varepsilon^2}t \,\middle|\, \tilde{x}\right] P_b\left[\tilde{x}\right],
\end{aligned}$$

where $\tilde{P}_m$ is the set of all possible paths of a random walk started at 0 on the integer lattice with m jumps. Since all the paths have the same probability we have $\mathbf{P}_b[\tilde{x}] = 2^{-m}$ for all $\tilde{x} \in P_m$. Furthermore, since $\tilde{T}_i$'s are independent of $\tilde{x}$ the probability $\mathbf{P}_b[N_t \geq m]$ remains unchanged if one conditions by a particular realization of the sequence $\tilde{x}$. Thus, using $\tilde{S}_m$ for the time of the mth jump of $\tilde{b}$,

$$\begin{aligned}
\mathbf{P}_b[N_t \geq m] &= \mathbf{P}_b\left[\tilde{T}_1 + \tilde{T}_2 + \cdots + \tilde{T}_m \leq \frac{\kappa}{\varepsilon^2}t \middle| x_0 = 0, x_1 = 1, \dots x_m = m\right] \\
&= \mathbf{P}_b\left[\tilde{T}_1 + \tilde{T}_2 + \cdots + \tilde{T}_m \leq \frac{\kappa}{\varepsilon^2}t \text{ and } x_0 = 0, x_1 = 1, \dots x_m = m\right] / 2^{-m} \\
&= 2^m \mathbf{P}_b\left[\tilde{T}_1 + \tilde{T}_2 + \cdots + \tilde{T}_m \leq \frac{\kappa}{\varepsilon^2}t \text{ and } b_{\tilde{T}_1 + \tilde{T}_2 + \cdots + \tilde{T}_m} = i \,\forall\, i \leq m\right] \\
&\leq 2^m \mathbf{P}_b\left[\tilde{S}_m \leq \frac{\kappa}{\varepsilon^2}t \text{ and } b_{\tilde{S}_m} = m\right] \\
&\leq 2^m \mathbf{P}_b\left[\sup_{s \leq \kappa\varepsilon^{-2}t} b_s \geq m\right].
\end{aligned}$$

The reflection principle for Brownian motion states that, for any $a > 0$ fixed, $\sup_{s \leq a} b_s$ and $|b_a|$ have the same distribution. Thus we obtain,

$$\begin{aligned}
\mathbf{P}_b[N_t \geq m] &\leq 2^m \mathbf{P}_b\left[\left|b_{\frac{\kappa}{\varepsilon^2}t}\right| \geq m\right] \\
&= 2^m \mathbf{P}_b\left[|Z| \geq \frac{m\varepsilon}{\sqrt{\kappa t}}\right] \leq 2^m \frac{1}{A} e^{-A^2/2}
\end{aligned}$$

with Z a standard normal variate and $A := m\varepsilon/\sqrt{\kappa t}$. Thus, with $m = t\alpha n \geq 1$ we are able to state the final result of this section.

Proposition 22 *If N_t is the number of jumps of the discretization $\tilde{b}$, defined in Section 4.1, of a Brownian motion b started at 0 with variance κ on the lattice $\varepsilon\mathbf{Z}$, then as soon as $t\alpha n \geq 1$ we have:*

$$\mathbf{P}_b[N_t \geq t\alpha n] \leq 2^{t\alpha n} \frac{1}{\sqrt{t}\,\alpha n \frac{\varepsilon}{\sqrt{\kappa}}} e^{-\frac{1}{2} t\alpha^2 n^2 \frac{\varepsilon^2}{\kappa}}.$$

Note that, while this result holds as soon as $t\alpha n \geq 1$, it only represents a tail estimate when α is larger than a certain value depending on $\kappa\varepsilon^{-2}$. When α is too small, the right-hand side in the above will be greater than 1, and the proposition will not claim anything. It is only in the tail estimate regime that this proposition will be used below, however. Still, checking that $t\alpha n$ exceeds 1 in the usage below is trivial, since we will only be using $n \geq 1$ and fixed $\alpha > 0$ not dependent on t: the condition is met trivially for $t > \alpha^{-1}$.

4.6 Final step

Now we can use Proposition 22 in the equation (31). Notice that we can only apply the result for $n \geq 2$. For those terms we have

$$\begin{aligned}
\sum_{n=2}^{\infty} e^{4K_Q\sqrt{n\alpha}\,t}\mathbf{P}_b\left[N_t \in [t\alpha(n-1), t\alpha n]\right] &\leq \sum_{n=1}^{\infty} e^{4K_Q\sqrt{(n+1)\alpha}\,t}\mathbf{P}_b\left[N_t \geq t\alpha n\right]\\
&\leq \sum_{n=1}^{\infty} e^{4K_Q\sqrt{(n+1)\alpha}\,t}2^{t\alpha n}\frac{1}{\sqrt{t}\,\alpha n\frac{\varepsilon}{\sqrt{\kappa}}}e^{-\frac{1}{2}t\alpha^2 n^2\frac{\varepsilon^2}{\kappa}}\\
&= \sum_{n=1}^{\infty} \frac{1}{\sqrt{t}\,\alpha n\frac{\varepsilon}{\sqrt{\kappa}}}e^{-\frac{1}{2}t\left(\alpha^2 n^2\frac{\varepsilon^2}{\kappa}-2\alpha n\log 2-8K_Q\sqrt{(n+1)\alpha}\right)}. \qquad (32)
\end{aligned}$$

We simply bound the term for $n = 1$ in (31) as follows:

$$e^{4K_Q\sqrt{\alpha}\,t}\mathbf{P}_b\left[N_t \in [0, t\alpha]\right] \leq e^{4K_Q\sqrt{\alpha}\,t}. \qquad (33)$$

We can also see that the general term of the series in (32) decays faster than geometrically; therefore the sum is dominated by a constant multiple of the term with $n = 1$.

Our purpose here is to choose our free parameter α to make the first term of the series in (32) dominated by a quantity that decays to 0 so that the only contribution in the Lyapunov exponent comes from (33). To this end choose α, so that:

$$-\frac{1}{2}t\left(\alpha^2\frac{\varepsilon^2}{\kappa} - 2\alpha\log 2 - 8K_Q\sqrt{\alpha}\right) \leq -\frac{1}{4}t\alpha^2\frac{\varepsilon^2}{\kappa},$$

or

$$\alpha^2\frac{\varepsilon^2}{\kappa} \geq 4\left(\alpha\log 2 + 4K_Q\sqrt{\alpha}\right).$$

Since we may insist on $\alpha < 1$, the right-hand side above is bounded above as follows:

$$4\left(\alpha\log 2 + 4K_Q\sqrt{\alpha}\right) \leq 4\sqrt{\alpha}\left(\log 2 + 4K_Q\right) = C\sqrt{\alpha},$$

with C a constant. Thus, taking $\alpha = \left(C\kappa\varepsilon^{-2}\right)^{2/3}$, with $\tilde{\lambda}_+$ the almost sure Lyapunov upper bound of the discretized $\tilde{u}$ of Section 4.1, we have proved

$$\tilde{\lambda}_+ \leq C_1\kappa^{\frac{1}{3}}\varepsilon^{-\frac{2}{3}},$$

with C_1 a constant depending only on Q, as long as $\kappa \leq \varepsilon^2$.

Combining this with the error estimate (24) of Section 4.2, we obtain our general upper bound theorem for arbitrary values of κ and ε.

Theorem 23 (Upper bound for the Lyapunov exponent) *Let u be the solution of the stochastic parabolic Anderson PDE* (1). *Under the Conditions* ***(E)****,* ***(E'-)****, and* ***(E")*** *we have almost surely:*

$$\limsup_{\substack{t\to\infty\\ t\in\mathbf{N}}} \frac{1}{t}\log u(t,x) \leq \mathbf{C}\kappa^{\frac{1}{3}}\varepsilon^{-\frac{2}{3}} + \mathbf{C}\delta^2(\varepsilon),$$

with $\mathbf{C}$ *a constant depending only on Q, for all $\kappa \leq \varepsilon^2$.*

Corollary 24 *By possibly adjusting the leading constant* $\mathbf{C}$ *by a universal positive factor less than* 1*, the previous theorem holds even if the* $\limsup$ *is taken over all times* $t \in \mathbf{R}_+$ *(removing the subscript* $t \in \mathbf{N}$*).*

Proof. The idea of the proof is identical to that of the corollary to Theorem 14. ■

5 Examples of specific bounds

In this last section, we assume **(E)**, **(E'-)**, and **(E")** hold.

5.1 Holder scale

In this paragraph, we assume the following.

(H) [**Hölder regularity scale**] For two constant $c, C > 0$, and a parameter $H \in (0,1)$, we have, in a neighborhood of 0,

$$\delta(r) \leq Cr^H$$
$$\Delta(r) \geq cr^H.$$

We refer to Section 1.3 for an explanation of how this is connected to W's spatial Hölder continuity. The lower bound Theorem 14 yields the following.

Corollary 25 *For some positive constant* C_Q *depending only on* Q*, for all small* κ *and all* x*, almost surely,*

$$\liminf_{t\to\infty} \frac{\log u(t,x)}{t} \geq C_Q \kappa^{H/(H+1)}.$$

Proof. From Condition **(H)** we calculate an upper bound on the unique solution t_0 to equation (21), finding

$$t_0 \leq c_u^2 c \kappa^{-H/(H+1)},$$

and the result follows immediately, with $C_Q = \left(\sqrt{32\pi} c_u^2 c\right)^{-1}$. ■

For an upper bound, we only need to optimize the value of ε in Theorem 23, now that we know that $\delta^2(\varepsilon)$ is of order ε^{2H}. Clearly, up to constant factors, it is optimal to choose ε so that the error and the main term in the upper bound are equal:

$$\varepsilon^{2H} = \kappa^{\frac{1}{3}} \varepsilon^{-\frac{2}{3}}.$$

This gives $\varepsilon = \kappa^{1/(6H+2)}$, so that the Lyapunov upper bound equals, up to a constant, $2\varepsilon^{2H} = 2\kappa^{H/(3H+1)}$. In other words, we have the following.

Corollary 26 *For some positive constant* C_Q *depending only on* Q*, for all small* κ *and all* x*, almost surely,*

$$\limsup_{t\to\infty} \frac{\log u(t,x)}{t} \leq C_Q \kappa^{\frac{H}{3H+1}}.$$

5.2 Logarithmic scale

In this paragraph, we assume the following.

(L) [**Logarithmic regularity scale**] For two constant $c, C > 0$, and a parameter $\beta > 1$, we have

$$\delta\left(r\right) \leq C\left(\log\frac{1}{r}\right)^{-\beta/2},$$

$$\Delta\left(r\right) \geq c\left(\log\frac{1}{r}\right)^{-\beta/2}.$$

We refer to Section 1.3, in particular Remark 1 and the paragraph following it, for an explanation of why we must have $\beta > 1$ and how this is connected to W's spatial modulus of continuity.

We trivially get the following result for upper and lower bounds, which we state together to emphasize the fact that the two bounds are of the same order, showing that in the logarithmic scale, our proofs are sharp.

Corollary 27 *For some positive constants c_Q and C_Q depending only on Q, for all small κ and all x, almost surely,*

$$c_Q\left(\log\frac{1}{\kappa}\right)^{-\beta} \leq \liminf_{t\to\infty}\frac{\log u(t,x)}{t} \leq \limsup_{t\to\infty}\frac{\log u(t,x)}{t} \leq C_Q\left(\log\frac{1}{\kappa}\right)^{-\beta}.$$

Proof. For the upper bound, again, we choose ε so that both of the quantities in Theorem 23 (and its corollary) are equal, i.e. choose ε so that

$$\varepsilon = \sqrt{\kappa}\left(\log\frac{1}{\varepsilon}\right)^{\frac{3}{2}\beta}. \tag{34}$$

With this choice, we have that the upper bound is commensurate with:

$$\varepsilon^{-2/3}\kappa^{1/3} = \left(\sqrt{\kappa}\left(\log\frac{1}{\varepsilon}\right)^{\frac{3}{2}\beta}\right)^{-2/3}\kappa^{1/3} = \left(\log\frac{1}{\varepsilon}\right)^{-\beta}.$$

Now in order to return to a formula involving only κ, it is sufficient to see that because of relation (34), we have $\varepsilon \ll \kappa^{1/2-\alpha}$ for any $\alpha > 0$.

For the lower bound, we use Theorem 14 (and its corollary) as follows. Letting t_0 be the unique solution to equation (21), we see that for small κ, t_0 has to be larger than 1 since no value less than or equal to 1 solves (21). As a consequence we have

$$\begin{aligned} t_0 &\leq Cc_u^2 2^\beta \log^\beta\left(\frac{1}{\kappa t_0}\right) \\ &\leq Cc_u^2 2^\beta \log^\beta\left(\frac{1}{\kappa}\right). \end{aligned}$$

The corollary follows. ■

This sharp result can also be related to the estimations in discrete space. It has been known since [6], and has been confirmed in [4] (where explicit constants were computed), that the Lyapunov upper and lower bounds for the stochastic Anderson model in $\mathbf{Z}^d$ are both of order

$$\left(\log\frac{1}{\kappa}\right)^{-1}.$$

In continuous space, since we must have $\beta > 1$ in order to even use the Feynman-Kac formula, we see that there is a fundamental quantitative difference between discrete and continuous space behaviors. The Anderson model in discrete space will always increase faster than the same model in continuous space, as long as some spatial regularity is assumed for the potential.

In order to further understand the above example of the logarithmic regularity scale, one can write $\delta(r) = \left(\log\frac{1}{r}\right)^{-1/2} f(r)$. In [18] it is shown that W is uniformly continuous in x if and only if $\lim_{r\to 0} f(r) = 0$, in which case f is, up to a non-random constant, an almost-sure uniform modulus of continuity of W in space. The case $\beta \leq 1$ in the logarithmic scale clearly does not satisfy this condition. The argument leading to the upper bound in the last corollary can be repeated in the general situation to show that the factor $f(\varepsilon)^2$ will always appear in any upper bound next to $\left(\log\varepsilon^{-1}\right)^{-1}$. Then, using again the relation (34) as above to return to κ, we obtain precisely the following.

Corollary 28 *Let $f(r)$ be an almost-sure modulus of continuity for W in space. For some positive constant C_Q depending only on Q, for all small κ and all x, almost surely,*

$$\limsup_{t\to\infty}\frac{\log u(t,x)}{t} \leq C_Q\frac{f^2\left(\sqrt{\kappa}\right)}{\log\frac{1}{\kappa}}.$$

This proves that the continuous-space exponential behavior is always of a lower order than the discrete space one, and that the ratio of the two upper bounds, $f^2\left(\sqrt{\kappa}\right)$, is precisely related to the almost-sure uniform modulus of continuity of W in space. This result makes no use of the fact that δ is in the logarithmic scale. While we cannot draw any conclusion for a lower bound in this general situation, Corollary 27 can still be reformulated using the representation $\delta(r) = \left(\log\frac{1}{r}\right)^{-1/2} f(r)$, if f is assumed to be large enough. We state this as the ultimate result of our article. It is an easy consequence of [18] and the calculations in the proofs of Theorems 14 and 23 and all corollaries above. The conditions it refers to in [18] are typically satisfied in all useful examples, including our logarithmic scale for $\beta > 1$. The result proves that there is a precise relation between the almost-sure Lyapunov exponent of the continuous-space Anderson model and the almost-sure spatial regularity of its potential.

Corollary 29 *Let f be an increasing function defined near 0 such that $f(r) \gg r^H$ for all $H > 0$, and $\lim_{r\to 0} f(r) = 0$, and f satisfies the technical conditions defined in [18].*

If $W(1,\cdot)$ admits a constant multiple of f for an exact almost-sure uniform modulus of continuity on any interval in $\mathbf{R}$*, then*

$$c_Q \frac{f^2\left(\sqrt{\kappa}\right)}{\log \frac{1}{\kappa}} \leq \liminf_{t\to\infty} \frac{\log u(t,x)}{t} \leq \limsup_{t\to\infty} \frac{\log u(t,x)}{t} \leq C_Q \frac{f^2\left(\sqrt{\kappa}\right)}{\log \frac{1}{\kappa}} \tag{35}$$

for some positive constants c_Q and C_Q depending only on Q, for all small κ and all x, almost surely.

Conversely, assume that $\delta(r) = \left(\log \frac{1}{r}\right)^{-1/2} g(r)$ and $\Delta(r) = \left(\log \frac{1}{r}\right)^{-1/2} h(r)$ for some g and h satisfying the technical conditions defined in [18], and such that $g(r), h(r) \gg r^H$ for all $H > 0$. If (35) holds almost surely for some x, then the ratios g/f and h/f are bounded away from 0 and $+\infty$, and $W(1,\cdot)$ admits a constant multiple of f for an exact almost-sure uniform modulus of continuity on any interval in $\mathbf{R}$.

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