Joint Continuity of the Local Times of Linear Fractional Stable Sheets

Antoine Ayache
UMR CNRS 8524, Laboratoire Paul Painlevé, Bât. M2, Université Lille 1, 59655 Villeneuve d'Ascq Cedex, FRANCE

François Roueff
Télécom Paris/CNRS LTCI, 46 rue Barrault, 75634 Paris Cedex 13, FRANCE

Yimin Xiao
Department of Statistics and Probability, Michigan State University, East Lansing, MI 48824, U.S.A.

Received *****; accepted after revision +++++

Presented by Jean-Pierre Kahane

Abstract

Linear fractional stable sheets (LFSS) are a class of random fields containing the class of fractional Brownian sheets (FBS) by allowing, in the linear fractional representation of the FBS, the random measure to be α-stable with $\alpha \in (0, 2]$. In this note, we extend some properties of the local time shown in [16] and [2] in the Gaussian case to the symmetric α-stable case. For any $N \geq 1$, an $(N, 1)$-LFSS is a real valued random field defined on $\mathbb{R}_+^N$. When $N = 1$, the process is called linear fractional stable motion (LFSM). For $N \geq 1$, an $(N, 1)$-LFSS is mainly parameterized by a multidimensional index $H = (H_1, \ldots, H_N) \in (0, 1)^N$. Let $N, d \geq 1$ be fixed, we consider a random field defined on $\mathbb{R}_+^N$ and taking its values in $\mathbb{R}^d$, an (N, d)-LFSS, whose components are d independent copies of the same $(N, 1)$-LFSS. We show that, if $d < H_1^{-1} + \cdots + H_N^{-1}$, then the (N, d)-LFSS with index H has a local time. Moreover, when the sample path of the LFSS is continuous, that is, for $\alpha < 2$, when $H_1, \ldots, H_N > 1/\alpha$, we show that the local time is jointly continuous.

Résumé

Bicontinuité du temps local du drap linéaire fractionnaire stable. Le drap linéaire fractionnaire stable (LFSS) est un champ aléatoire qui généralise le drap brownien fractionnaire en remplaçant la mesure gaussienne dans sa représentation linéaire fractionnaire par une mesure α-stable, $\alpha \in (0, 2]$. Dans cette note nous étendons certaines propriétés du temps local montrées dans [16] et [2] pour le cas gaussien au cas symétrique α-stable. Le $(N, 1)$-LFSS, $N \geq 1$, est défini sur $\mathbb{R}_+^N$ et prend ses valeurs dans $\mathbb{R}$, le cas $N = 1$ correspondant au mouvement linéaire fractionnaire stable (LFSM). Ce champ est principalement paramétré par $H = (H_1, \ldots, H_N) \in (0, 1)^N$. Nous considérons un champ aléatoire à valeurs dans $\mathbb{R}^d$, le (N, d)-LFSS, $N, d \geq 1$, défini en prenant d copies indépendantes d'un $(N, 1)$-LFSS. Nous montrons que, si $d < H_1^{-1} + \cdots + H_N^{-1}$, alors le (N, d)-LFSS de paramètre H admet un temps local. De plus, dans le cas où ses trajectoires sont continues, i.e., pour $\alpha < 2$, quand les paramètres vérifient $H_1, \ldots, H_N > 1/\alpha$, nous établissons la bicontinuité de ce temps local.

Preprint submitted to Elsevier Science *21 mars 2007*

Version française abrégée

Soient $\alpha \in (0,2]$ et $H = (H_1, \dots, H_N) \in (0,1)^N$. Nous considérons le drap linéaire fractionnaire α-stable (LFSS) $\{X_0(s),\, s \in \mathbb{R}_+^N\}$ défini par (1) et (2) où $\{Z_\alpha(s),\, s \in \mathbb{R}^N\}$ est le drap de Lévy symétrique α-stable et $x_+ = \max(x,0)$. Pour tout entier $d \geq 1$ un (N,d)-LFSS $X = \{X(t),\, t \in \mathbb{R}_+^N\}$ est défini par (3), où les composantes X_j, $j = 1, \dots, d$, sont des copies indépendantes de X_0. Il convient de noter que, pour tout $n = 1, \dots, N$, la restriction de X_0 à toute demi-droite parallèle au n-ième axe est un mouvement linéaire fractionnaire stable (LFSM) sur $\mathbb{R}_+$ de paramètre de Hurst H_n. Le processus X quant à lui satisfait une propriété d'auto-similarité en distribution, voir (4), où A est une matrice diagonale et a_ℓ, $\ell = 1, \dots, N$ sont ses coefficients diagonaux. Ces propriétés d'anisotropie et d'auto-similarité du LFSS peuvent entre autres avoir de l'importance du point de vue de la modélisation (voir [4] pour un exemple dans le cas gaussien).

Lorsque $\alpha = 2$ le LFSS est appelé drap brownien fractionnaire. Les propriétés fractales et asymptotiques de ce champ gaussien ont été étudiées dans [3], et la régularité de son temps local dans [16] puis dans [2]. L'objectif de cette note est d'étendre certains résultats sur le temps local au cas symétrique α-stable, complétant ainsi les résultats établis dans [1] sur la régularité des trajectoires.

Notre premier théorème concerne l'existence du temps local.

Theorem 0.1 *Soit $X = \left\{X(t),\, t \in \mathbb{R}_+^N\right\}$ un (N,d)-drap linéaire fractionnaire α-stable de paramètres $H \in (0,1)^N$ et $\alpha \in (0,2]$. Alors pour tout pavé borné $I \subset (0,\infty)^N$, X admet un temps local $L(\cdot, I)$ dans $L^2(\mathbb{R}^d \times \Omega, d\lambda_d \times d\mathbb{P})$ si et seulement si la condition (6) est vérifiée. Sous cette condition, $L(x,I)$ admet la transformée de Fourier (au sens L^2) :*

$$L(x,I) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-i\langle y,x\rangle} \int_I e^{i\langle y, X(s)\rangle}\, ds dy, \qquad x \in \mathbb{R}^d.$$

De plus, $L(x,\cdot)$ est diffuse pour p.t. $x \in \mathbb{R}^d$.

Pour montrer la bicontinuité de son temps local, nous supposerons que les trajectoires du champ $X = \{X(t),\, t \in \mathbb{R}_+^N\}$ sont continues, ce qui revient à imposer (8). Notre deuxième théorème établit cette bicontinuité.

Theorem 0.2 *Soit $X = \{X(t),\, t \in \mathbb{R}_+^N\}$ un (N,d)-drap linéaire fractionnaire α-stable de paramètres $H \in (0,1)^N$ et $\alpha \in (0,2)$. Si les conditions (6) et (8) sont vérifiées, alors pour tout pavé fermé $I \subset (0,\infty)^N$, X admet un temps local bicontinu sur I.*

1. Introduction

Let $0 < \alpha \leq 2$ and $H = (H_1, \dots, H_N) \in (0,1)^N$ be given. We define an α-stable field $X_0 = \{X_0(t), t \in \mathbb{R}_+^N\}$ with values in $\mathbb{R}$ by

$$X_0(t) = \int_{\mathbb{R}^N} h_H(t,s)\, Z_\alpha(ds), \qquad t \in \mathbb{R}_+^N, \tag{1}$$

Email addresses: `Antoine.Ayache@math.univ-lille1.fr` (Antoine Ayache), `roueff@tsi.enst.fr` (François Roueff), `xiao@stt.msu.edu` (Yimin Xiao).

where Z_α is a strictly α-stable random measure on $\mathbb{R}^N$ with Lebesgue measure as its control measure and $\beta(s)$ as its skewness intensity. If $\beta(s) \equiv 0$, then Z_α is a symmetric α-stable (SαS) random measure. In (1), the integrand $h_{_H}(t,s)$ is defined by

$$h_{_H}(t,s) = \kappa \prod_{\ell=1}^{N} \Big\{ (t_\ell - s_\ell)_+^{H_\ell - \frac{1}{\alpha}} - (-s_\ell)_+^{H_\ell - \frac{1}{\alpha}} \Big\}, \tag{2}$$

where $t_+ = \max\{t, 0\}$ and $\kappa > 0$ is a normalizing constant such that the scale parameter of $X_0(1)$, denoted by $\|X_0(1)\|_\alpha := \big[-\Re \log \mathbb{E}\big(e^{iX_0(1)}\big)\big]^{1/\alpha}$, equals 1. We refer to [13, Chapter 3] for more information on stable random measures and their integrals.

If $H_1 = \cdots = H_N = \frac{1}{\alpha}$, then the integral in (1) is understood as $Z_\alpha([0,t])$ for all $t \in \mathbb{R}^N_+$. Thus, in this case, X_0 is the ordinary stable sheet studied in [5]. In general, the random field X_0 is a strictly α-stable random field and is called a linear fractional α-stable sheet in $\mathbb{R}$ with index H. Note that, for every $\ell = 1, \ldots, N$, X_0 is a linear fractional stable motion in $\mathbb{R}$ of Hurst index H_ℓ along the direction of the ℓth axis; see [7,11,14,13,9,10] for various properties of linear fractional stable motion and fractional Brownian motion.

For simplicity we will only consider the case where X_0 is a SαS field in this note. Let $X_1, \ldots, X_d$ be d independent copies of X_0. We define an (N,d)-stable field $X = \{X(t), t \in \mathbb{R}^N_+\}$ by

$$X(t) = \big(X_1(t), \ldots, X_d(t)\big), \qquad t \in \mathbb{R}^N_+ \tag{3}$$

and refer it as an (N,d)-linear fractional stable sheet (LFSS, in short) with indices H and α. It is easy to verify that X has the following scaling property: For any $N \times N$ diagonal matrix $A = (a_{ij})$ with $a_{ii} = a_i > 0$ for all $1 \le i \le N$ and $a_{ij} = 0$ if $i \ne j$, we have

$$\big\{X(At),\, t \in \mathbb{R}^N_+\big\} \stackrel{d}{=} \Big\{ \prod_{\ell=1}^{N} a_\ell^{H_\ell} X(t),\ t \in \mathbb{R}^N_+ \Big\}, \tag{4}$$

where $X \stackrel{d}{=} Y$ means that the two processes have the same finite dimensional distributions. That is, X is an anisotropic and operator-self-similar random field. When $\alpha = 2$, X is an ordinary (N,d)-fractional Brownian sheet. While for $0 < \alpha < 2$, X has heavy-tailed distributions. These characteristics of X make it a potential model for various spatial objects as an alternative for anisotropic Gaussian fields (see [4]), which is the main motivation of our research.

The asymptotic and fractal properties of fractional Brownian sheets have been investigated in [3], and the regularities of the local times are considered in [16,2]. Note in passing that the regularities and other properties of the local times of the ordinary Brownian sheet were studied a long time ago (see, e.g., [15] and [5]). The objective of this note is to study the existence and joint continuity of the local times of linear fractional stable sheets and thus completing the regularity properties established in [1]. In particular, we prove that, when $\sum_{\ell=1}^N \frac{1}{H_\ell} > d$, X has a jointly continuous local time, which extends a recent result of [2]. Unlike in the Gaussian case, it is more difficult to study the local times of LFSS. For example, while the conditional distributions of Gaussian vectors remain Gaussian, not much information is available about the conditional distributions of stable random variables. Hence the methods in [16,2] can not be applied directly to LFSS and we have to rely on more specific techniques taking advantage of the representation (1).

2. Existence and joint continuity of local times

Existence is obtained by using the following lemma, which is an extension of Lemma 3.4 in [3] for fractional Brownian sheets.

Lemma 2.1 *For any constant $\varepsilon > 0$, there exist positive and finite constants $c_{2,1}$ and $c_{2,2}$ such that for all $s, t \in [\varepsilon, 1]^N$,*

$$c_{2,1} \sum_{\ell=1}^{N} |s_\ell - t_\ell|^{H_\ell} \le \|X_0(s) - X_0(t)\|_\alpha \le c_{2,2} \sum_{\ell=1}^{N} |s_\ell - t_\ell|^{H_\ell}. \tag{5}$$

Applying Lemma 2.1, we can easily establish

Theorem 2.2 *Let $X = \{X(t), t \in \mathbb{R}_+^N\}$ be an (N, d)-linear fractional stable sheet with indices $H \in (0,1)^N$ and $\alpha \in (0, 2]$. Then, for every finite interval $I \subset (0, \infty)^N$, X has a local time $L(\cdot, I)$ in $L^2(\mathbb{R}^d \times \Omega, d\lambda_d \times d\mathbb{P})$ if and only if*

$$\sum_{\ell=1}^{N} \frac{1}{H_\ell} > d\ . \tag{6}$$

If the latter condition holds, then $L(x, I)$ admits the following L^2-representation

$$L(x, I) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-i\langle y, x\rangle} \int_I e^{i\langle y, X(s)\rangle}\, ds dy, \qquad x \in \mathbb{R}^d. \tag{7}$$

Moreover, $L(x, \cdot)$ has no atoms for a.e. $x \in \mathbb{R}^d$.

The following theorem proves the joint continuity of the local times of the (N, d)-linear fractional stable sheets. For convenience, we further assume in the rest of this note that X has continuous paths, that is

$$\min(H_1, \dots, H_N) > 1/\alpha\ . \tag{8}$$

Theorem 2.3 *Let $X = \{X(t),\ t \in \mathbb{R}_+^N\}$ be an (N, d)-linear fractional α-stable sheet with index $H \in (0,1)^N$. If (6) and (8) hold, then for every closed interval $I \subset (0, \infty)^N$, X has a jointly continuous local time on I.*

In many aspects, the method of proving Theorem 2.3 is similar to that in [2] which extends those in [5] and [16]. The key step is to apply the Fourier analytic arguments to derive estimates on the moments of the local times. The main difference in the proofs of Theorem 2.3 and the joint continuity of fractional Brownian sheet in [2] is that, in this present note, we do not make direct use of sectorial local nondeterminism of LFSS. Moreover, for the stable case considered here, the conditioning arguments in [2] become ineffective and some results in [12] are needed for proving Lemma 2.4 below.

Lemma 2.4 *Suppose that the conditions of Theorem 2.3 hold. Let $\nu := d\left(\sum_{\ell=1}^N H_\ell^{-1}\right)^{-1} \in (0,1)$ and for any $r > 0$, let $\langle r\rangle$ denote the vector of $\mathbb{R}^N$ having all its coordinates equal to r. Then, for all integers $n \ge 1$ there exists a positive and finite constant $c_{2,3}(n)$ such that for all hypercubes $T = [a, a + \langle r\rangle] \subseteq I$, x, $y \in \mathbb{R}^d$ with $|x - y| \le 1$ and all $\gamma \in (0, (1-\nu)/\nu)$,*

$$\mathbb{E}\big[L(x,T)^n\big] \le c_{2,3}(n)\, r^{nN(1-\nu)},$$
$$\mathbb{E}\Big[\big(L(x,T) - L(y,T)\big)^{2n}\Big] \le c_{2,3}(n)\, |x-y|^{2n\gamma}\, r^{2nN[1-\nu(1+\gamma)]}.$$

3. Some proofs

Corresponding to LFSS X_0, we define the α-stable field $Y = \{Y(t), t \in \mathbb{R}_+^N\}$ with values in $\mathbb{R}$ by

$$Y(t)=\int_{[0,t]} h_{_H}(t,r)\, Z_\alpha(dr). \tag{9}$$

We call Y the linear fractional Liouville stable sheet of index H. It follows from (1) and (9) that for any integer $n\ge 2$ and $u^j\in\mathbb{R}$ ($j=1,\ldots,n$), we have

$$\Big\|\sum_{j=1}^n u^j\, X_0(t^j)\Big\|_\alpha \ge \Big\|\sum_{j=1}^n u^j\, Y(t^j)\Big\|_\alpha. \tag{10}$$

To proceed, we use the same argument as in [3, pp. 428–429] to decompose Y into a sum of independent stable processes. For every $t\in[\varepsilon,\infty)^N$, we write the rectangle $[0,t]$ as the following disjoint union:

$$[0,t]=[0,\varepsilon]^N\cup\bigcup_{\ell=1}^N R_\ell(t)\cup\Delta(\varepsilon,t), \tag{11}$$

where

$$R_\ell(t):=\big\{r\in[0,\infty)^N: 0\le r_i\le\varepsilon\ \text{ if } i\ne\ell \ \text{ and }\ \varepsilon<r_\ell\le t_\ell\big\} \tag{12}$$

and $\Delta(\varepsilon,t)$ can be written as a union of 2^N-N-1 sub-rectangles of $[0,t]$ with sides parallel to the axes. It follows from (9) and (11) that for every $t\in[\varepsilon,\infty)^N$,

$$\begin{aligned} Y(t) &= \int_{[0,\varepsilon]^N} h_{_H}(t,r)\, Z_\alpha(dr) + \sum_{\ell=1}^N \int_{R_\ell(t)} h_{_H}(t,r)\, Z_\alpha(dr) + \int_{\Delta(\varepsilon,t)} h_{_H}(t,r)\, Z_\alpha(dr)\\ &:= Y(\varepsilon,t)+\sum_{\ell=1}^N Y_\ell(t)+Z(\varepsilon,t). \end{aligned} \tag{13}$$

Since the processes $\{Y(\varepsilon,t),\, t\in\mathbb{R}_+^N\}$, $\{Y_\ell(t),\, t\in\mathbb{R}_+^N\}$ $(1\le\ell\le N)$ and $\{Z(\varepsilon,t),\, t\in\mathbb{R}_+^N\}$ are defined by the stochastic integrals with respect to Z_α over disjoint sets, they are independent. For our purpose, we can consider $\sum_{\ell=1}^N Y_\ell(t)$ as a local approximation to $Y(t)$. Together with (10), it is a useful ingredient to prove Lemma 2.1 and then to get higher moment estimates for the local times as in Lemma 2.4. However we will omit the details of the proofs of these two lemmas in this note.

Proof of Theorem 2.2 Without loss of generality, we may assume that $I=[\varepsilon,1]^N$ where $\varepsilon>0$. Let $\lambda_N\lfloor_I$ be the restriction of the Lebesgue measure λ_N on I. We denote by μ the image measure of $\lambda_N\lfloor_I$ under the mapping $t\mapsto X(t)$. Then the Fourier transform of μ is

$$\widehat{\mu}(\xi)=\int_I e^{i\langle\xi,X(t)\rangle}\, dt. \tag{14}$$

It follows from Fubini's theorem and (5) that

$$\begin{aligned} \mathbb{E}\int_{\mathbb{R}^d}\big|\widehat{\mu}(\xi)\big|^2\, d\xi &= \int_{\mathbb{R}^d}\int_I\int_I \mathbb{E}\Big(e^{i\langle\xi,X(s)-X(t)\rangle}\Big)\, dsdt\, d\xi\\ &= c\int_I\int_I \frac{1}{\|X_0(s)-X_0(t)\|_\alpha^d}\, dsdt\\ &\asymp \int_I\int_I \frac{1}{\big[\sum_{\ell=1}^N |s_\ell-t_\ell|^{H_\ell}\big]^d}\, dsdt, \end{aligned} \tag{15}$$

where $A\asymp B$ means A/B is bounded from below and above by absolute constants. The same argument as in [16, p.214] shows that the last integral is finite if and only if $d<\sum_{\ell=1}^N \frac{1}{H_\ell}$. Hence, the first part of

Theorem 2.2 follows from Theorem 21.9 in [6, p.36]. Moreover, in the latter case, $\widehat{\mu} \in L^2(\mathbb{R}^d)$ a.s. and (7) follows from the Plancherel theorem.

To prove that $L(x, \cdot)$ has no atoms, by Theorem 23.1 in [6, p.39], we only need to verify

$$\int_I \sup_{\eta>0} \frac{1}{\eta^d}\, \mathbb{P}\Big(\|X(t)-X(s)\| \le \eta\Big)\, ds < \infty \qquad \text{for a.e. } t \in \mathbb{R}_+^N, \tag{16}$$

where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^d$. This follows from the fact that the symmetric α-stable vector $\big(X(t)-X(s)\big)/\|X(t)-X(s)\|_\alpha$ has a bounded density uniformly in $t \neq s$ and the fact that

$$\int_I \frac{1}{\big[\sum_{\ell=1}^N |s_\ell - t_\ell|^{H_\ell}\big]^d}\, ds < \infty$$

whenever $\sum_{\ell=1}^N \frac{1}{H_\ell} > d$. The proof is finished. □

Proof of Theorem 2.3 Denote the lower-left vertex of the interval I by a_I. Then the joint continuity of $(x,t) \mapsto L(x, [a_I, a_I + t])$ follows from Lemma 2.4 and a multiparameter version of Kolmogorov continuity theorem, cf. [8]. □

Acknowledgement The research of Yimin Xiao was partially supported by the NSF grant DMS-0404729.

References

[1] Ayache, A., Roueff, F. and Xiao, Y. (2006), Local and asymptotic properties of linear fractional stable sheets. To appear in *C. R. Acad. Sci. Paris, Ser. I.*

[2] Ayache, A., Wu, D. and Xiao, Y. (2006), Joint continuity of the local times of fractional Brownian sheets. To appear in *Ann. IHP Proba. Stat.*

[3] Ayache, A. and Xiao, Y. (2005), Asmptotic properties and Hausdorff dimensions of fractional Brownian sheets. *J. Fourier Anal. Appl.* **11**, 407–439.

[4] Bonami, A. and Estrade, A. (2003), Anisotropic analysis of some Gaussian models. *J. Fourier Anal. Appl.* **9**, 215–236.

[5] Ehm, W. (1981), Sample function properties of multi-parameter stable processes. *Z. Wahrsch. verw Gebiete* **56**, 195–228.

[6] Geman, D. and Horowitz, J. (1980), Occupation densities. *Ann. Probab.* **8**, 1–67.

[7] Kahane, J.-P. (1985), *Some Random Series of Functions.* second edition, Cambridge Univeristy Press.

[8] Khoshnevisan, D. (2002), *Multiparameter Processes: An Introduction to Random Fields.* Springer, New York.

[9] Kôno, N. and Maejima, M. (1991), Hölder continuity of sample paths of some self-similar stable processes. *Tokyo J. Math.* **14**, 93–100.

[10] Kôno, N. and Shieh, N.-R. (1993), Local times and related sample path properties of certain self-similar processes. *J. Math. Kyoto Univ.* **33**, 51–64.

[11] Maejima, M. (1983), A self-similar process with nowhere bounded sample paths. *Z. Wahrsch. Verw. Gebiete* **65**, 115–119.

[12] Nolan, J. (1989), Local nondeterminsm and local times for stable processes. *Probab. Th. Rel. Fields* **82**, 387–410.

[13] Samorodnitsky, G. and Taqqu, M. S. (1994), *Stable non-Gaussian Random Processes: Stochastic Models with Infinite Variance.* Chapman & Hall, New York.

[14] Takashima, K. (1989), Sample path properties of ergodic self-similar processes. *Osaka Math. J.* **26**, 159–189.

[15] Walsh, J. (1978), The local time of the Brownian sheet. *SMF Astérisque* **52-53**, 47–61.

[16] Xiao, Y. and Zhang, T. (2002), Local times of fractional Brownian sheets. *Probab. Th. Rel. Fields* **124**, 204–226.