Properties of Strong Local Nondeterminism and Local Times of Stable Random Fields

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Abstract. We establish properties of strong local nondeterminism for several classes of α-stable random fields such as harmonizable-type fractional stable fields with stationary increments, harmonizable and linear fractional stable sheets. We apply these properties to study existence and joint continuity of the local times of stable random fields.

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1. Introduction

For $0 < \alpha \le 2$, a random field $X = \{X(t), t \in \mathbb{R}^N\}$ with values in $\mathbb{R}^d$ is called *an* (N, d, α)*-stable random field* if for all integers $n \ge 1$, $t^1, \dots, t^n \in \mathbb{R}^N$ and $u^1, \dots, u^n \in \mathbb{R}^d$, the linear combinations $\sum_{j=1}^n \langle u^j, X(t^j)\rangle$ are α-stable random variables, where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^d$. Stable random fields form an important subclass of infinitely divisible processes and have been studied by many authors. We refer to Samorodnitsky and Taqqu (1994) and the references therein for further information.

In this paper we study an (N, d, α)-stable random field $X = \{X(t), t \in \mathbb{R}^N\}$ defined by

$$X(t) = \big(X_1(t), \dots, X_d(t)\big), \qquad t \in \mathbb{R}^N, \tag{1.1}$$

where $X_1, \dots, X_d$ are independent copies of an α-stable random field $X_0 = \{X_0(t), t \in \mathbb{R}^N\}$ with values in $\mathbb{R}$.

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We further assume that $X_0 = \{X_0(t), t \in \mathbb{R}^N\}$ has the following stochastic integral representation

$$X_0(t) = \int_F f(t,x)\, M_\alpha(dx), \tag{1.2}$$

where M_α is a symmetric α-stable random measure on a measurable space $(F, \mathcal{F})$ with control measure m and $f(t,\cdot) : F \to \mathbb{R}$ $(t \in \mathbb{R}^N)$ is a family of measurable functions on F satisfying

$$\int_F |f(t,x)|^\alpha\, m(dx) < \infty, \qquad \forall\, t \in \mathbb{R}^N. \tag{1.3}$$

For any integer $n \geq 1$ and $t^1, \ldots, t^n \in \mathbb{R}^N$, the characteristic function of the joint distribution of $X_0(t^1), \ldots, X_0(t^n)$ is given by

$$\mathbb{E} \exp\bigg(i \sum_{j=1}^n u_j X_0(t^j) \bigg) = \exp\bigg(- \Big\| \sum_{j=1}^n u_j f(t^j, \cdot) \Big\|_{\alpha,m}^\alpha \bigg), \tag{1.4}$$

where $u_j \in \mathbb{R}$ $(1 \leq j \leq n)$ and $\|\cdot\|_{\alpha,m}$ is the $L^\alpha(F, \mathcal{F}, m)$ (quasi) norm. We write

$$\Big\| \sum_{j=1}^n u_j X_0(t^j) \Big\|_\alpha := \Big\| \sum_{j=1}^n u_j f(t^j, \cdot) \Big\|_{\alpha,m} \tag{1.5}$$

for the scale parameter of $\sum_{j=1}^n u_j X_0(t^j)$. From now on we will omit m from the subscript in (1.5).

We will also consider real-valued stable random fields X_0 represented by

$$X_0(t) = \mathrm{Re} \int_F f(t,x)\, \widetilde{M}_\alpha(dx), \tag{1.6}$$

where $\widetilde{M}_\alpha$ is a complex-valued, rotationally invariant α-stable random measure on a measurable space $(F, \mathcal{F})$ with control measure m and the complex-valued, measurable functions $f(t,\cdot)$ $(t \in \mathbb{R}^N)$ satisfy (1.3). See Chapter 6 of Samorodnitsky and Taqqu (1994) for definition of complex-valued stable random measures and their integrals.

The class of α-stable random fields with representation (1.2) or (1.6) is very broad. In particular, if a random field $X_0 = \{X_0(t), t \in \mathbb{R}^N\}$ is symmetric α-stable and separable in probability [that is, there is a countable subset $T_0 \subseteq \mathbb{R}^N$ such that for every $t \in \mathbb{R}^N$, there exists a sequence $\{t_k\} \subseteq T_0$ such that $X_0(t_k) \to X_0(t)$ in probability], then X_0 has a representation (1.2); see Theorems 13.2.1 and 13.2.2. in Samorodnitsky and Taqqu (1994) for details. At this point we should point out that, with little extra effort, the results in this paper hold for all strictly stable random fields with integral representations of the form (1.2) or (1.6).

It is known that many sample path properties of stable random fields represented by (1.2) or (1.6) can be derived from analytic properties of the functions $\{f(t,\cdot),\, t \in \mathbb{R}^N\}$. Closely related to the present paper, we mention that Nolan

(1988, 1989), Kôno and Shieh (1993), Shieh (1993) and Xiao (1995) studied Hausdorff dimensions, existence and joint continuity of the local times and intersection local times of stable random fields. One of the main technical tools in the aforementioned papers is the property of *local nondeterminism (LND)* of Nolan (1989), which is an extension of the local nondeterminism of Berman (1973) for Gaussian processes. Roughly speaking, for $t^1, \ldots, t^n \in \mathbb{R}^N$ close enough, the property of LND characterizes the dependence structure of the stable random variables $X_0(t^1), \ldots, X_0(t^n)$ in terms of the geometric properties of subspace of $L^\alpha(F, \mathcal{F}, m)$ generated by the functions $\{f(t^j, \cdot),\, 1 \le j \le n\}$.

As suggested in Xiao (2006), for studying many problems on stable random fields, it is useful (sometimes necessary) to strengthen the ideas of local nondeterminism introduced by Nolan (1989) so that we can describe the distributional properties of the stable random variables $X_0(t^1), \ldots, X_0(t^n)$ more precisely and more generally. As an example, Xiao (2006) provided a definition of *strong local nondeterminism (SLND)* for stable random fields; see Section 2. In the present paper, we continue this line of work by introducing two new properties of strong local nondeterminism, namely *strong local nondeterminism in metric* ρ and the *sectorial local nondeterminism* for stable random fields. These properties are capable of describing the anisotropic nature of stable random fields, and extend naturally the analogous properties of Gaussian random fields studied by Khoshnevisan, Wu and Xiao (2006), Khoshnevisan and Xiao (2007), Wu and Xiao (2007) and Xiao (2008). For convenience, we will simply refer to all of these properties of strong local nondeterminism as SLND properties.

We believe that, similar to the Gaussian case, SLND properties are useful for studying various sample path properties of (N, d, α)-stable random fields. As one of such applications, we apply SLND to establish the joint continuity and Hölder conditions for the local times of stable random fields. Further results can be found in Nolan and Xiao (2008).

The rest of this paper is organized as follows. In Section 2 we first recall the definitions of local nondeterminism for stable processes introduced by Nolan (1989) and strong local nondeterminism in Xiao (2006). Then we define *strong local nondeterminism in metric* ρ and *the sectorial local nondeterminism* for stable random fields, which are extensions of the analogous properties for Gaussian random fields introduced in Khoshnevisan and Xiao (2007), Wu and Xiao (2007) and Xiao (2008) respectively. By using a Pythagorean type theorem in the space $L^\alpha(m)$ $(1 < \alpha < 2)$ proved by Cheng, et al. (2003), we describe the underlying connection between finite dimensional distributions of stable random fields and the properties of strong local nondeterminism.

In Section 3, we provide some sufficient conditions for a stable random field to have the properties of strong local nondeterminism or sectorial local nondeterminism. These conditions are applicable to both isotropic or anisotropic stable random fields.

In Section 4 we study the existence and joint continuity of the local times of an α-stable random field $X = \{X(t), t \in \mathbb{R}^N\}$ with values in $\mathbb{R}^d$. Our arguments are based on the Fourier analytic methods initiated by Berman (1973) and further developed later by Pitt (1978), Geman and Horowitz (1980), Ehm (1981), Nolan (1988, 1989), Csörgő et al. (1995), Xiao (1997, 2007, 2008), just to mention a few. The new idea in this paper is to make use of the properties of strong local nondeterminism of a stable random field $X = \{X(t), t \in \mathbb{R}^N\}$ directly, without having to rely on the *approximate independence* of the increments $X(t^j)-X(t^{j-1})$ $(j = 1, \ldots, n)$ over suitably ordered time points $t^1 \le \cdots \le t^n$. This is the main difference between the work of Nolan (1988, 1989) and the present paper.

Throughout this paper, the underlying parameter space is $\mathbb{R}^N$ or $\mathbb{R}^N_+ = [0,\infty)^N$. We use $|\cdot|$ to denote the Euclidean norm in $\mathbb{R}^N$. The inner product and Lebesgue measure in $\mathbb{R}^N$ are denoted by $\langle \cdot, \cdot\rangle$ and λ_N, respectively. A typical parameter, $t \in \mathbb{R}^N$ is written as $t = (t_1, \ldots, t_N)$, or as $\langle c\rangle$ if $t_1 = \cdots = t_N = c$. For any $s, t \in \mathbb{R}^N$ such that $s_j < t_j$ $(j = 1, \ldots, N)$, $[s, t] = \prod_{j=1}^N [s_j, t_j]$ is called a closed interval (or a rectangle). We will let $\mathcal{A}$ denote the class of all closed intervals in $\mathbb{R}^N$. For two functions f and g, the notation $f(t) \asymp g(t)$ for $t \in T$ means that the function $f(t)/g(t)$ is bounded from below and above by positive constants that do not depend on $t \in T$.

We will use c and $c(n)$ to denote unspecified positive and finite constants, the latter depends on n. Both of them may not be the same in each occurrence. More specific constants in Section i are numbered as $c_{i,1}, c_{i,2}, \ldots$.

2. Properties of local nondeterminism for stable random fields

In this section, we start by recalling briefly the properties of local nondeterminism for stable random fields introduced by Nolan (1989) and discussed in Xiao (2006). Then we define the properties of strong local nondeterminism for anisotropic stable random fields. Finally we prove a lemma which will be useful for applying SLND properties to study local times of stable random fields. For simplicity, we only consider a real-valued α-stable random field $X_0 = \{X_0(t), t \in \mathbb{R}^N\}$.

The notion of local nondeterminism was first introduced by Berman (1973) for Gaussian processes (i.e., $\alpha = 2$ and $N = 1$). It was later extended by Pitt (1978) to Gaussian random fields $(N > 1)$ and by Nolan (1988, 1989) to SαS processes and random fields. Their definitions are expressed in terms of the increments of X_0. Hence an ordering in $\mathbb{R}^N$ is needed. For $N = 1$, there is a natural order in $\mathbb{R}$. For $N > 1$, Pitt (1978) defined that for any n points $t^1, \ldots, t^n \in \mathbb{R}^N$, $t^1 \preccurlyeq t^2 \preccurlyeq \cdots \preccurlyeq t^n$ if and only if

$$|t^j - t^{j-1}| \le |t^j - t^i| \quad \text{for all } \ 1 \le i < j \le n. \tag{2.1}$$

Note that the partial order defined by (2.1) is not unique. For any n points in $\mathbb{R}^N$ (including the case $N = 1$), there are at least n different ways to order them using (2.1). For example, one can pick any point and label it as t^n, then label the one which is the closest to t^n as t^{n-1}, and so on.

For the Gaussian case, the definitions of LND of Berman and Pitt are expressed in terms of the covariance function of X_0. Since, for an α-stable processes with $0 < \alpha < 2$, there is no covariance to measure the dependence, Nolan (1989) relied on the L^α-representations of symmetric α-stable random fields and the approximation properties of normed or quasi-normed linear spaces.

In order to state the definition of local nondeterminism for stable processes and random fields in Nolan (1989), we need some notation. For any integer $n \geq 2$ and $t^1, \ldots, t^n \in \mathbb{R}^N$, let $M^{n-1} := M(t^1, \ldots, t^{n-1})$ be the subspace of $L^\alpha(F, \mathcal{F}, m)$ spanned by $\{f(t^1, \cdot), \ldots, f(t^{n-1}, \cdot)\}$, and denote by $\|f(t^n, \cdot)|M^{n-1}\|_\alpha$ the L^α-distance from $f(t^n, \cdot)$ to M^{n-1}. That is,

$$\|f(t^n, \cdot)|M^{n-1}\|_\alpha = \inf\left\{\Big\|f(t^n, \cdot) - \sum_{j=1}^{n-1} u_j f(t^j, \cdot)\Big\|_\alpha \,:\, \forall\, u_1, \ldots, u_{n-1} \in \mathbb{R}\right\}. \tag{2.2}$$

Since M^{n-1} has finite dimension, the infimum in (2.2) is attained. In order to draw analogy with the Gaussian case, we abuse the notation and, from now on, write that for all $t^1, \ldots, t^n \in T$

$$\|X_0(t^n)|X_0(t^1), \ldots, X_0(t^{n-1})\|_\alpha := \|f(t^n, \cdot)|M^{n-1}\|_\alpha. \tag{2.3}$$

It can be viewed as the L^α-error of predicting $X_0(t^n)$, given $X_0(t^1), \ldots, X_0(t^{n-1})$.

Definition 2.1. Let $X_0 = \{X_0(t), t \in \mathbb{R}^N\}$ be a real-valued α-stable random field with representation (1.2) or (1.6), and let $T \in \mathcal{A}$ be a closed interval. Then X_0 is said to be locally nondeterministic on T if

$$\|X_0(t)\|_\alpha > 0 \quad \forall\, t \in T \quad \text{and} \quad \|X_0(s) - X_0(t)\|_\alpha > 0 \tag{2.4}$$

for all $s, t \in T$ with $|s - t|$ sufficiently small, and

$$\liminf \frac{\|X_0(t^n)|X_0(t^1), \ldots, X_0(t^{n-1})\|_\alpha}{\|X_0(t^n) - X_0(t^{n-1})\|_\alpha} > 0, \tag{2.5}$$

where the lim inf is taken over all $t^1, \ldots, t^n \in T$ that satisfy (2.1) with $|t^n - t^1| \to 0$.

For $N = 1$, (2.5) requires information for both $t^n \geq t^{n-1}$ and $t^n \leq t^{n-1}$. Hence Definition 2.1 is often referred to as *two-sided local nondeterminism.* Nolan (1989) proved that (2.5) is equivalent to each of the following two properties.

- [Characteristic function locally approximately independent increments] For every integer $n \geq 2$, there exists a constant $c_{2,1} = c_{2,1}(n) \geq 1$, depending on n only, such that for all $t^1, \ldots, t^n \in T$ satisfying (2.1),

$$\begin{aligned}
&\left|\mathbb{E}\left(e^{ic_{2,1}\, u_1 X_0(t^1)}\right) \prod_{j=2}^n \mathbb{E}\left(e^{ic_{2,1}\, u_j (X_0(t^j) - X_0(t^{j-1}))}\right)\right| \\
&\quad \leq \left|\mathbb{E}\exp\left\{i\left(u_1 X_0(t^1) + \sum_{j=2}^n u_j (X_0(t^j) - X_0(t^{j-1}))\right)\right\}\right| \qquad (2.6)\\
&\quad \leq \left|\mathbb{E}\left(e^{ic_{2,1}^{-1}\, u_1 X_0(t^1)}\right) \prod_{j=2}^n \mathbb{E}\left(e^{ic_{2,1}^{-1}\, u_j (X_0(t^j) - X_0(t^{j-1}))}\right)\right|
\end{aligned}$$

for all $u_j \in \mathbb{R}$ $(j = 1, \ldots, n)$.

- [Locally approximately independent increments] For every integer $n \geq 2$, there exists a constant $c_{2,2} = c_{2,2}(n) \geq 1$, depending on n only, such that for all $t^1, \ldots, t^n \in T$ satisfying (2.1),

$$\begin{aligned} c_{2,2}^{-1} & \bigg(\big\|u_1 X_0(t^1)\big\|_\alpha + \sum_{j=2}^n \big\|u_j \big(X_0(t^j) - X_0(t^{j-1})\big)\big\|_\alpha \bigg) \\ & \leq \bigg\| u_1 X_0(t^1) + \sum_{j=2}^n u_j \big(X_0(t^j) - X_0(t^{j-1})\big) \bigg\|_\alpha \qquad (2.7) \\ & \leq c_{2,2} \bigg(\big\|u_1 X_0(t^1)\big\|_\alpha + \sum_{j=2}^n \big\|u_j \big(X_0(t^j) - X_0(t^{j-1})\big)\big\|_\alpha \bigg) \end{aligned}$$

for all $u_j \in \mathbb{R}$ $(j = 1, \ldots, n)$.

As shown by Nolan (1988, 1989), Kôno and Shieh (1993), Shieh (1993) and Xiao (1995), these LND properties are useful for studying sample path properties such as joint continuity of local times of stable processes and stable random fields. However, there are two shortcomings with LND in Definition 2.1. First, as in the Gaussian case, the LND property is not useful for obtaining sharp uniform and/or local growth properties of the local times or self-intersection local times of stable random fields; see Dozzi and Soltani (1999, section 4) for related remarks. Secondly, (2.5) compares $\big\|X_0(t^n)\big|M^{n-1}\big\|_\alpha$ with $\|X_0(t^n) - X_0(t^{n-1})\|_\alpha$ and requires the ratio to stay positive when the points $t^1, \ldots, t^n$ get close. Similar to the Gaussian case [see, e.g., Ayache, Wu and Xiao (2008)], this is impossible for many stable random fields such as the stable sheet [Ehm (1981)], the linear and harmonizable fractional stable sheets [Xiao (2006)]. To address these issues, Xiao (2006) proposed to define the notions of strong local nondeterminism and sectorial local nondeterminism.

The following definition was given in Xiao (2006) and is useful for studying (approximately) isotropic α-stable random fields.

Definition 2.2. Let $X_0 = \{X_0(t), t \in \mathbb{R}^N\}$ be an α-stable field with representation (1.2) or (1.6). Let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a function such that $\phi(0) = 0$ and $\phi(r) > 0$ for $r > 0$. Then X_0 is said to be *strongly locally ϕ-nondeterministic* (SLϕND) on $T \in \mathcal{A}$ if, in addition to (2.4), there exists a constant $c_{2,3} > 0$ such that for all integers $n \geq 2$, all $t^1, \ldots, t^n \in T$,

$$\big\|X_0(t^n)\big|X_0(t^1), \ldots, X_0(t^{n-1})\big\|_\alpha^\alpha \geq c_{2,3}\, \phi\Big(\min_{0 \leq j \leq n-1} |t^n - t^j|\Big). \qquad (2.8)$$

Here and in the sequel, $t^0 = 0$.

In the above, "Strongly" refers to that fact that $c_{2,3}$ is independent of n and, moreover, if $\big\|X_0(s) - X_0(t)\big\|_\alpha^\alpha \asymp \phi(|s - t|)$ for all $s, t \in T$ such that $|s - t|$ is small, then (2.8) implies (2.5). In next section we will give some examples of stable random fields which are strongly ϕ-locally nondeterministic with $\phi(r) = r^\beta$

for some constant $\beta > 0$. More general conditions can be found in Nolan and Xiao (2008).

In recent years, several types of anisotropic stable random fields have arisen in theory and in applications. Important examples include the harmonizable and linear fractional stable sheets [cf. Xiao (2006)], and the operator-scaling stable random fields with stationary increments constructed by Biermé et al. (2007). In the following, we define three more properties of strong local nondeterminism for anisotropic stable random fields. They extend naturally analogous properties of fractional Brownian sheets and other anisotropic Gaussian random fields, and are expected to be a useful tool for studying sample path properties of stable random fields. In this regard, it is helpful to see Khoshnevisan, Wu and Xiao (2006), Khoshnevisan and Xiao (2007), Wu and Xiao (2007), Xiao (2007, 2008) for various applications of SLND properties in the Gaussian case.

Let $(H_1, \ldots, H_N) \in (0,1)^N$ be a fixed vector and denote by ρ the metric on $\mathbb{R}^N$ defined by

$$\rho(s,t) = \sum_{j=1}^{N} |s_j - t_j|^{H_j}, \qquad \forall\, s,t \in \mathbb{R}^N. \tag{2.9}$$

For any $r > 0$ and $t \in \mathbb{R}^N$, we denote by $B_\rho(t,r) = \{s \in \mathbb{R}^N : \rho(s,t) \le r\}$ the closed (or open) ball in the metric ρ.

Definition 2.3. Let $X_0 = \{X_0(t), t \in \mathbb{R}^N\}$ be an α-stable random field with representation (1.2) or (1.6) and satisfy (2.4).

(i) If there is a constant $c_{2,4} > 0$ such that for all $n \ge 2$ and all $t^1, \ldots, t^n \in T$,

$$\big\| X(t^n) \,|\, X(t^1), \ldots, X(t^{n-1})\big\|_\alpha \ge c_{2,4} \min_{0 \le k \le n-1} \rho(t^n, t^k). \tag{2.10}$$

Then we say that X_0 has the property of strong local nondeterminism in the metric space (T,ρ) (or SρLND on T).

(ii) If there is a constant $c_{2,5} > 0$ such that for all $n \ge 2$ and all $t^1, \ldots, t^n \in T$,

$$\big\| X_0(t^n) \,|\, X(t^1), \ldots, X(t^{n-1})\big\|_\alpha \ge c_{2,5} \sum_{j=1}^{N} \min_{0 \le k \le n-1} \big|t^n_j - t^k_j\big|^{H_j}, \tag{2.11}$$

then we say that X_0 has the property of sectorial local nondeterminism on T.

(iii) If there is a constant $c_{2,6} > 0$ such that for all $n \ge 2$ and all $t^1, \ldots, t^n \in T$ satisfying $t^k_\ell \le t^n_\ell$ for all $1 \le k \le n-1$ and some $1 \le \ell \le N$, we have

$$\big\| X_0(t^n) \,|\, X_0(t^1), \ldots, X_0(t^{n-1})\big\|_\alpha \ge c_{2,6} \min_{1 \le k \le n-1} (t^n_\ell - t^k_\ell)^{H_\ell}, \tag{2.12}$$

then we say that X_0 has the property of *one-sided* sectorial local nondeterminism on T.

The following are some remarks about Definitions 2.2 and 2.3.

Remark 2.4. • If $H_1 = \cdots = H_N$, then SLND in the metric ρ is equivalent to SϕLND with $\phi(r) = r^{\alpha H_1}$.

- It is clear that in Definition 2.3, (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii). In Section 4 we will prove joint continuity of local times for stable random fields satisfies Condition (ii) and also show that, for some problems, the weaker condition (iii) is enough.

In order to apply the properties of SLND to study sample path properties of stable random fields, it is desirable to be able to characterize the dependence structure and joint distributions of an α-stable random field by using their properties of strong local nondeterminism. However, it has been difficult to achieve this goal completely. This is because, unlike the case of Gaussian random fields where the geometry of Hilbert spaces has played important roles, properties of local nondeterminism for stable random fields carry less information and the geometry of the space $L^\alpha(m)$ is more complicated.

As a step toward this direction, we prove a useful connection between the joint distribution of $X_0(t^1), \ldots, X_0(t^n)$ and the strong local nondeterminism. This will be the main technical tool for us to study the joint continuity of local times in Section 4. Some new ideas may be needed in order to make further progress; see Conjecture 2.8 below.

Let us recall some geometric properties of the space $L^\alpha(m)$ $(1 < \alpha < 2)$. For any $x, y \in L^\alpha(m)$, x is said to be James orthogonal to y (written as $x \perp_\alpha y$) if $\|x + cy\|_\alpha \geq \|x\|_\alpha$ for all $c \in \mathbb{R}$; see Samorodnitsky and Taqqu (1994, p.97). Suppose M is a closed subspace of $L^\alpha(m)$. If $x \perp_\alpha y$ for all $y \in M$, then we will write $x \perp_\alpha M$. It is known [see, e.g., Köthe (1979)] that, for every $x \in L^\alpha(m)$ $(1 < \alpha < 2)$, there is a unique $y \in M$ such that

$$\|x - y\|_\alpha = \inf\big\{\|x - z\|_\alpha : z \in M\big\}. \tag{2.13}$$

The element y is called the metric projection of x into M and is denoted by $y = P_M x$. By (2.13), we have $x - P_M x \perp_\alpha M$.

As a consequence of the Pythagorean type theorem in the space $L^\alpha(m)$ $(1 < \alpha < 2)$ proved by Cheng, et al. (2003, Proposition 3.3), we have

Lemma 2.5. *Given a constant $1 < \alpha < 2$, there exists a positive constant $c_{2,7}$ depending on α only such that for all $x, y \in L^\alpha(m)$ satisfying $x \perp_\alpha y$ we have*

$$\big\|x + y\big\|_\alpha^\alpha \geq c_{2,7}\Big(\big\|x\big\|_\alpha^\alpha + \big\|y\big\|_\alpha^\alpha\Big). \tag{2.14}$$

Lemma 2.5 can be applied to prove the following useful lemma.

Lemma 2.6. *Assume $1 < \alpha < 2$. For all integers $n \geq 2$ there exists a positive constant $c(n)$ such that for all $t^1, \ldots, t^n \in \mathbb{R}^N$ and $u_1, \ldots, u_n \in \mathbb{R}$,*

$$\Big\| \sum_{j=1}^{n} u_j X_0(t^j) \Big\|_\alpha \geq c(n) \bigg(|v_1| \big\| X_0(t^1) \big\|_\alpha + \sum_{j=2}^{n} |v_j| \, \big\| X_0(t^j) \big| X_0(t^1), \ldots, X_0(t^{j-1}) \big\|_\alpha \bigg). \tag{2.15}$$

In the above,

$$(v_1, \ldots, v_n) = (u_1, \ldots, u_n)\, A, \tag{2.16}$$

where $A = (a_{ij})$ is an $n \times n$ lower triangle matrix (which depends on $t^1, \ldots, t^n$) with $a_{ii} = 1$ for all $1 \leq i \leq n$.

Remark 2.7. Roughly speaking, in (2.15) we expand $\big\| \sum_{j=1}^{n} u_j X_0(t^j) \big\|_\alpha$ by repeatedly "conditioning" $X_0(t^j)$, given $X_0(t^1), \ldots, X_0(t^{j-1})$. Moreover, this "conditioning" can be done in an arbitrary order of the random variables $X_0(t^1), \ldots, X_0(t^n)$. This observation will be useful in Section 4.

Proof of Lemma 2.6. This is proved by induction. For $n = 2$, let $t^1, t^2 \in \mathbb{R}^N$ and $u_1, u_2 \in \mathbb{R}$. Without loss of generality, we assume $u_2 \neq 0$. Let M^1 be the subspace generated by $X_0(t^1)$. Then the metric projection of $X_0(t^2)$ in M^1 can be written as $a_{21} X_0(t^1)$ for some $a_{21} \in \mathbb{R}$. Then $X_0(t^2) - a_{21} X_0(t^1) \perp_\alpha M^1$ and Lemma 2.6 implies

$$\begin{aligned} &\big\| u_1 X_0(t^1) + u_2 X_0(t^2) \big\|_\alpha \\ &= |u_2| \, \Big\| X_0(t^2) - a_{21} X_0(t^1) + \big(\frac{u_1}{u_2} + a_{21}\big) X_0(t^1) \Big\|_\alpha \\ &\geq c_{2,8} \Big(|u_2| \, \big\| X_0(t^2) \big| X_0(t^1) \big\|_\alpha + |u_1 + a_{21} u_2| \big\| X_0(t^1) \big\|_\alpha \Big). \end{aligned} \tag{2.17}$$

This proves (2.15) for $n = 2$. The rest of the proof is very similar and is omitted. □

Note that the inequality (2.15) may not be optimal. When $\alpha = 2$, the orthogonality in $L^2(m)$ implies that (2.15) is an equality with $c(n) = 1$ for all $n \geq 2$. Hence this answers a question in Xiao (2006, Remark 2.4) on strong local nondeterminism of Gaussian random fields. We expect that an analogous result remains partially true in $L^\alpha(m)$ for $1 < \alpha < 2$, and formulate the following conjecture.

Let m be a Borel measure on $\mathbb{R}^N \backslash \{0\}$ satisfying the condition

$$\int_{\mathbb{R}^N} (1 \wedge |x|^\alpha)\, m(dx) < \infty. \tag{2.18}$$

and let $f(t, \cdot)$ be exponential functions of the form

$$f(t, x) = 1 - e^{i\langle t, x \rangle}, \qquad t \in \mathbb{R}^N, \ x \in \mathbb{R}^N. \tag{2.19}$$

For $t^1, \ldots, t^n \in \mathbb{R}^N$, denote by $M(t^1, \ldots, t^n)$ the linear subspace of $L^\alpha(\mathbb{R}^N, m)$ generated by the functions $f(t^1, \cdot), \ldots, f(t^n, \cdot)$.

Conjecture 2.8. Given $1 < \alpha < 2$, there exists a constant $c_{2,9} \in (0,1)$, depending on α only, such that for all integers $n \geq 2$, $t^1, \ldots, t^n \in \mathbb{R}^N$, there is an element $g^* \in M(t^1, \ldots, t^{n-1})$ with the following property:

$$\big\|f(t^n, \cdot) - g\big\|_\alpha^\alpha \geq c_{2,9} \,\big\|f(t^n, \cdot) - g^*\big\|_\alpha^\alpha + \big\|g^* - g\big\|_\alpha^\alpha \tag{2.20}$$

for all $g \in M(t^1, \ldots, t^{n-1})$.

It is important that, in (2.20), the constant $c_{2,9}$ is independent of n and $t^1, \ldots, t^n$ and the coefficient of the term $\big\|g^* - g\big\|_\alpha^\alpha$ is 1. If this conjecture is indeed true, then we will not only be able to improve the results in Section 4 significantly, but also to investigate other fine properties such as the exact Hausdorff and packing measure functions of the trajectories of stable random fields. The latter problems are much more delicate than those on Hausdorff and packing dimensions of stable random fields considered in Shieh and Xiao (2007).

3. Spectral conditions for strong local nondeterminism

In this section we consider several types of α-stable random fields with $\alpha \in [1,2)$ and provide sufficient conditions for them to be strongly ρ-locally nondeterministic or sectorially locally nondeterministic. Our arguments extend those in Wu and Xiao (2007) and Xiao (2007, 2008). We expect that Theorems 3.1, 3.4, 3.5 and 3.7 still hold for the case $0 < \alpha < 1$, but we have not been able to prove these results.

3.1. Approximately isotropic stable random fields

Let $X_0 = \{X_0(t),\, t \in \mathbb{R}^N\}$ be a real-valued α-stable random field defined by

$$X_0(t) = \mathrm{Re} \int_{\mathbb{R}^N} \big(e^{i\langle t, x\rangle} - 1\big)\, \widetilde{M}_\alpha(dx), \tag{3.1}$$

where $\widetilde{M}_\alpha$ is a complex-valued rotationally invariant α-stable random measure on $\mathbb{R}^N$ with control measure m, which satisfies

$$\int_{\mathbb{R}^N} (1 \wedge |x|^\alpha)\, m(dx) < \infty. \tag{3.2}$$

This condition assures that stochastic integral in (3.1) is well-defined. In analogous terminology for stationary processes, the measure m is called the spectral measure of X_0. When m is absolutely continuous, its density is called the spectral density of X_0.

By (3.1), it can be verified that the stable random field X_0 has stationary increments and $X_0(0) = 0$. Denote the scale parameter of $X_0(t)$ by $\sigma(t) := \big\|X_0(t)\big\|_\alpha$. Then for all $h \in \mathbb{R}^N$,

$$\sigma^\alpha(h) = 2^{\alpha/2} \int_{\mathbb{R}^N} \big(1 - \cos\langle h, x\rangle\big)^{\alpha/2}\, m(dx). \tag{3.3}$$

Similar to the studies on Gaussian processes, this function plays an important role in studying sample path properties of stable random field X_0 defined by (3.1).

The class of stable random fields given by (3.1) is large, because m can be any measure on $\mathbb{R}^N$ satisfying (3.2). In particular, if Z is a stationary, harmonizable stable random field given by

$$Z(t) = \mathrm{Re} \int_{\mathbb{R}^N} e^{i\langle t,x\rangle}\, \widetilde{M}_\alpha(dx),$$

where the control measure of $\widetilde{M}_\alpha$ is a finite measure. Then X_0 defined by $X_0(t) = Z(t) - Z(0)$ is also of the form (3.1).

The following theorem provides a sufficient condition for X_0 to be strongly ϕ-locally nondeterministic with $\phi(r) = r^H$.

Theorem 3.1. *Let $X_0 = \{X_0(t),\, t \in \mathbb{R}^N\}$ be a real-valued α-stable random field defined by (3.1) with spectral density $f(x)$ and $1 \le \alpha < 2$. If there exist positive constants $H \in (0,1)$ and $c_{3,1}$ such that*

$$f(x) \le \frac{c_{3,1}}{|x|^{\alpha H + N}} \qquad \forall\, x \in \mathbb{R}^N \quad \text{with } \ |x| \ge c_{3,1}, \tag{3.4}$$

then there is a positive constant $c_{3,2}$ such that for all $n \ge 2$ and $t^1, \ldots, t^n \in \mathbb{R}^N$,

$$\big\|X(t^n)\big|X(t^1), \ldots, X_0(t^{n-1})\big\|_\alpha \ge c_{3,2} \min_{0 \le j \le n-1} |t^n - t^j|^H. \tag{3.5}$$

Proof. The Fourier analytic method for proving (3.5) is similar to that of Theorem 3.4 below. Since the condition (3.4) only provides information on the spectral density at infinity, a modification similar to the proof of Theorem 2.1 in Xiao (2007) is needed. Since more general results will be proved in Nolan and Xiao (2008), we omit the details. □

The following are two important examples of isotropic stable random fields which satisfy the condition (3.4).

Example 3.2. [Harmonizable fractional stable field] Let $H \in (0,1)$ and $\alpha \in (0,2]$ be given constants. The harmonizable fractional stable field $Y^H = \{Y^H(t), t \in \mathbb{R}^N\}$ with values in $\mathbb{R}$ is defined by

$$Y^H(t) = \mathrm{Re} \int_{\mathbb{R}^N} \frac{e^{i\langle t,x\rangle} - 1}{|x|^{H + \frac{N}{\alpha}}}\, \widetilde{M}_\alpha(dx), \tag{3.6}$$

where $\widetilde{M}_\alpha$ is a complex-valued rotationally invariant stable random measure on $\mathbb{R}^N$ with the N-dimensional Lebesgue measure as its control measure. It is easy to verify that the α-stable random field Y^H is H-self-similar with stationary increments. Namely, for all $c > 0$,

$$\big\{Y^H(ct), t \in \mathbb{R}^N\big\} \stackrel{d}{=} \big\{c^H\, Y^H(t), t \in \mathbb{R}^N\big\} \tag{3.7}$$

and for all $h \in \mathbb{R}^N$,

$$\big\{Y^H(t+h) - Y^H(h), t \in \mathbb{R}^N\big\} \stackrel{d}{=} \big\{Y^H(t) - Y^H(0), t \in \mathbb{R}^N\big\}. \tag{3.8}$$

In the above, $\stackrel{d}{=}$ denotes equality of all finite dimensional distributions. Moreover, Y^H is isotropic in the sense $Y^H(t) \stackrel{d}{=} |t|^H Y^H((1,0,\dots 0))$ for all $t \in \mathbb{R}^N$. Hence Y^H is an α-stable analogue of fractional Brownian motion. When $N = 1$, Y^H is a variant of the real harmonizable fractional stable motion (cf. Chapter 7 of Samorodnitsky and Taqqu (1994)).

The stable random field Y^H can be written in the form (3.1) and its spectral measure m has a density function which is given by

$$f_{H,\alpha}(x) = \frac{c(\alpha, H, N)}{|x|^{\alpha H+N}}, \tag{3.9}$$

where $c(\alpha, H, N) > 0$ is a normalizing constant such that the scale parameter of $Y^H(1)$ equals 1. A change of variables shows that $\sigma(t) = |t|^H$ for all $t \in \mathbb{R}^N$. When $1 \le \alpha < 2$, the local nondeterminism of Y^H was proved by Nolan (1989).

Example 3.3. [Fractional Riesz-Bessel α-stable motion] Consider the real-valued α-stable random field $Y^{\gamma,\eta} = \{Y^{\gamma,\eta}(t), t \in \mathbb{R}^N\}$ with representation (3.1) and spectral density

$$f_{\gamma,\eta}(x) = \frac{c(\alpha, \gamma, \eta, N)}{|x|^{2\gamma}(1+|x|^2)^{\eta}}, \tag{3.10}$$

where $c(\alpha, \gamma, \eta, N) > 0$ is a normalizing constant and γ and η are positive constants satisfying

$$\gamma + \eta > \frac{N}{2}, \qquad 0 < 2\gamma < \alpha + N.$$

This implies that condition (3.2) is satisfied. Since the spectral density $f_{\gamma,\eta}$ involves both the Fourier transforms of the Riesz kernel and the Bessel kernel, we call $Y^{\gamma,\eta}$ the fractional Riesz-Bessel α-stable motion with indices η and γ. When $\alpha = 2$, $Y^{\gamma,\eta}$ is the fractional Riesz-Bessel motion with indices γ and η defined by Anh et al. (1999), who have shown that these Gaussian random fields can be used for modeling simultaneously long range dependence and intermittency. The stable counterpart $Y^{\gamma,\eta}$ has the additional feature of having heavy-tailed distributions.

Since the spectral density $f_{\gamma,\eta}(x)$ is regularly varying at infinity of order $2(\gamma+\eta) > N$, by a variant of Theorem 1 in Pitman (1968) we can show that, if $2(\gamma+\eta) - N < \alpha$, then $\sigma(h)$ is regularly varying at 0 of order $(2(\gamma+\eta) - N)/\alpha$ and

$$\sigma(h) \sim |h|^{(2(\gamma+\eta)-N)/\alpha},$$

where $a(h) \sim b(h)$ means $a(h)/b(h) \to 1$ as $h \to 0$. Because of Theorem 3.1, we see that the results of this paper are applicable to the fractional Riesz-Bessel stable motion $Y^{\gamma,\eta}$.

3.2. Anisotropic stable fields with stationary increments

Now we deal with anisotropic stable random fields with stationary increments.

Theorem 3.4. *Let $X_0 = \{X_0(t),\, t \in \mathbb{R}^N\}$ be a real-valued α-stable random field defined by (*3.1*) with spectral density $f(x)$. Assume that there is a vector $H =$*

$(H_1, \ldots, H_N) \in (0,1)^N$ *such that*

$$f(x) \asymp \frac{1}{\big(\sum_{j=1}^N |x_j|^{H_j}\big)^{\alpha+Q}}, \qquad \forall\, x \in \mathbb{R}^N \backslash \{0\}, \tag{3.11}$$

where $Q = \sum_{j=1}^N \frac{1}{H_j}$. *Then there exist positive constants* $c_{3,3} \geq 1$ *and* $c_{3,4}$ *such that the following statements hold:*

(i) *For all* $s, t \in [0,1]^N$,

$$c_{3,3}^{-1}\ \rho(s,t) \leq \big\|X_0(s) - X_0(t)\big\|_\alpha \leq c_{3,3}\ \rho(s,t). \tag{3.12}$$

Recall that ρ *is the metric defined in (*2.9*).*

(ii) *If, in addition, we assume* $\alpha \in [1,2)$. *Then for all* $n \geq 2$ *and all* $t^1, \ldots, t^n \in \mathbb{R}^N$,

$$\big\|\ X_0(t^n) \mid X_0(t^1), \ldots, X_0(t^{n-1})\big\|_\alpha \geq c_{3,4} \min_{0 \leq k \leq n-1} \rho(t^n, t^k). \tag{3.13}$$

Proof. Let us first prove Part (i). For all $s, t \in \mathbb{R}^N$, we write

$$\begin{aligned} \big\|X_0(s) - X_0(t)\big\|_\alpha^\alpha &= \int_{\mathbb{R}^N} \big|e^{i\langle s,x\rangle} - e^{i\langle t,x\rangle}\big|^\alpha f(x)dx \\ &\asymp \int_{\mathbb{R}^N} \big(1 - \cos\langle s-t, x\rangle\big)^{\alpha/2}\ \frac{dx}{\rho(0,x)^{\alpha+Q}}. \end{aligned} \tag{3.14}$$

To evaluate the last integral we denote $t-s$ by $h = (h_1, \ldots, h_N)$ and make a change of variables $y_\ell = \rho(0,h)^{H_\ell^{-1}}\, x_\ell$ ($\ell = 1, \ldots, N$). Since $\rho(0,x) = \rho(0,y)/\rho(0,h)$, we derive

$$\begin{aligned} &\int_{\mathbb{R}^N} \big(1 - \cos\langle h, x\rangle\big)^{\alpha/2}\ \frac{dx}{\rho(0,x)^{\alpha+Q}} \\ &= \rho(0,h)^\alpha \int_{\mathbb{R}^N} \left|1 - \cos\left(\sum_{\ell=1}^N h_\ell\, \rho(0,h)^{-H_\ell^{-1}}\, y_\ell\right)\right|^\alpha \frac{dy}{\rho(0,y)^{\alpha+Q}}. \end{aligned} \tag{3.15}$$

Note that the point $(h_1\, \rho(0,h)^{-H_1^{-1}}, \ldots, h_N\, \rho(0,h)^{-H_N^{-1}}) \in \mathcal{S}_\rho = \{x \in \mathbb{R}^N : \rho(0,x) = 1\}$ which is a compact set, we see that the last integral in (3.15) is bounded from below and above by positive constants. Hence, (3.12) holds.

To prove Part (ii), we denote $r \equiv \min_{0 \leq k \leq n-1} \rho(t^n, t^k)$. It is sufficient to prove that for all $u_k \in \mathbb{R}$ ($1 \leq k \leq n-1$),

$$\left\|X_0(t^n) - \sum_{k=1}^{n-1} u_k\, X_0(t^k)\right\|_\alpha^\alpha \geq c_{3,4}\, r^\alpha \tag{3.16}$$

and $c_{3,4} > 0$ is a constant depending only on H and N.

By the stochastic integral representation (3.1) of X_0, the left hand side of (3.16) can be written as

$$\left\|X_0(t^n) - \sum_{k=1}^{n-1} u_k\, X_0(t^k)\right\|_\alpha^\alpha = \int_{\mathbb{R}^N} \left|e^{i\langle t^n, x\rangle} - \sum_{k=0}^{n-1} u_k\, e^{i\langle t^k,\, x\rangle}\right|^\alpha f(x)\, dx, \tag{3.17}$$

where $t^0 = 0$ and $u_0 = -1 + \sum_{k=1}^n u_k$.

Let $\delta(\cdot) : \mathbb{R}^N \to [0,1]$ be a function in $C^\infty(\mathbb{R}^N)$ such that $\delta(0) = 1$ and it vanishes outside the open ball $B_\rho(0,1)$ in the metric ρ. Denote by $\widehat{\delta}$ the Fourier transform of δ. Then $\widehat{\delta}(\cdot) \in C^\infty(\mathbb{R}^N)$ as well and $\widehat{\delta}(x)$ decays rapidly as $|x| \to \infty$.

Let E denote the diagonal matrix with $H_1^{-1}, \ldots, H_N^{-1}$ on its diagonal and let $\delta_r(t) = r^{-Q}\delta(r^{-E}t)$. Then the inverse Fourier transform and a change of variables yield

$$\delta_r(t) = (2\pi)^{-N} \int_{\mathbb{R}^N} e^{-i\langle t,x\rangle}\, \widehat{\delta}(r^E x)\, dx. \tag{3.18}$$

Since $\min\{\rho(t^n, t^k) : 0 \le k \le n-1\} \ge r$, we have $\delta_r(t^n - t^k) = 0$ for $k = 0, 1, \ldots, n-1$. This and (3.18) together imply that

$$\begin{aligned} J &:= \int_{\mathbb{R}^N} \bigg(e^{i\langle t^n,x\rangle} - \sum_{k=0}^{n-1} u_k\, e^{i\langle t^k,x\rangle} \bigg)\, e^{-i\langle t^n,x\rangle}\, \widehat{\delta}(r^E x)\, dx \\ &= (2\pi)^N \bigg(\delta_r(0) - \sum_{k=0}^{n-1} u_k\, \delta_r(t^n - t^k) \bigg) \\ &= (2\pi)^N\, r^{-Q}. \end{aligned} \tag{3.19}$$

Now let $\beta > 1$ be the constant such that $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. By Hölder's inequality, (3.17), (3.11) and the fact that $\rho(0, r^{-E}x) = r^{-1}\rho(0,x)$, we have

$$\begin{aligned} J &\le \bigg(\int_{\mathbb{R}^N} \Big| e^{i\langle t^n,x\rangle} - \sum_{k=0}^{n-1} u_k\, e^{i\langle t^k,x\rangle} \Big|^\alpha f(x)\, dx \bigg)^{1/\alpha} \\ &\qquad\qquad \times \bigg(\int_{\mathbb{R}^N} \frac{1}{f(x)^{\beta/\alpha}} \Big| \widehat{\delta}(r^E x) \Big|^\beta dx \bigg)^{1/\beta} \\ &= \Big\| X_0(t^n) - \sum_{k=0}^{n-1} u_k\, X_0(t^k) \Big\|_\alpha \bigg(\int_{\mathbb{R}^N} \frac{1}{r^Q f(r^{-E}\, x)^{\beta/\alpha}} \big|\widehat{\delta}(x)\big|^\beta dx \bigg)^{1/\beta} \\ &\le c \Big\| X_0(t^n) - \sum_{k=0}^{n-1} u_k\, X_0(t^k) \Big\|_\alpha \bigg(\int_{\mathbb{R}^N} \frac{\rho(0, r^{-E}\, x)^{(Q+\alpha)\beta/\alpha}}{r^Q} \big|\widehat{\delta}(x)\big|^\beta dx \bigg)^{1/\beta} \\ &= c_{3,5} \Big\| X_0(t^n) - \sum_{k=0}^{n-1} u_k\, X_0(t^k) \Big\|_\alpha \cdot r^{-Q-1}, \end{aligned} \tag{3.20}$$

where $c_{3,5} > 0$ is a constant which only depend on H, N and δ. It is clear that (3.13) follows from (3.19) and (3.20). This finishes the proof of the theorem. □

3.3. Harmonizable fractional stable sheets

For any given $0 < \alpha < 2$ and $H = (H_1, \ldots, H_N) \in (0,1)^N$, we define the harmonizable fractional stable sheet $\widetilde{Z}^H = \{\widetilde{Z}^H(t), t \in \mathbb{R}^N_+\}$ with values in $\mathbb{R}$ by

$$\widetilde{Z}^H(t) = \mathrm{Re} \int_{\mathbb{R}^N} \prod_{j=1}^N \frac{e^{it_j\lambda_j} - 1}{|\lambda_j|^{H_j + \frac{1}{\alpha}}} \widetilde{M}_\alpha(d\lambda), \tag{3.21}$$

where $\widetilde{Z}_\alpha$ is a complex-valued rotationally invariant α-stable random measure with Lebesgue control measure. From (3.21) it follows that $\widetilde{Z}^H$ has the following operator-scaling property: For any $N \times N$ diagonal matrix $E = (b_{ij})$ with $b_{ii} = b_i > 0$ for all $1 \le i \le N$ and $b_{ij} = 0$ if $i \ne j$, we have

$$\big\{\widetilde{Z}^H(Et),\, t \in \mathbb{R}^N\big\} \overset{d}{=} \left\{\left(\prod_{j=1}^N b_j^{H_j}\right) \widetilde{Z}^H(t),\ t \in \mathbb{R}^N\right\}. \tag{3.22}$$

Along each direction of $\mathbb{R}^N_+$, $\widetilde{Z}^H$ becomes a real-valued harmonizable fractional stable motion. When the indices $H_1, \ldots, H_N$ are not the same, $\widetilde{Z}^H$ has different scaling behavior along different directions.

Note that, unlike in Part (i) of Theorem 3.4 which holds for all $\alpha \in (0,2)$, we are only able to deal with the case when $\alpha \in [1,2)$ in the theorem below.

Theorem 3.5. *Suppose $\alpha \in [1,2)$. Then there exist positive constants $c_{3,6} \ge 1$ and $c_{3,7}$, depending on H and N only, such that*

(i) *For all $s, t \in [0,1]^N$,*

$$c_{3,6}^{-1}\ \rho(s,t) \le \big\|\widetilde{Z}^H(s) - \widetilde{Z}^H(t)\big\|_\alpha \le c_{3,6}\ \rho(s,t). \tag{3.23}$$

(ii) *For all positive integers $n \ge 2$ and all $t^1, \ldots, t^n \in [0,\infty)^N$, we have*

$$\Big\|\widetilde{Z}^H(t^n) \,\big|\, \widetilde{Z}^H(t^1), \ldots, \widetilde{Z}^H(t^{n-1})\Big\|_\alpha \ge c_{3,7} \sum_{j=1}^N \min_{0 \le k \le n-1} \big|t_j^n - t_j^k\big|^{H_j}. \tag{3.24}$$

Proof. We prove the upper bound in (3.23) by using induction on N [as in Ayache, Roueff and Xiao (2007c)] and it holds for all $\alpha \in (0, 2)$. Here is the main step:

$$
\begin{aligned}
&\big\|\widetilde{Z}^H(s) - \widetilde{Z}^H(t)\big\|_\alpha^\alpha \\
&= \int_{\mathbb{R}^N} \bigg| \prod_{j=1}^N \frac{e^{is_j\lambda_j}-1}{|\lambda_j|^{H_j+\frac{1}{\alpha}}} - \prod_{j=1}^N \frac{e^{it_j\lambda_j}-1}{|\lambda_j|^{H_j+\frac{1}{\alpha}}} \bigg|^\alpha d\lambda \\
&\le c \int_{\mathbb{R}^N} \prod_{j=1}^{N-1} \bigg| \frac{e^{is_j\lambda_j}-1}{|\lambda_j|^{H_j+\frac{1}{\alpha}}} \bigg|^\alpha \bigg| \frac{e^{is_N\lambda_N}-1}{|\lambda_N|^{H_N+\frac{1}{\alpha}}} - \frac{e^{it_N\lambda_N}-1}{|\lambda_N|^{H_N+\frac{1}{\alpha}}} \bigg|^\alpha d\lambda \\
&\qquad + c \int_{\mathbb{R}^N} \bigg| \prod_{j=1}^{N-1} \frac{e^{is_j\lambda_j}-1}{|\lambda_j|^{H_j+\frac{1}{\alpha}}} - \prod_{j=1}^{N-1} \frac{e^{it_j\lambda_j}-1}{|\lambda_j|^{H_j+\frac{1}{\alpha}}} \bigg|^\alpha \bigg| \frac{e^{it_N\lambda_N}-1}{|\lambda_N|^{H_N+\frac{1}{\alpha}}} \bigg|^\alpha d\lambda \\
&\le c_{3,8} \sum_{j=1}^N |s_j - t_j|^{\alpha H_j}.
\end{aligned} \tag{3.25}
$$

However, the method in Ayache, Roueff and Xiao (2007c) for proving the lower bound can not be applied to $\widetilde{Z}^H$. Because of this we have to assume $\alpha \in [1, 2)$ for the lower bound in Part (i), which we will show after proving (3.24).

The proof of (3.24) is a combination of the proofs of Theorem 3.4 in the above and Theorem 1 in Wu and Xiao (2007). Since there is no new ideas needed, we omit the details.

Finally, since $\big\|\widetilde{Z}^H(s) - \widetilde{Z}^H(t)\big\|_\alpha \ge \big\|\widetilde{Z}^H(s)\big|\widetilde{Z}^H(t)\big\|_\alpha$, the lower bound in (3.23) follows from (3.24) by setting $n = 2$, $t^1 = t$ and $t^2 = s$. □

Many properties of harmonizable fractional stable sheets are similar to those of their Gaussian counterpart, i.e., fractional Brownian sheets. See Shieh and Xiao (2007) for more information.

3.4. Linear fractional stable sheets

For any given $0 < \alpha < 2$ and $H = (H_1, \ldots, H_N) \in (0, 1)^N$, we define an α-stable random field $Z^H = \{Z^H(t), t \in \mathbb{R}_+^N\}$ with values in $\mathbb{R}$ by

$$
Z^H(t) = \int_{\mathbb{R}^N} g_H(t, s)\, M_\alpha(ds), \tag{3.26}
$$

where M_α is a symmetric α-stable random measure on $\mathbb{R}^N$ with Lebesgue control measure and

$$
g_H(t, s) = \kappa \prod_{\ell=1}^N \Big\{ \big((t_\ell - s_\ell)_+\big)^{H_\ell - 1/\alpha} - \big((-s_\ell)_+\big)^{H_\ell - 1/\alpha} \Big\}. \tag{3.27}
$$

In the above $\kappa > 0$ is a normalizing constant and $t_+ = \max\{t, 0\}$. Using (3.26) one can verify that the $(N, 1, \alpha)$-stable field Z^H is operator self-similar in the sense of (3.22), and along each direction of $\mathbb{R}_+^N$, Z^H becomes a real-valued linear fractional stable motion. $Z^H = \{Z(t), t \in \mathbb{R}_+^N\}$ is called an $(N, 1, \alpha)$-linear fractional stable sheet.

Note that, unlike the Gaussian case $\alpha = 2$, where both (3.21) and (3.26) determine (up to a constant) the same fractional Brownian sheet, the moving average and harmonizable fractional stable sheets with the same $\alpha \in (0, 2)$ and Hurst index H are different stable random fields. This is true even for $N = 1$; see Samorodnitsky and Taqqu (1994, page 358). Moreover, it is known that the regularity properties of linear fractional stable motion are very different from those of harmonizable fractional stable motion; see Maejima (1983) and Takashima (1989).

Ayache, Roueff and Xiao (2007a, b, c, 2008) have studied asymptotic properties, modulus of continuity, fractal dimensions and local times of linear fractional stable sheets. The methods there are different from the methods in this paper and rely on the non-anticipating nature of the representation (3.26).

The argument of Ayache, Roueff and Xiao (2007c) proves that Z^H satisfies the following *one-sided* form of sectorial local nondeterminism.

Proposition 3.6. *For any fixed positive number $\varepsilon \in (0, 1)$, there exist positive constants $c_{3,9} \ge 1$ and $c_{3,10}$, depending on ε, H and N only, with the following properties:*

(i) *For all $s, t \in [\varepsilon, 1]^N$,*

$$c_{3,9}^{-1}\ \rho(s, t) \le \left\|Z^H(s) - Z^H(t)\right\|_\alpha \le c_{3,9}\ \rho(s, t). \tag{3.28}$$

(ii) *For all positive integers $n \ge 2$, and all $t^1, \ldots, t^n \in [\varepsilon, \infty)^N$ such that $t_\ell^k \le t_\ell^n$ for some $1 \le \ell \le N$ and all $1 \le k \le n-1$, we have*

$$\left\|Z^H(t^n) \,\middle|\, Z^H(t^1), \ldots, Z^H(t^{n-1})\right\|_\alpha \ge c_{3,10} \min_{1 \le k \le n-1} \big(t_\ell^n - t_\ell^k\big)^{H_\ell}. \tag{3.29}$$

Proof. Part (i) is from Lemma 17 in Ayache, Roueff and Xiao (2007c), whose proof shows that Z^H can be decomposed as a sum of independent α-stable processes Y_ℓ $(1 \le \ell \le N)$ such that, in the ℓth direction, Y_ℓ behaves like a Liouville fractional stable motion of index H_ℓ. Then it is straightforward to derive (3.29) and we omit the technical details. □

In the following, we apply the Hausdorff-Young theorem to show that, for $\alpha \in (1, 2)$, LFSS Z^H satisfies the *two-sided* sectorial local nondeterminism. This strengthens Proposition 3.6 and also shows that the results in Section 4 are applicable to LFSS.

Theorem 3.7. *Suppose $\alpha \in (1, 2)$ and $\varepsilon \in (0, 1)$ are constants, and $\min_{1 \le j \le N} H_j > 1/\alpha$. Then there exists a positive constant $c_{3,11}$, depending on ε, H and N only, such that for all positive integers $n \ge 2$ and all $t^1, \ldots, t^n \in [\varepsilon, \infty)^N$, we have*

$$\left\|Z^H(t^n) \,\middle|\, Z^H(t^1), \ldots, Z^H(t^{n-1})\right\|_\alpha \ge c_{3,11} \sum_{j=1}^N \min_{0 \le k \le n-1} \big|t_j^n - t_j^k\big|^{H_j}. \tag{3.30}$$

Proof. We know that, for every $t \in (0,\infty)^N$, $g_{_H}(t,\cdot) \in L^\alpha(\mathbb{R}^N, dx)$. The key observation is that the Fourier transform of $g_{_H}(t,\cdot)$ is

$$\widehat{g_{_H}(t,\cdot)}(\lambda) = \prod_{j=1}^{N} \frac{e^{it_j\lambda_j}-1}{i\lambda_j}\,\frac{1}{|\lambda_j|^{H_j-\frac{1}{\alpha}}}. \tag{3.31}$$

Then the Hausdorff-Young theorem [cf. Lieb and Loss (1997, p. 121)] implies that the L^α-norm of linear combinations of $g_{_H}(t^k,\cdot)$ $(1 \le k \le n)$ is bounded from below by a constant times the L^β-norm of the Fourier transform, which is the corresponding linear combination of $\widehat{g_{_H}(t^k,\cdot)}$. In the above, $\beta > 2$ is the constant such that $\frac{1}{\alpha}+\frac{1}{\beta}=1$. The rest is similar to the proof of Theorem 3.5. See Ayache, Roueff and Xiao (2008) for more details. □

4. Local times and their joint continuity

In this section we apply the properties of sectorial local nondeterminism to study existence and joint continuity of the local times of stable random fields with values in $\mathbb{R}^d$. The main argument is still based on the Fourier analytic methods initiated by Berman (1973), and further developed in Pitt (1978), Geman and Horowitz (1980), Nolan (1988, 1989), Csörgő et al. (1995) and Xiao (1997, 2008). The new idea is to make use of strong local nondeterminism for estimating high moments of the local times.

We start by briefly recalling some aspects of the theory of local times. For excellent surveys on local times of random and deterministic vector fields, we refer to Geman and Horowitz (1980) and Dozzi (2002).

Let $X(t)$ be a Borel vector field on $\mathbb{R}^N$ with values in $\mathbb{R}^d$. For any Borel set $T \subseteq \mathbb{R}^N$, the occupation measure of X on T is defined as

$$\mu_{_T}(\bullet) = \lambda_N\big\{t \in T : X(t) \in \bullet\big\},$$

which is a Borel measure on $\mathbb{R}^d$. If $\mu_{_T}$ is absolutely continuous with respect to the Lebesgue measure λ_d, then $X(t)$ is said to have a *local time* on T. The local time, $L(\bullet, T)$, is defined as the Radon–Nikodým derivative of $\mu_{_T}$ with respect to λ_d, i.e.,

$$L(x,T) = \frac{d\mu_{_T}}{d\lambda_d}(x), \qquad \forall x \in \mathbb{R}^d.$$

In the above, x is the so-called *space variable*, and T is the *time variable*. Sometimes, we write $L(x,t)$ in place of $L(x,[0,t])$. It is clear that if X has local times on T, then for every Borel set $S \subseteq T$, $L(x,S)$ also exists.

It follows from Theorems 6.3 and 6.4 in Geman and Horowitz (1980) that the local times of X have a version, still denoted by $L(x,T)$, such that it is a kernel in the following sense:

(i). For each fixed $S \subseteq T$, the function $x \mapsto L(x,S)$ is Borel measurable in $x \in \mathbb{R}^d$.

(ii). For every $x \in \mathbb{R}^d$, $L(x, \cdot)$ is a Borel measure on $\mathcal{B}(T)$, the family of Borel subsets of T.

Moreover, $L(x, T)$ satisfies the following *occupation density formula*: For every measurable function $f : \mathbb{R}^d \to \mathbb{R}_+$,

$$\int_T f(X(t))\, dt = \int_{\mathbb{R}^d} f(x) L(x, T)\, dx. \tag{4.1}$$

Suppose we fix a rectangle $T = \prod_{i=1}^N [a_i, a_i + h_i]$ in $\mathcal{A}$. If we can choose a version of the local time, still denoted by $L\big(x, \prod_{i=1}^N [a_i, a_i + t_i]\big)$, such that it is a continuous function of $(x, t_1, \cdots, t_N) \in \mathbb{R}^d \times \prod_{i=1}^N [0, h_i]$, then X is said to have a *jointly continuous local time* on T. When a local time is jointly continuous, $L(x, \bullet)$ can be extended to be a finite Borel measure supported on the level set

$$X_T^{-1}(x) = \{t \in T : X(t) = x\}; \tag{4.2}$$

see Adler (1981) for details. Hence local times are useful in studying fractal properties of level sets and inverse images of the vector field X. See, for example, Ehm (1981), Monrad and Pitt (1987) and Xiao (1997, 2008).

In the rest of this paper, we study local times of stable random fields under some general conditions. Let $X = \{X(t), t \in \mathbb{R}^N\}$ be an (N, d, α) stable random field defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ by (1.1), where $X_1, \ldots, X_d$ are independent copies of an α-stable random field $X_0 = \{X_0(t), t \in \mathbb{R}^N\}$ with values in $\mathbb{R}$. Let $T \in \mathcal{A}$ be a fixed closed interval, and we will consider the existence, joint continuity and Hölder conditions for the local times of $X(t)$ when $t \in T$.

Many sample path properties of X can be determined by the properties of

$$\sigma(s, t) = \big\| X_0(s) - X_0(t) \big\|_\alpha, \qquad s, t \in T,$$

the scalar parameter of the increment $X_0(s) - X_0(t)$. We will assume that, for some vector $H = (H_1, \ldots, H_d) \in (0, 1)^N$, X_0 satisfies the following general conditions:

(S1). There exist positive constants $c_{4,1}, \ldots, c_{4,4}$ such that

$$c_{4,1} \le \sigma(t) := \sigma(0, t) \le c_{4,2}, \qquad \forall\, t \in T \tag{4.3}$$

and

$$c_{4,3}\, \rho(s, t) \le \sigma(s, t) \le c_{4,4}\, \rho(s, t) \quad \text{for all } \ s, t \in T. \tag{4.4}$$

Here ρ is the metric on $\mathbb{R}^N$ defined by (2.9).

(S2). The α-stable random field X_0 has representation (1.2) or (1.6). There exists a constant $c_{4,5} > 0$ such that for all integers $n \ge 2$ and all $t^1, \ldots, t^n \in T$,

$$\big\| X_0(t^n) \,\big|\, X_0(t^1), \ldots, X_0(t^{n-1}) \big\|_\alpha \ge c_{4,5} \sum_{j=1}^N \min_{0 \le k \le n-1} \big| t_j^n - t_j^k \big|^{H_j},$$

where $t_j^0 = 0$ for every $j = 1, \ldots, N$. That is, X_0 satisfies the (two-sided) sectorial local nondeterminism.

It is worthwhile to mention that Lemma 4.3 below holds under the *one-sided* sectorial local nondeterminism; see (iii) in Definition 2.3. For convenience, we state it as

(S3). The α-stable random field X_0 has representation (1.2) or (1.6). There exists a constant $c_{4,6} > 0$ such that for all integers $n \geq 2$ and all $t^1, \ldots, t^n \in T$ with the property that, for some $1 \leq \ell \leq N$, $t^k_\ell \leq t^n_\ell$ for all $1 \leq k \leq n-1$, we have

$$\big\|X_0(t^n)\,\big|\,X_0(t^1), \ldots, X_0(t^{n-1})\big\|_\alpha \geq c_{4,6} \min_{1\leq k\leq n-1} \big(t^n_\ell - t^k_\ell\big)^{H_\ell}.$$

First we consider the existence of the local times of stable random fields. It does not require X_0 to have representation (1.2) nor (1.6).

Theorem 4.1. *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be an (N, d, α)-stable random field defined by* (1.1) *and suppose X_0 satisfies Condition (S1) on $T \in \mathcal{A}$. Then X has a local time $L(x, T) \in L^2(\mathbb{P}\times\lambda_d)$ if and only if $d < \sum_{j=1}^N 1/H_j$. In the latter case, $L(x, T)$ admits the following L^2-representation:*

$$L(x, T) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-i\langle y,\, x\rangle} \int_T e^{i\langle y,\, X(s)\rangle} ds\, dy, \qquad \forall\, x \in \mathbb{R}^d. \tag{4.5}$$

Proof. The proof is similar to the Gaussian case; see Theorem 8.1 in Xiao (2008). Hence we only give a brief sketch. The Fourier transform of the occupation measure μ_T is

$$\widehat{\mu}_T(\xi) = \int_T e^{i\langle \xi,\, X(t)\rangle}\, dt.$$

By applying Fubini's theorem twice, we have

$$\mathbb{E} \int_{\mathbb{R}^d} \big|\widehat{\mu}_T(\xi)\big|^2\, d\xi = \int_T ds \int_T dt \int_{\mathbb{R}^d} \mathbb{E} \exp\Big(i\langle \xi,\, X(s) - X(t)\rangle\Big)\, d\xi. \tag{4.6}$$

We denote the right hand side of (4.6) by $\mathcal{J}(T)$. It follows from the Plancherel theorem that X has a local time $L(\cdot, T) \in L^2(\mathbb{P} \times \lambda_d)$ if and only if $\mathcal{J}(T) < \infty$; see Theorem 21.9 in Geman and Horowitz (1980), or Kahane (1985). Hence it is sufficient to prove

$$\mathcal{J}(T) < \infty \quad \text{ if and only if } \quad d < \sum_{j=1}^N \frac{1}{H_j}. \tag{4.7}$$

For this purpose, we use the independence of the coordinate processes of X and Condition (S1) to deduce

$$\mathcal{J}(T) = \int_T\int_T \frac{ds\, dt}{\|X_0(s) - X_0(t)\|_\alpha^d} \asymp \int_T\int_T \frac{ds\, dt}{\big(\sum_{j=1}^N |s_j - t_j|^{H_j}\big)^d}. \tag{4.8}$$

By using Lemma 8.6 in Xiao (2008), one can verify that the last integral in (4.8) is convergent if and only if $d < \sum_{j=1}^N 1/H_j$. This proves (4.7), and hence Theorem 4.1. □

The following result on the joint continuity of the local times of stable random fields is an extension of those proved by Xiao and Zhang (2002), Ayache, Wu and Xiao (2008), Xiao (2008), Wu and Xiao (2008) for Gaussian random fields.

Theorem 4.2. *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be an (N, d, α)-stable random field defined by (1.1) and we assume X_0 satisfies Conditions (S1) and (S2) on T. If $1 < \alpha < 2$ and $d < \sum_{j=1}^N 1/H_j$. Then X has a jointly continuous local time on T.*

To prove Theorem 4.2 we will, similarly to Ehm (1981), Nolan (1989), Xiao (1997, 2008), Ayache, Wu and Xiao (2008), first use the Fourier analytic arguments to derive estimates on the moments of the local times [see Lemmas 4.3 and 4.7 below] and then apply a multiparameter version of Kolmogorov continuity theorem [cf. Khoshnevisan (2002)]. We should point out that our method for proving the moment estimates in Lemmas 4.3 and 4.7 are quite different from those in the references mentioned above. It will be clear that the sectorial local nondeterminism [(S2) or (S3)] plays an essential role in the proofs of Lemmas 4.3 and 4.7.

Our starting point is the following identities about the moments of the local time and its increments from Geman and Horowitz (1980) [see also Pitt (1978)]: For all $x, y \in \mathbb{R}^d$, $I \in \mathcal{A}$ and all integers $n \geq 1$,

$$\begin{aligned} \mathbb{E}\Big[L(x,I)^n\Big] \quad = \quad & (2\pi)^{-nd} \int_{I^n} \int_{\mathbb{R}^{nd}} \exp\bigg(- i \sum_{j=1}^n \langle u^j, x\rangle\bigg) \\ & \times \mathbb{E} \exp\bigg(i \sum_{j=1}^n \langle u^j, X(t^j)\rangle\bigg) d\overline{u}\, d\overline{t} \end{aligned} \tag{4.9}$$

and for all even integers $n \geq 2$,

$$\begin{aligned} \mathbb{E}\Big[\big(L(x,I) - L(y,I)\big)^n\Big] =& (2\pi)^{-nd} \int_{I^n} \int_{\mathbb{R}^{nd}} \prod_{j=1}^n \Big[e^{-i\langle u^j, x\rangle} - e^{-i\langle u^j, y\rangle}\Big] \\ & \times \mathbb{E} \exp\bigg(i \sum_{j=1}^n \langle u^j, X(t^j)\rangle\bigg) d\overline{u}\, d\overline{t}, \end{aligned} \tag{4.10}$$

where $\overline{u} = (u^1, \ldots, u^n)$, $\overline{t} = (t^1, \ldots, t^n)$, and each $u^j \in \mathbb{R}^d$, $t^j \in I \subseteq (0, \infty)^N$. In the coordinate notation we then write $u^j = (u_1^j, \ldots, u_d^j)$.

For future use, we prove the following lemma under weaker conditions than those in Theorem 4.2. Without loss of generality, we assume from now on that

$$0 < H_1 \leq H_2 \leq \cdots \leq H_N < 1. \tag{4.11}$$

Lemma 4.3. *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be an (N, d, α)-stable random field defined by (1.1) and we assume X_0 satisfies Conditions (S1) and (S3) on T. Let $\tau \in \{1, \ldots, N\}$ be the integer such that*

$$\sum_{\ell=1}^{\tau-1} \frac{1}{H_\ell} \leq d < \sum_{\ell=1}^{\tau} \frac{1}{H_\ell} \tag{4.12}$$

with the convention that $\sum_{\ell=1}^{0} \frac{1}{H_\ell} := 0$*. Then for all integers* $n \ge 1$*, there exists a positive and finite constant* $c_{4,7} = c_{4,7}(n)$ *such that for all hypercubes* $I = [a, a + \langle r\rangle] \subseteq T$ *with side-length* $r \in (0,1)$ *and all* $x \in \mathbb{R}^d$

$$\mathbb{E}\big[L(x,I)^n\big] \le c_{4,7}\, r^{n\,\beta_\tau}, \tag{4.13}$$

where $\beta_\tau = \sum_{\ell=1}^{\tau} \frac{H_\tau}{H_\ell} + N - \tau - H_\tau d$*.*

Remark 4.4.

- If X is a Gaussian random field and (4.11), (4.12) hold, then $\beta_\tau = \sum_{\ell=1}^{\tau} \frac{H_\tau}{H_\ell} + N - \tau - H_\tau d$ is the Hausdorff dimension of the level set L_x [cf. Ayache and Xiao (2005), Xiao (2008)]. It can be proved that this remains true for harmonizable stable random fields in Sections 3.1–3.3. Details will have to be given elsewhere.
- We point out that the upper bound in (4.13) is not sharp because it is not known how $c_{4,7}$ relies on n. For Gaussian random fields with the property of strong local nondeterminism in the metric ρ, one can prove the following sharp inequality:

$$\mathbb{E}\big[L(x,T)^n\big] \le c_{4,8}^n\, (n!)^{N-\beta_\tau}\, r^{n\,\beta_\tau}. \tag{4.14}$$

In order to prove Lemma 4.3, we will make use of the following technical lemmas, both are from Ayache, Wu and Xiao (2008).

Lemma 4.5. *For any* $q \in [0, \sum_{\ell=1}^{N} H_\ell^{-1})$*, let* $\tau \in \{1, \ldots, N\}$ *be the integer such that*

$$\sum_{\ell=1}^{\tau-1} \frac{1}{H_\ell} \le q < \sum_{\ell=1}^{\tau} \frac{1}{H_\ell}. \tag{4.15}$$

Then there exists a positive constant $\delta_\tau \le 1$ *depending on* $(H_1, \ldots, H_N)$ *only such that for every* $\delta \in (0, \delta_\tau)$*, we can find* τ *real numbers* $p_\ell \ge 1$ $(1 \le \ell \le \tau)$ *satisfying the following properties:*

$$\sum_{\ell=1}^{\tau} \frac{1}{p_\ell} = 1, \qquad \frac{H_\ell\, q}{p_\ell} < 1, \qquad \forall\, \ell = 1, \ldots, \tau \tag{4.16}$$

and

$$(1-\delta) \sum_{\ell=1}^{\tau} \frac{H_\ell\, q}{p_\ell} \le H_\tau\, q + \tau - \sum_{\ell=1}^{\tau} \frac{H_\tau}{H_\ell}. \tag{4.17}$$

Lemma 4.6. *For all integers* $n \ge 1$*, positive numbers* a*,* r*,* $0 < b_j < 1$ *and an arbitrary* $s_0 \in [0, a/2]$*,*

$$\begin{aligned} &\int_{a \le s_1 \le \cdots \le s_n \le a+r} \prod_{j=1}^{n} (s_j - s_{j-1})^{-b_j}\, ds_1 \cdots ds_n \\ &\le c_{4,9}^n\, (n!)^{\frac{1}{n}\sum_{j=1}^{n} b_j - 1}\, r^{n - \sum_{j=2}^{n} b_j}, \end{aligned} \tag{4.18}$$

where $c_{4,9} > 0$ *is a constant depending on* a *and* b_j*'s only. In particular, if* $b_j = \alpha$ *for all* $j = 1, \ldots, n$*, then*

$$\begin{aligned}&\int_{a\le s_1\le\cdots\le s_n\le a+r}\prod_{j=1}^{n}(s_j - s_{j-1})^{-\alpha}\, ds_1\cdots ds_n\\ &\le c_{4,9}^n\,(n!)^{\alpha-1}\, r^{n(1-(1-\frac{1}{n})\alpha)}.\end{aligned} \tag{4.19}$$

Proof of Lemma 4.3. Since $X_1, \cdots, X_d$ are independent copies of X_0, it follows from (4.9) that for any interval $I = \prod_{j=1}^{N}[a_j, a_j + r_j] \subseteq T$ and all integers $n \ge 1$,

$$\mathbb{E}\big[L(x,I)^n\big] \le (2\pi)^{-nd}\int_{I^n}\prod_{k=1}^{d}\left\{\int_{\mathbb{R}^n}\exp\left(-\left\|\sum_{j=1}^{n}u_k^j\, X_0(t^j)\right\|_\alpha^\alpha\right)dU_k\right\}d\bar{t}, \tag{4.20}$$

where $U_k = (u_k^1, \cdots, u_k^n) \in \mathbb{R}^n$. It is clear that in order to bound the integral in $d\bar{t}$ it is sufficient to consider only the integral over $I^n_{\neq} = \{\bar{t} \in I^n : t^1, \ldots, t^n \text{ are distinct}\}$.

Let $\bar{t} \in I^n_{\neq}$ and $k \in \{1, \ldots, d\}$ be fixed and denote the inside integral in (4.20) by

$$\mathcal{J}_k = \int_{\mathbb{R}^n}\exp\left(-\left\|\sum_{j=1}^{n}u_k^j\, X_0(t^j)\right\|_\alpha^\alpha\right)dU_k. \tag{4.21}$$

For every $1 \le \ell \le N$, there exists a permutation π_ℓ of $\{1, \ldots, n\}$ such that

$$a_\ell \le t_\ell^{\pi_\ell(1)} \le t_\ell^{\pi_\ell(2)} \le \cdots \le t_\ell^{\pi_\ell(n)} \le a_\ell + r_\ell. \tag{4.22}$$

It follows from Lemma 2.6 and Condition (S3) that for every $1 \le \ell \le N$,

$$\left\|\sum_{j=1}^{n}u_k^j\, X_0(t^j)\right\|_\alpha^\alpha \ge c(n)\sum_{j=1}^{n}\big|v_{k,\ell}^j\big|^\alpha\,\big(t_\ell^{\pi_\ell(j)} - t_\ell^{\pi_\ell(j-1)}\big)^{\alpha H_\ell}, \tag{4.23}$$

where $c(n) > 0$ is a constant depending on n and

$$(v_{k,\ell}^1, \ldots, v_{k,\ell}^n) = (u_k^{\pi_\ell(1)}, \ldots, u_k^{\pi_\ell(n)})\, A_\ell \tag{4.24}$$

for an $n \times n$ lower triangle matrix $A_\ell = (a_{ij})$ with $a_{ii} = 1$ for all $1 \le i \le n$.

Summing up (4.23) over $\ell \in \{1, \ldots, N\}$ and combining it with (4.21), we obtain

$$\begin{aligned}\mathcal{J}_k &\le \int_{\mathbb{R}^n}\exp\left(-c(n)\sum_{\ell=1}^{N}\sum_{j=1}^{n}\big|v_{k,\ell}^j\big|^\alpha\,\big(t_\ell^{\pi_\ell(j)} - t_\ell^{\pi_\ell(j-1)}\big)^{\alpha H_\ell}\right)dU_k\\ &\le \int_{\mathbb{R}^n}\prod_{\ell=1}^{\tau}\exp\left(-c(n)\sum_{j=1}^{n}\big|v_{k,\ell}^j\big|^\alpha\,\big(t_\ell^{\pi_\ell(j)} - t_\ell^{\pi_\ell(j-1)}\big)^{\alpha H_\ell}\right)dU_k,\end{aligned} \tag{4.25}$$

where τ is the integer in (4.12). In order to estimate the last integral, we will separate the integrand so that the integration is taken with respect to the variables $(v_{k,\ell}^1, \ldots, v_{k,\ell}^n)$.

Since (4.12) holds, we apply Lemma 4.5 with $\delta = n^{-1}$ and $q = d$ to obtain τ positive numbers $p_1, \ldots, p_\tau \geq 1$ satisfying (4.16) and (4.17). It follows from (4.25), the generalized Hölder inequality and a change of variables by using (4.24) that

$$\begin{aligned} \mathcal{J}_k &\leq \prod_{\ell=1}^{\tau} \left[\int_{\mathbb{R}^n} \exp\biggl(-c(n) \sum_{j=1}^{n} \bigl| v_{j,\ell} \bigr|^{\alpha} \bigl(t_\ell^{\pi_\ell(j)} - t_\ell^{\pi_\ell(j-1)} \bigr)^{\alpha H_\ell} \biggr) \, dU_k \right]^{1/p_\ell} \\ &= c(n) \prod_{\ell=1}^{\tau} \prod_{j=1}^{n} \frac{1}{\bigl(t_\ell^{\pi_\ell(j)} - t_\ell^{\pi_\ell(j-1)} \bigr)^{H_\ell/p_\ell}}. \end{aligned} \tag{4.26}$$

Combining (4.20), (4.21), (4.25) and (4.26), we derive

$$\mathbb{E}\bigl[L(x,I)^n\bigr] \leq c(n) \sum_{\pi_1,\ldots,\pi_N} \int_{\Gamma(\pi_1,\ldots,\pi_N)} \prod_{\ell=1}^{\tau} \prod_{j=1}^{n} \frac{1}{\bigl(t_\ell^{\pi_\ell(j)} - t_\ell^{\pi_\ell(j-1)} \bigr)^{H_\ell d/p_\ell}} \, d\bar{t}. \tag{4.27}$$

In the above, the summation is taken over all permutations $\pi_1, \ldots, \pi_N$ of $\{1, \ldots, n\}$ and

$$\Gamma(\pi_1, \ldots, \pi_N) = \bigl\{ (t^1, \ldots, t^n) \in I^n : \ (4.22) \ \text{ holds for every } \ 1 \leq \ell \leq N \bigr\}.$$

To evaluate the integrals in (4.27), we first apply Lemma 4.6 to integrate $dt_\ell^1 dt_\ell^2 \ldots dt_\ell^n$ for $\ell = 1, \ldots, \tau$, and then continue to integrate $dt_\ell^1 dt_\ell^2 \ldots dt_\ell^n$ for $\ell = \tau + 1, \ldots, N$. This yields

$$\mathbb{E}\bigl[L(x,I)^n\bigr] \leq c(n) \prod_{\ell=1}^{\tau} r_\ell^{n(1-(1-\frac{1}{n})H_\ell d/p_\ell)} \cdot \prod_{\ell=\tau+1}^{N} r_\ell^n. \tag{4.28}$$

Now we consider the special case when $I = [a, a + \langle r \rangle]$ is a cube, i.e. $r_1 = \cdots = r_N = r$. It follows from (4.28) and (4.17) with $\delta = n^{-1}$ and $q = d$ that

$$\begin{aligned} \mathbb{E}\bigl[L(x,I)^n\bigr] &\leq c_{4,10}(n) \, r^{n\left(N - (1-n^{-1}) \sum_{\ell=1}^{\tau} H_\ell d/p_\ell\right)} \\ &\leq c_{4,10}(n) \, r^{n\beta_\tau}. \end{aligned} \tag{4.29}$$

This proves Lemma 4.3. □

Lemma 4.3 implies that for all $I \subseteq T$ and $n \geq 1$, $L(x, I) \in L^n(\mathbb{R}^d)$ a.s. [see Geman and Horowitz (1980, p. 42)]. Our next lemma estimates the moments of the increments of $L(x, I)$ in the space variable x.

Lemma 4.7. *Suppose the assumptions of Theorem* 4.2 *hold and assume* (4.12) *holds for some* $\tau \in \{1, \ldots, N\}$. *For all even integers* $n \geq 2$ *and all* $\gamma \in (0, 1)$ *small enough, there exists a positive and finite constant* $c_{4,11} = c_{4,11}(n)$ *such that for all hypercubes* $I = [a, a + \langle r \rangle] \subseteq T$, x, $y \in \mathbb{R}^d$ *with* $|x - y| \leq 1$,

$$\mathbb{E}\Bigl[\bigl(L(x,I) - L(y,I) \bigr)^n \Bigr] \leq c_{4,11} \, |x - y|^{n\gamma} \, r^{n(\beta_\tau - 2H_\tau \gamma)}. \tag{4.30}$$

Proof. Let $\gamma \in (0, 1)$ be a constant such that

$$\sum_{\ell=1}^{\tau-1} \frac{1}{H_\ell} \leq d + 2\gamma < \sum_{\ell=1}^{\tau} \frac{1}{H_\ell}. \tag{4.31}$$

Applying Lemma 4.5 with $\delta = 1/n$ and $q = d + 2\gamma$, we obtain τ positive numbers, still denoted by $p_1, \ldots, p_\tau \geq 1$, satisfying (4.16) and (4.17).

Note that by the elementary inequalities

$$|e^{iu} - 1| \leq 2^{1-\gamma}|u|^\gamma \qquad \text{for all } \ u \in \mathbb{R} \tag{4.32}$$

and $|u + v|^\gamma \leq |u|^\gamma + |v|^\gamma$, we see that for all $u^1, \ldots, u^n,\ x,\ y \in \mathbb{R}^d$,

$$\prod_{j=1}^n \left| e^{-i\langle u^j,\, x\rangle} - e^{-i\langle u^j,\, y\rangle} \right| \leq 2^{(1-\gamma)n}\ |x-y|^{n\gamma} \sum{}' \prod_{j=1}^n |u^j_{k_j}|^\gamma, \tag{4.33}$$

where the summation $\sum{}'$ is taken over all the sequences $(k_1, \cdots, k_n) \in \{1, \cdots, d\}^n$.

It follows from (4.10) and (4.33) that for every even integer $n \geq 2$,

$$\begin{aligned}
\mathbb{E}\Big[\big(L(x,I) - L(y,I)\big)^n\Big] &\leq (2\pi)^{-nd} 2^{(1-\gamma)n}\, |x-y|^{n\gamma} \\
&\quad \times \sum{}' \int_{I^n} \int_{\mathbb{R}^{nd}} \prod_{m=1}^n |u^m_{k_m}|^\gamma\, \mathbb{E} \exp\bigg(-i \sum_{j=1}^n \langle u^j,\, X(t^j)\rangle \bigg)\, d\overline{u}\, d\overline{t} \\
&\leq |x-y|^{n\gamma} \sum{}' \int_{I^n} d\overline{t} \\
&\quad \times \prod_{m=1}^n \bigg\{ \int_{\mathbb{R}^{nd}} |u^m_{k_m}|^{n\gamma}\ \exp\bigg(- \Big\| \sum_{j=1}^n \langle u^j,\, X(t^j)\rangle \Big\|_\alpha^\alpha \bigg)\, d\overline{u} \bigg\}^{1/n},
\end{aligned} \tag{4.34}$$

where the last inequality follows from the generalized Hölder inequality.

Now we fix a vector $\overline{k} = (k_1, \ldots, k_n) \in \{1, \cdots, d\}^n$ and n distinct points $t^1, \ldots, t^n \in I$. Let π_ℓ be the permutations of $\{1, \ldots, n\}$ as in the proof of Lemma 4.3. For simplicity of notation, we assume that they are all identities. In order words, we assume that for every $\ell \in \{1, \ldots, N\}$,

$$a_\ell \leq t^1_\ell \leq t^2_\ell \leq \cdots \leq t^n_\ell \leq a_\ell + r_\ell. \tag{4.35}$$

Let $\mathcal{M}(\overline{k}, \overline{t}, \gamma)$ be defined by

$$\mathcal{M}(\overline{k}, \overline{t}, \gamma) = \prod_{m=1}^n \bigg\{ \int_{\mathbb{R}^{nd}} |u^m_{k_m}|^{n\gamma}\ \exp\bigg(- \Big\| \sum_{j=1}^n \langle u^j,\, X(t^j)\rangle \Big\|_\alpha^\alpha \bigg)\, d\overline{u} \bigg\}^{1/n}. \tag{4.36}$$

In order to prove (4.30), it is sufficient to show that there exists a constant $c_{4,12}$ such that

$$\mathcal{M}(\overline{k}, \overline{t}, \gamma) \leq c_{4,12}(n) \prod_{\ell=1}^\tau \prod_{j=1}^n \frac{1}{\big|t^j_\ell - t^{j-1}_\ell\big|^{(H_\ell d + 2\gamma)/p_\ell}} \tag{4.37}$$

for every $\overline{k} = (k_1, k_2, \ldots, k_n) \in \{1, \cdots, d\}^n$. Then, as in the proof of Lemma 4.3, (4.30) will follow from (4.34), (4.37) and (4.19).

Now we proceed to prove (4.37). For concreteness, we assume $\overline{k} = \langle 1\rangle := (1, 1, \ldots, 1)$ and proceed to derive the desired upper bound for $\mathcal{M} := \mathcal{M}(\langle 1\rangle, \overline{t}, \gamma)$.

The method is the same for all the other sequences $\overline{k} \in \{1, \cdots, d\}^n$. By the independence of the coordinate processes, we have

$$\begin{aligned} \mathcal{M} = \prod_{m=1}^{n} \bigg\{ \int_{\mathbb{R}^n} |u_1^m|^{n\gamma} \exp\bigg(- \Big\| \sum_{j=1}^n u_1^j\, X_0(t^j) \Big\|_\alpha^\alpha \bigg)\, dU_1 \\ \times \prod_{k=2}^{d} \int_{\mathbb{R}^n} \exp\bigg(- \Big\| \sum_{j=1}^n u_k^j\, X_0(t^j) \Big\|_\alpha^\alpha \bigg)\, dU_k \bigg\}^{1/n}. \end{aligned} \tag{4.38}$$

Note that the integrals for $k \geq 2$ are $\mathcal{J}_k$ in the proof of Lemma 4.3 and one-sided local nondeterminism is sufficient to derive desired upper bounds for them. It only remains to estimate the first integral in (4.38), which will be denoted by $\mathcal{I}_m$. Here the extra factor $|u_1^m|^{n\gamma}$ makes things more complicated and we will use the (two-sided) sectorial local nondeterminism to deal with it.

In order to make an effective change of variables, we apply Lemma 2.6 by "conditioning" in the order $X_0(t^m), X_0(t^n), \ldots, X_0(t^{m+1}), X_0(t^{m-1}), \ldots, X_0(t^1)$. In this way, u_1^m is mapped into v_ℓ^n for every $\ell \in \{1, \ldots, N\}$. More precisely, we have

$$\begin{aligned} \Big\| \sum_{j=1}^n u_1^j X_0(t^j) \Big\|_\alpha \geq c(n) \bigg(|u_1^m|\, \big\| X_0(t^m) \big| X_0(t^i), i \neq m \big\|_\alpha \\ + \sum_{j=m+1}^{n} |v_\ell^j|\, \big\| X_0(t^j) \big| X_0(t^i), i \leq j-1, i \neq m \big\|_\alpha \\ + \sum_{j=1}^{m-1} |v_\ell^j|\, \big\| X_0(t^j) \big| X_0(t^i), i \leq j-1 \big\|_\alpha \bigg). \end{aligned} \tag{4.39}$$

Condition (S2) and (4.35) together imply the following three inequalities: For every $1 \leq \ell \leq N$, we have

$$\big\| X_0(t^m) \big| X_0(t^i), i \neq j \big\|_\alpha \geq c \min\Big\{ \big| t_\ell^{m+1} - t_\ell^m \big|^{H_\ell},\ \big| t_\ell^m - t_\ell^{m-1} \big|^{H_\ell} \Big\}, \tag{4.40}$$

$$\big\| X_0(t^{m+1}) \big| X_0(t^i), i \leq m-1 \big\|_\alpha \geq c\, \big| t_\ell^{m+1} - t_\ell^{m-1} \big|^{H_\ell} \tag{4.41}$$

and

$$\big\| X_0(t^j) \big| X_0(t^i), i \leq j-1 \big\|_\alpha \geq c\, \big| t_\ell^j - t_\ell^{j-1} \big|^{H_\ell} \tag{4.42}$$

for all $1 \leq j \leq n$ and $j \neq m, m+1$.

It follows from the generalized Hölder inequality, (4.39) – (4.42) and a change of variables by (4.24) [recall that $v_\ell^n = u_1^m$ for every ℓ] that

$$\begin{aligned}
\mathcal{I}_m \le \prod_{\ell=1}^{\tau} \Bigg\{ \int_{\mathbb{R}^n} |u_1^m|^{n\gamma} \exp\Big(-c(n)\,\big|u_1^m\big|^\alpha \min\big\{\big|t_\ell^{m+1}-t_\ell^m\big|^{\alpha H_\ell},\, \big|t_\ell^m - t_\ell^{m-1}\big|\big\}^{\alpha H_\ell}\Big) \\
\times \exp\Big(-c(n)\,\big|v_\ell^{m+1}\big|^\alpha\,\big|t_\ell^{m+1}-t_\ell^{m-1}\big|^{\alpha H_\ell}\Big) \\
\times \exp\Big(-c(n) \sum_{j=1,\, j\ne m,m+1}^{n} \big|v_\ell^j\big|^\alpha\,\big|t_\ell^j - t_\ell^{j-1}\big|^{\alpha H_\ell}\Big)\, dU_k \Bigg\}^{1/p_\ell} \\
= c(n) \prod_{\ell=1}^{\tau} \Bigg\{ \frac{1}{\min\big\{\big|t_\ell^{m+1}-t_\ell^m\big|^{H_\ell+n\gamma},\, \big|t_\ell^m-t_\ell^{m-1}\big|^{H_\ell+n\gamma}\big\}} \times \frac{1}{\big|t_\ell^{m+1}-t_\ell^{m-1}\big|^{H_\ell}} \\
\times \prod_{j=1, j\ne m,m+1}^{n} \frac{1}{\big|t_\ell^j - t_\ell^{j-1}\big|^{H_\ell}} \Bigg\}^{1/p_\ell}.
\end{aligned} \tag{4.43}$$

Since $\big|t_\ell^{m+1}-t_\ell^{m-1}\big| = \big|t_\ell^{m+1}-t_\ell^m\big| + \big|t_\ell^m - t_\ell^{m-1}\big|$, one can verify that

$$\mathcal{I}_m \le c(n) \prod_{\ell=1}^{\tau} \Bigg\{ \frac{1}{\big|t_\ell^{m+1}-t_\ell^m\big|^{n\gamma/p_\ell}\,\big|t_\ell^m-t_\ell^{m-1}\big|^{n\gamma/p_\ell}} \prod_{j=1}^{n} \frac{1}{\big|t_\ell^j-t_\ell^{j-1}\big|^{H_\ell/p_\ell}} \Bigg\}. \tag{4.44}$$

Combining (4.38), (4.26) and (4.44), we obtain

$$\mathcal{M} \le c_{4,12}(n) \prod_{\ell=1}^{\tau}\prod_{j=1}^{n} \frac{1}{\big|t_\ell^j - t_\ell^{j-1}\big|^{(H_\ell d+2\gamma)/p_\ell}}. \tag{4.45}$$

Hence we have verified (4.37).

It follows from (4.34), (4.36) and (4.37) that

$$\begin{aligned}
&\mathbb{E}\Big[\big(L(x,I)-L(y,I)\big)^n\Big] \\
&\le c_{4,13}(n)\,|x-y|^{n\gamma} \int_{\Gamma_n} \prod_{\ell=1}^{\tau}\prod_{j=1}^{n} \frac{1}{\big|t_\ell^j - t_\ell^{j-1}\big|^{(H_\ell d+2\gamma)/p_\ell}}\, d\bar{t},
\end{aligned} \tag{4.46}$$

where

$$\Gamma_n = \{(t^1,\dots,t^n) \in I^n :\ (4.35)\ \text{ holds for every }\ 1\le \ell\le N\}.$$

Similar to the proof of Lemma 4.3, we apply Lemma 4.6 to derive

$$\begin{aligned}
&\mathbb{E}\Big[\big(L(x,I)-L(y,I)\big)^n\Big] \\
&\le c_{4,14}\,|x-y|^{n\gamma} \prod_{\ell=1}^{\tau} r_\ell^{n(1-(1-\frac{1}{n})(H_\ell d+2\gamma)/p_\ell)} \cdot \prod_{\ell=\tau+1}^{N} r_\ell^n.
\end{aligned} \tag{4.47}$$

When $I = [a, a + \langle r \rangle]$, it follows from (4.47) and (4.17) that

$$\mathbb{E}\big[\big(L(x, I) - L(y, I)\big)^n\big] \le c_{4,11}(n)\, |x - y|^{n\gamma}\, r^{n(\beta_\tau - 2H_\tau\gamma)}. \tag{4.48}$$

This proves Lemma 4.7. □

Now we are ready to prove Theorem 4.2.

Proof of Theorem 4.2. It follows from Lemma 4.7 and the multiparameter version of Kolmogorov's continuity theorem [cf. Khoshnevisan (2002)] that, for every fixed interval $I \in \mathcal{A}$ such that $I \subseteq T$, X has almost surely a local time $L(x, I)$ that is continuous for all $x \in \mathbb{R}^d$.

The proof of the joint continuity of the local times is similar to that of Theorem 8.2 in Xiao (2008), which is included for completeness. For all $x, y \in \mathbb{R}^d$, $s, t \in T$ and all even integers $n \ge 1$, we have

$$\begin{aligned}\mathbb{E}\Big[\big(L(x,[\varepsilon,s]) - L(y,[\varepsilon,t])\big)^n\Big] \le 2^{n-1}\Big\{&\mathbb{E}\Big[\big(L(x,[\varepsilon,s]) - L(x,[\varepsilon,t])\big)^n\Big]\\ &+ \mathbb{E}\Big[\big(L(x,[\varepsilon,t]) - L(y,[\varepsilon,t])\big)^n\Big]\Big\}.\end{aligned} \tag{4.49}$$

Since the difference $L(x,[\varepsilon,s]) - L(x,[\varepsilon,t])$ can be written as a sum of finite number (which only depends on N) of terms of the form $L(x, I_j)$, where each $I_j \in \mathcal{A}$ is a closed subinterval of T with at least one edge length $\le |s-t|$. By further splitting these intervals into cubes of sides $\le |s-t|$, we can use Lemma 4.3 to bound the first term in (4.49). On the other hand, the second term in (4.49) can be dealt with using Lemma 4.7 as above. Consequently, for some $\gamma \in (0,1)$ small, the right hand side of (4.49) is bounded by $c_{4,15}(n)\,(|x-y| + |s-t|)^{n\gamma}$, where $n \ge 2$ is an arbitrary even integer. Therefore the joint continuity of the local times $L(x,t)$ follows again from the multiparameter version of Kolmogorov's continuity theorem. This finishes the proof of Theorem 4.2. □

Lemmas 4.3 and 4.7 also provide some information about the modulus of continuity of $L(x,t)$ as a function of x and t. For example, by modifying the proof of Theorem 8.10 in Xiao (2008) one can prove the following Hölder condition for the random measure $L(x,\cdot)$. We leave the details to an interested reader.

Theorem 4.8. *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be an (N, d, α)-stable random field defined by* (1.1) *and we assume X_0 satisfies Conditions (S1) and (S3) on T. Suppose $1 < \alpha < 2$ and $\tau \in \{1, \ldots, N\}$ is the integer so that* (4.12) *holds. Let L be the jointly continuous local time of X. Then, for every $0 < \eta < \beta_\tau$, there is a finite constant $c_{4,16}$ such that with probability one,*

$$\limsup_{r \to 0} \frac{L(x, U(t,r))}{r^\eta} \le c_{4,16} \tag{4.50}$$

holds for $L(x,\cdot)$-almost all $t \in T$. In the above, $U(t,r)$ is the open or closed ball [in the Euclidean metric] centered at t with radius r.

Ayache, Roueff and Xiao (2008) proved that if, in addition to the conditions in Theorem 4.8, X_0 is a linear fractional stable sheet Z^H, then

$$\limsup_{r\to 0} \frac{L(x, U(t,r))}{\varphi(r)} \le c_{4,17} \tag{4.51}$$

holds for $L(x,\cdot)$-almost all $t \in T$. Here $\varphi(r) = r^{\beta_\tau}\ \log\log(1/r)$.

Since the arguments in Ayache, Roueff and Xiao (2008) are based on some special properties of Z^H, it is not clear how to establish such results under the general condition (S2) or (S3). It would be interesting to pursue this problem.

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