# ON LOCAL TIMES OF ANISOTROPIC GAUSSIAN RANDOM FIELDS

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ABSTRACT. The joint continuity of local times is proved for a large class of anisotropic Gaussian random fields. Sharp local and global Hölder conditions for the local times under an anisotropic metric are established. These results are useful for studying sample path and fractal properties of the Gaussian fields.

## 1. Introduction

Gaussian random fields have been studied extensively in probability theory and applied in a wide range of scientific areas including physics, engineering, hydrology, biology, economics and finance. Two of the most important Gaussian random fields are respectively the Brownian sheet and fractional Brownian motion and they have been under active investigation for several decades. In recent years there has been an increased interest in using anisotropic Gaussian random fields as stochastic models in various scientific areas such as image processing, hydrology, geostatistics and spatial statistics, because many data sets from these areas have different geometric and probabilistic characteristics along different directions. See, for example, Davies and Hall (1999), Bonami and Estrade (2003), Benson, et al. (2008). Several classes of anisotropic Gaussian random fields have been introduced and studied for theoretical and application purposes. For instance, Kamont (1996) introduced fractional Brownian sheets and studied some of their regularity properties. Bonami and Estrade (2003), Biermé, Meerschaert and Scheffler (2007), Xue and Xiao (2009) constructed several classes of anisotropic Gaussian random fields with stationary increments and certain operator-scaling properties. Anisotropic Gaussian random fields also arise naturally in stochastic partial differential equations [see, e.g., Dalang (1999), Øksendal and Zhang (2000), Mueller and Tribe (2002), Hu and Nualart (2009)], and as spatial or spatiotemporal models in statistics [e.g., Christakos (2000), Gneiting (2002), Stein (2005)].

It is known that, compared with isotropic Gaussian fields such as fractional Brownian motion, the probabilistic and geometric properties of anisotropic Gaussian random fields are much richer [see Ayache and Xiao (2005), Wu and Xiao

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(2007, 2009a, 2009b), Xiao (2009), Xue and Xiao (2009)] and their estimation and prediction problems are more difficult to study.

This paper is concerned with regularity properties of local times of anisotropic Gaussian random fields. Even though the existence and joint continuity of local times have been established by Xiao and Zhang (2002) and Ayache, Wu and Xiao (2008) for fractional Brownian sheets, and by Xiao (2009) for a class of Gaussian random fields with stationary increments, several interesting problems remain to be resolved. In particular, it is of interest to provide explicit sufficient conditions for the joint continuity of local times of general Gaussian random fields and to determine the Hölder regularities of the local times. This paper is motivated by these two problems and our main results (Theorems 2.1, 3.1 and 3.2) not only improve significantly the results in Ayache, Wu and Xiao (2008) and Xiao (2009) but also provide a unified treatment in applying different forms of strong local nondeterminism to estimate high moments of local times. We believe that this later argument will find applications in studying other sample path properties of anisotropic random fields.

Now let us specify the class of Gaussian random fields to be considered in this paper. Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a Gaussian random field with values in $\mathbb{R}^d$ defined by

$$X(t) = (X_1(t), \ldots, X_d(t)), \tag{1.1}$$

where $X_1, \ldots, X_d$ are independent copies of a real-valued, centered anisotropic Gaussian random field $X_0 = \{X_0(t), t \in \mathbb{R}^N\}$ with $X_0(0) = 0$ a.s.

We call $X$ satisfying Condition A if there exists a constant vector $(H_1, \ldots, H_N)$ with all components in (0, 1) such that the random field $X_0$ satisfies the following Condition A1 and Condition A2:

A1. For any fixed positive number $\varepsilon \in (0,1)$ and any compact interval $T \subseteq [\varepsilon, \infty)^N$, there exist positive constants $\delta_0$, $c_{1,1} \geq 1$ such that

$$c_{1,1}^{-1}\, \rho^2(s,t) \leq \mathbb{E}\big[\big(X_0(t) - X_0(s)\big)^2\big] \leq c_{1,1}\, \rho^2(s,t) \tag{1.2}$$

for all $s, t \in T$ with $|s-t| \leq \delta_0$. In the above and sequel, $\rho$ is the anisotropic metric on $\mathbb{R}^N$ defined by

$$\rho(s,t) = \sum_{j=1}^{N} |s_j - t_j|^{H_j}, \qquad \forall s, t \in \mathbb{R}^N. \tag{1.3}$$

A2. For any fixed positive number $\varepsilon \in (0,1)$ and any compact interval $T \subseteq [\varepsilon, \infty)^N$, there exists a positive constant $c_{1,2}$ such that for all integers $n \geq 1$, and all $u, t^1, \ldots, t^n \in T$, we have

$$\mathrm{Var}\big(X_0(u) \mid X_0(t^1), \ldots, X_0(t^n)\big) \geq c_{1,2} \sum_{j=1}^{N} \min_{0 \leq k \leq n} \big|u_j - t_j^k\big|^{2H_j}, \tag{1.4}$$

where $t^0 = 0$.

We will call the vector $H \in (0,1)^N$ the (generalized) Hurst index of $X_0$. Without loss of generality, we assume that

$$0 < H_1 \leq H_2 \leq \cdots \leq H_N < 1. \tag{1.5}$$

Conditions A1 and A2 are the same as Conditions (C1) and (C3) in Xiao (2009). We note that, Condition A2 is the *weakest* among several forms of strong local nondeterminism in Xiao (2009) and, using the terminology in Khoshnevisan and Xiao (2007), is called the property of sectorial local nondeterminism. Pitt (1978) and Wu and Xiao (2007) proved respectively that multiparameter fractional Brownian motion and fractional Brownian sheets satisfy Condition A. We refer to Xiao (2009), Xue and Xiao (2009) for more examples of anisotropic Gaussian random fields which satisfy Condition A.

The main objective of this paper is to establish joint continuity and sharp Hölder conditions for the local times of Gaussian random fields that satisfy Condition A. This paper is organized as follows. In Section 2, we study joint continuity of the local times of the anisotropic Gaussian random fields satisfying Condition A. We prove that the necessary and sufficient condition for the existence of local times actually implies the joint continuity. The main argument in this section is different from that in Ayache, Wu and Xiao (2008) for fractional Brownian sheets and leads to better moment estimates. Section 3 is devoted to establish sharp local and global Hölder conditions for the local times of the fields. Our Theorems 3.1 and 3.2 show that the Hölder regularities of the local times of an anisotropic Gaussian random field satisfying Condition A can be more subtle than those of the Brownian sheet proved by Ehm (1981) and fractional Brownian motion proved by Xiao (1997) and Baraka, et al. (2009). In Section 4, we apply the regularity results on local times to study fractal properties of the level sets of $X$. Our results show that, due to the anisotropic nature of $X$, it is more convenient to characterize the regularity properties of the local times and the fractal properties of $X$ in terms of the metric $\rho$. Finally, in the Appendix we give several technical lemmas, which are used for proving the main lemmas in Section 2.

Throughout this paper, we use $\langle\cdot,\cdot\rangle$ and $|\cdot|$ to denote the ordinary scalar product and the Euclidean norm in $\mathbb{R}^p$ respectively, no matter what the value of the integer $p$ is. We use $\lambda_p$ to denote the Lebesgue measure in $\mathbb{R}^p$. A "time" index $t \in \mathbb{R}^p$ is written as $(t_1, \ldots, t_p)$, or as $\langle c\rangle$, if $t_1 = \cdots = t_p = c$. For any $s,\, t \in \mathbb{R}^p$ such that $s_j < t_j$ $(j = 1, \ldots, p)$, we define the closed interval (or rectangle) $[s,\, t] = \prod_{j=1}^p [s_j,\, t_j]$. We will use $\mathcal{A}$ to denote the class of all closed intervals $T \subset (0, \infty)^p$.

We will use $c$ to denote an unspecified positive and finite constant which may not be the same in each occurrence. More specific constants in Section $i$ are numbered as $c_{i,1}, c_{i,2}, \ldots$.

## 2. Joint continuity of the local times

In this section, we study the joint continuity of local times of Gaussian random field $X$ satisfying Condition A. We start by recalling the definition of local times and some basic facts from Geman and Horowitz (1980).

Let $Y(t)$ be a Borel vector field on $\mathbb{R}^p$ with values in $\mathbb{R}^q$. For any Borel set $T \subseteq \mathbb{R}^p$, the occupation measure of $Y$ on $T$ is defined as the following measure on $\mathbb{R}^q$:

$$\mu_T(\bullet) = \lambda_p\big\{t \in T : Y(t) \in \bullet\big\}.$$

If $\mu_T$ is absolutely continuous with respect to the Lebesgue measure $\lambda_q$, we say that $Y(t)$ has *local times* on $T$, and define its local time, $L(\bullet, T)$, as the Radon–Nikodým derivative of $\mu_T$ with respect to $\lambda_q$, i.e.,

$$L(x,T) = \frac{d\mu_T}{d\lambda_q}(x), \qquad \forall x \in \mathbb{R}^q.$$

In the above, $x$ is the so-called *space variable*, and $T$ is the *time variable.* Note that if $Y$ has local times on $T$ then for every Borel set $S \subseteq T$, $L(x,S)$ also exists.

By Theorem 6.4 of Geman and Horowitz (1980), one can choose a measurable version of $L(x,T)$ such that it satisfies the following *occupation density formula*: For every Borel set $T \subseteq \mathbb{R}^p$, and for every measurable function $f : \mathbb{R}^q \to \mathbb{R}_+$,

$$\int_T f(Y(t))\,dt = \int_{\mathbb{R}^q} f(x)L(x,T)\,dx. \tag{2.1}$$

Suppose we fix a rectangle $T = \prod_{i=1}^p [a_i, a_i + h_i] \subset \mathbb{R}^p$, where $a \in \mathbb{R}^p$ and $h \in \mathbb{R}^p_+$. If we can choose a version of the local time, still denoted by $L(x, \prod_{i=1}^p [a_i, a_i + t_i])$, such that it is a continuous function of $(x, t_1, \cdots, t_p) \in \mathbb{R}^q \times \prod_{i=1}^p [0, h_i]$, $Y$ is said to have a *jointly continuous local time* on $T$. When a local time is jointly continuous, $L(x, \cdot)$ can be extended to a finite Borel measure supported on the level set

$$Y_T^{-1}(x) = \{t \in T : Y(t) = x\}; \tag{2.2}$$

see Adler (1981) for details. This makes local times, besides of interest on their own right, a useful tool in studying fractal properties of $Y$.

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be an $(N, d)$ Gaussian random field defined by (1.1) with generalized Hurst index $H \in (0, 1)^N$ satisfying Condition A1. Xiao (2009) proved that for all intervals $T \in \mathcal{A}$, $X$ has a local time $\{L(x,T),\, x \in \mathbb{R}^d\}$ on $T$ and $L(\cdot, T) \in L^2(\mathbb{P} \times \lambda_d)$ if and only if $\sum_{\ell=1}^N \frac{1}{H_\ell} > d$. Furthermore, if it exists, the local time of $X$ admits the following $L^2$-representation

$$L(x,T) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-i\langle y,x\rangle} \int_T e^{i\langle y, X(t)\rangle}\,dtdy. \tag{2.3}$$

In the following, we will prove that if $X$ satisfies Condition A, then the existence condition $\sum_{\ell=1}^N \frac{1}{H_\ell} > d$ implies that $X$ has almost surely a jointly continuous local time on $T \in \mathcal{A}$.

**Theorem 2.1.** *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be an $(N, d)$ Gaussian random field defined by (1.1) satisfying Condition A with generalized Hurst index $H \in (0, 1)^N$. If $\sum_{\ell=1}^N \frac{1}{H_\ell} > d$, then for all intervals $T \in \mathcal{A}$, $X$ has almost surely a jointly continuous local time on $T$.*

The proof of Theorem 2.1 is based on high moment estimates for the local times and a multiparameter version of Kolmogorov's continuity theorem. We will make use of the following identities [cf. Geman and Horowitz (1980)]: For all $x, y \in \mathbb{R}^d$, any Borel set $D \subseteq \mathbb{R}^N$ and all integers $n \ge 1$,

$$\mathbb{E}\Big[L(x,D)^n\Big] = (2\pi)^{-nd} \int_{D^n} \int_{\mathbb{R}^{nd}} \exp\bigg(-i\sum_{j=1}^n \langle u^j, x\rangle\bigg) \times \mathbb{E}\exp\bigg(i\sum_{j=1}^n \langle u^j, X(t^j)\rangle\bigg) d\overline{u}\, d\overline{t} \tag{2.4}$$

and for all even integers $n \ge 2$,

$$\mathbb{E}\Big[(L(x,D)-L(y,D))^n\Big] = (2\pi)^{-nd} \int_{D^n} \int_{\mathbb{R}^{nd}} \prod_{j=1}^n \Big[e^{-i\langle u^j, x\rangle} - e^{-i\langle u^j, y\rangle}\Big] \times \mathbb{E}\exp\bigg(i\sum_{j=1}^n \langle u^j, X(t^j)\rangle\bigg) d\overline{u}\, d\overline{t}, \tag{2.5}$$

where $\overline{u} = (u^1, \ldots, u^n)$, $\overline{t} = (t^1, \ldots, t^n)$, and each $u^j \in \mathbb{R}^d$, $t^j \in D$. In the coordinate notation we then write $u^j = (u^j_1, \ldots, u^j_d)$.

To estimate the integrals in (2.4) and (2.5) we will make use of several technical lemmas. The following lemma is due to Cuzick and DuPreez (1982), and the others are given in the Appendix.

**Lemma 2.2.** *Let $Z_1, \ldots, Z_n$ be mean zero Gaussian random variables which are linearly independent. Then for any non-negative measurable function $g : \mathbb{R} \to \mathbb{R}_+$*

$$\int_{\mathbb{R}^n} g(v_1) \exp\Big[-\frac{1}{2}\mathrm{Var}\Big(\sum_{j=1}^n v_j Z_j\Big)\Big] dv_1 \cdots dv_n = \frac{(2\pi)^{(n-1)/2}}{(\det\mathrm{Cov}(Z_1, \cdots, Z_n))^{1/2}} \int_{-\infty}^{\infty} g\Big(\frac{v}{\sigma_1}\Big) e^{-v^2/2} dv,$$

*where $\sigma_1^2 = \mathrm{Var}(Z_1 | Z_2, \ldots, Z_n)$ is the conditional variance of $Z_1$ given $Z_2, \ldots, Z_n$.*

The determinant $\det\mathrm{Cov}(Z_1, \cdots, Z_n)$ can be evaluated by using the following well known expansion: For any Gaussian random vector $(Z_1, \ldots, Z_n)$,

$$\det\mathrm{Cov}(Z_1, \ldots, Z_n) = \mathrm{Var}(Z_1) \prod_{k=2}^n \mathrm{Var}(Z_k | Z_1, \ldots, Z_{k-1}). \tag{2.6}$$

Combined with the property of sectional local nondeterminism, (2.6) gives a lower bound for $\det\mathrm{Cov}(X_0(t^1), \ldots, X_0(t^n))$.

To state Lemma 2.3 and Lemma 2.4, we fix some notation. When $\sum_{\ell=1}^N \frac{1}{H_\ell} > d$, there exists $\tau \in \{1, 2, \ldots, N\}$ such that

$$\sum_{\ell=1}^{\tau-1} \frac{1}{H_\ell} \le d < \sum_{\ell=1}^{\tau} \frac{1}{H_\ell}, \tag{2.7}$$

with the convention that $\sum_1^0 (\cdot) \equiv 0$. Throughout, we denote

$$\alpha := \sum_{\ell=1}^N \frac{1}{H_\ell} - d, \qquad \eta_\tau := \tau + H_\tau d - \sum_{\ell=1}^{\tau} \frac{H_\tau}{H_\ell} \tag{2.8}$$

and we will distinguish three cases:

**Case 1**. $\sum\limits_{\ell=1}^{\tau-1}\frac{1}{H_\ell} < d < \sum\limits_{\ell=1}^{\tau}\frac{1}{H_\ell}$.

**Case 2**. $\sum\limits_{\ell=1}^{\tau-1}\frac{1}{H_\ell} = d < \sum\limits_{\ell=1}^{\tau}\frac{1}{H_\ell}$ and $H_{\tau-1} = H_\tau$.

**Case 3**. $\sum\limits_{\ell=1}^{\tau-1}\frac{1}{H_\ell} = d < \sum\limits_{\ell=1}^{\tau}\frac{1}{H_\ell}$ and $H_{\tau-1} < H_\tau$.

From now on, for any $u \in \mathbb{R}^N$ and $r > 0$, the open and closed $\rho$-balls (in $\mathbb{R}^N$) center at $u$ with radius $R$ are defined by

$$B_\rho(u,R) := \{t \in \mathbb{R}^N :\ \rho(t,u) < R\}, \quad \overline{B}_\rho(u,R) := \{t \in \mathbb{R}^N :\ \rho(t,u) \le R\}.$$

**Lemma 2.3.** *Suppose the assumptions of Theorem 2.1 hold. Then, for every $T \in \mathcal{A}$, there exists a positive and finite constant $c_{_{2,1}}$, which depends on $N$, $d$, $H$ and $T$ only, such that for all $r \in (0, 1/e)$, $D := \overline{B}_\rho(a,r) \subseteq T$, all $x \in \mathbb{R}^d$ and all integers $n \ge 1$, we have*

$$\mathbb{E}\left[L(x,D)^n\right] \le \begin{cases} c_{_{2,1}}^n\,(n!)^{\eta_\tau}\,r^{n\alpha} & \text{in Cases 1 and 2,} \\ c_{_{2,1}}^n\,(n!)^{\eta_\tau}\,\big(\log(e+n)\big)^n\,r^{n\alpha} & \text{in Case 3.} \end{cases} \tag{2.9}$$

*Proof.* Even though the proof of Lemma 2.3 follows the same spirit of the proofs of Lemma 2.5 in Xiao (1997) and Lemma 3.4 in Wu and Xiao (2009a), there are some subtle differences. Hence we give a complete proof. In particular, we provide a direct way to estimate the integrals in (2.12) below.

It follows from Eq. (2.4) and the fact that $X_1, \ldots, X_d$ are independent copies of $X_0$ that for all integers $n \ge 1$,

$$\begin{aligned}\mathbb{E}\big[L(x,D)^n\big] &\le (2\pi)^{-nd}\int_{D^n}\prod_{k=1}^{d}\bigg\{\int_{\mathbb{R}^n}\exp\bigg[-\frac{1}{2}\mathrm{Var}\Big(\sum_{j=1}^{n}u_k^j\,X_0(t^j)\Big)\bigg]\,d\overline{u}_k\bigg\}\,d\overline{t} \\ &= (2\pi)^{-nd/2}\int_{D^n}\Big[\mathrm{detCov}\left(X_0(t^1),\ldots,X_0(t^n)\right)\Big]^{-\frac{d}{2}}\,d\overline{t},\end{aligned} \tag{2.10}$$

where $\overline{u}_k = (u_k^1,\ldots,u_k^n) \in \mathbb{R}^n$, $\overline{t} = (t^1,\ldots,t^n)$ and the equality follows from the fact that for any positive definite $n \times n$ matrix $\Gamma$,

$$\int_{\mathbb{R}^n}\frac{[\det(\Gamma)]^{1/2}}{(2\pi)^{n/2}}\,\exp\Big(-\frac{1}{2}x'\Gamma x\Big)\,dx = 1. \tag{2.11}$$

By using Condition A2, (2.6) and (2.7), we derive from (2.10) that

$$\begin{aligned}\mathbb{E}\big[L(x,D)^n\big] &\le c_{_{2,2}}^n\int_{D^n}\prod_{j=1}^{n}\bigg[\sum_{\ell=1}^{N}\min_{0\le s\le j-1}|t_\ell^j - t_\ell^s|^{2H_\ell}\bigg]^{-\frac{d}{2}}\,d\overline{t} \\ &\le c_{_{2,2}}^n\int_{D^n}\prod_{j=1}^{n}\bigg[\sum_{\ell=1}^{\tau}\min_{0\le s\le j-1}|t_\ell^j - t_\ell^s|^{2H_\ell}\bigg]^{-\frac{d}{2}}\,d\overline{t}.\end{aligned} \tag{2.12}$$

To estimate the last integral in (2.12), we will integrate in the order of $dt_1^n, \ldots, dt_N^n, \ldots, dt_1^1, \ldots, dt_N^1$. In Case 1, if $\tau = 1$, which implies that $H_1 d < 1$, we apply

Part (i) of Lemma 5.3 to derive

$$\begin{aligned}\int_D \frac{dt_1^n\cdots dt_N^n}{\Big(\min\limits_{0\le s\le n-1}|t_1^n-t_1^s|^{2H_1}\Big)^{d/2}} &= (2r)^{\sum_{\ell=2}^N \frac{1}{H_\ell}} \int_{a_1-r^{\frac{1}{H_1}}}^{a_1+r^{\frac{1}{H_1}}} \frac{dt_1^n}{\min\limits_{0\le s\le n-1}|t_1^n-t_1^s|^{H_1 d}}\\ &\le c_{2,3}\, n^{H_1 d}\, r^{\sum_{\ell=1}^N \frac{1}{H_\ell}-d}\\ &= c_{2,3}\, n^{\eta_1}\, r^{\alpha}.\end{aligned} \tag{2.13}$$

If $\tau>1$, since $H_1 d>1$, we apply Part (i) of Lemma 5.2 with $p=1$ and $A=\sum\limits_{\ell=2}^{\tau}\min\limits_{0\le s\le n-1}|t_\ell^n-t_\ell^s|^{2H_\ell}$ at first to derive

$$\begin{aligned}&\int_{a_1-r^{\frac{1}{H_1}}}^{a_1+r^{\frac{1}{H_1}}} \frac{dt_1^n}{\Big(\min\limits_{0\le s\le n-1}|t_1^n-t_1^s|^{2H_1}+\sum\limits_{\ell=2}^{\tau}\min\limits_{0\le s\le n-1}|t_\ell^n-t_\ell^s|^{2H_\ell}\Big)^{d/2}}\\ &\quad\le \frac{c_{2,4}\, n}{\Big(\sum\limits_{\ell=2}^{\tau}\min\limits_{0\le s\le n-1}|t_\ell^n-t_\ell^s|^{H_\ell}\Big)^{d-\frac{1}{H_1}}}.\end{aligned} \tag{2.14}$$

Actually, since $H_{\tau-1}\big(d-\sum_{\ell=1}^{\tau-2}\frac{1}{H_\ell}\big)>1$, we can apply Part (i) of Lemma 5.2 repeatedly for $\tau-1$ many times to get

$$\begin{aligned}&\int_D \frac{dt_1^n\cdots dt_N^n}{\Big(\sum\limits_{\ell=1}^{\tau}\min\limits_{0\le s\le n-1}|t_\ell^n-t_\ell^s|^{2H_\ell}\Big)^{d/2}}\\ &\quad\le c_{2,5}\, n^{\tau-1} r^{\sum_{\ell=\tau+1}^N \frac{1}{H_\ell}} \int_{a_\tau-r^{\frac{1}{H_\tau}}}^{a_\tau+r^{\frac{1}{H_\tau}}} \frac{dt_\tau^n}{\Big(\min\limits_{0\le s\le n-1}|t_\tau^n-t_\tau^s|^{H_\tau}\Big)^{d-\sum_{\ell=1}^{\tau-1}\frac{1}{H_\ell}}}.\end{aligned} \tag{2.15}$$

Notice that $H_\tau\big(d-\sum_{\ell=1}^{\tau-1}\frac{1}{H_\ell}\big)<1$, by applying Part (i) of Lemma 5.3 to the last integral, we derive

$$\begin{aligned}\int_D \frac{dt_1^n\cdots dt_N^n}{\Big(\sum\limits_{\ell=1}^{\tau}\min\limits_{0\le s\le n-1}|t_\ell^n-t_\ell^s|^{2H_\ell}\Big)^{d/2}} &\le c_{2,6}\, n^{\tau-1+H_\tau\left(d-\sum_{\ell=1}^{\tau-1}\frac{1}{H_\ell}\right)}\, r^{\sum_{\ell=1}^N\frac{1}{H_\ell}-d}\\ &= c_{2,6}\, n^{\eta_\tau} r^{\alpha}.\end{aligned} \tag{2.16}$$

By iterating the above procedure for integrating $dt_1^{n-1},\ldots,dt_N^{n-1}$ and so on, we obtain (2.9) for Case 1.

Now we consider Cases 2 and 3. Exactly like what we did for Case 1, we will integrate in the order of $dt_1^n,\ldots,dt_N^n,\ldots,dt_1^1,\ldots,dt_N^1$. This time we can repeatedly apply Part (i) of Lemma 5.2 for $\tau-2$ times (notice that $\tau>1$ in this case) to have

$$\begin{aligned}
&\int_T \frac{dt_1^n \cdots dt_N^n}{\Big(\sum\limits_{\ell=1}^{\tau} \min\limits_{0\le s\le n-1} |t_\ell^n - t_\ell^s|^{2H_\ell}\Big)^{d/2}} \\
&\le \int_{a_\tau - r^{\frac{1}{H_\tau}}}^{a_\tau + r^{\frac{1}{H_\tau}}} \int_{a_{\tau-1} - r^{\frac{1}{H_{\tau-1}}}}^{a_{\tau-1} + r^{\frac{1}{H_{\tau-1}}}} \frac{c_{2,7}\, n^{\tau-2} r^{\sum_{\ell=\tau+1}^N \frac{1}{H_\ell}}\, dt_{\tau-1}^n\, dt_\tau^n}{\Big(\sum_{\ell=\tau-1}^{\tau} \min\limits_{0\le s\le n-1} |t_\ell^n - t_\ell^s|^{H_\ell}\Big)^{d-\sum_{\ell=1}^{\tau-2}\frac{1}{H_\ell}}}.
\end{aligned} \tag{2.17}$$

Let $\kappa \in (0,1)$ be a constant. Notice that $H_{\tau-1}\big(d - \sum_{\ell=1}^{\tau-2} \frac{1}{H_\ell}\big) = 1$, applying Part (ii) of Lemma 5.2 with $A = \min\limits_{0\le s\le n-1} |t_\tau^n - t_\tau^s|^{2H_\tau}$ and then applying Part (ii) of Lemma 5.3, we obtain

$$\begin{aligned}
&\int_{a_\tau - r^{\frac{1}{H_\tau}}}^{a_\tau + r^{\frac{1}{H_\tau}}} \int_{a_{\tau-1} - r^{\frac{1}{H_{\tau-1}}}}^{a_{\tau-1} + r^{\frac{1}{H_{\tau-1}}}} \frac{c_{2,7}\, n^{\tau-2} r^{\sum_{\ell=\tau+1}^N \frac{1}{H_\ell}}\, dt_{\tau-1}^n\, dt_\tau^n}{\Big(\sum_{\ell=\tau-1}^{\tau} \min\limits_{0\le s\le n-1} |t_\ell^n - t_\ell^s|^{H_\ell}\Big)^{d-\sum_{\ell=1}^{\tau-2}\frac{1}{H_\ell}}} \\
&\le c_{2,8}\, n^{\tau-1} r^{\sum_{\ell=\tau+1}^N \frac{1}{H_\ell}} \\
&\quad \times \int_{a_\tau - r^{\frac{1}{H_\tau}}}^{a_\tau + r^{\frac{1}{H_\tau}}} \log\left[e + \left(\big(\min_{0\le s\le n-1} |t_\tau^n - t_\tau^s|^{2H_\tau}\big)^{-\frac{1}{2H_{\tau-1}}} \frac{r^{\frac{1}{H_{\tau-1}}}}{n}\right)^{\kappa}\right] dt_\tau^n \\
&\le c_{2,9}\, n^{\tau-1} r^{\sum_{\ell=\tau}^N \frac{1}{H_\ell}} \log\big(e + n^{H_\tau - H_{\tau-1}}\big),
\end{aligned} \tag{2.18}$$

where we have used the fact that $H_\tau \ge H_{\tau-1}$ and $\kappa \in (0,\,1)$.

By iterating the procedure and integrating $dt_1^{n-1}, \ldots, dt_N^{n-1}, \ldots, dt_1^1, \ldots, dt_N^1$, we obtain that

$$\begin{aligned}
\mathbb{E}\big[L(x,D)^n\big] &\le c_{2,1}^n\, (n!)^{\tau-1}\, r^{n\sum_{\ell=\tau}^N \frac{1}{H_\ell}} \prod_{j=1}^n \log\big(e + j^{H_\tau - H_{\tau-1}}\big) \\
&= c_{2,1}^n\, (n!)^{\eta_\tau}\, r^{n\alpha} \prod_{j=1}^n \log\big(e + j^{H_\tau - H_{\tau-1}}\big),
\end{aligned} \tag{2.19}$$

where we have used the fact that $\eta_\tau = \tau - 1$ and $\alpha = \sum_{\ell=\tau}^N \frac{1}{H_\ell}$ in Cases 2 and 3. If $H_\tau = H_{\tau-1}$ then (2.19) yields (2.9) for Case 2. Finally, we note that in Case 3 (where $H_{\tau-1} < H_\tau$),

$$\prod_{j=1}^n \log\big(e + j^{H_\tau - H_{\tau-1}}\big) \le \prod_{j=1}^n \log(e+j) \le \big(\log(e+n)\big)^n. \tag{2.20}$$

Hence, in Case 3, (2.9) follows from (2.19) and (2.20). This finishes the proof of Lemma 2.3. □

The following lemma estimates the higher moments of the increments of the local times of $X$. Combined with a multiparameter version of Kolmogorov's continuity theorem, it immediately implies the existence of a continuous version of $x \mapsto L(x,D)$.

**Lemma 2.4.** *Suppose the assumptions of Theorem 2.1 hold. Then, there exists a positive constant $c_{2,10}$, depending on $N$, $d$, $H$ and $T$ only, such that for all $r \in (0,\,1)$ small, $D := \overline{B}_\rho(a,r) \subseteq T$, all $x,\, y \in \mathbb{R}^d$ with $|x-y| \le 1$, all even integers $n \ge 1$ and all $\gamma \in (0,\,1)$ small enough, we have*

$$\begin{aligned}&\mathbb{E}\left[\big(L(x,D)-L(y,D)\big)^n\right]\\&\quad\le \begin{cases} c_{2,10}^n\,(n!)^{\eta_\tau+(2H_\tau+1)\gamma}\, r^{n(\alpha-\gamma)} & \text{in Cases 1 and 2,}\\ c_{2,10}^n\,(n!)^{\eta_\tau+(2H_\tau+1)\gamma}\,\big(\log(e+n)\big)^n\, r^{n(\alpha-\gamma)} & \text{in Case 3.}\end{cases}\end{aligned} \tag{2.21}$$

*where $\alpha$ and $\eta_\tau$ are the same as that defined in Eq (2.8).*

*Proof.* Let $\gamma \in (0,1)$ be a small constant whose value will be determined later. Note that by the elementary inequalities $|e^{iu}-1| \le 2^{1-\gamma}|u|^\gamma$ for all $u \in \mathbb{R}$ and $|u+v|^\gamma \le |u|^\gamma + |v|^\gamma$, we see that for all $u^1,\dots,u^n,\ x,\ y \in \mathbb{R}^d$,

$$\prod_{j=1}^n \left|e^{-i\langle u^j,x\rangle} - e^{-i\langle u^j,y\rangle}\right| \le 2^{(1-\gamma)n}\ |x-y|^{n\gamma}\ {\sum}' \prod_{j=1}^n |u^j_{k_j}|^\gamma, \tag{2.22}$$

where the summation $\sum'$ is taken over all the sequences $(k_1,\dots,k_n) \in \{1,\dots,d\}^n$.

It follows from (2.5) and (2.22) that for every even integer $n \ge 2$,

$$\begin{aligned}&\mathbb{E}\Big[(L(x,\,D)-L(y,D))^n\Big]\\&\quad\le (2\pi)^{-nd}2^{(1-\gamma)n}\,|x-y|^{n\gamma}\\&\qquad\times {\sum}' \int_{D^n}\int_{\mathbb{R}^{nd}} \prod_{m=1}^n |u^m_{k_m}|^\gamma\,\mathbb{E}\exp\bigg(-i\sum_{j=1}^n \langle u^j,\,X(t^j)\rangle\bigg)\,d\overline{u}\,d\overline{t}\\&\quad\le c_{2,11}^n|x-y|^{n\gamma}{\sum}'\int_{D^n} d\overline{t}\\&\qquad\times\prod_{m=1}^n\left\{\int_{\mathbb{R}^{nd}}|u^m_{k_m}|^{n\gamma}\,\exp\bigg[-\frac12\mathrm{Var}\bigg(\sum_{j=1}^n\langle u^j,\,X(t^j)\rangle\bigg)\bigg]\,d\overline{u}\right\}^{1/n},\end{aligned} \tag{2.23}$$

where the last inequality follows from the generalized Hölder inequality.

Now we fix a vector $\overline{k} = (k_1,k_2,\dots,k_n) \in \{1,\dots,d\}^n$ and $n$ points $t^1,\dots,t^n \in D$ such that all the coordinates of $t^1,\dots,t^n$ are distinct [the set of such points has full $nN$-dimensional Lebesgue measure]. Let $\mathcal{M} = \mathcal{M}(\overline{k},\overline{t},\gamma)$ be defined by

$$\mathcal{M} = \prod_{m=1}^n\left\{\int_{\mathbb{R}^{nd}}|u^m_{k_m}|^{n\gamma}\,\exp\bigg[-\frac12\mathrm{Var}\bigg(\sum_{j=1}^n\langle u^j,\,X(t^j)\rangle\bigg)\bigg]\,d\overline{u}\right\}^{1/n}. \tag{2.24}$$

Note that $X_\ell$ $(1 \le \ell \le d)$ are independent copies of $X_0$. By Condition A2, the random variables $X_\ell(t^j)$ $(1 \le \ell \le d, 1 \le j \le n)$ are linearly independent. Hence

Lemma 2.2 gives

$$\begin{aligned}
&\int_{\mathbb{R}^{nd}} |u_{k_m}^m|^{n\gamma} \exp\left[-\frac{1}{2}\mathrm{Var}\Big(\sum_{j=1}^n \langle u^j, X(t^j)\rangle\Big)\right] d\overline{u}\\
&\quad= \frac{(2\pi)^{(nd-1)/2}}{\big[\det\mathrm{Cov}\big(X_0(t^1),\ldots,X_0(t^n)\big)\big]^{d/2}} \int_{\mathbb{R}} \Big(\frac{v}{\sigma_m}\Big)^{n\gamma} e^{-\frac{v^2}{2}}\, dv \\
&\quad\le \frac{c_{2,12}^n\,(n!)^\gamma}{\big[\det\mathrm{Cov}\big(X_0(t^1),\ldots,X_0(t^n)\big)\big]^{d/2}}\,\frac{1}{\sigma_m^{n\gamma}},
\end{aligned} \tag{2.25}$$

where $\sigma_m^2$ is the conditional variance of $X_{k_m}(t^m)$ given $X_i(t^j)$ ($i \neq k_m$ or $i = k_m$ but $j \neq m$), and the last inequality follows from Stirling's formula.

Combining (2.24) and (2.25) we obtain

$$\mathcal{M} \le \frac{c_{2,13}^n\,(n!)^\gamma}{\big[\det\mathrm{Cov}\big(X_0(t^1),\ldots,X_0(t^n)\big)\big]^{d/2}} \prod_{m=1}^n \frac{1}{\sigma_m^\gamma}. \tag{2.26}$$

In order to control the second product in (2.26), we use again the independence of the coordinate processes of $X$ and Condition A2 to derive

$$\sigma_m^2 = \mathrm{Var}\Big(X_{k_m}(t^m)\Big|X_{k_m}(t^j),\ j\neq m\Big) \ge c_{2,14}\sum_{\ell=1}^N \min_{j\neq m}\big\{|t_\ell^m - t_\ell^j|^{2H_\ell}\big\}. \tag{2.27}$$

For any $n$ points $\{t^1,\ldots,t^n\} \in D^n$, we define a permutation $\pi_\tau$ of $\{1,2,\ldots,n\}$ such that

$$t_\tau^{\pi_\tau(1)} \le \cdots \le t_\tau^{\pi_\tau(n)}. \tag{2.28}$$

Then, by (2.27), we have

$$\begin{aligned}
\prod_{m=1}^n \frac{1}{\sigma_m^\gamma} &\le \prod_{m=1}^n \frac{1}{c_{2,15}\,\big[|t_\tau^{\pi_\tau(m)} - t_\tau^{\pi_\tau(m-1)}|^{H_\tau} \wedge |t_\tau^{\pi_\tau(m+1)} - t_\tau^{\pi_\tau(m)}|^{H_\tau}\big]^\gamma}\\
&\le c_{2,15}^{-n}\prod_{m=1}^n \frac{1}{|t_\tau^{\pi_\tau(m)} - t_\tau^{\pi_\tau(m-1)}|^{q_m H_\tau\gamma}},
\end{aligned} \tag{2.29}$$

where $q_m \in \{0,\,1,\,2\}$ satisfy that $\sum_{m=1}^n q_m = n$.

It follows from (2.26), (2.6) and (2.29) that

$$\begin{aligned}
\mathcal{M} &\le \frac{c_{2,12}^n\,(n!)^\gamma}{\big[\det\mathrm{Cov}\big(X_0(t^1),\ldots,X_0(t^n)\big)\big]^{d/2}} \prod_{m=1}^n \frac{1}{\sigma_m^\gamma}\\
&= \frac{c_{2,12}^n\,(n!)^\gamma}{\big[\det\mathrm{Cov}\big(X_0(t^{\pi_\tau(1)}),\ldots,X_0(t^{\pi_\tau(n)})\big)\big]^{d/2}} \prod_{m=1}^n \frac{1}{\sigma_m^\gamma}\\
&\le \frac{c_{2,16}^n\,(n!)^\gamma}{\prod_{j=1}^n \Big[\sum_{\ell=1}^\tau \min_{0\le s\le j-1} |t_\ell^{\pi_\tau(j)} - t_\ell^{\pi_\tau(s)}|^{H_\ell}\Big]^d} \prod_{m=1}^n \frac{1}{|t_\tau^{\pi_\tau(m)} - t_\tau^{\pi_\tau(m-1)}|^{q_m H_\tau \gamma}}.
\end{aligned} \tag{2.30}$$

As in the proof of Eq. (2.9), we will prove Eq. (2.21) by cases. we will integrate in the order of $dt_1^{\pi_\tau(n)}, \ldots, dt_N^{\pi_\tau(n)}, \ldots, dt_1^{\pi_\tau(1)}, \ldots, dt_N^{\pi_\tau(1)}$. In Case 1, after applying Part (i) of Lemma 5.2 (with $p = 1$) $\tau - 1$ many times, we have

$$\begin{aligned} &\int_D \frac{dt_1^{\pi_\tau(n)} \cdots dt_N^{\pi_\tau(n)}}{\Big[\sum_{\ell=1}^{\tau} \min_{0\le s\le n-1} |t_\ell^{\pi_\tau(n)} - t_\ell^{\pi_\tau(s)}|^{H_\ell}\Big]^d \big|t_\tau^{\pi_\tau(n)} - t_\tau^{\pi_\tau(n-1)}\big|^{q_n H_\tau \gamma}} \\ &\quad \le \int_{a_\tau - r^{\frac{1}{H_\tau}}}^{a_\tau + r^{\frac{1}{H_\tau}}} \frac{c_{2,17}\, n^{\tau-1} r^{\sum_{\ell=\tau+1}^N \frac{1}{H_\ell}}\, dt_\tau^{\pi_\tau(n)}}{\min_{0\le s\le n-1} \big|t_\tau^{\pi_\tau(n)} - t_\tau^{\pi_\tau(s)}\big|^{H_\tau\left(d - \sum_{\ell=1}^{\tau-1} \frac{1}{H_\ell} + q_n\gamma\right)}}. \end{aligned} \tag{2.31}$$

For $\gamma > 0$ sufficiently small so that $H_\tau\big(d - \sum_{\ell=1}^{\tau-1} \frac{1}{H_\ell} + 2\gamma\big) < 1$, we apply Part (i) of Lemma 5.3 to the last integral to obtain that

$$\begin{aligned} &\int_D \frac{dt_1^{\pi_\tau(n)} \cdots dt_N^{\pi_\tau(n)}}{\Big[\sum_{\ell=1}^{\tau} \min_{0\le s\le n-1} \big|t_\ell^{\pi_\tau(n)} - t_\ell^{\pi_\tau(s)}\big|^{H_\ell}\Big]^d \big|t_\tau^{\pi_\tau(n)} - t_\tau^{\pi_\tau(n-1)}\big|^{q_n H_\tau \gamma}} \\ &\quad \le c_{2,18}\, n^{\tau-1+H_\tau\left(d-\sum_{\ell=1}^{\tau-1}\frac{1}{H_\ell}\right)+q_n H_\tau\gamma}\, r^{\sum_{\ell=1}^N \frac{1}{H_\ell} - d - q_n\gamma} \\ &\quad \le c_{2,18} n^{\eta_\tau + 2H_\tau\gamma} r^{\alpha - q_n\gamma}, \end{aligned} \tag{2.32}$$

where, in deriving the last inequality, we have used the fact that $q_n \le 2$.

Repeating the above procedure yields

$$\begin{aligned} \int_{D^n} \mathcal{M}(\bar{k}, \bar{t}, \gamma)\, d\bar{t} &\le c_{2,19}^n\, (n!)^{\eta_\tau + (2H_\tau+1)\gamma} r^{n\alpha - \sum_{m=1}^n q_m\gamma} \\ &= c_{2,19}^n\, (n!)^{\eta_\tau + (2H_\tau+1)\gamma} r^{n(\alpha-\gamma)}. \end{aligned} \tag{2.33}$$

Combining Eq. (2.23) with Eq. (2.33), we prove (2.21) in Case 1.

For Cases 2 and 3, we can repeatedly apply Part (i) of Lemma 5.2 for $\tau - 2$ times (notice that $\tau > 1$ in this case) to derive

$$\begin{aligned} &\int_D \frac{dt_1^{\pi_\tau(n)} \cdots dt_N^{\pi_\tau(n)}}{\Big[\sum_{\ell=1}^{\tau} \min_{0\le s\le n-1} |t_\ell^{\pi_\tau(n)} - t_\ell^{\pi_\tau(s)}|^{H_\ell}\Big]^d \big|t_\tau^{\pi_\tau(n)} - t_\tau^{\pi_\tau(n-1)}\big|^{q_n H_\tau \gamma}} \\ &\le c_{2,20}\, n^{\tau-2} r^{\sum_{\ell=\tau+1}^N \frac{1}{H_\ell}} \int_{a_\tau - r^{\frac{1}{H_\tau}}}^{a_\tau + r^{\frac{1}{H_\tau}}} dt_\tau^{\pi_\tau(n)} \\ &\quad \int_{a_{\tau-1} - r^{\frac{1}{H_{\tau-1}}}}^{a_{\tau-1} + r^{\frac{1}{H_{\tau-1}}}} \frac{|t_\tau^{\pi_\tau(n)} - t_\tau^{\pi_\tau(n-1)}|^{-q_n H_\tau\gamma} dt_{\tau-1}^{\pi_\tau(n)}}{\Big[\sum_{\ell=\tau-1}^{\tau} \min_{0\le s\le n-1} |t_\ell^{\pi_\tau(n)} - t_\ell^{\pi_\tau(s)}|^{H_\ell}\Big]^{d - \sum_{\ell=1}^{\tau-2}\frac{1}{H_\ell}}}. \end{aligned} \tag{2.34}$$

Let $\kappa \in (0, 1)$ be a constant and $\gamma \in (0, 1)$ small such that $2H_\tau\gamma < 1$. Notice that $H_{\tau-1}\big(d - \sum_{\ell=1}^{\tau-2} \frac{1}{H_\ell}\big) = 1$, applying Part (ii) of Lemma 5.2 with

$$A = \min_{0\le s\le n-1} |t_\tau^n - t_\tau^s|^{2H_\tau}$$

and then applying Lemma 5.4, we have

$$\begin{aligned}
&\int_D \frac{dt_1^{\pi_\tau(n)} \cdots dt_N^{\pi_\tau(n)}}{\Big[\sum_{\ell=1}^{\tau} \min_{0\le s\le n-1} |t_\ell^{\pi_\tau(n)} - t_\ell^{\pi_\tau(s)}|^{H_\ell}\Big]^d |t_\tau^{\pi_\tau(n)} - t_\tau^{\pi_\tau(n-1)}|^{q_n H_\tau \gamma}} \\
&\le \int_{a_\tau - r^{\frac{1}{H_\tau}}}^{a_\tau + r^{\frac{1}{H_\tau}}} \frac{c_{2,21}\, n^{\tau-1} r^{\sum_{\ell=\tau+1}^N \frac{1}{H_\ell}} |t_\tau^{\pi_\tau(n)} - t_\tau^{\pi_\tau(n-1)}|^{-q_n H_\tau \gamma}\, dt_\tau^{\pi_\tau(n)}}{\log\Big[e + \Big(\big(\min_{0\le s\le n-1} |t_\tau^{\pi_\tau(n)} - t_\tau^{\pi_\tau(s)}|^{H_\tau}\big)^{-\frac{1}{H_{\tau-1}}} \frac{r}{n}\Big)^{\kappa}\Big]} \qquad (2.35)\\
&\le c_{2,22}\, n^{\tau-1+q_n H_\tau \gamma}\, r^{\sum_{\ell=\tau}^N \frac{1}{H_\ell} - q_n\gamma} \log\big[e + n^{H_\tau - H_{\tau-1}}\big] \\
&\le c_{2,22}\, n^{\tau-1+2H_\tau \gamma}\, r^{\sum_{\ell=\tau}^N \frac{1}{H_\ell} - q_n\gamma} \log\big[e + n^{H_\tau - H_{\tau-1}}\big],
\end{aligned}$$

where we have used the fact that $q_n \le 2$, $H_\tau \ge H_{\tau-1}$ and $\kappa \in (0,1)$.

Repeating the above procedure yields

$$\begin{aligned}
\int_{D^n} \mathcal{M}(\overline{k}, \overline{t}, \gamma)\, d\overline{t} &\le c_{2,23}^n\, (n!)^{\eta_\tau + (2H_\tau+1)\gamma} r^{n\alpha - \sum_{m=1}^n q_m \gamma} \prod_{j=1}^n \log\big[e + j^{H_\tau - H_{\tau-1}}\big] \\
&= c_{2,23}^n\, (n!)^{\eta_\tau + (2H_\tau+1)\gamma} r^{n(\alpha-\gamma)} \prod_{j=1}^n \log\big[e + j^{H_\tau - H_{\tau-1}}\big].
\end{aligned} \qquad (2.36)$$

Combining Eq. (2.23) with Eq. (2.36) and (2.20), we prove (2.21) in Cases 2 and 3. This finishes the proof of Lemma 2.4. □

Now we are ready to prove Theorem 2.1.

*Proof.* The joint continuity of the local time of $X$ follows from the moment estimates in Lemmas 2.3, 2.4 and a multiparameter version of Kolmogorov's continuity theorem [cf. Khoshnevisan (2002)]. Since the proof is similar to that of Theorem 3.1 in Ayache, Wu and Xiao (2008) [see also the proof of Theorem 8.2 in Xiao (2009)], we omit the details. □

We end this section with the following two technical lemmas, which will be useful in the next section.

**Lemma 2.5.** *Under the conditions of Lemma 2.3, there exist positive and finite constants $c_{2,24}$ and $c_{2,25}$, depending on $N$, $d$, $H$ and $T$ only, such that the following hold:*
*(i). For all $D = \overline{B}_\rho(a,r) \subseteq T$ with $r \in (0,1)$, $x \in \mathbb{R}^d$ and all integers $n \ge 1$,*

$$\mathbb{E}\Big[L\big(x + X(a), D\big)^n\Big] \le \begin{cases} c_{2,24}^n\, (n!)^{\eta_\tau}\, r^{n\alpha} & \text{in Cases 1 and 2,} \\ c_{2,24}^n\, (n!)^{\eta_\tau} \big(\log(e+n)\big)^n\, r^{n\alpha} & \text{in Case 3.} \end{cases} \qquad (2.37)$$

*where $\eta_\tau$ and $\alpha$ are defined in (2.8).*

*(ii). For all $D = \overline{B}_\rho(a,r) \subseteq T$ with $r \in (0,1)$, $x$, $y \in \mathbb{R}^d$ with $|x-y| \le 1$, all even integers $n \ge 1$ and all $\gamma \in \big(0, 1 \wedge \frac{\alpha}{2}\big)$,*

$$\begin{aligned}&\mathbb{E}\Big[\big(L(x+X(a),D) - L(y+X(a),D)\big)^n\Big]\\ &\le \begin{cases} c_{2,25}^n\,(n!)^{\eta_\tau+(2H_\tau+1)\gamma}\, r^{n(\alpha-\gamma)} & \text{in Cases 1 and 2,}\\ c_{2,25}^n\,(n!)^{\eta_\tau+(2H_\tau+1)\gamma}\,\big(\log(e+n)\big)^n\, r^{n(\alpha-\gamma)} & \text{in Case 3.}\end{cases}\end{aligned} \tag{2.38}$$

*Proof.* For each fixed $a \in T$, we define the Gaussian random field $Y = \{Y(t), t \in \mathbb{R}^N_+\}$ with values in $\mathbb{R}^d$ by $Y(t) = X(t) - X(a)$. It follows from (2.1) that if $X$ has a local time $L(x,S)$ on any Borel set $S$, then $Y$ also has a local time $\tilde{L}(x,S)$ on $S$ and, moreover, $L(x+X(a),S) = \tilde{L}(x,S)$. It can be verified that the random field $Y$ satisfies Condition A. By applying Lemmas 2.3 and 2.4 to the Gaussian field $Y$, we derive that both (2.37) and (2.38) hold. □

The following lemma is a consequence of Lemma 2.5 and Chebyshev's inequality. The proof is standard, hence omitted.

**Lemma 2.6.** *Assume the conditions of Lemma 2.3 hold. For any constants $b_1 > 0$ and $b_2 > 0$, there exist positive constants $c_{2,26}$ and $c_{2,27}$ (depending on $N$, $d$, $H$ and $T$ only) such that the following hold:*
*(i) For all $D = \overline{B}_\rho(a,r) \subseteq T$ with radius $r \in (0,1)$, $x \in \mathbb{R}^d$ and $u > 1$, we have*

$$\mathbb{P}\Big\{L\big(x+X(a),\, D\big) \ge\, c_{2,26}\, r^\alpha\, u^{\eta_\tau}\Big\} \le \exp\big(-b_1\, u\big) \tag{2.39}$$

*in Cases 1 and 2; and*

$$\mathbb{P}\Big\{L\big(x+X(a),\, D\big) \ge\, c_{2,26}\, r^\alpha\, u^{\eta_\tau}\log(e+u)\Big\} \le \exp\big(-b_1\, u\big) \tag{2.40}$$

*in Case 3.*
*(ii) For all $D = \overline{B}_\rho(a,r) \subseteq T$ with $r \in (0,1)$, $|x-y| \le 1$ and $\gamma \in \Big(0, 1 \wedge \frac{1}{2}\big(\sum_{\ell=1}^{\tau} \frac{1}{H_\ell} - d\big)\Big)$,*

$$\begin{aligned}\mathbb{P}\Big\{&\Big|L\big(x+X(a),\, D\big) - L\big(y+X(a),\, D\big)\Big|\\ &\qquad \ge\, c_{2,27}\, |x-y|^\gamma\, r^{\alpha-\gamma}\, u^{\eta_\tau+(1+2H_\tau)\gamma}\Big\} \le \exp\big(-b_2\, u\big)\end{aligned} \tag{2.41}$$

*in Cases 1 and 2; and*

$$\begin{aligned}\mathbb{P}\Big\{&\Big|L\big(x+X(a),\, D\big) - L\big(y+X(a),\, D\big)\Big|\\ &\qquad \ge\, c_{2,27}\, |x-y|^\gamma\, r^{\alpha-\gamma}\, u^{\eta_\tau+(1+2H_\tau)\gamma}\log(e+u)\Big\} \le \exp\big(-b_2\, u\big)\end{aligned} \tag{2.42}$$

*in Case 3.*

We conclude this section by the exponential integrability of $L(x,D)$, which is a direct consequence of Lemma 2.3. We omit its proof here.

**Theorem 2.7.** *Assume the conditions of Lemma 2.3 hold and $D_1 = \overline{B}_\rho(a,1) \subset T$. Then there exists a positive constant $\delta$, depending on $N$, $d$, $H$ and $T$ only, such that the following hold:*

(i). *In Cases 1 and 2, $\mathbb{E}\big(e^{\delta L(x,D_1)^{\eta_\tau^{-1}}}\big) < \infty$ for every $x \in \mathbb{R}^d$.*
(ii). *In Case 3, $\mathbb{E}\big(e^{\delta\psi(L(x,D_1))}\big) < \infty$ for every $x \in \mathbb{R}^d$, where*

$$\psi(x) = \big(x/\log x\big)^{1/(\tau-1)}.$$

## 3. Hölder conditions for the local times

In this section we investigate the local and uniform asymptotic behavior of the local time $L(x,D)$ at $x$ and the maximum local time $L^*(D) = \max_{x\in\mathbb{R}^d} L(x,D)$ as $r \to 0$, where $D = \overline{B}_\rho(a,r) \subset T$. The results are then applied to study the sample path properties of $X$.

**3.1. Hölder Conditions for $L(x,\bullet)$.** By applying Lemma 2.6 and the Borel-Cantelli lemma, one can easily derive the following law of the iterated logarithm for the local time $L(x,\cdot)$: There exists a positive constant $c_{3,1}$ such that for every $x \in \mathbb{R}^d$ and $t \in [\varepsilon,\infty)^N$,

$$\begin{aligned}
&\limsup_{r\to 0} \frac{L(x,\overline{B}_\rho(t,r))}{\varphi_1(r)} \le c_{3,1} \qquad \text{in Cases 1 and 2,}\\
&\limsup_{r\to 0} \frac{L(x,\overline{B}_\rho(t,r))}{\varphi_2(r)} \le c_{3,1} \qquad \text{in Case 3,}
\end{aligned} \tag{3.1}$$

where

$$\begin{aligned}
\varphi_1(r) &= r^\alpha \,\big(\log\log(1/r)\big)^{\eta_\tau},\\
\varphi_2(r) &= r^\alpha \,\big(\log\log(1/r)\big)^{\eta_\tau} \log\log\log(1/r)
\end{aligned} \tag{3.2}$$

for $r > 0$ small enough. We believe the rate function $\varphi_1(r)$ is sharp in Cases 1 and 2, but the question in Case 3 seems to be more subtle. In the special case of fractional Brownian motion, laws of the iterated logarithm for the local times have been obtained recently by Baraka and Mountford (2008), Baraka, et al. (2009), and Chen et al. (2009).

It follows from Fubini's theorem that, with probability one, (3.1) holds for $\lambda_N$-almost all $t \in [\varepsilon,\infty)^N$. Now we prove a stronger version of this result, which is useful in determining the exact $\rho$-Hausdorff measure of the level set $X^{-1}(x) = \{t \in \mathbb{R}^N : X(t) = x\}$.

**Theorem 3.1.** *For any fixed $x \in \mathbb{R}^d$, let $L(x,\cdot)$ be the local time of $X$ at $x$ which is a random measure supported on $\Gamma_x$. Then there exists a positive and finite constant $c_{3,2}$ independent of $x$ such that with probability 1,*

$$\begin{aligned}
&\limsup_{r\to 0} \frac{L(x,\overline{B}_\rho(t,r))}{\varphi_1(r)} \le c_{3,2} \qquad \textit{in Cases 1 and 2,}\\
&\limsup_{r\to 0} \frac{L(x,\overline{B}_\rho(t,r))}{\varphi_2(r)} \le c_{3,2} \qquad \textit{in Case 3}
\end{aligned} \tag{3.3}$$

*for $L(x,\cdot)$-almost all $t \in T$, where $\varphi_1(r)$ and $\varphi_2(r)$ are defined in (3.2).*

*Proof.* For any $x \in \mathbb{R}^d$ and every integer $k > 0$, we consider the random measure $L_k(x, \bullet)$ on the Borel subsets $C$ of $T$ defined by

$$\begin{aligned} L_k(x, C) &= \int_C (2\pi k)^{d/2} \exp\Big( - \frac{k\,|X(t) - x|^2}{2}\Big)\, dt \\ &= \int_C \int_{\mathbb{R}^d} \exp\Big( - \frac{|u|^2}{2k} + i\langle u, X(t) - x\rangle\Big) du\, dt. \end{aligned} \tag{3.4}$$

Then, by the occupation density formula (2.1) and the continuity of the function $y \mapsto L(y, C)$, one can verify that almost surely $L_k(x, C) \to L(x, C)$ as $k \to \infty$ for every Borel set $C \subset B$.

For every integer $m \geq 1$, denote $f_m(t) = L\big(x, \overline{B}_\rho(t, 2^{-m})\big)$. From the proof of Theorem 2.1 we can see that almost surely the functions $f_m(t)$ are continuous and bounded. Hence we have almost surely, for all integers $m$, $n \geq 1$,

$$\int_T [f_m(t)]^n\ L(x, dt) = \lim_{k\to\infty} \int_T [f_m(t)]^n\ L_k(x, dt). \tag{3.5}$$

It follows from (3.5), (3.4) and the proof of Proposition 3.1 of Pitt (1978) that for every positive integer $n \geq 1$,

$$\begin{aligned} \mathbb{E} \int_T [f_m(t)]^n\ L(x, dt) = \Big(\frac{1}{2\pi}\Big)^{(n+1)d} \int_T \int_{\overline{B}_\rho(t, 2^{-m})^n} &\int_{\mathbb{R}^{(n+1)d}} \exp\Big( - i \sum_{j=1}^{n+1} \langle x, u^j\rangle\Big) \\ &\times \mathbb{E} \exp\Big( i \sum_{j=1}^{n+1} \langle u^j, X(s^j)\rangle\Big)\, d\overline{u} d\overline{s}, \end{aligned} \tag{3.6}$$

where $\overline{u} = (u^1, \ldots, u^{n+1}) \in \mathbb{R}^{(n+1)d}$ and $\overline{s} = (t, s^1, \ldots, s^n)$. Similar to the proof of (2.9) we have that the right hand side of Eq. (3.6) is at most

$$\begin{aligned} & c_{3,3}^n \int_T \int_{\overline{B}_\rho(t, 2^{-m})^n} \frac{d\overline{s}}{\Big[\mathrm{detCov}\big(X_0(t), X_0(s^1), \ldots, X_0(s^n)\big)\Big]^{d/2}} \\ & \leq \begin{cases} c_{3,4}^n\ (n!)^{\eta_\tau}\, 2^{-nm\alpha} & \text{in Cases 1 and 2,} \\ c_{3,4}^n\ (n!)^{\eta_\tau}\, (\log n)^n\, 2^{-nm\alpha} & \text{in Cases 3,} \end{cases}. \end{aligned} \tag{3.7}$$

where $c_{3,4}$ is a positive finite constant depending on $N$, $d$, $H$, and $T$ only.

In Cases 1 and 2, let $\gamma > 0$ be a constant whose value will be determined later. We consider the random set

$$B_m(\omega) = \big\{ t \in T : \, f_m(t) \geq \gamma \varphi_1(2^{-m}) \big\}\,.$$

Denote by $\mu_\omega$ the restriction of the random measure $L(x, \cdot)$ on $T$, that is, $\mu_\omega(E) = L(x,\, E \cap T)$ for every Borel set $E \subset \mathbb{R}^N_+$. Now we take $n = \lfloor \log m \rfloor$, where $\lfloor y \rfloor$ denotes the integer part of $y$. Then by applying (3.7) and by Stirling's formula, we have

$$\begin{aligned}\mathbb{E}\mu_\omega(B_m) &\le \frac{\mathbb{E}\int_T [f_m(t)]^n\, L(x,dt)}{[\gamma\varphi_1(2^{-m})]^n}\\ &\le \frac{c_{3,4}^n\,(n!)^{\eta_\tau}\,2^{-mn\,\alpha}}{\gamma^n 2^{-mn\,\alpha}(\log m)^{n\eta_\tau}} \le m^{-2},\end{aligned} \tag{3.8}$$

provided $\gamma > 0$ is chosen large enough, say, $\gamma \ge c_{3,4}\, e^2 := c_{3,2}$. This implies that

$$\mathbb{E}\left(\sum_{m=1}^{\infty} \mu_\omega(B_m)\right) < \infty.$$

Therefore, with probability 1 for $\mu_\omega$ almost all $t \in T$, we have

$$\limsup_{m\to\infty} \frac{L(x, \overline{B}_\rho(t, 2^{-m}))}{\varphi_1(2^{-m})} \le c_{3,2}. \tag{3.9}$$

For any $r > 0$ small enough, there exists an integer $m$ such that $2^{-m} \le r < 2^{-m+1}$ and (3.9) is applicable. Since $\varphi_1(r)$ is increasing near $r = 0$, (3.3) for Cases 1 and 2 follows from (3.9) and a monotonicity argument.

The proof of Case 3 is almost identical to the above, therefore it is omitted here. □

**3.2. Hölder Conditions for $L^*(\bullet)$.** The following theorem establishes sharp Hölder conditions for the maximum local times $L^*(D) = \sup_{x\in\mathbb{R}^d} L(x, D)$ of $X$ as the radius of the $\rho$-ball $D$ approaches to 0 under $\rho$-metric.

**Theorem 3.2.** *Let $X = \{X(t),\ t \in \mathbb{R}^N\}$ be an anisotropic Gaussian random field in $\mathbb{R}^d$ satisfying Condition A. Then for every $T \in \mathcal{A}$ there exist positive constants $c_{3,5}$ and $c_{3,6}$ such that for every $a \in T$,*

$$\begin{aligned}&\limsup_{r\to0} \frac{L^*(\overline{B}_\rho(a,r))}{\varphi_1(r)} \le c_{3,5}, \quad \text{a.s.} \quad \textit{in Cases 1 and 2,}\\ &\limsup_{r\to0} \frac{L^*(\overline{B}_\rho(a,r))}{\varphi_2(r)} \le c_{3,5}, \quad \text{a.s.} \quad \textit{in Case 3,}\end{aligned} \tag{3.10}$$

*and*

$$\begin{aligned}&\limsup_{r\to0}\sup_{a\in T} \frac{L^*(\overline{B}_\rho(a,r))}{\Phi_1(r)} \le c_{3,6}, \quad \text{a.s.} \quad \textit{in Cases 1 and 2,}\\ &\limsup_{r\to0}\sup_{a\in T} \frac{L^*(\overline{B}_\rho(a,r))}{\Phi_2(r)} \le c_{3,6}, \quad \text{a.s.} \quad \textit{in Case 3,}\end{aligned} \tag{3.11}$$

*where*

$$\begin{aligned}\Phi_1(r) &= r^\alpha \big(\log(1/r)\big)^{\eta_\tau},\\ \Phi_2(r) &= r^\alpha \big(\log(1/r)\big)^{\eta_\tau} \log\log\big(1/r\big).\end{aligned} \tag{3.12}$$

For proving Theorem 3.2, we will make use of the following lemma, which is a consequence of Lemma 2.1 in Talagrand (1995) and Condition A1.

**Lemma 3.3.** *There exist positive constants $c_{3,7}$ and $c_{3,8}$ such that for all $D = \overline{B}_\rho(a,h)$ with $h \in (0,1)$ and all $u > c_{3,7}h$, we have*

$$\mathbb{P}\Big\{ \sup_{t\in D} \big|X(t)-X(a)\big| \ge u \Big\} \le \exp\bigg( - c_{3,8}\, \Big(\frac{u}{h}\Big)^2 \bigg). \tag{3.13}$$

*Proof.* As in Ehm (1981) and Xiao (1997), the proof of Theorem 3.2 is based on Lemma 2.6 and a chaining argument. Hence we will only sketch a proof of (3.10) for Cases 1 and 2, indicating the necessary modifications.

Recall that $\varphi_1(r) = r^\alpha \big(\log\log r^{-1}\big)^{\eta_\tau}$ for $r \in (0, 1/e)$. In order to prove (3.10) it is sufficient to show that for every $a \in T$,

$$\limsup_{n\to\infty} \frac{L^*(C_n)}{\varphi_1(2^{-n})} \le c_{3,9}, \qquad \text{a.s.}, \tag{3.14}$$

where $C_n = \overline{B}_\rho(a, 2^{-n})$ for $n \ge 1$.

We divide the proof of (3.14) into four steps.

(a) Pick $u = 2^{-n}\sqrt{2c_{3,10}^{-1}\log n}$ in Lemma 3.3, we have

$$\mathbb{P}\left\{ \sup_{t\in C_n} \big|X(t)-X(a)\big| \ge 2^{-n}\sqrt{2c_{3,10}^{-1}\log n} \right\} \le \exp(-2\log n) = n^{-2}. \tag{3.15}$$

Hence the Borel-Cantelli lemma implies that a.s. $\exists\, n_1 = n_1(\omega)$ such that

$$\sup_{t\in C_n} \big|X(t)-X(a)\big| \le 2^{-n}\sqrt{2c_{3,10}^{-1}\log n}, \qquad \text{for all } \ n \ge n_1. \tag{3.16}$$

(b) Let $\theta_n = 2^{-n}\big(\log\log 2^n\big)^{-(1+2H_\tau)}$ for all $n \ge 1$, and define

$$G_n = \left\{ x \in \mathbb{R}^d : \, |x| \le 2^{-n}\sqrt{2\, c_{3,10}^{-1}\log n} \ \text{ with } \ x = \theta_n p \text{ for some } \ p \in \mathbb{Z}^d \right\}.$$

Then, at least when $n$ is large enough, the cardinality of $G_n$ satisfies

$$\sharp G_n \le c_{3,11}\, (\log n)^{(2+2H_\tau)d}. \tag{3.17}$$

It follows from (2.39) (take $b_1 = 2$) that we can choose a constant $c > 0$ such that for all integer $n$ large enough,

$$\mathbb{P}\left\{ \max_{x\in G_n} L\left(x + X(a),\, C_n\right) \ge c^{\eta_\tau}\, \varphi_1(2^{-n}) \right\} \le c_{3,11}\, \big(\log n\big)^{(2+2H_\tau)d} n^{-2}. \tag{3.18}$$

Since the right hand side of (3.18) is summable, the Borel-Cantelli lemma implies that almost surely $\exists\, n_2 = n_2(\omega)$ such that

$$\max_{x\in G_n} L\left(x + X(a), C_n\right) \le c^{\eta_\tau}\, \varphi_1(2^{-n}), \qquad \text{for all } n \ge n_2. \tag{3.19}$$

(c) Given integers $n,\, k \ge 1$ and $x \in G_n$, we define

$$F(n,k,x) = \bigg\{ y \in \mathbb{R}^d : \, y = x + \theta_n \sum_{j=1}^k \varepsilon_j\, 2^{-j}, \ \varepsilon_j \in \{0,1\}^d \ \text{ for } \ 1 \le j \le k \bigg\}.$$

A pair of points $y_1,\, y_2 \in F(n,k,x)$ is said to be linked if $y_2 - y_1 = \theta_n \varepsilon 2^{-k}$ for some $\varepsilon \in \{0,1\}^d$. We choose $\gamma > 0$ small such that (2.41) in Lemma 2.6 holds. Consider the event $F_n$ defined by

$$\begin{aligned} F_n = \bigcup_{x\in G_n} \bigcup_{k=1}^{\infty} \bigcup_{y_1,y_2} \Big\{ & \big|L(y_1 + X(a),\, C_n) - L(y_2 + X(a),\, C_n)\big| \\ & \geq 2^{-n(\alpha-\gamma)}\, |y_1 - y_2|^{\gamma} \big(c\, k \log n\big)^{\eta_\tau + (1+2H_\tau)\gamma} \Big\}, \end{aligned} \tag{3.20}$$

where $\bigcup_{y_1,y_2}$ signifies the union over all the linked pairs $y_1,\, y_2 \in F(n,k,x)$ [note that there are at most $2^{kd}3^d$ linked pairs in $F(n,k,x)$] and where $c > 0$ is a constant such that (2.41) holds with $b_2 = 2$.

Consequently we derive that for all $n > 2^d$ ,

$$\begin{aligned} \mathbb{P}\{F_n\} &\leq c_{3,12} \big(\log n\big)^{(2+2H_\tau)d} \sum_{k=1}^{\infty} 2^{dk} \exp\big(-2k \log n\big) \\ &= c_{3,13} \big(\log n\big)^{(2+2H_\tau)d}\, \frac{2^d/n^2}{1 - 2^d/n^2}. \end{aligned} \tag{3.21}$$

Since $\sum_{n=1}^{\infty} \mathbb{P}\{F_n\} < \infty$, the Borel-Cantelli lemma implies that almost surely, $F_n$ occurs only finitely many times.

(d) Fix an integer $n$ together with some $y \in \mathbb{R}^d$ that satisfies

$$|y| \leq 2^{-n} \sqrt{2c_{3,10}^{-1} \log n},$$

we can represent $y$ in the form $y = \lim_{k\to\infty} y_k$ with

$$y_k = x + \theta_n \sum_{j=1}^{k} \varepsilon_j 2^{-j}, \tag{3.22}$$

where $y_0 = x \in G_n$ and $\varepsilon_j \in \{0,1\}^d$ for $j = 1, \ldots, k$.

Since the local time $L$ is jointly continuous, by expansion (3.22) and the triangular inequality, we see that on the event $F_n^c$,

$$\begin{aligned} &\big|L(y + X(a),\, C_n) - L(x + X(a),\, C_n)\big| \\ &\leq \sum_{k=1}^{\infty} \big|L(y_k + X(a),\, C_n) - L(y_{k-1} + X(a),\, C_n)\big| \\ &\leq \sum_{k=1}^{\infty} 2^{-n(\alpha-\gamma)}\, |y_k - y_{k-1}|^{\gamma} \big(ck \log n\big)^{\eta_\tau + (1+2H_\tau)\gamma} \\ &\leq c_{3,14}\, \varphi_1(2^{-n}). \end{aligned} \tag{3.23}$$

We combine (3.19) and (3.23) to get that for $n$ large enough,

$$\sup_{|x| \leq 2^{-n}\sqrt{2c_{3,10}^{-1}\log n}} L\big(x + X(a),\, C_n\big) \leq c_{3,15}\, \varphi_1(2^{-n}). \tag{3.24}$$

Since $L^*(C_n) = \sup\big\{L(x,\, C_n) : x \in \overline{X(C_n)}\big\}$, (3.14) follows from (3.24). This proves (3.10) for Cases 1 and 2. □

The Hölder conditions for the local times of $X$ are closely related to the irregularity of the sample path of $X$. In the following, we apply Theorem 3.2 to derive results about the degree of oscillation of the sample paths of $X$, which improves Theorem 4.5 of Ayache, Wu and Xiao (2008) for fractional Brownian sheets.

**Theorem 3.4.** *Let $X = \{X(t),\, t \in \mathbb{R}^N_+\}$ be an $(N, d)$-Gaussian random field satisfying Condition A and let $T \in \mathcal{A}$ be a fixed interval. Then there exists a constant $c_{3,16} > 0$ such that for every $a \in T$,*

$$\liminf_{r\to 0} \sup_{t\in \overline{B}_\rho(a,r)} \frac{|X(t)-X(a)|}{r\big(\log\log r^{-1}\big)^{-H_1}} \ge c_{3,16}, \qquad a.s. \tag{3.25}$$

*and*

$$\liminf_{r\to 0} \inf_{a\in T} \sup_{t\in \overline{B}_\rho(a,r)} \frac{|X(t)-X(a)|}{r\big(\log r^{-1}\big)^{-H_1}} \ge c_{3,16}, \qquad a.s. \tag{3.26}$$

*Proof.* It is sufficient to prove the results for $d = 1$. Note that $H_1 < 1$, Theorem 3.2 is always applicable for $d = 1$ with $\tau = 1$, which belongs to Case 1. For any $D \subset T$, we have

$$\lambda_N(D) = \int_{\overline{X_0(D)}} L(x, D)\, dx \le L^*(D) \times \sup_{u,v\in D} \big|X_0(u) - X_0(v)\big|. \tag{3.27}$$

By taking $D = \overline{B}_\rho(s, r)$ we see that (3.25) follows immediately from (3.27) and (3.10). Similarly, (3.26) follows from (3.27) and (3.11). □

It would be interesting to study if the equalities in (3.25) and (3.26) hold. These problems are closely related to the small ball probability of Gaussian random fields which satisfy Condition A and are non-trivial. For some partial results for fractional Brownian sheets, see Mason and Shi (2001). Recently Luan and Xiao (2010) proved a Chung's law of the iterated logarithm for a class of anisotropic Gaussian random fields which satisfy the property of strong local nondeterminism in metric $\rho$ [i.e., Condition (C3′) in Xiao (2009)]. In that case, the power of $\log\log r^{-1}$ is different from the one in (3.25).

## 4. $\rho$-Hausdorff dimension of the level sets

In this section we apply the results on local times to study fractal properties of the level set $X^{-1}(x) = \{t \in \mathbb{R}^N : X(t) = x\}$ of $X$. For this purpose we first collect some results on Hausdorff measure and Hausdorff dimension in the metric space $(\mathbb{R}^N, \rho)$, where $\rho$ is defined by (1.3).

**4.1. Hausdorff measures in $(\mathbb{R}^N, \rho)$.** For some $\delta_0 > 0$, let $\mathfrak{F}$ be the class of non-decreasing, right continuous functions $\varphi : (0, \delta_0) \to \mathbb{R}_+$ which satisfy $\varphi(0+) = 0$ and the *doubling condition*, that is, there exists a finite constant $c_{4,1} > 0$ for which

$$\frac{\varphi(2s)}{\varphi(s)} \le c_{4,1} \qquad \text{for } 0 < s < \frac{1}{2}\delta_0.$$

For any $\varphi \in \mathfrak{F}$, the $\varphi$-Hausdorff measure in the metric $\rho$ is defined by

$$\varphi\text{-}m_\rho(E) = \lim_{\delta\to 0} \inf \left\{ \sum_{n=1}^{\infty} \varphi(2r_n) : \ E \subseteq \bigcup_{n=1}^{\infty} B_\rho(t^n, r_n),\, r_n \le \delta \right\}, \qquad \forall E \subset \mathbb{R}^N. \tag{4.1}$$

When $\varphi(s) = s^\beta$ for $\beta > 0$, $\varphi$-$m_\rho(E)$ is called the $\beta$-dimensional Hausdorff measure of $E$, and is denoted by $\mathcal{H}_\rho^\beta(E)$. The Hausdorff dimension of $E$ in the metric $\rho$ (or simply, $\rho$-Hausdorff dimension of $E$) is defined by

$$\dim_{\mathcal{H}}^\rho E = \inf\{\beta > 0 : \mathcal{H}_\rho^\beta(E) = 0, \}. \tag{4.2}$$

It has been shown by Wu and Xiao (2007) and Xiao (2009) that $\rho$-Hausdorff dimension is useful for studying fractal properties of anisotropic random fields. Next we present a useful tool for studying the exact Hausdorff measure of anisotropic fractals.

Let $\mu$ be a finite Borel measure on $\mathbb{R}^N$. For any $t \in \mathbb{R}^N$, the *$\rho$-upper $\varphi$-density* of $\mu$ at $t$ is defined by

$$\overline{D}_\mu^{\varphi,\rho}(t) = \limsup_{r\downarrow 0} \frac{\mu\big(B_\rho(t,r)\big)}{\varphi(r)}. \tag{4.3}$$

The connection between $\overline{D}_\mu^{\varphi,\rho}(t)$ and $\varphi$-$m_\rho$ is given by the following theorem, which generalizes the classical result of Rogers and Taylor (1961) in the Euclidean metric.

**Theorem 4.1.** *For any $\varphi \in \mathfrak{F}$, there is a positive constant $c_{4,2} \ge 1$ (depending on $c_{4,1}$ only) such that for any finite Borel measure $\mu$ on $\mathbb{R}^N$ and any Borel set $E \subset \mathbb{R}^N$*

$$c_{4,2}^{-1} \varphi\text{-}m_\rho(E) \inf_{t\in E} \overline{D}_\mu^{\varphi,\rho}(t) \le \mu(E) \le c_{4,2}\, \varphi\text{-}m_\rho(E) \sup_{t\in E} \overline{D}_\mu^{\varphi,\rho}(t). \tag{4.4}$$

*Proof.* In the special case of $\varphi(s) = s^\beta$ ($\beta > 0$), (4.4) is Theorem 1.5.13 in Edgar (1998). Since $\varphi$ satisfies the doubling condition, the same proof of Edgar (1998) goes through with little modification. There is no need to reproduce the proof. □

**4.2. Uniform Hausdorff dimension results.** The Hausdorff dimension (in the Euclidean metric) of the level sets of a Gaussian field $X$ that satisfies Condition A has been derived in Xiao (2009). More generally, Biermé, et al. (2009) have determined the Hasdorff dimension of the inverse image $X^{-1}(F)$ for all Borel sets $F \subset \mathbb{R}^d$.

By using a similar argument we can prove that, if $\sum_{\ell=1}^N \frac{1}{H_\ell} > d$, then for every $x \in \mathbb{R}^d$,

$$\dim_{\mathcal{H}}^\rho X^{-1}(x) = \sum_{\ell=1}^N \frac{1}{H_\ell} - d \tag{4.5}$$

with positive probability [which depends on $x$]. In the following, we prove a "uniform" result for the $\rho$-Hausdorff dimension of the level sets. Compared with Theorem 7.1 in Xiao (2009), the formula (4.5) is much simpler, which shows that $\dim_{\mathcal{H}}^\rho$ is naturally adapted to the anisotropy of $X$.

**Theorem 4.2.** *Let $X = \{X(t),\, t \in \mathbb{R}^N\}$ be an $(N,d)$ Gaussian random field satisfying Condition A. If $\sum_{\ell=1}^N \frac{1}{H_\ell} > d$, then for every $T \in \mathcal{A}$ almost surely*

$$\dim_{\mathcal{H}}^{\rho}\big(X^{-1}(x) \cap T\big) = \sum_{\ell=1}^{N} \frac{1}{H_\ell} - d, \qquad \forall\, x \in \mathcal{O}, \tag{4.6}$$

*where $\mathcal{O}$ is the random set defined by $\mathcal{O} = \{x \in \mathbb{R}^d : L(x,T) > 0\}$.*

For proving Theorem 4.2, we will make use of the following lemmas. Lemma 4.3 on the modulus of continuity of $X$ is derived from Condition A1 by using standard methods [cf. e.g., Xiao (2009)]. More precise results on the exact uniform and local moudli of continuity of $X$ have been proved recently by Meerschaert, Wang and Xiao (2010) under Condition A.

**Lemma 4.3.** *Let $X_0 = \{X_0(t),\, t \in \mathbb{R}^N\}$ be a real valued, centered Gaussian random field satisfying Condition A1. Then, for any compact interval $T \in \mathcal{A}$, there exists a positive constant $c_{4,3}$ such that*

$$\limsup_{|h|\to 0} \frac{\sup_{t\in T, s\in[0,h]} |X_0(t+s) - X_0(t)|}{\rho(0,h)\sqrt{\log(1+\rho(0,h)^{-1})}} \le c_{4,3} \qquad a.s. \tag{4.7}$$

The following is a Frostman-type lemma in the metric space $(\mathbb{R}^N, \rho)$, see Xiao (2009).

**Lemma 4.4.** *For any Borel set $E \subset \mathbb{R}^N$, $\mathcal{H}_\rho^\beta(E) > 0$ if and only if there exist a Borel probability measure $\mu$ on $E$ and a positive constant $c$ such that $\mu\big(B_\rho(s,r)\big) \le c\, r^\beta$ for all $s \in \mathbb{R}^N$ and $r > 0$.*

**Lemma 4.5.** *Let $X = \{X(t),\, t \in \mathbb{R}^N\}$ be a centered Gaussian random field with values in $\mathbb{R}^d$ and let $T \in \mathcal{A}$. If for any $\varepsilon > 0$, there exists a positive random variable $\delta$ such that a.s.*

$$|X(s) - X(t)| \le \rho(s,t)^{1-\varepsilon} \qquad \forall\, s,t \in T \quad \text{with } \rho(s,t) \le \delta, \tag{4.8}$$

*and $X$ has a local time $L(x,T)$ which is almost surely bounded in $x$. Then almost surely*

$$\dim_{\mathcal{H}}^{\rho}\big(X^{-1}(x) \cap T\big) \le \sum_{\ell=1}^{N} \frac{1}{H_\ell} - d, \qquad \forall\, x \in \mathbb{R}^N. \tag{4.9}$$

*Proof.* We divide $T$ into upright sub-intervals $C_{n,k}$ of side length $2^{-n/H_j}$ $(j = 1, \cdots, N)$. Then the number of such intervals is at most $c2^{nQ}$ and the $\rho$-diameter of $C_{n,k}$ is $c2^{-n}$. Let us fix $\omega \in \Omega$ such that (4.8) holds and $L(x,T)$ is bounded.

For any $x \in \mathbb{R}^d$, if $X^{-1}(x) \cap C_{n,k} \ne \emptyset$, then (4.8) implies that, for $n$ large enough,

$$X(C_{n,k}) \subset B_\rho(x, 2^{-n(1-\varepsilon)}). \tag{4.10}$$

Denote by $N_n(x)$ the number of intervals $C_{n,k}$ that satisfy (4.10). Then the occupation density formula (2.1) and the boundedness of $L(\cdot,T)$ imply that

$$N_n(x) 2^{-Qn} \le \int_{B_\rho(x, 2^{-n(1-\varepsilon)})} L(y,T)\, dy \le K\, 2^{-n(1-\varepsilon)d}, \tag{4.11}$$

where $K$ is a random variable. Thus, $N_n(x) \le K\, 2^{-n[Q-(1-\varepsilon)d]}$ a.s. Since the family $\{C_{n,k} : X^{-1}(x) \cap C_{n,k} \neq \emptyset\}$ forms a covering of $X^{-1}(x) \cap T$ by $\rho$-balls of radius $2^{-n}$, we derive that $\dim_{\mathcal{H}}^{\rho}\big(X^{-1}(x) \cap T\big) \le Q - (1-\varepsilon)d$ a.s. Since $\varepsilon > 0$ is arbitrary, (4.9) follows. □

Now, we are ready to prove Theorem 4.2.

*Proof.* The upper bound in (4.6) follows from Lemmas 4.3 and 4.5. Next we prove that a.s.

$$\dim_{\mathcal{H}}^{\rho}\big(X^{-1}(x) \cap T\big) \ge \sum_{\ell=1}^{N} \frac{1}{H_\ell} - d \tag{4.12}$$

for all $x \in \mathcal{O}$. To this end, note that $L(x, \cdot)$ is a finite and positive Borel measure on $X^{-1}(x) \cap T$ for every $x \in \mathcal{O}$. Hence Theorem 3.2 and Lemma 4.4 together yield (4.12). This finishes the proof. □

**4.3. Exact Hausdorff measure functions.** As an application of Theorem 3.1, we present a partial result on the exact Hausdorff measure of the level set.

**Theorem 4.6.** *Assume that $\sum_{\ell=1}^{N} \frac{1}{H_\ell} > d$ and $T \in \mathcal{A}$. Then there exists a positive constant $c_{4,4}$ such that with probability 1*

$$\varphi_1\text{-}m_\rho\big(X^{-1}(x) \cap T\big) \ge c_{4,4} L(x,T) \qquad \textit{in Cases 1 and 2,} \tag{4.13}$$

*and*

$$\varphi_2\text{-}m_\rho\big(X^{-1}(x) \cap T\big) \ge c_{4,4} L(x,T) \qquad \textit{in Case 3}\ . \tag{4.14}$$

*Proof.* We only prove (4.13). If $L(x,T) = 0$, it holds automatically. If $L(x,T) > 0$, then $L(x,\cdot)$ is a finite Borel measure on $X^{-1}(x) \cap T$. Hence (4.13) follows from Theorem 3.1 and Theorem 4.1. □

## 5. Appendix

In this Appendix, we provide some technical lemmas that are used in Section 2 for proving the main moment estimates. Lemma 5.1 is similar to Lemma 8.6 in Xiao (2009) whose proof is elementary. Lemma 5.2 and Lemma 5.3 are extensions of Lemma 2.3 in Xiao (1997), which were proved in Wu and Xiao (2009a). Lemma 5.4 is a further generalization of Lemma 5.3, which is useful for estimating the moments of increments of the local times.

**Lemma 5.1.** *Let $\beta$, $\gamma$ and $p$ be positive constants, then for all $A \in (0,1)$*

$$\int_0^1 \frac{r^{p-1}}{\big(A + r^\gamma\big)^\beta}\, dr \asymp \begin{cases} A^{\frac{p}{\gamma}-\beta} & \textit{if } \beta\gamma > p, \\ \log\big(1 + A^{-1/\gamma}\big) & \textit{if } \beta\gamma = p, \\ 1 & \textit{if } \beta\gamma < p. \end{cases} \tag{5.1}$$

*In the above, $f(A) \asymp g(A)$ means that the ratio $f(A)/g(A)$ is bounded from below and above by positive constants that do not depend on $A \in (0,1)$.*

Even though, in this paper, Lemmas 5.2–5.4 are only applied for $p = 1$, we state and prove them in their general forms which will be useful elsewhere.

**Lemma 5.2.** *Let $\beta$, $\gamma$ and $p$ be positive constants such that $\gamma\beta \ge p$.*

(i). *If $\gamma\beta > p$, then there exists a constant $c_{5,1} > 0$ whose value depends on $\gamma$, $\beta$ and $p$ only such that for all $A \in (0,1)$, $r > 0$, $u^* \in \mathbb{R}^p$, all integers $n \geq 1$ and all distinct $u_1, \dots, u_n \in O_p(u^*, r)$ we have*

$$\int_{O_p(u^*,r)} \frac{du}{\Big[A + \min\limits_{1\leq j\leq n} |u-u_j|^\gamma\Big]^\beta} \leq c_{5,1}\, n\, A^{\frac{p}{\gamma}-\beta}, \tag{5.2}$$

*where $O_p(u^*, r)$ denotes a $p$-dimensional ball centered at $u^*$ with radius $r$.*

(ii). *If $\gamma\beta = p$, then for any $\kappa \in (0,1)$ there exists a constant $c_{5,2} > 0$ whose value depends on $\gamma$, $\beta$, $\kappa$ and $p$ only such that for all $A \in (0,1)$, $r > 0$, $u^* \in \mathbb{R}^p$, all integers $n \geq 1$ and all distinct $u_1, \dots, u_n \in O_p(u^*, r)$ we have*

$$\int_{O_p(u^*,r)} \frac{du}{\Big[A + \min\limits_{1\leq j\leq n} |u-u_j|^\gamma\Big]^\beta} \leq c_{5,2}\, n \log\bigg[e + \Big(A^{-1/\gamma}\frac{r}{n^{1/p}}\Big)^\kappa\bigg]. \tag{5.3}$$

**Lemma 5.3.** *Let $\beta > 0$ be a constant and let $p \geq 1$ be an integer such that $\beta < p$. Then the following statements hold.*

(i). *For all $r > 0$, $u^* \in \mathbb{R}^p$, for any integer $n \geq 1$ and any distinct $u_1, \dots, u_n \in O_p(u^*, r)$ we have*

$$\int_{O_p(u^*,r)} \frac{du}{\min\limits_{1\leq j\leq n} |u-u_j|^\beta} \leq c_{5,3}\, n^{\frac{\beta}{p}}\, r^{p-\beta}, \tag{5.4}$$

*where $c_{5,3} > 0$ is a constant whose value depends on $\beta$ and $p$ only.*

(ii). *For all constants $r > 0$ and $K > 0$, all $u^* \in \mathbb{R}^p$, integers $n \geq 1$ and any distinct $u_1, \dots, u_n \in O_p(u^*, r)$ we have*

$$\begin{aligned} &\int_{O_p(u^*,r)} \log\Big[e + K\Big(\min_{1\leq j\leq n}|u-u_j|\Big)^{-\beta}\Big]\, du \\ &\qquad \leq c_{5,4}\, r^p \log\bigg[e + K\Big(\frac{r}{n^{1/p}}\Big)^{-\beta}\bigg], \end{aligned} \tag{5.5}$$

*where $c_{5,4} > 0$ is a constant whose value depends on $\beta$ and $p$ only.*

**Lemma 5.4.** *Let $\alpha$, $\beta > 0$ be two constants and let $p \geq 1$ be an integer such that $\beta < p$ and $\alpha < p$. Then for all $r > 0$, $K > 0$, $u^* \in \mathbb{R}^p$, for any integer $n \geq 1$ and all points $u_1, \dots, u_n \in O_p(u^*, r)$ we have*

$$\begin{aligned} &\int_{O_p(u^*,r)} \frac{\log\Big[e + K\big(\min\limits_{1\leq j\leq n}|u-u_j|\big)^{-\beta}\Big]}{\min\limits_{1\leq j\leq n}|u-u_j|^\alpha}\, du \\ &\qquad \leq c_{5,5}\, n^{\frac{\alpha}{p}}\, r^{p-\alpha} \log\bigg[e + K\Big(\frac{r}{n^{1/p}}\Big)^{-\beta}\bigg], \end{aligned} \tag{5.6}$$

*where $c_{5,5} > 0$ is a constant whose value depends on $\alpha$, $\beta$ and $p$ only.*

*Proof.* The idea of the proof is similar to that of Xiao (1997) [see also the proof of Lemma 2.4 in Wu and Xiao (2009a)]. We omit it here. □

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