

PACKING DIMENSION RESULTS FOR ANISOTROPIC GAUSSIAN RANDOM FIELDS

ANNE ESTRADE, DONGSHENG WU, AND YIMIN XIAO

ABSTRACT. Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a Gaussian random field with values in \mathbb{R}^d defined by

$$X(t) = (X_1(t), \dots, X_d(t)), \quad \forall t \in \mathbb{R}^N,$$

where X_1, \dots, X_d are independent copies of a centered real-valued Gaussian random field X_0 . We consider the case when X_0 is anisotropic and study the packing dimension of the range $X(E)$, where $E \subseteq \mathbb{R}^N$ is a Borel set. For this purpose we extend the original notion of packing dimension profile due to Falconer and Howroyd [11] to the anisotropic metric space (\mathbb{R}^N, ρ) , where $\rho(s, t) = \sum_{j=1}^N |s_j - t_j|^{H_j}$ and $(H_1, \dots, H_N) \in (0, 1)^N$ is a given vector. The extended notion of packing dimension profile is of independent interest.

1. Introduction

Fractal dimensions such as Hausdorff dimension, box-counting dimension and packing dimension are useful tools in studying fractals (see, e.g., Falconer [10]), as well as in characterizing roughness or irregularity of stochastic processes and random fields. We refer to Taylor [24] and Xiao [30] for extensive surveys on results and techniques for investigating fractal properties of Markov processes, and to Adler [1], Kahane [16], Khoshnevisan [17] and Xiao [31, 32] for geometric results for Gaussian random fields.

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a Gaussian random field with values in \mathbb{R}^d . For any set $E \subseteq \mathbb{R}^N$, let $X(E) = \{X(t), t \in E\}$ and $\text{Gr}X(E) = \{(t, X(t)) : t \in E\}$ be the range and graph of X respectively. It is known that if X is a fractional Brownian motion or the Brownian sheet, the packing dimensions of $X([0, 1]^N)$ and $\text{Gr}X([0, 1]^N)$ coincide with their Hausdorff dimensions. However, when $E \subseteq \mathbb{R}^N$ is an arbitrary Borel set, significant difference between the Hausdorff and packing dimensions of the image $X(E)$ may appear. Talagrand and Xiao [23] proved that, even for such “nice” Gaussian random fields as fractional Brownian motion and the Brownian sheet, the Hausdorff and packing dimensions of $X(E)$ can be different because they depend on different aspects of the fractal structure of E . Xiao [29] further showed that the packing dimension of $X(E)$ is determined by the packing dimension profiles introduced by Falconer and Howroyd [11] (see Section 2 for the definition).

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On the other hand, as noted in Xiao [31, 33], the fractal dimensions of the range $X([0, 1]^N)$ and graph $\text{Gr}X([0, 1]^N)$ themselves become more involved when X is a general Gaussian random field. To be more specific, let $X = \{X(t), t \in \mathbb{R}^N\}$ be a Gaussian random field with values in \mathbb{R}^d defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ by

$$X(t) = (X_1(t), \dots, X_d(t)), \quad \forall t \in \mathbb{R}^N, \quad (1.1)$$

where X_1, \dots, X_d are independent copies of a real-valued, centered Gaussian random field $X_0 = \{X_0(t), t \in \mathbb{R}^N\}$. When X_0 is at least approximately isotropic in the sense that

$$\mathbb{E}[(X_0(s) - X_0(t))^2] \asymp \phi(\|t - s\|), \quad \forall s, t \in [0, 1]^N, \quad (1.2)$$

where $\phi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a nondecreasing and continuous function with $\phi(0) = 0$ and $\|\cdot\|$ (here and throughout the paper) is the Euclidean norm, and where $f(x) \asymp g(x)$ for $x \in T$ means that the function $f(x)/g(x)$ is bounded from below and above by positive and finite constants that do not depend on $x \in T$, Xiao [31] introduced an upper index α^* and a lower index α_* for ϕ at 0 (see Section 2 for their definitions) and proved that

$$\dim_{\text{H}} X([0, 1]^N) = \min \left\{ d, \frac{N}{\alpha^*} \right\}, \quad \text{a.s.} \quad (1.3)$$

and

$$\dim_{\text{H}} \text{Gr}X([0, 1]^N) = \min \left\{ \frac{N}{\alpha^*}, N + (1 - \alpha^*)d \right\}, \quad \text{a.s.}, \quad (1.4)$$

where $\dim_{\text{H}} E$ denotes Hausdorff dimension of E . Xiao [33] showed that the packing dimensions of $X([0, 1]^N)$ and $\text{Gr}X([0, 1]^N)$ are determined by the lower index α_* of ϕ . Namely,

$$\dim_{\text{P}} X([0, 1]^N) = \min \left\{ d, \frac{N}{\alpha_*} \right\}, \quad \text{a.s.} \quad (1.5)$$

and

$$\dim_{\text{P}} \text{Gr}X([0, 1]^N) = \min \left\{ \frac{N}{\alpha_*}, N + (1 - \alpha_*)d \right\}, \quad \text{a.s.}, \quad (1.6)$$

where $\dim_{\text{P}} E$ denotes the packing dimension of E . There are many interesting examples of Gaussian random fields with stationary increments which satisfy (1.2) with $\alpha_* < \alpha^*$. Such examples can be constructed by choosing appropriate spectral measures; see Section 2 for more details. The results (1.3)–(1.6) show that, similarly to the well-known cases of Lévy processes (see Pruitt and Taylor [21]), the Hausdorff dimensions of $X([0, 1]^N)$ and $\text{Gr}X([0, 1]^N)$ may be different from their packing dimensions.

In recent years, there has been a lot of interest in studying anisotropic random fields such as fractional Brownian sheets or solution to the stochastic heat equation. Ayache and Xiao [2], Wu and Xiao [27, 28] and Xiao [32] have shown that, when X_0 is anisotropic, the Hausdorff dimensions of the range and graph of the Gaussian random field X defined by (1.1) can be very different from the approximately

isotropic case. In particular, the notion of Hausdorff dimension on \mathbb{R}^N equipped with the anisotropic metric ρ defined by

$$\rho(s, t) = \sum_{j=1}^N |s_j - t_j|^{H_j}, \quad \forall s, t \in \mathbb{R}^N \quad (1.7)$$

is needed in order to determine the Hausdorff dimension of $X(E)$. In the above and in the sequel, $H = (H_1, \dots, H_N) \in (0, 1)^N$ is a fixed vector.

The main objective of this paper is to study the packing dimension of the range $X(E)$ for a class of anisotropic Gaussian random fields defined as in (1.1). In particular, we determine the packing dimension of the range $X([0, 1]^N)$ when (1.2) is replaced by Condition (C) below and estimate the packing dimension of $X(E)$ for a general Borel set $E \subset \mathbb{R}^N$. For this latter purpose, we first extend the ideas in Falconer and Howroyd [11] and introduce packing dimension profiles in the metric space (\mathbb{R}^N, ρ) . For comparison purposes we also determine the Hausdorff dimensions of the $X([0, 1]^N)$ and $\text{Gr}X([0, 1]^N)$ and show that they are determined by the upper index α^* and (H_1, \dots, H_N) .

The rest of the paper is organized as follows. In Section 2 we recall some basic facts about Gaussian random fields and construct several interesting examples of isotropic and anisotropic Gaussian random fields, including those with different upper and lower indices. We also recall the definition of packing dimension profile of Falconer and Howroyd [11]. In Section 3 we provide the definition and some basic properties of packing dimension in the metric space (\mathbb{R}^N, ρ) , and extend the packing dimension profiles of Falconer and Howroyd [11] to (\mathbb{R}^N, ρ) . Results in this section may have applications beyond the scope of the present paper. For example, they may be useful for studying self-affine fractals. We should mention that another extended notion of packing dimension profiles has also been developed by Khoshnevisan, Schilling and Xiao [18] for studying the packing dimension of the range of a Lévy process. In Section 4, we determine the packing dimension of $X(E)$, where E can either be $[0, 1]^N$ or a general Borel set. We prove the upper bound by using a standard covering argument. The method for proving the lower bound for the packing dimension is potential-theoretic. It can be viewed as an analogue of the classical and powerful ‘‘capacity argument’’ (based on the Frostman theorem) for Hausdorff dimension computation. In the Appendix, we provide proofs for some technical results in Section 3 and determine the Hausdorff dimensions of $X([0, 1]^N)$ and $\text{Gr}X([0, 1]^N)$. These latter results should be compared with the packing dimension of $X([0, 1]^N)$ given in Theorem 4.1.

We will use K to denote an unspecified positive constant which may differ in each occurrence.

2. Preliminaries

2.1. Anisotropic Gaussian random fields. Let $X = \{X(t), t \in \mathbb{R}^N\}$ be an (N, d) -Gaussian random field defined by (1.1). To demonstrate the main differences in the fractal dimension properties between the isotropic and anisotropic cases, we assume that the real-valued centered Gaussian random field $X_0 = \{X_0(t), t \in \mathbb{R}^N\}$ satisfies $X_0(0) = 0$ and the following Condition (C):

- (C) Let $\phi : [0, \delta_0) \rightarrow [0, \infty)$ be a non-decreasing, right continuous function with $\phi(0) = 0$. For every compact interval $T \subset \mathbb{R}^N$, there exist positive constants δ_0 and $K \geq 1$ such that

$$K^{-1} \phi^2(\rho(s, t)) \leq \mathbb{E}[(X_0(t) - X_0(s))^2] \leq K \phi^2(\rho(s, t)) \quad (2.1)$$

for all $s, t \in T$ with $\rho(s, t) \leq \delta_0$, where ρ is the metric defined in (1.7).

The upper index of ϕ at 0 is defined by

$$\alpha^* = \inf \left\{ \beta \geq 0 : \lim_{r \rightarrow 0} \frac{\phi(r)}{r^\beta} = \infty \right\} \quad (2.2)$$

with the convention $\inf \emptyset = \infty$. Analogously, the lower index of ϕ at 0 is defined by

$$\alpha_* = \sup \left\{ \beta \geq 0 : \lim_{r \rightarrow 0} \frac{\phi(r)}{r^\beta} = 0 \right\} \quad (2.3)$$

with the convention $\sup \emptyset = 0$.

When $X_0 = \{X_0(t), t \in \mathbb{R}^N\}$ satisfies (1.2), Condition (C) holds with $H_1 = \dots = H_N = 1$ (or ρ is replaced by the Euclidean metric) and the above upper and lower indices α^* and α_* coincide with those defined in Xiao [31, 33]. When X_0 has stationary and isotropic increments, α^* and α_* coincide with the upper and lower indices of $\sigma(h)$ (which is a function of $\|h\|$), where

$$\sigma^2(h) = \mathbb{E}[(X_0(t+h) - X_0(t))^2], \quad \forall h \in \mathbb{R}^N. \quad (2.4)$$

However, the class of Gaussian random fields with $\alpha^* = \alpha_*$ in this paper is wider than the index- α Gaussian fields in Adler [1] or Khoshnevisan [17].

As in Xiao [31, 32, 33], many interesting examples of Gaussian random fields satisfying Condition (C) have stationary increments. Hence we collect some basic facts about them. Suppose $X_0 = \{X_0(t), t \in \mathbb{R}^N\}$ has stationary increments and continuous covariance function $R(s, t) = \mathbb{E}[X(s)X(t)]$. Then, according to Yaglom [34], $R(s, t)$ can be represented as

$$R(s, t) = \int_{\mathbb{R}^N} (e^{i\langle s, \lambda \rangle} - 1)(e^{-i\langle t, \lambda \rangle} - 1) \Delta(d\lambda) + \langle s, \Sigma t \rangle, \quad (2.5)$$

where $\langle x, y \rangle$ is the ordinary scalar product in \mathbb{R}^N , Σ is an $N \times N$ non-negative definite matrix and $\Delta(d\lambda)$ is a nonnegative symmetric measure on $\mathbb{R}^N \setminus \{0\}$ (i.e., $\Delta(A) = \Delta(-A)$ for all $\mathcal{B}(\mathbb{R}^N \setminus \{0\})$) which satisfies

$$\int_{\mathbb{R}^N} \frac{\|\lambda\|^2}{1 + \|\lambda\|^2} \Delta(d\lambda) < \infty. \quad (2.6)$$

If $\Delta(d\lambda)$ is a spherically (or radially) symmetric measure on $\mathbb{R}^N \setminus \{0\}$, then X_0 is isotropic, this follows from (2.8) below.

The measure Δ in (2.5) is called the *spectral measure* of X . It follows from (2.5) that X has the following stochastic integral representation:

$$X_0(t) = \int_{\mathbb{R}^N} (e^{i\langle t, \lambda \rangle} - 1) W(d\lambda) + \langle \mathbf{Y}, t \rangle, \quad (2.7)$$

where \mathbf{Y} is an N -dimensional Gaussian random vector with mean 0 and covariance matrix Σ , $W(d\lambda)$ is a centered complex-valued Gaussian random measure which is independent of \mathbf{Y} and satisfies

$$\mathbb{E}(W(A)\overline{W(B)}) = \Delta(A \cap B) \quad \text{and} \quad W(-A) = \overline{W(A)}$$

for all Borel sets $A, B \subseteq \mathbb{R}^N$. Since the linear term $\langle \mathbf{Y}, t \rangle$ in (2.7) will not have any effect on fractal dimensions of the range and graph of X , we will simply assume $\mathbf{Y} = 0$. Consequently, we have

$$\sigma^2(h) = \mathbb{E}[(X_0(t+h) - X_0(t))^2] = 2 \int_{\mathbb{R}^N} (1 - \cos \langle h, \lambda \rangle) \Delta(d\lambda). \quad (2.8)$$

It is important to observe that the incremental-variance function $\sigma^2(h)$ in (2.8) is a negative definite function in the sense of I. J. Schoenberg and thus can be viewed as the characteristic exponent of a symmetric infinitely divisible distribution. See Berg and Forst [3] for more information on negative definite functions. This connection makes it possible for us to construct interesting examples of Gaussian random fields with various properties on $\sigma^2(h)$ (e.g. to have different upper and lower indices) by modifying the constructions of Lévy processes with various prescribed properties on their characteristic (or Laplace) exponents. See section 2.2 below.

We remark that the class of Gaussian random fields satisfying Condition (C) is large. It not only includes fractional Brownian sheets of index $H = (H_1, \dots, H_N)$, the operator-scaling Gaussian fields with stationary increments in Xiao [33] and solutions to the stochastic heat equation (in all these cases, $\phi(r) = r$), but also the following subclass that can be constructed from general subordinators (cf. Bertoin [5] or Sato [22]).

Proposition 2.1. *Let ϕ be the Laplace exponent of a subordinator $T = \{T(r), r \geq 0\}$ and let σ_1^2 be a negative definite function on \mathbb{R}^N . Then $\sigma^2(u) = \phi(\sigma_1^2(u))$ is also a negative definite function. In particular, there is a centered Gaussian random field X_0 with stationary increments such that $X_0(0) = 0$ and $\mathbb{E}[(X_0(s) - X_0(t))^2] = \phi(\sigma_1^2(t - s))$ for all $s, t \in \mathbb{R}^N$.*

Proof. For completeness, we provide a proof by using the subordination argument for Lévy processes; see e.g. Bertoin [5] or Sato [22]. Let $Y = \{Y(r), r \geq 0\}$ be a symmetric Lévy process with values in \mathbb{R}^N and characteristic exponent $\sigma_1^2(u)$ ($u \in \mathbb{R}^N$). We assume that Y and T are independent. Then a conditioning argument shows that the subordinated process Z defined by $Z(r) = Y(T(r))$ for $r \geq 0$ is also a Lévy process with values in \mathbb{R}^N whose characteristic function is given by

$$\mathbb{E}\left(e^{i\langle u, Z(r) \rangle}\right) = \mathbb{E}\left(e^{-T(r)\sigma_1^2(u)}\right) = e^{-r\phi(\sigma_1^2(u))}, \quad \forall u \in \mathbb{R}^N.$$

Thus $\sigma^2(u) = \phi(\sigma_1^2(u))$ is negative definite. \square

Since ϕ may have different upper and lower indices (see Example 2.7 below) and σ_1^2 can be chosen to be the incremental variance of any anisotropic Gaussian field with stationary increments, Proposition 2.1 produces a quite large class of Gaussian random fields that satisfy Condition (C) with $0 < \alpha_* < \alpha^* \leq 1$.

Sample path continuity of Gaussian fields is well studied and there are several ways to determine modulus of continuity of Gaussian random fields; see, e.g., Dudley [9] and Marcus and Rosen [20] for a review. The following lemma is a consequence of Corollary 2.3 in Dudley [9]. It will be useful for deriving upper bounds for the Hausdorff and packing dimensions of the range and graph.

Lemma 2.2. *Assume $X_0 = \{X_0(t), t \in \mathbb{R}^N\}$ is a real-valued centered Gaussian random field that satisfies the upper bound in (2.1). If the upper and lower indices of ϕ at 0 satisfy $0 < \alpha_* \leq \alpha^* \leq 1$, then for every compact interval $T \subset \mathbb{R}^N$, there exists a finite constant K such that*

$$\limsup_{\delta \rightarrow 0} \frac{\sup_{s, t \in T: \rho(s, t) \leq \delta} |X_0(s) - X_0(t)|}{f(\delta)} \leq K, \quad \text{a.s.}, \quad (2.9)$$

where $f(h) = \phi(h) |\log \phi(h)|^{1/2}$.

2.2. Examples of Gaussian fields with different upper and lower indices.

Xiao [31] showed that Gaussian random fields with stationary increments and different upper and lower indices can be constructed by choosing appropriately the spectral measures Δ in (2.5). Similar approach has been applied in the literature to construct Lévy processes with different upper and lower Blumenthal-Gettoor indices [7]. See Pruitt and Taylor [21] and the references therein for more information. We remark that for studying local properties of a Gaussian random field one is interested in the behavior of $\sigma^2(h)$ near $h = 0$, while Blumenthal and Gettoor's indices are concerned with the asymptotic behavior of $\sigma^2(h)$ as $\|h\| \rightarrow \infty$. Nevertheless one can modify the constructions of Lévy processes to obtain negative definite functions $\sigma^2(h)$ with prescribed properties near $h = 0$.

The following proposition is useful for constructing Gaussian random field with prescribed upper and lower indices.

Proposition 2.3. *Let $X_0 = \{X_0(t), t \in \mathbb{R}^N\}$ be a mean zero, real-valued Gaussian random field with stationary increments and $X_0(0) = 0$. Assume that the spectral measure Δ of X_0 has a density function f that satisfies the following condition:*

$$0 < \liminf_{\|\lambda\| \rightarrow \infty} \frac{\beta_N \|\lambda\|^N f(\lambda)}{\Delta\{\xi : \|\xi\| \geq \|\lambda\|\}} \leq \limsup_{\|\lambda\| \rightarrow \infty} \frac{\beta_N \|\lambda\|^N f(\lambda)}{\Delta\{\xi : \|\xi\| \geq \|\lambda\|\}} < 2, \quad (2.10)$$

where $\beta_1 = 2$ and for $N \geq 2$, $\beta_N = \mu(S^{N-1})$ is the area (i.e., the $(N-1)$ -dimensional Lebesgue measure) of S^{N-1} . Then

$$0 < \liminf_{\|h\| \rightarrow 0} \frac{\sigma^2(h)}{\psi(\|h\|)} \leq \limsup_{\|h\| \rightarrow 0} \frac{\sigma^2(h)}{\psi(\|h\|)} < \infty, \quad (2.11)$$

where $\psi(r) = \Delta\{\xi : \|\xi\| \geq r^{-1}\}$ and $\psi(0) = 0$. Moreover, ψ is regularly varying at 0 with index 2α if and only if

$$2\alpha = \lim_{r \rightarrow \infty} \frac{r^N \int_{S^{N-1}} f(r\theta) \mu(d\theta)}{\Delta\{\xi : |\xi| \geq r\}}. \quad (2.12)$$

Proof. This is a consequence of Remark 2.4 and Theorem 2.5 in Xiao [31]. \square

Remark 2.4. Here are some remarks about Proposition 2.3.

- When $N = 1$, the condition (2.10) is due to Berman [4], who showed that the corresponding Gaussian process X_0 is locally nondeterministic. Xiao [31] extended it to $N \geq 2$ and proved that it is a sufficient condition for X_0 to have the property of strong local nondeterminism.
- Proposition 2.3 implies that, under condition (2.10), X_0 is approximately isotropic and the upper and lower indices of σ at 0 are the same as those of $\psi(r)$, which is determined by the tail-measure of Δ . It is not difficult to construct a measure Δ such that the ratio in the right side of (2.12) has different liminf and limsup behavior, which will imply that $\psi(r)$ has different upper and lower indices.

In the following we provide three examples of Gaussian random fields with different upper and lower indices. Example 2.5 is similar to Example 2.10 in Xiao [31], which does not require (2.10). Example 2.6 is more general and is due to Clausel [8] who called it *infinity scale fractional Brownian motion (ISFBM)*. In her terminology, the Gaussian field X_0 in Example 2.5 is essentially a lacunary fractional Brownian motion with lower Hurst index $\underline{H} = \alpha_1$ and lacunary index $\tau = (\alpha_2(1 - \alpha_1))/(\alpha_1(1 - \alpha_2))$. The Gaussian random fields in both examples are isotropic with stationary increments. For an (N, d) -Gaussian random field associated with this X_0 , (1.3) - (1.6) show that the range and graph have different Hausdorff and packing dimensions.

Example 2.5. For any given constants $0 < \alpha_1 < \alpha_2 < 1$ and any increasing sequence $\{b_k, k \geq 0\}$ of real numbers such that $b_0 = 0$ and $b_k \rightarrow \infty$ as $k \rightarrow \infty$, define the function f on \mathbb{R}^N by

$$f(\lambda) = \begin{cases} \|\lambda\|^{-(2\alpha_1+N)} & \text{if } \|\lambda\| \in (b_{2k}, b_{2k+1}], \\ \|\lambda\|^{-(2\alpha_2+N)} & \text{if } \|\lambda\| \in (b_{2k+1}, b_{2k+2}]. \end{cases} \quad (2.13)$$

Using $\Delta(d\lambda) = f(\lambda)d\lambda$ as spectral measure, we obtain from (2.7) an isotropic Gaussian random field X_0 with stationary increments. By using (2.8) one can verify that there exists a positive constant $K \geq 1$ such that

$$K^{-1} \|h\|^{2\alpha_2} \leq \sigma^2(h) := \tilde{\sigma}^2(\|h\|) \leq K \|h\|^{2\alpha_1} \quad (2.14)$$

for all $h \in \mathbb{R}^N$ with $\|h\| \leq 1$.

Now we choose a strictly increasing sequence $\{b_k\}$ such that for all $k \geq 0$,

$$b_{2k+1} \geq 2b_{2k} \quad \text{and} \quad b_{2k+2} > b_{2k+1}^{\frac{\alpha_2(1-\alpha_1)}{\alpha_1(1-\alpha_2)}}. \quad (2.15)$$

This can be done inductively. We prove that $\alpha_* = \alpha_1$ and $\alpha^* = \alpha_2$, where α^* and α_* are defined by (2.2) and (2.3) with ϕ replace by $\tilde{\sigma}$ (see also [31]).

Let $r_n = (2b_{2n})^{-1}$. Then $r_n \downarrow 0$ as $n \rightarrow \infty$. We pick $t_n \in \mathbb{R}^N$ such that $\|t_n\| = r_n$, then use (2.8), a change of variable and the isotropy of $f(\lambda)$ to derive

$$\begin{aligned} \tilde{\sigma}^2(r_n) &\geq 2 \int_{\|\lambda\| \in (b_{2n}, b_{2n+1}]} (1 - \cos \langle t_n, \lambda \rangle) \frac{d\lambda}{\|\lambda\|^{2\alpha_1+N}} \\ &= K \int_{b_{2n}}^{b_{2n+1}} (1 - \cos(r_n x)) \frac{dx}{x^{2\alpha_1+1}} \\ &\geq K \int_{b_{2n}}^{2b_{2n}} r_n^2 x^2 \frac{dx}{x^{2\alpha_1+1}} \\ &= K r_n^{2\alpha_1}, \end{aligned} \tag{2.16}$$

where in the last two steps we have also used the fact that $b_{2n+1} \geq 2b_{2n}$ and the elementary inequality $1 - \cos x \geq x^2/4$ for $x \in (-1, 1)$. Combining (2.14) and (2.16) yields $\alpha_* = \alpha_1$.

Next we verify that $\alpha^* = \alpha_2$. For this purpose, we take $r_n = b_{2n+1}^{-1}$ (or any r_n which satisfies $b_{2n+2}^{-\alpha_1/\alpha_2} \leq r_n \leq b_{2n+1}^{-(1-\alpha_1)/(1-\alpha_2)}$). By using (2.8) and a change of variable we get

$$\begin{aligned} \tilde{\sigma}^2(r_n) &= K \left\{ \sum_{k=0}^{\infty} \int_{b_{2k}}^{b_{2k+1}} (1 - \cos(r_n x)) \frac{dx}{x^{2\alpha_1+1}} \right. \\ &\quad \left. + \sum_{k=0}^{\infty} \int_{b_{2k+1}}^{b_{2k+2}} (1 - \cos(r_n x)) \frac{dx}{x^{2\alpha_2+1}} \right\}. \end{aligned} \tag{2.17}$$

The second sum is at most $K r_n^{2\alpha_2}$. To estimate the first sum, we break it into two parts.

$$\begin{aligned} \sum_{k=0}^n \int_{b_{2k}}^{b_{2k+1}} (1 - \cos(r_n x)) \frac{dx}{x^{2\alpha_1+1}} &\leq \int_0^{b_{2n+1}} r_n^2 x^2 \frac{dx}{x^{2\alpha_1+1}} \\ &= \frac{r_n^2}{2 - 2\alpha_1} b_{2n+1}^{2-2\alpha_1} \leq K r_n^{2\alpha_2}, \end{aligned} \tag{2.18}$$

thanks to our choice of r_n . Similarly, by the second condition in (2.15) we have

$$\begin{aligned} \sum_{k=n+1}^{\infty} \int_{b_{2k}}^{b_{2k+1}} (1 - \cos(r_n x)) \frac{dx}{x^{2\alpha_1+1}} &\leq 2 \int_{b_{2n+2}}^{\infty} \frac{dx}{x^{2\alpha_1+1}} \\ &\leq K b_{2n+2}^{-2\alpha_1} \leq K r_n^{2\alpha_2}. \end{aligned} \tag{2.19}$$

Combining (2.17) - (2.19) and (2.14) yields $\alpha^* = \alpha_2$.

Example 2.6. This example is due to Clausel [8]. Let $\mathbf{H} = \{H_j, j \geq 0\}$ be a sequence of reals numbers such that

$$0 < \liminf_{j \rightarrow \infty} H_j \leq \limsup_{j \rightarrow \infty} H_j < 1$$

and let $\sigma = \{\sigma_j, j \geq 0\}$ be a bounded sequence of positive real numbers. The real valued Gaussian field $\{B_{\mathbf{H}, \sigma}(t), t \in \mathbb{R}^N\}$ with stationary increments defined by the

following harmonizable representation

$$B_{\mathbf{H},\sigma}(t) = \sum_{\ell=0}^{\infty} \sigma_{\ell} \int_{D_{\ell}} \frac{e^{i\langle t,\lambda \rangle} - 1}{\|\lambda\|^{H_{\ell} + \frac{N}{2}}} W(d\lambda) \quad (2.20)$$

is called the infinity scale fractional Brownian motion with Hurst indices $\mathbf{H} = \{H_j, j \geq 0\}$ and amplitudes $\sigma = \{\sigma_j, j \geq 0\}$ in [8]. In (2.20), $D_0 = \{\lambda \in \mathbb{R}^N : \|\lambda\| < 1\}$ and for $D_{\ell} = \{\lambda \in \mathbb{R}^N : 2^{\ell-1} \leq \|\lambda\| < 2^{\ell}\}$ for all $\ell \geq 1$.

For simplicity we take $\sigma_j \equiv 1$ for all $j \geq 0$. Similarly to Example 2.5 we can verify that $\alpha_* = \liminf_{j \rightarrow \infty} H_j := \underline{H}$ always holds. In order to obtain $\alpha^* = \limsup_{j \rightarrow \infty} H_j$, we need an extra condition. Let $\bar{H} = \limsup_{j \rightarrow \infty} H_j$. For every $\varepsilon \in (0, \underline{H})$, let

$$T_1 = \inf\{j : H_j \geq \bar{H} - \varepsilon\}, \quad T_2 = \inf\{j > T_1 : H_j < \bar{H} - \varepsilon\},$$

and for all $k \geq 1$ we define inductively

$$T_{2k+1} = \inf\{j > T_{2k} : H_j \geq \bar{H} - \varepsilon\}$$

and

$$T_{2k+2} = \inf\{j > T_{2k+1} : H_j < \bar{H} - \varepsilon\}.$$

If we assume that

$$T_{2k+2} > \frac{(\bar{H} - \varepsilon)(1 - \underline{H} + \varepsilon)}{(\underline{H} - \varepsilon)(1 - \bar{H} + \varepsilon)} T_{2k+1}, \quad (2.21)$$

then we can verify as in Example 2.5 that $\alpha^* = \limsup_{j \rightarrow \infty} H_j$. We omit the details and just remark that the condition (2.21) governs the distribution of the integers j such that $H_j > \bar{H} - \varepsilon$ and plays a similar role as the second condition in (2.15). Such a condition can not be eliminated in order to have $\alpha_* < \alpha^*$. For example, if $0 < \alpha_1 < \alpha_2$ and we take $H_j = \alpha_1$ when j is odd and $H_j = \alpha_2$ when j is even, then $\alpha_* = \alpha^* = \alpha_1$.

Our third example constructs a subordinator whose Laplace exponent ϕ has different upper and lower indices at 0. The method is a modification of the construction of a subordinator due to Hawkes and Pruitt [13] who were interested in the asymptotic properties of $\phi(u)$ as $u \rightarrow \infty$. Combined with Proposition 2.1, this example provides a class of Gaussian random fields which satisfy Condition (C) with $\alpha_* = 1/3$ and $\alpha^* = 1/2$.

Example 2.7. We define the Lévy measure ν of a subordinator $T = \{T(r), r \geq 0\}$ to be the discrete measure $\nu(\{x_k\}) = p_k$, where

$$p_k = 2^{-2^k} \quad \text{and} \quad x_k = p_k^{-2}, \quad \forall k \geq 1.$$

The corresponding Laplace exponent is

$$\phi(u) = \sum_{k=1}^{\infty} (1 - e^{-ux_k}) p_k. \quad (2.22)$$

Notice that $p_{k+1} = p_k^2$ for all $k \geq 1$ and, if $p_{n+1} \leq u \leq p_n$ then $1 \leq ux_n \leq p_n^{-1}$. By splitting the sum in (2.22) over $k \leq n-1$ and $k \geq n$, we derive that

$$\phi(u) \asymp u \sum_{k=1}^{n-1} x_k p_k + \sum_{k=n}^{\infty} p_k \asymp u p_{n-1}^{-1} + p_n \quad (2.23)$$

for $p_{n+1} \leq u \leq p_n$. We claim that the upper index of ϕ at 0 is $\alpha^* = 1/2$ and its lower index is $\alpha_* = 1/3$.

In order to show $\alpha_* = 1/3$, we derive from (2.23) and the fact $p_n = p_{n+1}^{1/2}$ that if $p_{n+1} \leq u \leq p_n^{3/2}$ then

$$\phi(u) \leq K(p_n^{3/2} \cdot p_n^{-1/2} + p_n) \leq K u^{1/2} \leq K u^{1/3};$$

and if $p_n^{3/2} \leq u \leq p_n$ then

$$\phi(u) \leq K(p_n \cdot p_n^{-1/2} + p_n) \leq K p_n^{1/2} \leq K u^{1/3}.$$

Thus, $\phi(u) \leq K u^{1/3}$ for all $0 \leq u \leq p_1$. Moreover, (2.23) implies

$$\phi(p_n^{3/4}) \geq K^{-1}(p_n^{3/4} \cdot p_n^{-1/2} + p_n) \geq K^{-1} p_n^{1/4} = K^{-1} (p_n^{3/4})^{1/3}.$$

This proves that $\alpha_* = 1/3$.

Similarly, (2.23) and an elementary calculus argument shows that there exists a constant $K > 1$ such that

$$\phi(u) \geq K^{-1}(u p_n^{-1/2} + p_n) \geq (2K)^{-1} u^{1/2}, \quad \forall u \in [p_{n+1}, p_n].$$

Moreover, $\phi(p_{n+1}) \leq K p_n = K (p_{n+1})^{1/2}$. Consequently, we have $\alpha^* = 1/2$. This verifies the above claims.

2.3. Packing dimension and packing dimension profile. Packing dimension and packing measure on $(\mathbb{R}^N, \|\cdot\|)$ were introduced in the early 1980s by Tricot [26] and Taylor and Tricot [25] as dual concepts to Hausdorff dimension and Hausdorff measure. The notion of packing dimension profiles was introduced by Falconer and Howroyd [11] for computing the packing dimension of orthogonal projections. Their definition of packing dimension profiles is based on potential-theoretical approach. Later Howroyd [14] defined another packing dimension profile from the point of view of box-counting dimension. Recently, Khoshnevisan and Xiao [19] proved that the packing dimension profiles of Falconer and Howroyd [11] and Howroyd [14] are the same.

For any $\varepsilon > 0$ and any bounded set $E \subset \mathbb{R}^N$, let $N(E, \varepsilon)$ be the smallest number of balls of radius ε needed to cover E . The upper box-counting dimension of E is defined as

$$\overline{\dim}_B E = \limsup_{\varepsilon \rightarrow 0} \frac{\log N(E, \varepsilon)}{-\log \varepsilon}$$

and the packing dimension of E is defined as

$$\dim_P E = \inf \left\{ \sup_n \overline{\dim}_B E_n : E \subset \bigcup_{n=1}^{\infty} E_n \right\}, \quad (2.24)$$

see Tricot [26] or Falconer ([10], p.45). It is well known that $0 \leq \dim_H E \leq \dim_P E \leq \overline{\dim}_B E \leq N$ for every set $E \subset \mathbb{R}^N$.

For a finite Borel measure μ on \mathbb{R}^N , its packing dimension is defined by

$$\dim_{\text{p}}\mu = \inf \{ \dim_{\text{p}}E : \mu(E) > 0 \text{ and } E \subset \mathbb{R}^N \text{ is a Borel set} \}. \quad (2.25)$$

Falconer and Howroyd [11] defined the s -dimensional packing dimension profile of μ as

$$\text{Dim}_s\mu = \sup \{ \beta \geq 0 : \liminf_{r \rightarrow 0} \frac{F_s^\mu(x, r)}{r^\beta} = 0 \text{ for } \mu\text{-a.a. } x \in \mathbb{R}^N \}, \quad (2.26)$$

where, for any $s > 0$, $F_s^\mu(x, r)$ is the s -dimensional potential of μ defined by

$$F_s^\mu(x, r) = \int_{\mathbb{R}^N} \min \{ 1, r^s \|y - x\|^{-s} \} d\mu(y). \quad (2.27)$$

Falconer and Howroyd [11] showed that

$$0 \leq \text{Dim}_s\mu \leq s \quad \text{and} \quad \text{Dim}_s\mu = \dim_{\text{p}}\mu \quad \text{if } s \geq N. \quad (2.28)$$

Note that the identity in (2.28) provides the following equivalent characterization of $\dim_{\text{p}}\mu$ in terms of the potential $F_N^\mu(x, r)$:

$$\dim_{\text{p}}\mu = \sup \left\{ \beta \geq 0 : \liminf_{r \rightarrow 0} \frac{F_N^\mu(x, r)}{r^\beta} = 0 \text{ for } \mu\text{-a.a. } x \in \mathbb{R}^N \right\}. \quad (2.29)$$

For any Borel set $E \subseteq \mathbb{R}^N$, the s -dimensional packing dimension profile of E is defined by

$$\text{Dim}_sE = \sup \{ \text{Dim}_s\mu : \mu \in \mathcal{M}_c^+(E) \}, \quad (2.30)$$

where $\mathcal{M}_c^+(E)$ denotes the family of finite Borel measures with compact support in E . It follows from (2.28) that $0 \leq \text{Dim}_sE \leq s$ and $\text{Dim}_sE = \dim_{\text{p}}E$ if $s \geq N$. This last fact gives a measure-theoretic characterization of $\dim_{\text{p}}E$ in terms of packing dimension profiles.

3. Packing Dimension and Packing Dimension Profile on Anisotropic Metric Spaces

Ordinary Hausdorff and packing dimension (i.e. those in the Euclidean metric) may not be able to characterize the Hausdorff and packing dimensions of the images of anisotropic random fields, and the notion of Hausdorff dimension on the metric space (\mathbb{R}^N, ρ) is needed; see Wu and Xiao [27, 28] and Xiao [32]. In this section, we define packing measure, packing dimension and packing dimension profiles on the metric space (\mathbb{R}^N, ρ) . The latter is an extension of the notion of packing dimension profiles of Falconer and Howroyd [11] to (\mathbb{R}^N, ρ) . We believe it will have applications beyond scope of this paper.

Throughout this paper, let

$$B_\rho(x, r) := \{ y \in \mathbb{R}^N : \rho(y, x) < r \}.$$

For any $\beta > 0$ and $E \subseteq \mathbb{R}^N$, the β -dimensional packing measure of E in the metric ρ is defined by

$$\mathcal{P}_\rho^\beta(E) = \inf \left\{ \sum_n \overline{\mathcal{P}}_\rho^\beta(E_n) : E \subseteq \bigcup_n E_n \right\}, \quad (3.1)$$

where the infimum is taken over all countable coverings $\{E_n\}$ of E and where

$$\overline{\mathcal{P}}_\rho^\beta(E) = \limsup_{\delta \rightarrow 0} \left\{ \sum_{n=1}^{\infty} (2r_n)^\beta : \{B_\rho(x_n, r_n)\} \text{ are disjoint, } x_n \in E, r_n \leq \delta \right\}. \quad (3.2)$$

The packing dimension of E is defined by

$$\dim_{\mathbb{P}}^\rho E = \inf \{ \beta > 0 : \overline{\mathcal{P}}_\rho^\beta(E) = 0 \}. \quad (3.3)$$

It can be verified directly that $\dim_{\mathbb{P}}^\rho$ is σ -stable: for any sequence of sets $E_n \subseteq \mathbb{R}^N$, we have

$$\dim_{\mathbb{P}}^\rho \left(\bigcup_{n=1}^{\infty} E_n \right) = \sup_n \dim_{\mathbb{P}}^\rho E_n. \quad (3.4)$$

Similarly to the Euclidean case studied by Tricot [26] (see also Falconer [10]), the packing dimension in (\mathbb{R}^N, ρ) can also be defined through the upper box-counting dimension. For any $\varepsilon > 0$ and any bounded set $E \subseteq \mathbb{R}^N$, let $N_\rho(E, \varepsilon)$ be the smallest number of balls of radius ε (in the metric ρ) needed to cover E . The upper box-counting dimension (in the metric ρ) of E is defined as

$$\overline{\dim}_{\mathbb{B}}^\rho E = \limsup_{\varepsilon \rightarrow 0} \frac{\log N_\rho(E, \varepsilon)}{-\log \varepsilon}.$$

The following proposition is an extension of a result of Tricot [26].

Proposition 3.1. *For any set $E \subseteq \mathbb{R}^N$, we have*

$$\dim_{\mathbb{P}}^\rho E = \inf \left\{ \sup_n \overline{\dim}_{\mathbb{B}}^\rho E_n : E \subseteq \bigcup_{n=1}^{\infty} E_n \right\}. \quad (3.5)$$

Proof. The proof is a slight modification of the proof of that in Tricot (1982) for the Euclidean metric, see also Falconer (1990, p.45). For completeness we give it in the Appendix. \square

Throughout this paper, let $Q := \sum_{j=1}^N H_j^{-1}$. It follows from the definition of $\dim_{\mathbb{H}}^\rho$ (cf. Xiao [32]), (3.2) and Proposition 3.1 that for every set $E \subseteq \mathbb{R}^N$,

$$0 \leq \dim_{\mathbb{H}}^\rho E \leq \dim_{\mathbb{P}}^\rho E \leq \overline{\dim}_{\mathbb{B}}^\rho E \leq Q. \quad (3.6)$$

Moreover, if E has non-empty interior, then $\dim_{\mathbb{H}}^\rho E = \dim_{\mathbb{P}}^\rho E = Q$.

For a finite Borel measure μ on \mathbb{R}^N , similarly to (2.25) we define its packing dimension in metric ρ as

$$\dim_{\mathbb{P}}^\rho \mu = \inf \{ \dim_{\mathbb{P}}^\rho E : \mu(E) > 0 \text{ and } E \subseteq \mathbb{R}^N \text{ is a Borel set} \}. \quad (3.7)$$

The following proposition gives a characterization of $\dim_{\mathbb{P}}^\rho \mu$ in terms of the local dimension of μ . It is obtained by applying Lemma 4.1 (cf. (4.7)) of Hu and Taylor [15] to $\dim_{\mathbb{P}}^\rho$.

Proposition 3.2. *Let μ be a finite Borel measure on \mathbb{R}^N . Then*

$$\dim_{\mathbb{P}}^\rho \mu = \sup \left\{ \beta > 0 : \liminf_{r \rightarrow 0} \frac{\mu(B_\rho(x, r))}{r^\beta} = 0 \text{ for } \mu\text{-a.a. } x \in \mathbb{R}^N \right\}. \quad (3.8)$$

Extending the definition of Falconer and Howroyd [11], we define the following s -dimensional packing dimension profile of μ in metric ρ as

$$\text{Dim}_s^\rho \mu = \sup \left\{ \beta \geq 0 : \liminf_{r \rightarrow 0} \frac{F_{s,\rho}^\mu(x,r)}{r^\beta} = 0 \text{ for } \mu\text{-a.a. } x \in \mathbb{R}^N \right\}, \quad (3.9)$$

where, for any $s > 0$, $F_{s,\rho}^\mu(x,r)$ is the s -dimensional potential of μ in metric ρ defined by

$$F_{s,\rho}^\mu(x,r) = \int_{\mathbb{R}^N} \min \left\{ 1, \frac{r^s}{\rho(x,y)^s} \right\} d\mu(y). \quad (3.10)$$

The following proposition is analogous to Proposition 2 in Falconer and Howroyd [11] and we provide a proof in the Appendix.

Proposition 3.3. *Let μ be a finite Borel measure on \mathbb{R}^N and let $s \in (0, Q]$. Then for μ -almost all $x \in \mathbb{R}^N$ the following holds: If*

$$\liminf_{r \rightarrow 0} r^{-s} \mu(B_\rho(x,r)) < \infty, \quad (3.11)$$

then for all $0 \leq t < s$,

$$\liminf_{r \rightarrow 0} r^{-t} F_{Q,\rho}^\mu(x,r) = 0. \quad (3.12)$$

To prove a similar result as Proposition 18 in Falconer and Howroyd [11], we define a local variant of Dim_s^ρ by

$$p_{x,\rho}(s) = \sup \{ t \geq 0 : \liminf_{r \rightarrow 0} r^{-t} F_{s,\rho}^\mu(x,r) = 0 \}, \quad \forall x \in \mathbb{R}^N.$$

Note that, by writing $\mu(B_\rho(x,r))$ as $b_{x,\rho}(r)$,

$$F_{s,\rho}^\mu(x,r) = b_{x,\rho}(r) + r^s \int_r^\infty h^{-s} db_{x,\rho}(h) = sr^s \int_r^\infty h^{-s-1} b_{x,\rho}(h) dh. \quad (3.13)$$

For $0 \leq s \leq t$, we have $\mu(\mathbb{R}^N) \geq F_{s,\rho}^\mu(x,r) \geq F_{t,\rho}^\mu(x,r)$, which gives us that

$$0 \leq p_{x,\rho}(s) \leq p_{x,\rho}(t). \quad (3.14)$$

Since we also have

$$\mu(\mathbb{R}^N) \geq F_{s,\rho}^\mu(x,r) \geq r^s \int_r^\infty h^{-s} db_{x,\rho}(h)$$

and $\int_r^\infty h^{-s} db_{x,\rho}(h)$ increases as r decreases and is positive for sufficiently small r , we obtain that

$$p_{x,\rho}(s) \leq s. \quad (3.15)$$

By noting that $F_{s,\rho}^\mu(x,r) \geq b_{x,\rho}(r) = \mu(B_\rho(x,r))$, we prove

$$p_{x,\rho}(s) \leq \sup \{ t \geq 0 : \liminf_{r \rightarrow 0} r^{-t} \mu(B_\rho(x,r)) = 0 \}. \quad (3.16)$$

By the same token as that of the proof of Proposition 16 in Falconer and Howroyd [11], we also can derive that for $0 < s \leq t < \infty$,

$$p_{x,\rho}(s) \geq \frac{p_{x,\rho}(t)}{1 + (1/s - 1/t)p_{x,\rho}(t)}. \quad (3.17)$$

Clearly, (3.17) and (3.15) are equivalent to the following: $p_{x,\rho}(0) = 0$ and for all $0 < s \leq t < \infty$,

$$0 \leq \frac{1}{p_{x,\rho}(s)} - \frac{1}{s} \leq \frac{1}{p_{x,\rho}(t)} - \frac{1}{t}. \quad (3.18)$$

By Proposition 3.3, we have that for μ -almost all $x \in \mathbb{R}^N$,

$$p_{x,\rho}(Q) \geq \sup\{t \geq 0 : \liminf_{r \rightarrow 0} r^{-t} \mu(B_\rho(x, r)) = 0\}. \quad (3.19)$$

Combining (3.19) with (3.16), (3.14) and (3.15), we have that for μ -almost all $x \in \mathbb{R}^N$ and for all $t \geq Q$,

$$p_{x,\rho}(t) = p_{x,\rho}(Q) \leq Q. \quad (3.20)$$

Proposition 3.4. *For any finite Borel measure μ on \mathbb{R}^N ,*

$$0 \leq \text{Dim}_s^\rho \mu \leq s \text{ and } \text{Dim}_s^\rho \mu = \dim_p^\rho \mu \text{ if } s \geq Q. \quad (3.21)$$

Furthermore, $\text{Dim}_s^\rho \mu$ is continuous in s .

Proof. This follows immediately from (3.18), the definitions of $\text{Dim}_s^\rho \mu$ (cf. (3.9)) and Proposition 3.2. \square

Note that the identity in (3.21) provides the following equivalent characterization of $\dim_p^\rho \mu$ in terms of the potential $F_{Q,\rho}^\mu(x, r)$, where $Q = \sum_{j=1}^N H_j^{-1}$:

$$\dim_p^\rho \mu = \sup \left\{ \beta \geq 0 : \liminf_{r \rightarrow 0} \frac{F_{Q,\rho}^\mu(x, r)}{r^\beta} = 0 \text{ for } \mu\text{-a.a. } x \in \mathbb{R}^N \right\}.$$

For any Borel set $E \subseteq \mathbb{R}^N$, the s -dimensional packing dimension profile of E in the metric ρ is defined by

$$\text{Dim}_s^\rho E = \sup \{ \text{Dim}_s^\rho \mu : \mu \in \mathcal{M}_c^+(E) \}, \quad (3.22)$$

where $\mathcal{M}_c^+(E)$ denotes the family of finite Borel measures with compact support in E . It follows from (3.21) that

$$0 \leq \text{Dim}_s^\rho E \leq s \quad \text{and} \quad \text{Dim}_s^\rho E = \dim_p^\rho E \quad \text{if } s \geq Q. \quad (3.23)$$

4. Packing Dimension Results

Now we consider the packing dimensions of the range of an (N, d) -Gaussian random field. We will assume throughout the rest of this paper that

$$0 < H_1 \leq \dots \leq H_N < 1. \quad (4.1)$$

Recall that $Q = \sum_{j=1}^N \frac{1}{H_j}$.

4.1. Packing dimension of $X([0, 1]^N)$. First we consider the packing dimension of $X([0, 1]^N)$. The following result shows that it is determined by the lower index of ϕ and (H_1, \dots, H_N) .

Theorem 4.1. *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be the Gaussian random field in \mathbb{R}^d defined by (1.1). We assume that the associated random field X_0 satisfies Condition (C). Suppose ϕ is such that $0 < \alpha_* \leq \alpha^* < 1$ and satisfies one of the following two conditions: For all sufficiently small $\varepsilon > 0$, there exists a constant K such that*

$$\int_0^N \left(\frac{1}{\phi(x)} \right)^{d-\varepsilon} x^{Q-1} dx \leq K \quad (4.2)$$

or

$$\int_1^{N/a} \left(\frac{\phi(a)}{\phi(ax)} \right)^{d-\varepsilon} x^{Q-1} dx \leq K a^{-\varepsilon} \quad \text{for all } a \in (0, 1]. \quad (4.3)$$

Then with probability 1,

$$\dim_{\text{p}} X([0, 1]^N) = \min \left\{ d; \sum_{j=1}^N \frac{1}{\alpha_* H_j} \right\}. \quad (4.4)$$

We will prove that with probability 1, $\min \left\{ d; \sum_{j=1}^N \frac{1}{\alpha_* H_j} \right\}$ is an upper bound and a lower bound for $\dim_{\text{p}} X([0, 1]^N)$ separately. The upper bound is proved by using the modulus of continuity of X and a covering argument, and the proof of the lower bound is based on the potential-theoretic approach to packing dimension (see (2.29)) of finite Borel measures.

For any Borel measure μ on \mathbb{R}^N , the image measure of μ under the mapping $t \mapsto f(t)$ is defined by

$$(\mu \circ f^{-1})(B) := \mu(\{t \in \mathbb{R}^N : f(t) \in B\}) \quad \text{for all Borel sets } B \subset \mathbb{R}^d.$$

The following lemma was proved in Xiao [29], which relates $\dim_{\text{p}} f(E)$ with the packing dimensions of the image measures.

Lemma 4.2. *Let $E \subset \mathbb{R}^N$ be an analytic set. Then for any continuous function $f : \mathbb{R}^N \rightarrow \mathbb{R}^d$*

$$\dim_{\text{p}} f(E) = \sup \left\{ \dim_{\text{p}} (\mu \circ f^{-1}) : \mu \in \mathcal{M}_c^+(E) \right\}. \quad (4.5)$$

We are now ready to prove Theorem 4.1.

Proof. We first prove the upper bound in (4.4). Since $\dim_{\text{p}} X([0, 1]^N) \leq d$ a.s., it is sufficient to show that $\dim_{\text{p}} X([0, 1]^N) \leq Q/\alpha_*$ a.s. For any $\varepsilon \in (0, \alpha_*)$, Lemma 2.2 implies that $X(t)$ satisfies almost surely the following uniform Hölder condition

$$\|X(s) - X(t)\| \leq K(\omega) \rho(s, t)^{\alpha_* - \varepsilon}, \quad \forall s, t \in [0, 1]^N.$$

Hence a standard covering argument (e.g., Xiao [32]) shows that

$$\overline{\dim}_{\text{B}} X([0, 1]^N) \leq Q/(\alpha_* - \varepsilon) \quad \text{a.s.}$$

This implies $\dim_{\text{p}} X([0, 1]^N) \leq Q/(\alpha_* - \varepsilon)$ a.s. Letting $\varepsilon \downarrow 0$ along the sequence of rational numbers yields the desired upper bound.

Now we proceed to prove the lower bound in (4.4). By Lemma 4.2, we have $\dim_{\mathbb{P}} X([0, 1]^N) \geq \dim_{\mathbb{P}}(m_N \circ X^{-1})$ a. s., where m_N is the restriction of the N -dimensional Lebesgue measure on $[0, 1]^N$. Hence it is sufficient to show that

$$\dim_{\mathbb{P}}(m_N \circ X^{-1}) \geq \min \left\{ d, \frac{Q}{\alpha_*} \right\}, \quad \text{a.s.} \quad (4.6)$$

For simplicity of notation, we will, from now on, denote the image measure $m_N \circ X^{-1}$ by μ_X .

Note that, for every fixed $s \in \mathbb{R}^N$, Fubini's theorem implies

$$\begin{aligned} \mathbb{E} F_d^{\mu_X}(X(s), r) &= \mathbb{E} \int_{\mathbb{R}^d} \min \{1, r^d \|v - X(s)\|^{-d}\} d\mu_X(v) \\ &= \int_{[0, 1]^N} \mathbb{E} \min \{1, r^d \|X(t) - X(s)\|^{-d}\} dt. \end{aligned} \quad (4.7)$$

The last integrand in (4.7) can be written as

$$\begin{aligned} &\mathbb{E} \min \{1, r^d \|X(t) - X(s)\|^{-d}\} \\ &= \mathbb{P}\{\|X(t) - X(s)\| \leq r\} + \mathbb{E}\{r^d \|X(t) - X(s)\|^{-d} \cdot \mathbf{1}_{\{\|X(t) - X(s)\| \geq r\}}\}. \end{aligned} \quad (4.8)$$

By Condition (C), we obtain that for all $s, t \in [0, 1]^N$ and $r > 0$,

$$\mathbb{P}\{\|X(t) - X(s)\| \leq r\} \leq K \min \left\{ 1, \frac{r^d}{\phi(\rho(t, s))^d} \right\}. \quad (4.9)$$

Denote the distribution of $X(t) - X(s)$ by $\Gamma_{s,t}(\cdot)$. Let ν be the image measure of $\Gamma_{s,t}(\cdot)$ under the mapping $T : z \mapsto \|z\|$ from \mathbb{R}^d to \mathbb{R}_+ . Then the second term in (4.8) can be written as

$$\begin{aligned} \int_{\mathbb{R}^d} \frac{r^d}{\|z\|^d} \mathbf{1}_{\{\|z\| \geq r\}} \Gamma_{s,t}(dz) &= \int_r^\infty \frac{r^d}{u^d} \nu(du) \\ &\leq d \int_r^\infty \frac{r^d}{u^{d+1}} \mathbb{P}\{\|X(t) - X(s)\| \leq u\} du, \end{aligned} \quad (4.10)$$

where the last inequality follows from an integration-by-parts formula.

Hence, by (4.9) and (4.10) we see that, to within a constant, the second term in (4.8) is bounded by

$$\begin{aligned} &r^d \int_r^\infty \frac{1}{u^{d+1}} \min \left\{ 1, \left(\frac{u}{\phi(\rho(t, s))} \right)^d \right\} du \\ &\leq K \begin{cases} 1 & \text{if } r \geq \phi(\rho(t, s)), \\ \left(\frac{r}{\phi(\rho(t, s))} \right)^d \log \left(\frac{\phi(\rho(t, s))}{r} \right) & \text{if } r < \phi(\rho(t, s)). \end{cases} \end{aligned} \quad (4.11)$$

It follows from (4.8), (4.9), (4.10) and (4.11) that for any $0 < \varepsilon < 1$ and $s, t \in [0, 1]^N$,

$$\mathbb{E} \min \{1, r^d \|X(t) - X(s)\|^{-d}\} \leq K \min \left\{ 1, \left(\frac{r}{\phi(\rho(t, s))} \right)^{d-\varepsilon} \right\}. \quad (4.12)$$

Combining (4.7) and (4.12) we derive

$$\mathbb{E}F_d^{\mu_x}(X(s), r) \leq K \int_{[0,1]^N} \min \left\{ 1, \left(\frac{r}{\phi(\rho(0, t-s))} \right)^{d-\varepsilon} \right\} dt. \quad (4.13)$$

Let us consider the diagonal matrix $D = \text{diag}(1/H_1, \dots, 1/H_N)$. Then, $t \mapsto \rho(0, t)$ is D -homogeneous function in the sense of Definition 2.6 of Biermé, et al. [6], that is $\rho(0, r^D t) = r \rho(0, t)$ for all $r > 0$, where $r^D := \exp(\log(r)D)$. By a variable substitution to the integral in (4.13) (see Proposition 2.3 in Biermé, et al. [6]), we obtain

$$\begin{aligned} \mathbb{E}F_d^{\mu_x}(X(s), r) &\leq K \int_0^N \min \left\{ 1, \left(\frac{r}{\phi(x)} \right)^{d-\varepsilon} \right\} x^{Q-1} dx \\ &= K \left\{ \int_0^{\phi^{-1}(r)} x^{Q-1} dx + \int_{\phi^{-1}(r)}^N \left(\frac{r}{\phi(x)} \right)^{d-\varepsilon} x^{Q-1} dx \right\} \\ &:= I_1 + I_2. \end{aligned} \quad (4.14)$$

In the above, $\phi^{-1}(x) = \inf\{y : \phi(y) > x\}$ is the right-continuous inverse function of ϕ . It can be seen that ϕ^{-1} is non-decreasing and satisfies $\phi(\phi^{-1}(x)) = x$ and $\lim_{x \rightarrow 0} \phi^{-1}(x) = 0$.

Let us estimate I_1 and I_2 . Clearly, we have

$$I_1 = K [\phi^{-1}(r)]^Q. \quad (4.15)$$

To estimate I_2 , we distinguish two cases. If ϕ satisfies (4.2), then for all $r > 0$ small enough, we derive

$$I_2 \leq K r^{d-\varepsilon} \int_0^N \left(\frac{1}{\phi(x)} \right)^{d-\varepsilon} x^{Q-1} dx \leq K r^{d-\varepsilon}. \quad (4.16)$$

On the other hand, if ϕ satisfies (4.3), then we make a change of variable $x = \phi^{-1}(r)y$ to derive that for all $r > 0$ small enough,

$$\begin{aligned} I_2 &\leq K [\phi^{-1}(r)]^Q \int_1^{N/\phi^{-1}(r)} \frac{r^{d-\varepsilon}}{\phi(\phi^{-1}(r)y)^{d-\varepsilon}} y^{Q-1} dy \\ &\leq K [\phi^{-1}(r)]^{Q-\varepsilon}. \end{aligned} \quad (4.17)$$

It follows from (4.14), (4.15), (4.16) and (4.17) that for all $r > 0$ small enough,

$$\mathbb{E}F_d^{\mu_x}(X(s), r) \leq K \left\{ [\phi^{-1}(r)]^{Q-\varepsilon} + r^{d-\varepsilon} \right\}. \quad (4.18)$$

Now for any $0 < \gamma < \min\{d, Q/\alpha_*\}$, we choose $\varepsilon > 0$ small so that

$$\gamma < \frac{Q-2\varepsilon}{\alpha_*} \quad \text{and} \quad \gamma < d-\varepsilon. \quad (4.19)$$

By the first inequality in (4.19), we see that there exists a sequence $\rho_n \rightarrow 0$ such that

$$\phi(\rho_n) \geq \rho_n^{(Q-2\varepsilon)/\gamma} \quad \text{for all integers } n \geq 1. \quad (4.20)$$

We choose a sequence $\{r_n, n \geq 1\}$ of positive numbers such that $\phi^{-1}(r_n) = \rho_n$. Then $\phi(\rho_n) = r_n$ and $\lim_{n \rightarrow \infty} r_n = 0$.

By Fatou's lemma and (4.18) we obtain that for every $s \in [0, 1]^N$,

$$\begin{aligned} \mathbb{E} \left(\liminf_{r \rightarrow 0} \frac{F_d^{\mu_X}(X(s), r)}{r^\gamma} \right) &\leq K \liminf_{n \rightarrow \infty} \frac{[\phi^{-1}(r_n)]^{Q-\varepsilon} + r_n^{d-\varepsilon}}{r_n^\gamma} \\ &\leq K \liminf_{n \rightarrow \infty} \left\{ \frac{\rho_n^{Q-\varepsilon}}{\phi(\rho_n)^\gamma} + \phi(\rho_n)^{d-\gamma-\varepsilon} \right\} = 0. \end{aligned}$$

In deriving the last equality, we have made use of (4.19) and (4.20).

By using Fubini's theorem again, we see that almost surely,

$$\liminf_{r \rightarrow 0} \frac{F_d^{\mu_X}(X(s), r)}{r^\gamma} = 0 \quad \text{for } m_N\text{-a.a. } s \in \mathbb{R}^N.$$

This and (2.29) together imply $\dim_{\mathbb{P}} \mu_X \geq \gamma$ almost surely. Since γ can be arbitrarily close to $\min\{d, Q/\alpha_*\}$, we have proved (4.6). This finishes the proof of Theorem 4.1. \square

4.2. Packing dimension of $X(E)$. To determine the packing dimension of $X(E)$, we will make use of the following lemma, which is a generalization of Lemma 2.2 in Xiao [29].

Lemma 4.3. *Let T be any compact interval in \mathbb{R}^N and let $g : T \rightarrow \mathbb{R}^d$ be a continuous function satisfying the following condition: For some constant $\alpha \in (0, 1]$ and any $\varepsilon \in (0, \alpha)$, there exists a constant $K > 0$ such that*

$$|g(x) - g(y)| \leq K \rho(x, y)^{\alpha-\varepsilon}, \quad \forall x, y \in T. \quad (4.21)$$

Then for any finite Borel measure μ on \mathbb{R}^N with support contained in T , we have

$$\dim_{\mathbb{P}} \mu_g \leq \frac{1}{\alpha} \text{Dim}_{\alpha d}^\rho \mu, \quad (4.22)$$

where $\mu_g = \mu \circ g^{-1}$ is the image measure of μ .

Proof. We first prove that for any $\varepsilon \in (0, \alpha)$, we have

$$\dim_{\mathbb{P}} \mu_g \leq \frac{1}{\alpha - \varepsilon} \text{Dim}_{(\alpha - \varepsilon)d}^\rho \mu. \quad (4.23)$$

Take any $\gamma < \dim_{\mathbb{P}} \mu_g$, by (2.29) we have

$$\liminf_{r \rightarrow 0} r^{-\gamma} \int_{\mathbb{R}^d} \min\{1, r^d \|v - u\|^{-d}\} \mu_g(dv) = 0 \quad \mu_g\text{-a.a. } u \in \mathbb{R}^d,$$

that is, for μ -almost all $x \in \mathbb{R}^N$,

$$\liminf_{r \rightarrow 0} r^{-\gamma} \int_T \min\{1, r^d \|g(y) - g(x)\|^{-d}\} \mu(dy) = 0. \quad (4.24)$$

By (4.21) we have

$$\min\{1, r^d \|g(y) - g(x)\|^{-d}\} \geq K \min\{1, r^d \rho(x, y)^{-(\alpha-\varepsilon)d}\}. \quad (4.25)$$

It follows from (4.24) and (4.25) that for μ -almost all $x \in \mathbb{R}^N$,

$$\liminf_{r \rightarrow 0} r^{-(\alpha-\varepsilon)\gamma} \int_{\mathbb{R}^N} \min\{1, r^{(\alpha-\varepsilon)d} \rho(x, y)^{-(\alpha-\varepsilon)d}\} \mu(dy) = 0, \quad (4.26)$$

which implies, by the definition (3.9), that $\text{Dim}_{(\alpha-\varepsilon)d}^\rho \mu \geq (\alpha - \varepsilon)\gamma$. Since $\gamma < \dim_{\mathbb{P}} \mu_g$ is arbitrary, we have proved (4.23). Letting $\varepsilon \downarrow 0$ and applying Proposition 3.4, we have proved (4.22). \square

Theorem 4.4. *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be the Gaussian random field in \mathbb{R}^d defined by (1.1). We assume that the associated random field X_0 satisfies Condition (C) and $0 < \alpha_* \leq \alpha^* < 1$. Let μ be any finite Borel measure on \mathbb{R}^N . Then with probability 1,*

$$\frac{1}{\alpha^*} \text{Dim}_{\alpha^*d}^\rho \mu \leq \dim_{\mathbb{P}} \mu_X \leq \frac{1}{\alpha_*} \text{Dim}_{\alpha_*d}^\rho \mu. \quad (4.27)$$

Proof. By following the first half of the proof of Theorem 3.1 in Xiao [29], and by Lemmas 2.2 and 4.3, we derive that

$$\dim_{\mathbb{P}} \mu_X \leq \frac{1}{\alpha_*} \text{Dim}_{\alpha_*d}^\rho \mu \quad \text{a.s.} \quad (4.28)$$

To prove the reverse inequality, by Fubini's Theorem, for any $s \in \mathbb{R}^N$,

$$\begin{aligned} \mathbb{E} [F_d^{\mu_X}(X(s), r)] &= \int_{\mathbb{R}^N} \mathbb{E} [\min \{1, r^d \|X(t) - X(s)\|^{-d}\}] \mu(dt) \\ &\leq K \int_{\mathbb{R}^N} \min \{1, r^{d-\varepsilon} \rho(s, t)^{-\alpha^*(d-\varepsilon)}\} \mu(dt) \end{aligned} \quad (4.29)$$

where the last inequality follows from (4.12).

For any $\gamma < \text{Dim}_{\alpha^*d}^\rho \mu$, by Proposition 3.4, there exists $\varepsilon > 0$ such that $\gamma \leq \text{Dim}_{\alpha^*(d-\varepsilon)}^\rho \mu$. It follows from (3.9) that

$$\liminf_{r \rightarrow 0} r^{-\frac{\gamma}{\alpha^*}} \int_{\mathbb{R}^N} \min \{1, r^{d-\varepsilon} \rho(s, t)^{-\alpha^*(d-\varepsilon)}\} \mu(dt) = 0 \quad \text{for } \mu\text{-a.a. } s \in \mathbb{R}^N. \quad (4.30)$$

By (4.29) and (4.30), we have that for μ -almost all $s \in \mathbb{R}^N$

$$\begin{aligned} &\mathbb{E} \left[\liminf_{r \rightarrow 0} r^{-\frac{\gamma}{\alpha^*}} F_d^{\mu_X}(X(s), r) \right] \\ &\leq K \liminf_{r \rightarrow 0} r^{-\frac{\gamma}{\alpha^*}} \int_{\mathbb{R}^N} \min \{1, r^{d-\varepsilon} \rho(s, t)^{-\alpha^*(d-\varepsilon)}\} \mu(dt) = 0. \end{aligned}$$

By applying Fubini's Theorem, we see that with probability 1

$$\liminf_{r \rightarrow 0} r^{-\frac{\gamma}{\alpha^*}} F_d^{\mu_X}(X(s), r) = 0 \quad \text{for } \mu\text{-a.a. } s \in \mathbb{R}^N,$$

which implies $\dim_{\mathbb{P}} \mu_X \geq \frac{\gamma}{\alpha^*}$ a.s. Since γ can be arbitrarily close to $\text{Dim}_{\alpha^*d}^\rho \mu$, we have

$$\dim_{\mathbb{P}} \mu_X \geq \frac{1}{\alpha^*} \text{Dim}_{\alpha^*d}^\rho \mu \quad \text{a.s.} \quad (4.31)$$

Combining (4.28) and (4.31), we prove Theorem 4.4. \square

The following theorem determines the packing dimension of the image $X(E)$ for an arbitrary analytic set $E \subseteq [0, 1]^N$ when $\alpha^* = \alpha_*$.

Theorem 4.5. *If, in additions to the assumptions in Theorem 4.4, $0 < \alpha_* = \alpha^* < 1$, then for every analytic set $E \subseteq [0, 1]^N$, we have that*

$$\dim_{\mathbb{P}} X(E) = \frac{1}{\alpha} \text{Dim}_{\alpha_d}^{\rho} E \quad \text{a.s.},$$

where $\alpha := \alpha^* = \alpha_*$.

Proof. By Theorem 4.4, we have that for any finite Borel measure μ on \mathbb{R}^N ,

$$\dim_{\mathbb{P}} \mu_X = \frac{1}{\alpha} \text{Dim}_{\alpha_d}^{\rho} \mu \quad \text{a.s.} \quad (4.32)$$

The rest of the proof of Theorem 4.5 is reminiscent to the proof of Theorem 4.1 in Xiao [29], with the help of (4.32). We omit it here. \square

Remark 4.6. When $\alpha^* \neq \alpha_*$, the problem of determining the packing dimension of $X(E)$, where $E \subseteq \mathbb{R}^N$ is a Borel set, remains open. In order to solve this problem, a more general form of packing dimension profile needs to be introduced. A promising approach is to combine the method in Section 3 with that in Khoshnevisan, Schilling and Xiao [18]. Since this more general approach does not rely on upper or lower indices of ϕ and may not provide an explicit formula as obtained in Theorem 4.1, we will develop it elsewhere.

5. Appendix

In this Appendix, we provide proofs for Propositions 3.1 and 3.3. Also for comparison purpose and completeness, we determine the Hausdorff dimensions of the range $X([0, 1]^N)$ and graph $\text{Gr}X([0, 1]^N)$.

5.1. Proofs of Propositions 3.1 and 3.3. The proof of Proposition 3.1 is a slight modification of the proof of that in Tricot (1982) for the Euclidean metric, see also Falconer (1990, p.45). It is included for completeness.

Proof of Proposition 3.1 First, we prove that for $E \subseteq \mathbb{R}^N$,

$$\dim_{\mathbb{P}}^{\rho} E \leq \overline{\dim}_{\mathbb{B}}^{\rho} E. \quad (5.1)$$

In fact, for any fixed $\gamma < \beta < \dim_{\mathbb{P}}^{\rho} E$, we have $\mathcal{P}_{\rho}^{\beta}(E) = \infty$ and thus $\overline{\mathcal{P}}_{\rho}^{\beta}(E) = \infty$. Therefore, for a given $0 < \delta \leq 1$, there exists a family of disjoint $\{B_{\rho}(x_i, r_i)\}$, where $x_i \in E$ and $r_i \leq \delta$, such that $1 \leq \sum_{i=1}^{\infty} (2r_i)^{\beta}$. Suppose, for every nonnegative integer k , there are n_k ρ -balls satisfying $2^{-k-2} < r_i \leq 2^{-k-1}$, then $1 \leq \sum_{k=0}^{\infty} n_k 2^{-k\beta}$, which implies that there exists an k_0 such that $n_{k_0} > 2^{k_0\gamma} (1 - 2^{\gamma-\beta})$. Furthermore, each of these n_{k_0} ρ -balls contains a ρ -ball centered in E with radius $2^{-k_0-2} \leq \delta$. Let $N_{\rho}(E, \varepsilon)$ be the largest number of disjoint ρ -balls centered in E with radius ε , then

$$N_{\rho}(E, 2^{-k_0-2}) (2^{-k_0-2})^{\gamma} \geq n_{k_0} (2^{-k_0-2})^{\gamma} > 2^{-2\gamma} (1 - 2^{\gamma-\beta}), \quad (5.2)$$

where $2^{-k_0-2} \leq \delta$. Therefore, $\limsup_{\delta \downarrow 0} N_{\rho}(E, \delta) \delta^{\gamma} > 0$, which implies that for every $\gamma < \dim_{\mathbb{P}}^{\rho} E$ we have $\overline{\dim}_{\mathbb{B}}^{\rho} E \geq \gamma$. This proves (5.1).

Now we are ready to prove (3.5). If $E \subseteq \bigcup_n E_n$, by (3.4) and (5.1), we have

$$\dim_{\mathbb{P}}^{\rho} E \leq \sup_n \dim_{\mathbb{P}}^{\rho} E_n \leq \sup_n \overline{\dim}_{\mathbb{B}}^{\rho} E_n, \quad (5.3)$$

which proves

$$\dim_{\mathbb{P}}^{\rho} E \leq \inf \left\{ \sup_n \overline{\dim}_{\mathbb{B}}^{\rho} E_n : E \subseteq \bigcup_{n=1}^{\infty} E_n \right\}. \quad (5.4)$$

Conversely, if $\beta > \dim_{\mathbb{P}}^{\rho} E$, then $s^{\beta-p_{\rho}}(E) = 0$. Hence there exists a sequence $\{E_n\}$ such that $E \subseteq \bigcup E_n$ and $\sum_{n=1}^{\infty} \overline{\mathcal{P}}_{\rho}^{\beta}(E_n) < \infty$. By (3.2), we have that $N_{\rho}(E_n, \delta)\delta^{\beta}$ is bounded when δ is sufficiently small. Therefore, for each n , $\overline{\dim}_{\mathbb{B}}^{\rho} E_n \leq \beta$, which implies

$$\dim_{\mathbb{P}}^{\rho} E \geq \inf \left\{ \sup_n \overline{\dim}_{\mathbb{B}}^{\rho} E_n : E \subseteq \bigcup_{n=1}^{\infty} E_n \right\}. \quad (5.5)$$

Combining (5.4) and (5.5) yields (3.5). \square

In order to prove Proposition 3.3, we will make use of the following lemma, which is an extension of Corollary 2.3 of Falconer and Mattila [12] (see also Lemma 1 of Falconer and Howroyd [11]) to the metric space (\mathbb{R}^N, ρ) . Since the proof essentially follows the same idea as the proofs of Lemma 2.1 and Lemma 2.2 of Falconer and Mattila [12], we omit it.

Lemma 5.1. *Let $0 < a < 1$ and $\varepsilon > 0$. For every finite Borel measure μ on \mathbb{R}^N the following holds for μ -almost all x : If $r > 0$ is sufficiently small, then for all h with $r^a \leq h \leq 1$ we have*

$$\mu(B_{\rho}(x, h)) \leq \left(\frac{4h}{r}\right)^{Q(1+\varepsilon)} \mu(B_{\rho}(x, r)). \quad (5.6)$$

Proof of Proposition 3.3 We fix $0 \leq t < s$. Choose $\varepsilon > 0$ and $0 < a < 1$ such that

$$Q\varepsilon < s - t \quad \text{and} \quad Q(1+\varepsilon)(1-a) < s - t. \quad (5.7)$$

Suppose $x \in \mathbb{R}^N$ is such that the conclusion of Lemma 5.1 and (3.11) hold. Again, write $\mu(B_{\rho}(x, r))$ as $b_{x,\rho}(r)$, then by (3.13), Lemma 5.1 and (3.11) we derive

$$\begin{aligned} F_{Q,\rho}^{\mu}(x, r) &= Qr^Q \int_r^{\infty} h^{-Q-1} b_{x,\rho}(h) dh \\ &\leq Qr^Q \int_r^{r^a} h^{-Q-1} b_{x,\rho}(r^a) dh + Qr^Q \int_{r^a}^1 h^{-Q-1} (b_{x,\rho}(r)(4h/r)^{Q(1+\varepsilon)}) dh \\ &\quad + Qr^Q \int_1^{\infty} h^{-Q-1} \mu(\mathbb{R}^N) dh \\ &\leq (4r^{a-1})^{Q(1+\varepsilon)} b_{x,\rho}(r) + 4^{Q(1+\varepsilon)} (Q\varepsilon)^{-1} r^{-Q\varepsilon} b_{x,\rho}(r) + r^Q \mu(\mathbb{R}^N). \end{aligned}$$

By (3.11), there exists a finite constant $K > 0$ such that

$$\liminf_{r \rightarrow 0} r^{-s} \mu(B_{\rho}(x, r)) \leq K.$$

Hence for some finite constant K and arbitrary small $r > 0$,

$$F_{Q,\rho}^{\mu}(x, r) \leq K (r^{s-Q(1+\varepsilon)(1-a)} + r^{s-Q\varepsilon} + r^Q).$$

Therefore, by (5.7) and by noting that $t < s \leq Q$, we have (3.12) as required. \square

5.2. Hausdorff Dimension Results. The following is an extension of Theorem 6.1 in Xiao [32], which shows that the Hausdorff dimensions of $X([0, 1]^N)$ and $\text{Gr}X([0, 1]^N)$ are determined by the upper index of ϕ and (H_1, \dots, H_N) .

Theorem 5.2. *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be an (N, d) -Gaussian field satisfying Condition (C) on $I = [0, 1]^N$ and let $0 < \alpha_* \leq \alpha^* \leq 1$ be the lower and upper indices of ϕ . Then, with probability 1,*

$$\dim_{\text{H}} X([0, 1]^N) = \min \left\{ d; \sum_{j=1}^N \frac{1}{\alpha^* H_j} \right\} \quad (5.8)$$

and

$$\begin{aligned} & \dim_{\text{H}} \text{Gr}X([0, 1]^N) \\ &= \min \left\{ \sum_{j=1}^k \frac{H_k}{H_j} + N - k + (1 - \alpha^* H_k)d, 1 \leq k \leq N; \sum_{j=1}^N \frac{1}{\alpha^* H_j} \right\} \\ &= \begin{cases} \sum_{j=1}^N \frac{1}{\alpha^* H_j}, & \text{if } \sum_{j=1}^N \frac{1}{\alpha^* H_j} \leq d, \\ \sum_{j=1}^k \frac{H_k}{H_j} + N - k + (1 - \alpha^* H_k)d, & \text{if } \sum_{j=1}^{k-1} \frac{1}{\alpha^* H_j} \leq d < \sum_{j=1}^k \frac{1}{\alpha^* H_j}, \end{cases} \end{aligned} \quad (5.9)$$

where $\sum_{j=1}^0 \frac{1}{H_j} := 0$.

Proof. Since the proofs of the lower bounds in (5.8) and (5.9) are based on the standard capacity argument and are similar to the proof of Theorem 6.1 in Xiao [32], we will not give the details. Instead, we only provide a sketch of the proof of upper bounds in (5.8) and (5.9).

For any $\gamma' < \gamma < \alpha^*$, it follows from (2.2) that there exists a sequence $r_n \rightarrow 0$ such that $\phi(r_n) \leq r_n^\gamma$. By Lemma 2.2, we derive that almost surely for all n large enough

$$\sup_{s, t \in [0, 1]^N: \rho(s, t) \leq r_n} \|X(s) - X(t)\| \leq r_n^{\gamma'}. \quad (5.10)$$

For each fixed n large enough, we divide $[0, 1]^N$ into r_n^{-Q} cubes $C_{n,i}$ ($i = 1, \dots, r_n^{-Q}$) in the metric ρ . (Note that $C_{n,i}$ is a rectangle with side-length $r_n^{H_j^{-1}}$ ($j = 1, \dots, N$)). It follows from (5.10) that each $X(C_{n,i})$ can be covered by a ball of radius $r_n^{\gamma'}$ in \mathbb{R}^d . This implies that $\dim_{\text{H}} X([0, 1]^N) \leq \frac{1}{\gamma'} \sum_{j=1}^N \frac{1}{H_j}$ a.s. Since $\gamma' < \alpha^*$ is arbitrary, we have

$$\dim_{\text{H}} X([0, 1]^N) \leq \min \left\{ d, \sum_{j=1}^N \frac{1}{\alpha^* H_j} \right\} \quad \text{a.s.}$$

This proves (5.8). The proof of the upper bound in (5.9) is similar and hence omitted. Finally the last equality in (5.9) follows from Lemma 6.2 in Xiao [32], or can be verified directly. This finishes the proof of Theorem 5.2. \square

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ANNE ESTRADE: MAP5, UNIVERSITÉ PARIS DESCARTES, 45 RUE DES SAINTS-PÈRES, 75006 PARIS, FRANCE.

E-mail address: anne.estrade@parisdescartes.fr

URL: <http://www.math-info.univ-paris5.fr/~estradea>

DONGSHENG WU: DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF ALABAMA IN HUNTSVILLE, HUNTSVILLE, AL 35899, U.S.A.

E-mail address: dongsheng.wu@uah.edu

URL: <http://webpages.uah.edu/~dw0001>

YIMIN XIAO: DEPARTMENT OF STATISTICS AND PROBABILITY, MICHIGAN STATE UNIVERSITY, EAST LANSING, MI 48824, U.S.A.

E-mail address: xiao@stt.msu.edu

URL: <http://www.stt.msu.edu/~xiaoyimi>