Continuity in the Hurst Index of the Local Times of Anisotropic Gaussian Random Fields

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Abstract

Let $\big\{\{X^H(t),\, t \in \mathbb{R}^N\},\, H \in (0,\,1)^N\big\}$ be a family of (N,d)-anisotropic Gaussian random fields with generalized Hurst indices $H = (H_1, \ldots, H_N) \in (0,1)^N$. Under certain general conditions, we prove that the local time of $\{X^{H^0}(t),\, t \in \mathbb{R}^N\}$ is jointly continuous whenever $\sum_{\ell=1}^N 1/H_\ell^0 > d$. Moreover we show that, when H approaches H^0, the law of the local times of $X^H(t)$ converges weakly [in the space of continuous functions] to that of the local time of X^{H^0}. The latter theorem generalizes the result of Jolis and Viles (2007) for one-parameter fractional Brownian motions with values in $\mathbb{R}$ to a wide class of (N,d)-Gaussian random fields. The main argument of this paper relies on the recently developed sectorial local nondeterminism for anisotropic Gaussian random fields.

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1 Introduction

Gaussian random fields have been extensively studied in probability theory and applied in many scientific areas including physics, engineering, hydrology, biology, economics, just to mention a few. Since many data sets from various areas such as image processing, hydrology, geostatistics and spatial statistics have anisotropic nature in the sense that they have different geometric and probabilistic characteristics along different directions, many authors have proposed to apply anisotropic Gaussian random fields as more realistic models. See, for example, Davies and Hall (1999), Christakos (2000), Bonami and Estrade (2003) and Benson, et al. (2006).

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Several classes of anisotropic Gaussian random fields have been introduced and studied for theoretical and application purposes. For example, Kamont (1996) introduced fractional Brownian sheets and studied some of their regularity properties. Benassi et al. (1997) and Bonami and Estrade (2003) considered some anisotropic Gaussian random fields with stationary increments. Anisotropic Gaussian random fields also arise naturally in stochastic partial differential equations [see, e.g., Dalang (1999), Øksendal and Zhang (2000), Mueller and Tribe (2002), Nualart (2006)], in studying the most visited sites of symmetric Markov processes [Eisenbaum and Khoshnevisan (2002)], and as spatial or spatiotemporal models in statistics [e.g., Christakos (2000), Gneiting (2002), Stein (2005)].

Many of these anisotropic Gaussian random fields are governed by their generalized Hurst indices $H \in (0,\, 1)^N$ [see Section 2 for the definition of a generalized Hurst index]. People often have to use a statistical estimate of the index H in practice since the exact value of the index is unknown in general. Therefore, a justification of the use of a model is needed in application with an unknown H. Motivated by this purpose, Jolis and Viles (2007) investigated the continuity in law with respect to the Hurst parameter of the local time of real-valued fractional Brownian motions. They proved that the law of the local times of the fractional Brownian motions with Hurst index α converges weakly to that of the local time of fractional Brownian motion with Hurst index α_0, when α tends to α_0. However, the method they developed there depends heavily on the one-parameter setting and the explicit covariance structure of fractional Brownian motion. It seems hard to apply the method of Jolis and Viles (2007) to Gaussian random fields, where "time" parameters are vectors and their covariance structures are more complicated in general.

The main objective of this paper is to provide a general method for studying the continuity of the laws of the local times of Gaussian random fields. More precisely, we prove that, under some mild conditions, the law [in the space of continuous functions] of the local times of (N, d)-anisotropic Gaussian random fields with generalized Hurst indices H converges weakly to that of the local time of an (N, d)-anisotropic Gaussian field with index H^0, when H approaches H^0. Our result generalizes the result of Jolis and Viles (2007) for real-valued fractional Brownian motion to a wide class of (N, d)-anisotropic Gaussian random fields, including fractional Brownian sheets, anisotropic Gaussian fields with stationary increments and the spatio-temporal models in Gneiting (2002) and Stein (2005). The main ingredient we use in our proof is the recently developed properties of sectorial local nondeterminism for anisotropic Gaussian random fields, see Xiao (2007a, 2007b) and Wu and Xiao (2007).

The rest of this paper is organized as follows. Section 2 states the general condition (i.e., Condition A below) on Gaussian random fields under investigation. We show that these conditions are satisfied by several classes of Gaussian random fields which are of importance in theory and/or in applications. In Section 3, we recall the definition of local times of vector fields and prove the existence and joint continuity of the local times of Gaussian random fields satisfying Condition A. The key estimate for this paper is stated as Lemma 3.2. In Section 4, we prove the tightness of the laws of local time $\{L^H\}$ as H belongs to a neighborhood of a fixed index $H^0 \in (0, 1)^N$. In Section 5, we study the convergence in law of local times of the family of Gaussian random fields satisfying Condition A. Finally, we give the proof of our key lemma, Lemma 3.2, in Section 6.

Throughout this paper, we use $\langle \cdot, \cdot \rangle$ and $|\cdot|$ to denote the ordinary scalar product and the Euclidean norm in $\mathbb{R}^m$ respectively, no matter the value of the integer m. Unspecified positive and finite constants in Section i will be numbered as $c_{i,1}$, $c_{i,2}\dots$.

2 General assumptions and examples

For a fixed vector $H = (H_1, \dots, H_N) \in (0, 1)^N$, let $X_0^H = \{X_0^H(t), t \in \mathbb{R}^N\}$ be a real-valued, centered Gaussian random field with $X_0^H(0) = 0$ a.s. Denote

$$\sigma^2(s, t; H) = \mathbb{E}\big[X_0^H(s) - X_0^H(t)\big]^2, \qquad s, t \in \mathbb{R}^N. \tag{2.1}$$

Let $I \subseteq \mathbb{R}^N$ be a closed interval in $\mathbb{R}^N$. We call a family of Gaussian random fields $\{X_0^H, H \in (0, 1)^N\}$ satisfies *Condition A* on I if the following three conditions hold:

Condition A1. For all $s, t \in I$, $\sigma^2(s, t; H)$ is continuous in $H \in (0, 1)^N$.

Condition A2. There exist positive continuous functions (in H) $c_{1,1}(H), \dots, c_{1,4}(H)$ such that for all $s, t \in I$

$$c_{1,1}(H) \le \sigma^2(0, t; H) \le c_{1,2}(H), \tag{2.2}$$

and

$$c_{1,3}(H) \sum_{\ell=1}^{N} |s_\ell - t_\ell|^{2H_\ell} \le \sigma^2(s, t; H) \le c_{1,4}(H) \sum_{\ell=1}^{N} |s_\ell - t_\ell|^{2H_\ell}. \tag{2.3}$$

Condition A3. There exists a positive continuous function (in H) $c_{1,5}(H)$ such that for all integers $n \ge 1$, all $u, t^1, \dots, t^n \in I$,

$$\mathrm{Var}\big(X_0^H(u) \big| X_0^H(t^1), \dots, X_0^H(t^n)\big) \ge c_{1,5}(H) \sum_{\ell=1}^{N} \min_{0 \le k \le n} |u_\ell - t_\ell^k|^{2H_\ell}, \tag{2.4}$$

where $t_\ell^0 = 0$ for all $\ell = 1, \dots, N$.

As in Xiao (2007b), an anisotropic Gaussian random field is said to have the property of sectorial local nondeterminism on I if Condition A3 is fulfilled.

Throughout this paper, we will call the vector $H \in (0,1)^N$ the (generalized) Hurst index of X_0^H. Without loss of generality, we may assume that

$$0 < H_1 \le \dots \le H_N < 1. \tag{2.5}$$

Let $X^H = \{X^H(t), t \in \mathbb{R}^N\}$ be an (N, d)-anisotropic Gaussian random field with Hurst index H defined by

$$X^H(t) = \big(X_1^H(t), \dots, X_d^H(t)\big), \quad t \in \mathbb{R}^N, \tag{2.6}$$

where $X_1^H, \ldots, X_d^H$ are d independent copies of X_0^H. We still call a family of Gaussian random fields $\{X^H,\, H \in (0,\,1)^N\}$ satisfies Condition A on I if the corresponding real-valued family $\{X_0^H,\, H \in (0,\,1)^N\}$ satisfies Condition A on I.

Under Condition A2, the Gaussian random field X^H has a version whose sample paths are a.s. continuous on I. Hence, throughout this paper, we will tacitly assume that the sample paths $X^H(t)$ are a.s. continuous on I. For simplicity of notation, later on we will further assume that $I = [\varepsilon,\, \varepsilon + 1]^N$, where $\varepsilon \in (0,\,1)$ is a fixed constant.

For a fixed index $H \in (0,\,1)^N$, Xiao (2007b) studied sample path properties of an anisotropic Gaussian random field X^H satisfying Condition A2 and Condition A3′ [see Eq. (2.15) for a definition of Condition A3′], where he established results on the modulus of continuity, small ball probabilities, fractal dimensions, hitting probabilities and local times for X^H. The emphasis of the present paper is different and we focus on continuity of the laws of the functionals of X^H as $H \in (0,1)^N$ varies.

In the following we provide some important examples of families of Gaussian random fields which satisfy Condition A. They cover both isotropic and anisotropic Gaussian random fields, as well as the stationary spatial and spatiotemporal Gaussian models constructed in Gneiting (2002) and Stein (2005).

2.1 Fractional Brownian sheets

For a given vector $H = (H_1, \ldots, H_N) \in (0,1)^N$, a real-valued fractional Brownian sheet $B_0^H = \{B_0^H(t), t \in \mathbb{R}_+^N\}$ with index H is a centered Gaussian random field with covariance function given by

$$\mathbb{E}\Big[B_0^H(s)B_0^H(t)\Big] = \prod_{\ell=1}^{N} \frac{1}{2}\Big(s_\ell^{2H_\ell} + t_\ell^{2H_\ell} - |s_\ell - t_\ell|^{2H_\ell}\Big), \qquad \forall\; s,t \in \mathbb{R}_+^N. \tag{2.7}$$

An (N,d)-fractional Brownian sheet $B^H = \{B^H(t) :\; t \in \mathbb{R}_+^N\}$ is defined by

$$B^H(t) = \big(B_1^H(t), \ldots, B_d^H(t)\big), \qquad \forall\; t \in \mathbb{R}_+^N, \tag{2.8}$$

where $B_1^H, \ldots, B_d^H$ are d independent copies of B_0^H. Because of (2.7), B^H can be seen as generalizations of one-parameter fractional Brownian motion and the Brownian sheet.

Fractional Brownian sheets have become a typical representative of anisotropic Gaussian random fields since they were first introduced by Kamont (1996). In particular, we believe the methods developed for fractional Brownian sheets can be adapted for studying many spatial and spatiotemporal models with separable covariance structures [see, e.g., Christakos (2000)].

Many authors have studied the probabilistic, statistical and sample path properties of fractional Brownian sheets. Related to the problems considered in this paper, we mention that Xiao and Zhang (2002) and Ayache, Wu and Xiao (2008) studied the existence and joint continuity of the local times of fractional Brownian sheet B^H.

Proposition 2.1 *The family of (N,d)-fractional Brownian sheets $\{B^H,\, H \in (0,1)^N\}$ satisfies Condition A.*

Proof Eq. (2.7) implies that for all $s, t \in I$, $\sigma^2(s,t,H)$ is continuous in $H \in (0,1)^N$. Hence Condition A1 is satisfied. On the other hand, Conditions A2 and A3 follows respectively from the proofs of Lemma 8 in Ayache and Xiao (2005) and Theorem 1 in Wu and Xiao (2007). We omit the details. □

2.2 Gaussian random fields with stationary increments

Let $\eta = \{\eta(t),\, t \in \mathbb{R}^N\}$ be a real-valued centered Gaussian random field with $\eta(0) = 0$. We assume that η has stationary increments and continuous covariance function $R(s,t) = \mathbb{E}\,[\eta(s)\eta(t)]$. According to Yaglom (1957), $R(s,t)$ can be represented as

$$R(s,t) = \int_{\mathbb{R}^N} \big(e^{i\langle s,\lambda\rangle} - 1\big)\big(e^{-i\langle t,\lambda\rangle} - 1\big)\,\Delta(d\lambda) + \langle s, \Sigma t\rangle, \tag{2.9}$$

where Σ is an $N \times N$ nonnegative definite matrix and $\Delta(d\lambda)$ is a nonnegative symmetric measure on $\mathbb{R}^N \setminus \{0\}$ satisfying

$$\int_{\mathbb{R}^N} \frac{|\lambda|^2}{1+|\lambda|^2}\,\Delta(d\lambda) < \infty. \tag{2.10}$$

The measure Δ and its density (if it exists) $f(\lambda)$ are called the *spectral measure* and *spectral density* of η, respectively.

It follows from (2.9) that η has stochastic integral representation:

$$\big\{\eta(t),\, t \in \mathbb{R}^N\big\} \stackrel{d}{=} \bigg\{ \int_{\mathbb{R}^N} \big(e^{i\langle t,\lambda\rangle} - 1\big)\,W(d\xi) + \langle \mathbf{Y}, t\rangle,\, t \in \mathbb{R}^N \bigg\}, \tag{2.11}$$

where $\stackrel{d}{=}$ denotes equality in all finite dimensional distributions. In the right-hand side of (2.11), $\mathbf{Y}$ is a Gaussian random variable with mean 0 and $W(d\lambda)$ is a centered complex valued Gaussian random measure which is independent of $\mathbf{Y}$ and satisfies

$$\mathbb{E}\Big[W(A)\overline{W(B)}\Big] = \Delta(A \cap B) \quad \text{and} \quad W(-A) = \overline{W(A)}$$

for all Borel sets A, $B \subseteq \mathbb{R}^N$. From now on, we will assume $\mathbf{Y} = 0$, which is equivalent to assuming $\Sigma = 0$ in (2.9). Therefore, we have

$$\sigma_\eta^2(h) = \mathbb{E}\Big[\big(\eta(t+h) - \eta(t)\big)^2\Big] = 2\int_{\mathbb{R}^N} \big(1 - \cos\langle h, \lambda\rangle\big)\,\Delta(d\lambda). \tag{2.12}$$

Eq. (2.11) provides a useful way for constructing Gaussian random fields with stationary increments by choosing the spectral measure Δ. In particular, for $\alpha \in (0,1)$, if Δ has a density function f given by

$$f_\alpha(\lambda) = \frac{1}{|\lambda|^{2\alpha+N}}, \qquad \forall \lambda \in \mathbb{R}^N \backslash \{0\}, \tag{2.13}$$

then $\eta = \{\eta(t),\, t \in \mathbb{R}^N\}$ is a real-valued fractional Brownian motion of index α, which is an isotropic Gaussian random field and will be denoted by η^α. Another interesting example of isotropic Gaussian random fields is the fractional Riesz-Bessel motion with indices β and γ introduced by Anh et al. (1999), whose spectral density is given by

$$f_{\gamma,\beta}(\lambda) = \frac{1}{|\lambda|^{2\gamma}(1+|\lambda|^2)^{\beta}}, \qquad \forall \lambda \in \mathbb{R}^N \backslash \{0\}, \tag{2.14}$$

where γ and β are constants satisfying $\beta + \gamma > \frac{N}{2}$ and $0 < \gamma < 1 + \frac{N}{2}$. Anh et al. (1999) showed that these Gaussian random fields can be used for modeling simultaneously long range dependence and intermittency; and Xiao (2007a) studied their sample path properties.

The following example covers a wide class of Gaussian random fields that satisfy Condition A. In fact we will prove a stronger result that these Gaussian random fields satisfy Conditions A1, A2 and the following

Condition A3′. There exists a positive function $c_{1,6}(H)$ which is continuous in $H \in (0,1)^N$ such that for all integers $n \geq 1$, all u, $t^1, \ldots, t^n \in I$,

$$\mathrm{Var}\big(X_0^H(u) \big| X_0^H(t^1), \ldots, X_0^H(t^n)\big) \geq c_{1,6}(H) \min_{0 \leq k \leq n} \sum_{\ell=1}^N |u_\ell - t_\ell^k|^{2H_\ell}, \tag{2.15}$$

where $t^0 = 0$.

Following Xiao (2007b), an anisotropic Gaussian random field satisfying Condition A3′ is said to have the property of strong local nondeterminism in the metric $\rho(s,t) = \sum_{j=1}^N |s_j - t_j|^{H_j}$. Clearly, Condition A3′ implies Condition A3, but the converse does not hold. Consequently, if a family of anisotropic Gaussian random fields satisfies Conditions A1, A2 and A3′, then it satisfies Condition A.

Proposition 2.2 *Let $\big\{\eta^H = \{\eta^H(t), t \in \mathbb{R}^N\}, H = (H_1, \ldots, H_N) \in (0,1)^N\big\}$ be a family of real-valued centered Gaussian random fields with stationary increments and spectral densities $\big\{f(\lambda; H), H \in (0,1)^N\big\}$. Suppose $f(\lambda;H)$ is continuous in $H \in (0,1)^N$ and*

$$f(\lambda; H) \asymp \frac{1}{\big(\sum_{j=1}^N |\lambda_j|^{H_j}\big)^{2+Q}}, \qquad \forall \lambda \in \mathbb{R}^N \backslash \{0\}, \tag{2.16}$$

where $Q = \sum_{\ell=1}^N H_\ell^{-1}$, and where two functions $q(t;H) \asymp r(t;H)$ for $t \in T$ means that there are positive continuous functions $c_{1,7}(H)$ and $c_{1,8}(H)$ in H such that $c_{1,7}(H) \leq q(t;H)/r(t;H) \leq c_{1,8}(H)$ for all $t \in T$. Then the family of Gaussian random fields $\big\{\eta^H, H \in (0,1)^N\big\}$ satisfies Conditions A1, A2 and A3′.

Proof By (2.12), we can write

$$\sigma^2_{\eta^H}(s,t;H) = \sigma^2_{\eta^H}(s-t;H) = 2\int_{\mathbb{R}^N} \big(1 - \cos\langle s-t, \lambda\rangle\big) f(\lambda;H)\, d\lambda. \tag{2.17}$$

By the continuity of $f(\lambda;H)$ in H, (2.16) and the Dominated Convergence Theorem, one can verify that Condition A1 is satisfied.

In order to verify Conditions A2 and A3, we first derive an appropriate upper bound for $\sigma^2_{\eta^H}(h;H)$ $(h \in \mathbb{R}^N)$ which implies the upper bounds in (2.2) and (2.3), and then prove Condition A3$'$, which also provides the desired lower bounds in (2.2) and (2.3).

Because of (2.16) we may, without of loss of generality, assume $h_\ell \ge 0$ for all $\ell = 1, \ldots, N$. By (2.16), (2.17) and a change of variables $\nu_\ell = \big(\sum_{j=1}^N h_j^{H_j}\big)^{H_\ell^{-1}} \lambda_\ell$ $(\ell = 1, \ldots, N)$, we obtain,

$$\begin{aligned}
\sigma^2_{\eta^H}(h) &\le 2c_{1,8}(H) \int_{\mathbb{R}^N} \frac{1-\cos\langle h, \lambda\rangle}{\big(\sum_{j=1}^N |\lambda_j|^{H_j}\big)^{2+Q}}\, d\lambda \\
&= 2c_{1,8}(H) \int_{\mathbb{R}^N} \frac{1-\cos\Big(\sum_{\ell=1}^N \big(\sum_{j=1}^N h_j^{H_j}\big)^{-H_\ell^{-1}} h_\ell \nu_\ell\Big)}{\big(\sum_{j=1}^N |\nu_j|^{H_j}\big)^{2+Q}}\, d\nu \left(\sum_{j=1}^N h_j^{H_j}\right)^2.
\end{aligned} \tag{2.18}$$

Since $\frac{h_\ell^{H_\ell}}{\sum_{j=1}^N h_j^{H_j}} \le 1$ for all $\ell = 1, \ldots, N$ and the function $x \mapsto \cos x$ is decreasing in $(0, \frac{\pi}{2})$, we derive that

$$\begin{aligned}
&\int_{\mathbb{R}^N} \frac{1-\cos\Big(\sum_{\ell=1}^N \big(\sum_{j=1}^N h_j^{H_j}\big)^{-H_\ell^{-1}} h_\ell \nu_\ell\Big)}{\big(\sum_{j=1}^N |\nu_j|^{H_j}\big)^{2+Q}}\, d\nu \\
&\le \int_{|\nu|<\pi/2} \frac{1-\cos\big(\sum_{j=1}^N |\nu_j|\big)}{\big(\sum_{j=1}^N |\nu_j|^{H_j}\big)^{2+Q}}\, d\nu + \int_{|\nu|\ge\pi/2} \frac{2}{\big(\sum_{j=1}^N |\nu_j|^{H_j}\big)^{2+Q}}\, d\nu.
\end{aligned} \tag{2.19}$$

It can be verified that the last two integrals are convergent [see, e.g., Lemmas 6.3 and 6.4 in Xiao (2007b)]. Combining (2.18), (2.19) and the elementary inequality $\big(\sum_{j=1}^N h_j^{H_j}\big)^2 \le N \sum_{j=1}^N h_j^{2H_j}$, we obtain

$$\sigma^2_{\eta^H}(h) \le c_{1,9}(H) \sum_{j=1}^N h_j^{2H_j}, \tag{2.20}$$

where

$$c_{1,9}(H) = 2N\, c_{1,8}(H) \bigg(\int_{|\nu|<\pi/2} \frac{1-\cos\big(\sum_{j=1}^N |\nu_j|\big)}{\big(\sum_{j=1}^N |\nu_j|^{H_j}\big)^{2+Q}}\, d\nu + \int_{|\nu|\ge\pi/2} \frac{2}{\big(\sum_{j=1}^N |\nu_j|^{H_j}\big)^{2+Q}}\, d\nu \bigg),$$

which is a positive continuous function of $H \in (0,\, 1)^N$. Therefore, the upper bounds in Condition A2 follow from (2.20).

Next, we prove that the family $\{\eta^H, H \in (0,1)^N\}$ satisfies Condition A3$'$. The key technique in our derivation is based on the Fourier analytic argument in Kahane (1985, Chapter 18); see Xiao (2007a, 2007b) and Wu and Xiao (2007) for further information. By following the proof of Theorem 3.2 in Xiao (2007b) line by line, one can verify that for all integers $n \ge 1$, all $u,\, t^1, \ldots, t^n \in I$,

$$\mathrm{Var}\big(\eta^H(u) | \eta^H(t^1), \ldots, \eta^H(t^n)\big) \ge c_{1,10}(H) \min_{0\le k\le n} \sum_{\ell=1}^N |u_\ell - t_\ell^k|^{2H_\ell}, \tag{2.21}$$

where $t^0 = 0$ and $c_{1,10}(H)$ can be chosen as

$$c_{1,10}(H) := \frac{c}{c_{1,7}(H)} \int_{\mathbb{R}^N} \left(\sum_{j=1}^N |\lambda_j|^{H_j} \right)^{2+Q} \left|\widehat{\delta}(\lambda)\right|^2 d\lambda.$$

In the above, $\widehat{\delta}$ is the Fourier transform of a $C^\infty(\mathbb{R}^N)$ function δ such that $\delta(0) = 1$ and $\delta(t) \equiv 0$ for all $t \in \mathbb{R}^N$ with $\rho(0,t) = \sum_{j=1}^N |t_j|^{H_j} \geq 1$. Hence $\widehat{\delta}(\cdot) \in C^\infty(\mathbb{R}^N)$ as well and $\widehat{\delta}(\lambda)$ decays rapidly as $|\lambda| \to \infty$. This implies that $c_{1,10}(H)$ is a positive continuous function in $H \in (0,1)^N$. Consequently, we prove Condition A3$'$.

Finally, we can use the lower bound in Condition A3$'$ with $n = 1$ by choosing $u = t$, $t^1 = 0$ and $u = t$, $t^1 = s$, respectively, to serve as the lower bounds in Condition A2. This finishes the proof of Proposition 2.2. □

Remark 2.3 It follows from (2.13) and (2.14) that the spectral density functions of fractional Brownian motion and fractional Riesz-Bessel motion (with $0 < \gamma + \beta - \frac{N}{2} < 1$) satisfy the spectral conditions in Proposition 2.2. Therefore, both families of fractional Brownian motions and fractional Riesz-Bessel motions satisfy Condition A. When the index $\alpha \in (0,1)$ is fixed, Pitt (1978) proved that fractional Brownian motion with Hurst index α is strongly local nondeterministic, i.e., for all integers $n \geq 1$, all $u, t^1, \ldots, t^n \in I$,

$$\operatorname{Var}\big(\eta^\alpha(u) \big| \eta^\alpha(t^1), \ldots, \eta^\alpha(t^n)\big) \geq c_{1,11}(\alpha) \min_{0 \leq k \leq n} |u - t^k|^{2\alpha}, \tag{2.22}$$

where $t^0 = 0$ and $c_{1,11}(\alpha)$ is a positive constant depending on α. A similar result was proved by Xiao (2007a) for fractional Riesz-Bessel motion. However their results do not provide any information on whether the constants in the lower bounds are continuous in the indices α, β and γ. In this sense, Proposition 2.2 strengthens the results of Pitt (1978) and Xiao (2007a).

Remark 2.4 Anisotropic Gaussian random fields with the above type of spectral density functions arise naturally in probability theory and its applications. See Robeva and Pitt (2004) for their relevance to the solution of the stochastic heat equation; Bonami and Estrade (2003), Benson, et al. (2006) and Biermé et al. (2007) for their applications in modeling bone and aquifer structures; Gneiting (2002) and Stein (2005) for stationary nonseparable spatial and spatiotemporal Gaussian models.

3 Local times and their joint continuity

In this section, we briefly recall some aspects of the theory of local times in general at first and then study the existence and joint continuity of the local times of Gaussian random fields satisfying Condition A. For excellent surveys on local times of random and/or deterministic vector fields, we refer to Geman and Horowitz (1980) and Dozzi (2002).

Let $Y(t)$ be a Borel vector field on $\mathbb{R}^N$ with values in $\mathbb{R}^d$. For any Borel set $T \subseteq \mathbb{R}^N$, the occupation measure of Y on T is defined as the following measure on $\mathbb{R}^d$:

$$\mu_T(\bullet) = \lambda_N\big\{t \in T : Y(t) \in \bullet\big\},$$

where λ_N denotes the Lebesgue measure in $\mathbb{R}^N$.

If μ_T is absolutely continuous with respect to the Lebesgue measure λ_d, we say that $Y(t)$ has *local times* on T, and define its local times, $L(\bullet, T)$, as the Radon–Nikodým derivative of μ_T with respect to λ_d, i.e.,

$$L(x,T) = \frac{d\mu_T}{d\lambda_d}(x), \qquad \forall x \in \mathbb{R}^d.$$

In the above, x is the so-called *space variable*, and T is the *time variable*. Note that if Y has local times on T then for every Borel set $S \subseteq T$, $L(x,S)$ also exists.

By standard martingale and monotone class arguments, one can deduce that the local times have a measurable modification that satisfies the following *occupation density formula* [see Geman and Horowitz (1980, Theorem 6.4)]: For every Borel set $T \subseteq \mathbb{R}^N$, and for every measurable function $f : \mathbb{R}^d \to \mathbb{R}_+$,

$$\int_T f(Y(t))\, dt = \int_{\mathbb{R}^d} f(x) L(x,T)\, dx. \tag{3.1}$$

Suppose we fix a rectangle $T = \prod_{i=1}^N [a_i, a_i + h_i] \subseteq \mathbb{R}^N$, where $a \in \mathbb{R}^N$ and $h \in \mathbb{R}^N_+$. If we can choose a version of the local time, still denoted by $L(x, \prod_{i=1}^N [a_i, a_i + t_i])$, such that it is a continuous function of $(x, t_1, \cdots, t_N) \in \mathbb{R}^d \times \prod_{i=1}^N [0, h_i]$, Y is said to have a *jointly continuous local time* on T. When a local time is jointly continuous, $L(x, \cdot)$ can be extended to be a finite Borel measure supported on the level set

$$Y_T^{-1}(x) = \{t \in T : Y(t) = x\}; \tag{3.2}$$

see Adler (1981) for details. This makes local times, besides of interest on their own right, a useful tool for studying fractal properties of Y.

It follows from (25.5) and (25.7) in Geman and Horowitz (1980) that for all $x, y \in \mathbb{R}^d$, $T \subseteq \mathbb{R}^N$ a closed interval and all integers $n \geq 1$,

$$\begin{aligned} \mathbb{E}\Big[L(x,T)^n\Big] \; &= \; (2\pi)^{-nd} \int_{T^n} \int_{\mathbb{R}^{nd}} \exp\Big(- i \sum_{j=1}^n \langle u^j, x\rangle \Big) \\ &\qquad \times \mathbb{E} \exp\Big(i \sum_{j=1}^n \langle u^j, Y(t^j)\rangle \Big) d\overline{u}\; d\overline{t} \end{aligned} \tag{3.3}$$

and for all even integers $n \geq 2$,

$$\begin{aligned} \mathbb{E}\Big[(L(x,T) - L(y,T))^n\Big] = &(2\pi)^{-nd} \int_{T^n} \int_{\mathbb{R}^{nd}} \prod_{j=1}^n \Big[e^{-i\langle u^j, x\rangle} - e^{-i\langle u^j, y\rangle} \Big] \\ &\times \mathbb{E} \exp\Big(i \sum_{j=1}^n \langle u^j, Y(t^j)\rangle \Big) d\overline{u}\; d\overline{t}, \end{aligned} \tag{3.4}$$

where $\overline{u} = (u^1, \ldots, u^n)$, $\overline{t} = (t^1, \ldots, t^n)$, and each $u^j \in \mathbb{R}^d$, $t^j \in T$. In the coordinate notation we then write $u^j = (u_1^j, \ldots, u_d^j)$.

Let $\{X^H, H \in (0,1)^N\} = \{\{X^H(t), t \in \mathbb{R}^N\}, H \in (0,1)^N\}$ be a family of (N,d)-Gaussian random fields. For a fixed index $H^0 \in (0,1)^N$ such that $\sum_{\ell=1}^N \frac{1}{H_\ell^0} > d$, Xiao (2007b) proved the following results on the existence and joint continuity of the local times of Gaussian random field X^{H^0}:

(i) If X^{H^0} satisfies Condition A2 (for H^0), then X^{H^0} has a local time $L^{H^0}(x,T) \in L^2(\mathbb{P}\times\lambda_d)$.

(ii) If X^{H^0} satisfies Conditions A2 and A3$'$ (for H^0), then X^{H^0} has a jointly continuous local time on T.

The main result of this section is the following Theorem 3.1. It shows that, under Conditions A2 and A3 [instead of A3$'$], the above conclusions still hold for all $H \in (0,1)^N$ which are close to H^0. Moreover, we can bound the moments of the local times of X^H in terms of H^0. Hence, Theorem 3.1 strengthens and extends the results of Ayache, Wu and Xiao (2008) for fractional Brownian sheets and Theorem 8.2 in Xiao (2007b).

We set up some notation. Let $H^0 \in (0,\,1)^N$ be an index satisfying

$$\sum_{\ell=1}^N \frac{1}{H_\ell^0} > d. \tag{3.5}$$

With the convention $\sum_{\ell=1}^0 \frac{1}{H_\ell^0} := 0$, we can see that there exists an integer $\tau_0 \in \{1,\ldots,N\}$ such that

$$\sum_{\ell=1}^{\tau_0-1} \frac{1}{H_\ell^0} \le d < \sum_{\ell=1}^{\tau_0} \frac{1}{H_\ell^0}. \tag{3.6}$$

Define

$$\beta_{\tau_0} = \sum_{\ell=1}^{\tau_0} \frac{H_{\tau_0}^0}{H_\ell^0} + N - \tau_0 - H_{\tau_0}^0\, d. \tag{3.7}$$

Then it can be easily verified that $\beta_{\tau_0} \in (N-\tau_0,\, N-\tau_0+1]$, where $\beta_{\tau_0} = N-\tau_0+1$ if and only if $\sum_{\ell=1}^{\tau_0-1} \frac{1}{H_\ell^0} = d$; and, if $\tau_0 = N$, then $\beta_{\tau_0} = H_N^0(\sum_{\ell=1}^N \frac{1}{H_\ell^0} - d) > 0$.

Distinguishing two cases $\sum_{\ell=1}^{\tau_0-1} \frac{1}{H_\ell^0} < d$ and $\sum_{\ell=1}^{\tau_0-1} \frac{1}{H_\ell^0} = d$, we see that we can choose a positive number $\delta_0 < \min\{H_\ell^0, 1-H_\ell^0 : 1 \le \ell \le N\}$, which depends on d and H^0 only, with the following property: For all indices $H \in \prod_{\ell=1}^N [H_\ell^0 - \delta_0,\, H_\ell^0 + \delta_0] \subseteq (0,\,1)^N$, there is an integer $\tau \in \{\tau_0 - 1, \tau_0\}$ such that

$$\sum_{\ell=1}^{\tau-1} \frac{1}{H_\ell} \le d < \sum_{\ell=1}^{\tau} \frac{1}{H_\ell}. \tag{3.8}$$

Moreover, if we denote

$$\beta_\tau = \sum_{\ell=1}^{\tau} \frac{H_\tau}{H_\ell} + N - \tau - H_\tau\, d, \tag{3.9}$$

then $\beta_\tau \in (N-\tau_0,\, N-\tau_0+2]$. It is useful to note that, even though τ varies with H, its value depends only on H^0 and, β_τ is always bounded from below and above by positive constants depending only on H^0. In the sequel, δ_0 and τ_0 will always be the constants defined above.

Theorem 3.1 *Let $\{X^H, H \in (0,1)^N\} = \{\{X^H(t), t \in \mathbb{R}^N\}, H \in (0,1)^N\}$ be a family of (N,d)-Gaussian random fields with Hurst indices H satisfying Conditions A2 and A3 on $I = [\varepsilon, 1+\varepsilon]^N$. Let $H^0 \in (0,1)^N$ be a Hurst index satisfying (3.5). Then for every $H \in \prod_{\ell=1}^N [H_\ell^0 - \delta_0, H_\ell^0 + \delta_0]$, X^H has a local time $L^H(x,I) \in L^2(\mathbb{P} \times \lambda_d)$, which admits the following L^2-representation*

$$L^H(x,I) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-i\langle y,x\rangle} \int_I e^{i\langle y, X^H(s)\rangle}\, ds dy, \qquad \forall x \in \mathbb{R}^d. \tag{3.10}$$

Furthermore, the Gaussian random field $X^H = \{X^H(t), t \in \mathbb{R}^N\}$ has almost surely a jointly continuous local time on I.

For simplicity of notation, we have assumed that $I = [\varepsilon, 1+\varepsilon]^N$. This does not lose any generality. We will further denote $L^H(x,t) := L^H(x, [\varepsilon, \varepsilon + t])$. By Theorem 3.1, $L^H(x,t)$ is a continuous function on $\mathbb{R}^d \times [0,1]^N$. Hence we will view $L^H(x,t)$ as an element in $C(\mathbb{R}^d \times [0,1]^N, \mathbb{R})$ and $L^H(x,\cdot)$ as a finite Borel measure. Here and in the sequel, for any integers p, q and Borel set $S \subseteq \mathbb{R}^p$, $C(S, \mathbb{R}^q)$ denotes the space of continuous functions from S to $\mathbb{R}^q$, endowed with the topology of uniform convergence on compact subsets of S.

The proof of Theorem 3.1 is based on the following lemma which extends the inequalities in Lemma 8.4 and Lemma 8.8 of Xiao (2007b). It will also play an essential rôle in Section 4 for proving the tightness of the laws of the local times of $\{X^H, H \in (0,1)^N\}$.

Lemma 3.2 *Suppose the assumptions of Theorem 3.1 hold. Then, for all $H \in \prod_{\ell=1}^N [H_\ell^0 - \delta_0, H_\ell^0 + \delta_0]$, there exist positive and finite constants $c_{3,1}$ and $c_{3,2}$ depending on N, d, H^0 and I only, such that for all hypercubes $T = [a, a + \langle r\rangle] \subseteq I$ with side-length $r \in (0,1)$ the following estimates hold:*

(1). for all $x \in \mathbb{R}^d$ and all integers $n \geq 1$,

$$\mathbb{E}\left[L^H(x,T)^n\right] \leq c_{3,1}^n\, (n!)^{N-\beta_\tau}\, r^{n\,\beta_\tau}, \tag{3.11}$$

where β_τ is defined in (3.9).

(2). for all $x, y \in \mathbb{R}^d$ with $|x-y| \leq 1$, all even integers $n \geq 1$ and all $\gamma \in (0,1)$ small enough,

$$\mathbb{E}\left[\left(L^H(x,T) - L^H(y,T)\right)^n\right] \leq c_{3,2}^n\, (n!)^{N-\beta_\tau+(1+H_\tau)\gamma}\, |x-y|^{n\gamma}\, r^{n(\beta_\tau - H_\tau\gamma)}. \tag{3.12}$$

The moment estimates (3.11) and (3.12) are a lot more precise than what we actually need in this paper. We expect that they may be useful for some other purposes. For example, one can apply them to show that, for every fixed $x \in \mathbb{R}^d$, there is an event of positive probability (which only depends on H^0, N, d and x) such that the Hausdorff dimension of the level set $\dim_{\mathrm{H}}\big((X^H)^{-1}(\{x\}) \cap I\big)$ of X^H tends to $\dim_{\mathrm{H}}\big((X^{H^0})^{-1}(\{x\}) \cap I\big)$ as $H \to H^0$. Note that this result can not be derived directly from the Hausdorff dimension result for the level set of X^H in Xiao (2007b, Theorem 7.1), where $H \in (0,1)^N$ is fixed.

The proof of Lemma 3.2 makes use of Fourier analytic arguments and the property of sectorial local nondeterminism. We will defer the lengthy proof of Lemma 3.2 to Section 6.

Proof of Theorem 3.1 Since (3.8) holds for all $H \in \prod_{\ell=1}^N [H_\ell^0 - \delta_0,\, H_\ell^0 + \delta_0]$, the proof of the first part of Theorem 3.1 (i.e., the existence and (3.10)) is the same as that of Theorem 8.1 in Xiao (2007b) and is omitted.

On the other hand, the proof of the joint continuity of the local time of X^H is similar to that of Theorem 8.2 in Xiao (2007b) [see also the proof of Theorem 3.1 in Ayache, Wu and Xiao (2008)]. Because of its usefulness for proving the tightness in the next section, we include it here. Observe that for all $x, y \in \mathbb{R}^d$, $s, t \in [0,1]^N$ and all even integers $n \ge 1$, we have

$$\begin{aligned} \mathbb{E}\Big[\big(L^H(x,s) - L^H(y,t)\big)^n\Big] \le 2^{n-1}\bigg\{&\mathbb{E}\Big[\big(L^H(x,s) - L^H(x,t)\big)^n\Big] \\ &+ \mathbb{E}\Big[\big(L^H(x,t) - L^H(y,t)\big)^n\Big]\bigg\}. \end{aligned} \tag{3.13}$$

Since $L^H(x,\cdot)$ is a finite Borel measure, the difference $L^H(x,s) - L^H(x,t) = L^H(x,[\varepsilon,\varepsilon+s]) - L^H(x,[\varepsilon,\varepsilon+t])$ can be written as a sum of finite number (only depends on N) of terms of the form $L^H(x,T_j)$, where each T_j is a closed subinterval of I with at least one edge length $\le |s-t|$. By further splitting these intervals into cubes of sides $\le |s-t|$, we can use (3.11) to bound the first term in (3.13). On the other hand, the second term in (3.13) can be dealt with using (3.12) as above. Consequently, there exist some constants $\gamma \in (0,1)$ and n_0 such that for all $x, y \in \mathbb{R}^d$, $s, t \in [0,1]^N$ and all even integers $n \ge n_0$,

$$\mathbb{E}\Big[\big(L^H(x,s) - L^H(y,t)\big)^n\Big] \le c_{3,3}^n \,\big(|x-y| + |s-t|\big)^{n\gamma}. \tag{3.14}$$

Therefore the joint continuity of the local times $L^H(x,t)$ follows from the multiparameter version of Kolmogorov's continuity theorem [cf. Khoshnevisan (2002)]. This finishes the proof. □

4 Tightness

In this section, for any index $H^0 \in (0,1)^N$ satisfying (3.5), we prove the tightness of the laws of $\big\{L^H(x,t),\, H \in \prod_{\ell=1}^N [H_\ell^0 - \delta_0,\, H_\ell^0 + \delta_0]\big\}$ in $C\big([-D,\, D]^d \times [0,1]^N, \mathbb{R}\big)$ for all $D > 0$.

For this purpose, we will make use of the following tightness criterion which is a consequence of Corollary 16.9 in Kallenberg (2002).

Lemma 4.1 *Let $\{Z^{(p)}, p \ge 1\}$ with $Z^{(p)} = \{Z^{(p)}(t), t \in \mathbb{R}^M\}$ be a sequence of continuous random fields with values in $\mathbb{R}^q$. Assume that $K \subseteq \mathbb{R}^M$ is a compact interval and $u \in K$ is a fixed point. If there exist some positive constants $c_{4,1}$, b_1, b_2 and b_3 such that*

$$\mathbb{E}\Big[\big|Z^{(p)}(u)\big|^{b_1}\Big] \le c_{4,1} \qquad \forall\, p \ge 1 \tag{4.1}$$

and

$$\mathbb{E}\Big[\big|Z^{(p)}(s) - Z^{(p)}(t)\big|^{b_2}\Big] \le c_{4,1}\, |s-t|^{M+b_3} \qquad \forall\, s,t \in K \;\; \text{and} \;\; \forall\, p \ge 1. \tag{4.2}$$

Then $\{Z^{(p)}, p \ge 1\}$ is tight in $C(K, \mathbb{R}^q)$.

Proposition 4.2 *Let $\{X^H,\, H \in (0,\,1)^N\}$ be a family of (N,d)-Gaussian random fields satisfying Condition A. Let $H^0 \in (0,\,1)^N$ be a Hurst index satisfying (3.5) and $\delta_0 > 0$ be the corresponding constant defined before. Then the laws of $\{L^H(x,t),\, H \in \prod_{\ell=1}^N [H_\ell^0 - \delta_0,\, H_\ell^0 + \delta_0]\}$ in $C\big([-D,\,D]^d \times [0,1]^N, \mathbb{R}\big)$ is tight for all $D > 0$.*

Proof Note that, for all $H \in \prod_{\ell=1}^N [H_\ell^0 - \delta_0,\, H_\ell^0 + \delta_0]$, $L^H(0,0) = 0$ almost surely. Hence, by Lemma 4.1, it is sufficient to prove that there exist positive constants $c_{4,2}$ and $\gamma \in (0,1)$ such that for all even integers n large and all $H \in \prod_{\ell=1}^N [H_\ell^0 - \delta_0,\, H_\ell^0 + \delta_0]$,

$$\mathbb{E}\Big[\big(L^H(x,s) - L^H(y,t)\big)^n\Big] \le c_{4,2}^n \,\big(|x-y| + |s-t|\big)^{n\gamma}. \tag{4.3}$$

This is similar to (3.14) and the only difference is that the constants $c_{4,2}$ and γ are independent of $H \in \prod_{\ell=1}^N [H_\ell^0 - \delta_0,\, H_\ell^0 + \delta_0]$. By (3.13) we only need to verify that the upper bounds appearing in the moment estimates (3.11) and (3.12) can be taken to be independent of the index H provided $H \in \prod_{\ell=1}^N [H_\ell^0 - \delta_0,\, H_\ell^0 + \delta_0]$.

Recall that, by our choice of the constant δ_0, β_τ is bounded from below and above by positive constants depending only on H^0. That is, there exist positive constants $0 < \beta' < \beta''$ such that $\beta_\tau \in [\beta',\, \beta'']$ for all $H \in \prod_{\ell=1}^N [H_\ell^0 - \delta_0,\, H_\ell^0 + \delta_0]$. Hence by Lemma 3.2, we can choose $\gamma \in (0,1)$ small enough such that

$$\mathbb{E}\left[L^H(x,T)^n\right] \le c_{3,1}^n \,(n!)^{N-\beta'}\, r^{n\beta'}, \tag{4.4}$$

and

$$\mathbb{E}\left[\big(L^H(x,T) - L^H(y,T)\big)^n\right] \le c_{3,2}^n \,(n!)^{N-\beta'+(1+H_N^0+\delta_0)\gamma}\, |x-y|^{n\gamma}\, r^{n\left(\beta'-(H_N^0+\delta_0)\gamma\right)} \tag{4.5}$$

for all intervals $T = [a, a+\langle r\rangle] \subseteq I$ and all $H \in \prod_{\ell=1}^N [H_\ell^0 - \delta_0,\, H_\ell^0 + \delta_0]$. This finishes the proof of Proposition 4.2. □

5 Convergence in law

In this section, we establish the continuity of the laws of the local times of X^H in the Hurst index $H \in (0,1)^N$. For this purpose, we will make use of the following result, which is an extension of Proposition 4.2 in Jolis and Viles (2007).

Proposition 5.1 *Let $\{\{Y_n(t), t \in \mathbb{R}^N\},\, n \ge 1\}$ be a family of (N,d)-random fields satisfying the following conditions:*

(1). $\{Y_n\}$ converges in law to Y in $C\big(I, \mathbb{R}^d\big)$ as $n \to \infty$.

(2). Both families $\{Y_n\}$ and Y have local times L_n and L, which are jointly continuous in x and t.

(3). The family of local times L_n converges in law to a random field Z in $C\big([-D,\, D]^d \times [0,\, 1]^N, \mathbb{R}\big)$ as $n \to \infty$.

Then, for all points $(x^1, t^1), \ldots, (x^m, t^m) \in [-D,\, D]^d \times [0,\, 1]^N$, we have

$$\big(Z(x^1, t^1), \ldots, Z(x^m, t^m)\big) \stackrel{\mathcal{L}}{=} \big(L(x^1, t^1), \ldots, L(x^m, t^m)\big). \tag{5.1}$$

Proof When $N = 1$ and $d = 1$, this is proved by Jolis and Viles (2007). The key ingredient in their proof is the occupation density formula (3.1). Extending their proof to the multiparameter case is straightforward and is omitted. □

Theorem 5.2 *Let $\{X^H,\, H \in (0,\, 1)^N\}$ be a family of (N, d)-Gaussian random fields satisfying Condition A and let $H^0 \in (0, 1)^N$ be a Hurst index satisfying (3.5). Then the family of local times $\{L^H,\, H \in (0,\, 1)^N\}$ of $\{X^H\}$ converges in law to the local time L^{H^0} of X^{H^0} in $C\big(\mathbb{R}^d \times [0, 1]^N, \mathbb{R}\big)$ as $H \to H^0$.*

Proof It follows from Proposition 16.6 in Kallenberg (2002) that it is sufficient to prove that, for all constants $D > 0$, the family of local times $\{L^H,\, H \in (0,\, 1)^N\}$ converge in law to the local time L^{H^0} in $C\big([-D,\, D]^d \times [0, 1]^N, \mathbb{R}\big)$ as $H \to H^0$.

By Proposition 4.2, we see that the laws of $\{L^H(x, t),\, H \in \prod_{\ell=1}^N [H_\ell^0 - \delta_0,\, H_\ell^0 + \delta_0]\}$ is tight in $C\big([-D,\, D]^d \times [0, 1]^N,\, \mathbb{R}\big)$ for all $D > 0$. Hence it only remains to prove the convergence of finite dimensional distributions. This can be done by applying Proposition 5.1.

Take an arbitrary sequence $\{H^n\} \subset \prod_{\ell=1}^N [H_\ell^0 - \delta_0,\, H_\ell^0 + \delta_0]$ converging to H^0 as $n \to \infty$. First we verify that, as $n \to \infty$, the sequence $\{X^{H^n},\, n \geq 1\}$ of Gaussian random fields converges in law to X^{H^0} in $C(I, \mathbb{R}^d)$. In fact, Condition A2 implies that for any fixed point $u \in I$ and all integers $m \geq 2$,

$$\sup_{n \geq 1} \mathbb{E}\Big(\big|X^{H^n}(u)\big|^{2m}\Big) \leq c_{5,1}^m. \tag{5.2}$$

and

$$\mathbb{E}\Big(\big|X^{H^n}(u) - X^{H^n}(v)\big|^{2m}\Big) \leq c_{5,2}^m \left(\sum_{\ell=1}^N |u_\ell - v_\ell|^{2H_\ell}\right)^m \qquad \forall\, u,\, v, \in I. \tag{5.3}$$

Hence Lemma 4.1 implies that the family of laws of $\{X^H,\, H \in \prod_{\ell=1}^N [H_\ell^0 - \delta_0,\, H_\ell^0 + \delta_0]\}$ is tight in $C(I, \mathbb{R}^d)$. On the other hand, Condition A1 implies that

$$\lim_{H \to H^0} \mathbb{E}\left[X_j^H(u) X_k^H(v)\right] = \mathbb{E}\left[X_j^{H^0}(u) X_k^{H^0}(v)\right], \qquad \forall j,\, k = 1, \ldots, d,\ \forall u,\, v \in I, \tag{5.4}$$

which implies the convergence of the finite dimensional distributions of $\{X^H,\, H \in (0, 1)^N\}$ as $H \to H^0$. This verifies Condition (1) in Proposition 5.1. Condition (2) in Proposition 5.1 follows from Theorem 3.1.

If for any sequence $\{H^n\} \subset \prod_{\ell=1}^N [H_\ell^0 - \delta_0,\, H_\ell^0 + \delta_0]$ converging to H^0 as $n \to \infty$ such that

$$L^{H^n}(x,t) \xrightarrow{\mathcal{L}} Z(x,t) \quad \text{in } C\big([-D,\, D]^d \times [0,\, 1]^N, \mathbb{R}\big) \text{ as } n \to \infty, \tag{5.5}$$

for some random field Z. Then, by Proposition 5.1, we have that for all fixed points (x^1, t^1), $\ldots, (x^m, t^m)$,

$$\big(Z(x^1,t^1), \ldots, Z(x^m,t^m)\big) \stackrel{\mathcal{L}}{=} \big(L^{H^0}(x^1,t^1), \ldots, L^{H^0}(x^m,t^m)\big), \tag{5.6}$$

which gives us that $\mathcal{L}(Z) = \mathcal{L}(L^{H^0})$ in $C\big([-D,\, D]^d \times [0,\, 1]^N, \mathbb{R}\big)$. This finishes the proof of Theorem 5.2. □

6 Proof of Lemma 3.2

The proof of Lemma 3.2 follows the same spirit of the proofs of Lemma 3.7 and Lemma 3.10 of Ayache, Wu and Xiao (2008), where only fractional Brownian sheets were considered. In order to extend their argument to Gaussian random fields satisfying Condition A and to prove that the constants $c_{3,1}$ and $c_{3,2}$ are independent of H, we need to make several modifications and rely completely on the sectorial local nondeterminism A3.

We will make use of following lemmas. Among them, Lemma 6.1 is essentially due to Cuzick and DuPreez (1982) [see also Khoshnevisan and Xiao (2007)], Lemma 6.2 is from Ayache, Wu and Xiao (2008). Lemma 6.3 is a direct consequence of Condition A3 and tells us that the Gaussian random field X_0^H has the one-sided strong local nondeterminism along every direction.

Lemma 6.1 *Let $Z_1, \ldots, Z_n$ be mean zero Gaussian variables which are linearly independent, then for any nonnegative Borel function $g : \mathbb{R} \to \mathbb{R}_+$,*

$$\begin{aligned}\int_{\mathbb{R}^n} & g(v_1) \exp\Big[-\frac{1}{2}\mathrm{Var}\Big(\sum_{j=1}^n v_j Z_j\Big)\Big]\, dv_1 \cdots dv_n \\ &= \frac{(2\pi)^{(n-1)/2}}{(\mathrm{detCov}(Z_1, \cdots, Z_n))^{1/2}} \int_{-\infty}^{\infty} g\Big(\frac{v}{\sigma_1}\Big)\, e^{-v^2/2}\, dv,\end{aligned}$$

where $\sigma_1^2 = \mathrm{Var}(Z_1|Z_2, \ldots, Z_n)$ is the conditional variance of Z_1 given $Z_2, \ldots, Z_n$.

Lemma 6.2 *For any $q \in [0, \sum_{\ell=1}^N H_\ell^{-1})$, let $\tau \in \{1, \ldots, N\}$ be the integer such that*

$$\sum_{\ell=1}^{\tau-1} \frac{1}{H_\ell} \le q < \sum_{\ell=1}^{\tau} \frac{1}{H_\ell} \tag{6.1}$$

with the convention that $\sum_{\ell=1}^0 \frac{1}{H_\ell} := 0$. Then there exists a positive constant $\delta_\tau \le 1$ depending on $(H_1, \ldots, H_N)$ only such that for every $\delta \in (0, \delta_\tau)$, we can find τ real numbers $p_\ell \ge 1$ $(1 \le \ell \le \tau)$ satisfying the following properties:

$$\sum_{\ell=1}^{\tau} \frac{1}{p_\ell} = 1, \qquad \frac{H_\ell\, q}{p_\ell} < 1, \qquad \forall\, \ell = 1, \ldots, \tau \tag{6.2}$$

and

$$(1-\delta)\sum_{\ell=1}^{\tau}\frac{H_\ell\, q}{p_\ell}\le H_\tau\, q+\tau-\sum_{\ell=1}^{\tau}\frac{H_\tau}{H_\ell}. \tag{6.3}$$

Furthermore, if we denote $\alpha_\tau := \sum_{\ell=1}^{\tau}\frac{1}{H_\ell}-q>0$, *then for any positive number* $\rho\in\big(0,\frac{\alpha_\tau}{2\tau}\big)$, *there exists an* $\ell_0\in\{1,\dots,\tau\}$ *such that*

$$\frac{H_{\ell_0}q}{p_{\ell_0}}+2H_{\ell_0}\rho<1. \tag{6.4}$$

Lemma 6.3 *Let* X_0^H *be an* $(N,1)$*-Gaussian random field satisfying Condition A3, and let* $\ell\in\{1,\dots,N\}$ *be fixed. For any integer* $n\ge 2$, $t^1,\dots,t^n\in I$ *such that*

$$t_\ell^1\le t_\ell^2\le\cdots\le t_\ell^n$$

we have

$$\mathrm{Var}\Big(X_0^H(t^n)\Big|X_0^H(t^k):1\le k\le n-1\Big)\ge c_{1,5}(H)\,|t_\ell^n-t_\ell^{n-1}|^{2H_\ell}, \tag{6.5}$$

where $c_{1,5}(H)$ *is the positive continuous function defined in Condition A3.*

Lemma 6.4 below is a refinement of Lemma 3.6 in Ayache, Wu and Xiao (2008). It is important to note that the constant $c_{6,1}$ in (6.6) is independent of $\{b_j,\,1\le j\le n\}$.

Lemma 6.4 *Let* $a>0$ *and* $0<\underline{b}<\overline{b}<1$ *be given constants. There exists a positive constant* $c_{6,1}$ *such that for all integers* $n\ge 1$, *real numbers* $0<r\le 1$, $b_j\in[\underline{b},\,\overline{b}]$ *and an arbitrary* $s_0\in[0,a/2]$,

$$\int_{a\le s_1\le\cdots\le s_n\le a+r}\prod_{j=1}^{n}(s_j-s_{j-1})^{-b_j}\,ds_1\cdots ds_n\le c_{6,1}^n\,(n!)^{\frac{1}{n}\sum_{j=1}^n b_j-1}r^{n-\sum_{j=2}^n b_j}, \tag{6.6}$$

In particular, if $b_j=\alpha\in[\underline{b},\,\overline{b}]$ *for all* $j=1,\dots,n$, *then*

$$\int_{a\le s_1\le\cdots\le s_n\le a+r}\prod_{j=1}^{n}(s_j-s_{j-1})^{-\alpha}\,ds_1\cdots ds_n\le c_{6,1}^n\,(n!)^{\alpha-1}\,r^{n(1-(1-\frac{1}{n})\alpha)}. \tag{6.7}$$

Proof Clearly, we only need to prove (6.6). By integrating the integral in (6.6) in the order of $ds_n, ds_{n-1},\cdots,ds_1$, by using a change of variable in each step to construct Beta functions, and by applying the relationship between Beta and Gamma functions, we derive

$$\begin{aligned}
&\int_{a\le s_1\le\cdots\le s_n\le a+r}\prod_{j=1}^{n}(s_j-s_{j-1})^{-b_j}\,ds_1\cdots ds_n\\
&=\frac{1}{1-b_n}\cdot\frac{\Gamma(2-b_n)\prod_{j=2}^{n-1}\Gamma(1-b_j)}{\Gamma\big(n-\sum_{j=2}^n b_j\big)}\int_a^{a+r}(a+r-s_1)^{n-1-\sum_{j=2}^n b_j}(s_1-s_0)^{-b_1}\,ds_1\\
&\le\frac{\prod_{j=1}^n\Gamma(1-b_j)}{\Gamma\big(n-\sum_{j=2}^n b_j\big)}\left(\frac{a}{2}\right)^{-b_1}\int_a^{a+r}(a+r-s_1)^{n-1-\sum_{j=2}^n b_j}\,ds_1\\
&=\frac{\prod_{j=1}^n\Gamma(1-b_j)}{\Gamma\big(1+n-\sum_{j=2}^n b_j\big)}\left(\frac{a}{2}\right)^{-b_1}r^{n-\sum_{j=2}^n b_j}.
\end{aligned}\tag{6.8}$$

Since the Gamma function $\Gamma(x)$ is continuous on $[1-\overline{b},\, 1-\underline{b}]$, there is a finite constant $c > 0$ such that $\Gamma(1-b_j) \le c$ for all $1 \le j \le n$. The inequality (6.6) follows from (6.8) and Stirling's formula. $\square$

Now we are ready to prove Lemma 3.2.

Proof of Lemma 3.2 For any Gaussian random field X_0^H satisfying Condition A3, there exists a positive constant

$$c_{6,2} := \min_{H \in \prod_{\ell=1}^N [H_\ell^0 - \delta_0,\, H_\ell^0 + \delta_0]} c_{1,5}(H),$$

which depends on H^0 and δ_0 only, such that for all integers $n \ge 1$, all $u, t^1, \ldots, t^n \in I$,

$$\mathrm{Var}\big(X_0^H(u) \big| X_0^H(t^1), \ldots, X_0^H(t^n)\big) \ge c_{6,2} \sum_{\ell=1}^N \min_{0 \le k \le n} |u_\ell - t_\ell^k|^{2H_\ell}, \tag{6.9}$$

where $t^0 = 0$. Meanwhile, by Lemma 6.3, we have that for $H \in \prod_{\ell=1}^N [H_\ell^0 - \delta_0,\, H_\ell^0 + \delta_0]$, $\ell \in \{1, \ldots, N\}$ fixed, and for any integer $n \ge 2$, $t^1, \ldots, t^n \in I$ such that

$$t_\ell^1 \le t_\ell^2 \le \cdots \le t_\ell^n$$

we have

$$\mathrm{Var}\Big(X_0^H(t^n) \Big| X_0^H(t^k) : 1 \le k \le n-1\Big) \ge c_{6,2}\, |t_\ell^n - t_\ell^{n-1}|^{2H_\ell}, \tag{6.10}$$

Now we proceed to prove (3.11). We will start with an arbitrary closed interval $T = \prod_{\ell=1}^N [a_\ell,\, a_\ell + r_\ell] \subseteq I$. It follows from (3.3) and the fact that $X_1^H, \ldots, X_d^H$ are independent copies of X_0^H that for all integers $n \ge 1$,

$$\mathbb{E}\big[L(x,T)^n\big] \le (2\pi)^{-nd} \int_{T^n} \prod_{k=1}^d \bigg\{ \int_{\mathbb{R}^n} \exp\bigg[-\frac{1}{2} \mathrm{Var}\Big(\sum_{j=1}^n u_k^j\, X_0^H(t^j) \Big) \bigg] dU_k \bigg\}\, d\bar{t}, \tag{6.11}$$

where $U_k = (u_k^1, \ldots, u_k^n) \in \mathbb{R}^n$. Fix $k = 1, \ldots, d$ and denote the inner integral in (6.11) by $\mathfrak{J}_k$. Since (3.8) holds, we apply Lemma 6.2 with $\delta = n^{-1}$ and $q = d$ to obtain τ positive numbers $p_1, \ldots, p_\tau \ge 1$ satisfying (6.2) and (6.3). Then for all points $t^1, \ldots, t^n \in T$ such that $t_\ell^1, \ldots, t_\ell^n$ are all distinct for every $1 \le \ell \le N$ [the set of such points has full (nN)-dimensional Lebesgue measure] we have

$$\begin{aligned} \mathfrak{J}_k &= c_{6,3}^n \Big[\mathrm{detCov}\left(X_0^H(t^1), \ldots, X_0^H(t^n) \right) \Big]^{-\frac{1}{2}} \\ &= c_{6,3}^n \prod_{\ell=1}^\tau \Big[\mathrm{detCov}\left(X_0^H(t^1), \ldots, X_0^H(t^n) \right) \Big]^{-\frac{1}{2p_\ell}}, \end{aligned} \tag{6.12}$$

where the first equality follows from the fact that for any positive definite $q \times q$ matrix Γ,

$$\int_{\mathbb{R}^q} \frac{[\det(\Gamma)]^{1/2}}{(2\pi)^{u/2}} \exp\Big(-\frac{1}{2} x' \Gamma x \Big)\, dx = 1 \tag{6.13}$$

and the second equality follows from (6.2).

Combining (6.11) and (6.12) yields

$$\mathbb{E}\big[L(x,T)^n\big] \le c_{6,4}^n \int_{T^n} \prod_{\ell=1}^{\tau} \Big[\mathrm{detCov}(X_0^H(t^1), \ldots, X_0^H(t^n))\Big]^{-\frac{d}{2p_\ell}} d\bar{t}. \tag{6.14}$$

To evaluate the integral in (6.14), we will first integrate $[dt_\ell^1 \ldots dt_\ell^n]$ for $\ell = 1, \ldots, \tau$. To this end, we will make use of the following fact about multivariate normal distributions: For any Gaussian random vector $(Z_1, \ldots, Z_n)$,

$$\mathrm{detCov}(Z_1, \ldots, Z_n) = \mathrm{Var}(Z_1) \prod_{j=2}^{n} \mathrm{Var}(Z_j | Z_1, \ldots, Z_{j-1}). \tag{6.15}$$

By the above fact and (6.10), we can derive that for every $\ell \in \{1, \ldots, \tau\}$ and for all $t^1, \ldots, t^n \in T = \prod_{\ell=1}^{N} [a_\ell,\, a_\ell + r_\ell]$ satisfying

$$a_\ell \le t_\ell^{\pi_\ell(1)} \le t_\ell^{\pi_\ell(2)} \le \cdots \le t_\ell^{\pi_\ell(n)} \le a_\ell + r_\ell \tag{6.16}$$

for some permutation π_ℓ of $\{1, \ldots, N\}$, we have

$$\mathrm{detCov}\big(X_0^H(t^1), \ldots, X_0^H(t^n)\big) \ge c_{6,5}^n \prod_{j=1}^{n} \big(t_\ell^{\pi_\ell(j)} - t_\ell^{\pi_\ell(j-1)}\big)^{2H_\ell}, \tag{6.17}$$

where $t_\ell^{\pi_\ell(0)} := \varepsilon$, and where $c_{6,5}$ is a constant depends on N, H^0, δ_0 and I only. We have chosen $\varepsilon < \frac{1}{2} \min\{a_\ell,\, 1 \le \ell \le N\}$ so that Lemma 6.4 is applicable.

It follows from (6.16) and (6.17) that

$$\begin{aligned} &\int_{[a_\ell,\, a_\ell + r_\ell]^n} \Big[\mathrm{detCov}(X_0^H(t^1), \ldots, X_0^H(t^n))\Big]^{-\frac{d}{2p_\ell}} dt_\ell^1 \cdots dt_\ell^n \\ &\le \sum_{\pi_\ell} c^n \int_{a_\ell \le t_\ell^{\pi_\ell(1)} \le \cdots \le t_\ell^{\pi_\ell(n)} \le a_\ell + r_\ell} \prod_{j=1}^{n} \frac{1}{\big(t_\ell^{\pi_\ell(j)} - t_\ell^{\pi_\ell(j-1)}\big)^{H_\ell d/p_\ell}} \, dt_\ell^1 \cdots dt_\ell^n \\ &\le c_{6,6}^n \, (n!)^{H_\ell d/p_\ell} \, r_\ell^{n\left(1 - (1 - \frac{1}{n}) H_\ell d/p_\ell\right)}. \end{aligned} \tag{6.18}$$

In the above, the last inequality follows from (6.7).

Combining (6.14), (6.18) and continuing to integrate $[dt_\ell^1 \ldots dt_\ell^n]$ for $\ell = \tau + 1, \ldots, N$, we obtain

$$\mathbb{E}\big[L(x,T)^n\big] \le c_{6,7}^n \, (n!)^{\sum_{\ell=1}^{\tau} H_\ell d/p_\ell} \prod_{\ell=1}^{\tau} r_\ell^{n(1 - (1 - \frac{1}{n}) H_\ell d/p_\ell)} \cdot \prod_{\ell=\tau+1}^{N} r_\ell^n. \tag{6.19}$$

Now we consider the special case when $T = [a, a + \langle r \rangle]$, i.e. $r_1 = \cdots = r_N = r$. Eq. (6.19) and (6.3) with $\delta = n^{-1}$ and $q = d$ together yield

$$\begin{aligned} \mathbb{E}\big[L(x,T)^n\big] &\le c_{6,8}^n \, (n!)^{\sum_{\ell=1}^{\tau} H_\ell d/p_\ell} \, r^{n\left(N - (1 - n^{-1}) \sum_{\ell=1}^{\tau} H_\ell d/p_\ell\right)} \\ &\le c_{3,1}^n \, (n!)^{N - \beta_\tau} \, r^{n\beta_\tau}. \end{aligned} \tag{6.20}$$

This proves (3.11).

We prove estimate (3.12) next. Let $\gamma \in (0, 1 \wedge \frac{\alpha_\tau}{2\tau})$ be a constant, depending on H^0 only. Note that by the elementary inequalities

$$|e^{iu} - 1| \le 2^{1-\gamma} |u|^\gamma \qquad \text{for all} \ \ u \in \mathbb{R} \tag{6.21}$$

and $|u+v|^\gamma \le |u|^\gamma + |v|^\gamma$, we see that for all $u^1, \ldots, u^n, \ x, \ y \in \mathbb{R}^d$,

$$\prod_{j=1}^n \Big| e^{-i\langle u^j, x\rangle} - e^{-i\langle u^j, y\rangle} \Big| \le 2^{(1-\gamma)n} \ |x-y|^{n\gamma} \sum{}' \prod_{j=1}^n |u^j_{k_j}|^\gamma, \tag{6.22}$$

where the summation $\sum{}'$ is taken over all the sequences $(k_1, \cdots, k_n) \in \{1, \cdots, d\}^n$.

It follows from (3.4) and (6.22) that for every even integer $n \ge 2$,

$$\begin{aligned}
\mathbb{E}\Big[\big(L(x,T) - L(y,T)\big)^n\Big] &\le (2\pi)^{-nd} 2^{(1-\gamma)n} \, |x-y|^{n\gamma} \\
&\quad \times \sum{}' \int_{T^n} \int_{\mathbb{R}^{nd}} \prod_{m=1}^n |u^m_{k_m}|^\gamma \, \mathbb{E} \exp\bigg(-i \sum_{j=1}^n \langle u^j, \, X^H(t^j)\rangle \bigg) \, d\overline{u} \, d\overline{t} \\
&\le c_{6,9}^n |x-y|^{n\gamma} \sum{}' \int_{T^n} \, d\overline{t} \\
&\quad \times \prod_{m=1}^n \bigg\{ \int_{\mathbb{R}^{nd}} |u^m_{k_m}|^{n\gamma} \, \exp\bigg[-\frac{1}{2} \mathrm{Var}\bigg(\sum_{j=1}^n \langle u^j, \, X^H(t^j)\rangle \bigg) \bigg] \, d\overline{u} \bigg\}^{1/n},
\end{aligned} \tag{6.23}$$

where the last inequality follows from the generalized Hölder inequality.

Now we fix a vector $\overline{k} = (k_1, \ldots, k_n) \in \{1, \cdots, d\}^n$ and n points $t^1, \ldots, t^n \in T$ such that $t^1_\ell, \ldots, t^n_\ell$ are all distinct for every $1 \le \ell \le N$. Let $\mathcal{M} = \mathcal{M}(\overline{k}, \overline{t}, \gamma)$ be defined by

$$\mathcal{M} = \prod_{m=1}^n \bigg\{ \int_{\mathbb{R}^{nd}} |u^m_{k_m}|^{n\gamma} \, \exp\bigg[-\frac{1}{2} \mathrm{Var}\bigg(\sum_{j=1}^n \langle u^j, \, X^H(t^j)\rangle \bigg) \bigg] \, d\overline{u} \bigg\}^{1/n}. \tag{6.24}$$

Note that the coordinate fields X^H_ℓ $(1 \le \ell \le N)$ are independent copies of X^H_0. By Condition A3, the random variables $X^H_\ell(t^j)$ $(1 \le \ell \le N, 1 \le j \le n)$ are linearly independent. Hence Lemma 6.1 gives

$$\begin{aligned}
&\int_{\mathbb{R}^{nd}} |u^m_{k_m}|^{n\gamma} \, \exp\bigg[-\frac{1}{2} \mathrm{Var}\bigg(\sum_{j=1}^n \langle u^j, X^H(t^j)\rangle \bigg) \bigg] \, d\overline{u} \\
&\qquad = \frac{(2\pi)^{(nd-1)/2}}{\big[\mathrm{detCov}\big(X^H_0(t^1), \ldots, X^H_0(t^n)\big)\big]^{d/2}} \int_{\mathbb{R}} \Big(\frac{v}{\sigma_m}\Big)^{n\gamma} e^{-\frac{v^2}{2}} \, dv \\
&\qquad \le \frac{c_{6,10}^n \, (n!)^\gamma}{\big[\mathrm{detCov}\big(X^H_0(t^1), \ldots, X^H_0(t^n)\big)\big]^{d/2}} \, \frac{1}{\sigma_m^{n\gamma}},
\end{aligned} \tag{6.25}$$

where σ_m^2 is the conditional variance of $X^H_{k_m}(t^m)$ given $X^H_i(t^j)$ $(i \ne k_m$ or $i = k_m$ but $j \ne m)$, and the last inequality follows from Stirling's formula.

Combining (6.24) and (6.25) we obtain

$$\mathcal{M} \le \frac{c_{6,11}^n\,(n!)^\gamma}{\big[\det\mathrm{Cov}\big(X_0^H(t^1),\ldots,X_0^H(t^n)\big)\big]^{d/2}} \prod_{m=1}^n \frac{1}{\sigma_m^\gamma}. \tag{6.26}$$

For $\delta = 1/n$ and $q = d$, let p_ℓ ($\ell = 1,\ldots,\tau$) be the constants as in Lemma 6.2. Observe that, since $\gamma \in \big(0, \frac{\alpha_\tau}{2\tau}\big)$, there exists an $\ell_0 \in \{1,\ldots,\tau\}$ such that

$$\frac{H_{\ell_0} d}{p_{\ell_0}} + 2H_{\ell_0}\gamma < 1. \tag{6.27}$$

It follows from (6.26) and (6.2) that

$$\mathcal{M} \le c_{6,12}^n\,(n!)^\gamma \prod_{\ell=1}^\tau \frac{1}{\big[\det\mathrm{Cov}\big(X_0^H(t^1),\ldots,X_0^H(t^n)\big)\big]^{d/(2p_\ell)}} \prod_{m=1}^n \frac{1}{\sigma_m^\gamma}. \tag{6.28}$$

The second product in (6.28) will be treated as a "perturbation" factor and will be shown to be small when integrated. For this purpose, we use again the independence of the coordinate processes of X^H and (6.9) [cf. Condition A3] to derive

$$\begin{aligned}\sigma_m^2 &= \mathrm{Var}\Big(X_{k_m}^H(t^m)\Big|X_{k_m}^H(t^j),\ j\ne m\Big)\\ &\ge c_{6,13}^2 \sum_{\ell=1}^N \min\big\{|t_\ell^m - t_\ell^j|^{2H_\ell}:\ j\ne m\big\}.\end{aligned} \tag{6.29}$$

For any n points $t^1,\ldots,t^n \in T$, let $\pi_1,\ldots,\pi_N$ be N permutations of $\{1,2,\ldots,n\}$ such that for every $1\le \ell\le N$,

$$t_\ell^{\pi_\ell(1)} \le t_\ell^{\pi_\ell(2)} \le \cdots \le t_\ell^{\pi_\ell(n)}. \tag{6.30}$$

Then, by (6.29) and (6.30) we have

$$\begin{aligned}\prod_{m=1}^n \frac{1}{\sigma_m^\gamma} &\le \prod_{m=1}^n \frac{1}{c_{6,14}\ \sum_{\ell=1}^N \big[\big(t_\ell^{\pi_\ell(m)} - t_\ell^{\pi_\ell(m-1)}\big)\wedge\big(t_\ell^{\pi_\ell(m+1)} - t_\ell^{\pi_\ell(m)}\big)\big]^{H_\ell\gamma}}\\ &\le \prod_{m=1}^n \frac{1}{c_{6,14}\ \big[\big(t_{\ell_0}^{\pi_{\ell_0}(m)} - t_{\ell_0}^{\pi_{\ell_0}(m-1)}\big)\wedge\big(t_{\ell_0}^{\pi_{\ell_0}(m+1)} - t_{\ell_0}^{\pi_{\ell_0}(m)}\big)\big]^{H_{\ell_0}\gamma}}\\ &\le c_{6,14}^{-n} \prod_{m=1}^n \frac{1}{\big(t_{\ell_0}^{\pi_{\ell_0}(m)} - t_{\ell_0}^{\pi_{\ell_0}(m-1)}\big)^{q_{\ell_0}^m H_{\ell_0}\gamma}},\end{aligned} \tag{6.31}$$

for some $(q_{\ell_0}^1,\ldots,q_{\ell_0}^n) \in \{0,1,2\}^n$ satisfying $\sum_{m=1}^n q_{\ell_0}^m = n$ and $q_{\ell_0}^1 = 0$. That is, we will only need to consider the contribution of σ_m in the ℓ_0-th direction.

So far we have obtained all the ingredients for bounding the integral in (6.23) and the rest of the proof is quite similar to the proof of (3.11). It follows from (6.28) and (6.31) that

$$\begin{aligned}\int_{T^n} \mathcal{M}(\bar{k},\bar{t},\gamma)\,d\bar{t} \le c_{6,15}^n\,(n!)^\gamma \int_{T^n} &\prod_{\ell=1}^\tau \frac{1}{\big[\det\mathrm{Cov}\big(X_0^H(t^1),\ldots,X_0^H(t^n)\big)\big]^{d/(2p_\ell)}}\\ &\times \prod_{m=1}^n \frac{1}{\big(t_{\ell_0}^{\pi_{\ell_0}(m)} - t_{\ell_0}^{\pi_{\ell_0}(m-1)}\big)^{q_{\ell_0}^m H_{\ell_0}\gamma}}\,d\bar{t}.\end{aligned} \tag{6.32}$$

To evaluate the above integral, we will first integrate $[dt_\ell^1 \dots dt_\ell^n]$ for every $\ell = 1, \dots, \tau$. Let us first consider $\ell = \ell_0$. By using (6.17), (6.15), (6.6) and, thanks to (6.27) and the nature of $q_{\ell_0}^m$, we see that

$$\begin{aligned}
&\int_{[a_{\ell_0}, a_{\ell_0}+r_{\ell_0}]^n} \frac{1}{\big[\mathrm{detCov}(X_0^H(t^1), \dots, X_0^H(t^n))\big]^{d/(2p_{\ell_0})}} \\
&\qquad \times \prod_{m=1}^{n} \frac{1}{\big(t_{\ell_0}^{\pi_{\ell_0}(m)} - t_{\ell_0}^{\pi_{\ell_0}(m-1)}\big)^{q_{\ell_0}^m H_{\ell_0}\gamma}}\, dt_{\ell_0}^1 \cdots dt_{\ell_0}^n \\
&\leq \sum_{\pi_{\ell_0}} c_{6,16}^n \int_{a_{\ell_0} \leq t_{\ell_0}^{\pi_{\ell_0}(1)} \leq \cdots \leq t_{\ell_0}^{\pi_{\ell_0}(n)} \leq a_{\ell_0}+r_{\ell_0}} \qquad (6.33) \\
&\qquad \prod_{m=1}^{n} \Big(t_{\ell_0}^{\pi_{\ell_0}(m)} - t_{\ell_0}^{\pi_{\ell_0}(m-1)}\Big)^{-\big(H_{\ell_0} d/p_{\ell_0} + q_{\ell_0}^m H_{\ell_0}\gamma\big)}\, dt_{\ell_0}^1 \cdots dt_{\ell_0}^n \\
&\leq c_{6,17}^n\, (n!)^{H_{\ell_0} d/p_{\ell_0} + H_{\ell_0}\gamma}\, r_{\ell_0}^{n\left[1-(1-\frac{1}{n})H_{\ell_0}\, d/p_{\ell_0} - H_{\ell_0}\gamma\right]}. \qquad (6.34)
\end{aligned}$$

In the above, $t_{\ell_0}^{\pi_{\ell_0}(0)} = \varepsilon$ as in the proof of (3.11) and the last inequality follows from (6.6).

Meanwhile, recall that, for every $\ell \neq \ell_0$ ($\ell \in \{1, \dots, \tau\}$), we have shown in (6.18) that

$$\begin{aligned}
&\int_{[a_\ell, a_\ell+r_\ell]^n} \Big[\mathrm{detCov}(X_0^H(t^1), \dots, X_0^H(t^n))\Big]^{-\frac{d}{2p_\ell}}\, dt_\ell^1 \cdots dt_\ell^n \\
&\qquad \leq c_{6,6}^n\, (n!)^{H_\ell d/p_\ell}\, r_\ell^{n\left(1-(1-\frac{1}{n})H_\ell d/p_\ell\right)}.
\end{aligned} \tag{6.35}$$

Finally, we proceed to integrate $[dt_\ell^1 \dots dt_\ell^n]$ for $\ell = \tau+1, \dots, N$. It follows from the above that

$$\begin{aligned}
\int_{T^n} \mathcal{M}(\bar{k}, \bar{t}, \gamma)\, d\bar{t} &\leq c_{4,22}^n\, (n!)^{\sum_{\ell=1}^{\tau} H_\ell\, d/p_\ell + H_{\ell_0}\gamma + \gamma} \\
&\quad \times r_{\ell_0}^{n\left[1-(1-\frac{1}{n})H_{\ell_0}\, d/p_{\ell_0} - H_{\ell_0}\gamma\right]} \times \prod_{\ell \neq \ell_0}^{\tau} r_\ell^{n\left[1-(1-\frac{1}{n})H_\ell\, d/p_\ell\right]} \prod_{\ell=\tau+1}^{N} r_\ell^n.
\end{aligned} \tag{6.36}$$

In particular, if $r_1 = \cdots = r_N = r \leq 1$, we combine (6.23) and (6.36) to obtain

$$\begin{aligned}
&\mathbb{E}\Big[\big(L(x,T) - L(y,T)\big)^n\Big] \\
&\leq c_{6,18}^n\, |x-y|^{n\gamma}\, (n!)^{\sum_{\ell=1}^{\tau} H_\ell\, d/p_\ell + H_{\ell_0}\gamma + \gamma} \cdot r^{n\left(N-(1-\frac{1}{n})\sum_{\ell=1}^{\tau} H_\ell\, d/p_\ell - H_{\ell_0}\gamma\right)} \\
&\leq c_{3,2}^n\, (n!)^{N-\beta_\tau+(1+H_\tau)\gamma}\, |x-y|^{n\gamma}\, r^{n(\beta_\tau - H_\tau\gamma)}.
\end{aligned} \tag{6.37}$$

The last inequality follows from the fact that $H_{\ell_0} \leq H_\tau$ and Lemma 6.2. This finishes the proof of (3.12). □

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