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# Geometric Properties of Fractional Brownian Sheets

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*ABSTRACT.* *Let* $B^H = \{B^H(t), t \in \mathbb{R}_+^N\}$ *be an* $(N,d)$*-fractional Brownian sheet with Hurst index* $H = (H_1, \ldots, H_N) \in (0,1)^N$*. Our objective of the present paper is to characterize the anisotropic nature of* $B^H$ *in terms of* $H$*. We prove the following results:*

***(1)*** $B^H$ *is sectorially locally nondeterministic.*

***(2)*** *By introducing a notion of "dimension" for Borel measures and sets, which is suitable for describing the anisotropic nature of* $B^H$*, we determine* $\dim_{\mathcal{H}} B^H(E)$ *for an arbitrary Borel set* $E \subset (0,\infty)^N$*. Moreover, when* $B^{\langle\alpha\rangle}$ *is an* $(N,d)$*-fractional Brownian sheet with index* $\langle\alpha\rangle = (\alpha, \ldots, \alpha)$ $(0 < \alpha < 1)$*, we prove the following uniform Hausdorff dimension result for its image sets: If* $N \le \alpha d$*, then with probability one,*

$$\dim_{\mathcal{H}} B^{\langle\alpha\rangle}(E) = \frac{1}{\alpha} \dim_{\mathcal{H}} E \quad \text{for all Borel sets } E \subset (0,\infty)^N.$$

***(3)*** *We provide sufficient conditions for the image* $B^H(E)$ *to be a Salem set or to have interior points.*

*The results in **(2)** and **(3)** describe the geometric and Fourier analytic properties of* $B^H$*. They extend and improve the previous theorems of Mountford [35], Khoshnevisan and Xiao [29] and Khoshnevisan, Wu and Xiao [28] for the Brownian sheet, and Ayache and Xiao [5] for fractional Brownian sheets.*

## 1. Introduction

Gaussian processes and Gaussian random fields have been extensively studied and applied in many areas to model phenomena having self-similarity

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and long memory properties. One of the most important Gaussian fields is fractional Brownian motion $X = \{X(t),\ t \in \mathbb{R}^N\}$, which is a centered $(N,d)$-Gaussian random field with covariance function

$$\mathbb{E}[X_i(s)X_j(t)] = \frac{1}{2}\delta_{ij}\Big(|s|^{2\alpha} + |t|^{2\alpha} - |s-t|^{2\alpha}\Big), \quad \forall s,t \in \mathbb{R}^N, \tag{1.1}$$

where $0 < \alpha < 1$ is a constant and $\delta_{ij} = 1$ if $i = j$ and 0 if $i \neq j$, and where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^N$. When $N = d = 1$, it was first introduced by Mandelbrot and Van-Ness [31] as a moving-average Gaussian process. It can be verified from (1.1) that $X$ is $\alpha$-self-similar and has stationary increments in the strong sense; see Section 8.1 of Samorodnitsky and Taqqu [40]. In particular, $X$ is isotropic in the sense that $X(s) - X(t)$ depends only on the Euclidean distance $|s-t|$.

Many data sets from various scientific areas such as image processing, hydrology, geostatistics and spatial statistics have anisotropic nature in the sense that they have different geometric and probabilistic characteristics along different directions, hence fractional Brownian motion is not adequate for modelling such phenomena. Many people have proposed to apply anisotropic Gaussian random fields as more realistic models. See, for example, Davies and Hall [13], Bonami and Estrade [11], Benson, et al. [8].

Several different classes of anisotropic Gaussian random fields have been introduced for theoretical and application purposes. For example, Kamont [26] introduced fractional Browian sheets [see the definition below] and studied some of their regularity properties. Benassi, et al. [7] and Bonami and Estrade [11] considered some anisotropic Gaussian random fields with stationary increments. More recently, Biermé, et al. [10] constructed a large class of operator self-similar Gaussian or stable random fields with stationary increments. Anisotropic Gaussian random fields also arise in stochastic partial differential equations [see, e.g., Mueller and Tribe [36], Øksendal and Zhang [38], Nualart [37]]; and in studying the most visited sites of symmetric Markov processes [Eisenbaum and Khoshnevisan [16]]. Hence it is of importance in both theory and applications to investigate the probabilistic and statistical properties of such random fields. However, systematic studies of anisotropic Gaussian random fields have only been started recently, which have shown to have significantly different properties from those of fractional Brownian motion, or their isotropic counterparts.

This paper is concerned with sample path properties of fractional Brownian sheets. We believe that a good understanding of them will help us to better understand anisotropic Gaussian random fields in general. In fact, some methods for studying sample path properties of fractional Brownian sheets have proved to be useful for other anisotropic random fields; see Wu and Xiao [43], Xiao [48] for more information.

For a given vector $H = (H_1, \ldots, H_N) \in (0,1)^N$, an $(N,1)$-fractional Brownian sheet $B_0^H = \{B_0^H(t), t \in \mathbb{R}^N\}$ with Hurst index $H$ is a real-valued, centered Gaussian random field with covariance function given by

$$\mathbb{E}\big[B_0^H(s)B_0^H(t)\big] = \prod_{j=1}^{N} \frac{1}{2}\Big(|s_j|^{2H_j} + |t_j|^{2H_j} - |s_j - t_j|^{2H_j}\Big), \quad s, t \in \mathbb{R}^N. \tag{1.2}$$

It follows from (1.2) that $B_0^H(t) = 0$ a.s. for every $t \in \mathbb{R}^N$ with at least one zero coordinate. When $H_1 = \cdots = H_N = \alpha \in (0, 1)$, we will write $H = \langle \alpha \rangle$.

Let $B_1^H, \ldots, B_d^H$ be $d$ independent copies of $B_0^H$. Then the $(N, d)$-fractional Brownian sheet with Hurst index $H = (H_1, \ldots, H_N)$ is the Gaussian random field $B^H = \{B^H(t) : \ t \in \mathbb{R}^N\}$ with values in $\mathbb{R}^d$ defined by

$$B^H(t) = \big(B_1^H(t), \ldots, B_d^H(t)\big), \quad t \in \mathbb{R}^N. \tag{1.3}$$

It follows from (1.2) that $B^H$ is operator-self-similar in the sense that for all constants $c > 0$,

$$\big\{B^H(c^A t),\, t \in \mathbb{R}^N\big\} \stackrel{d}{=} \Big\{c^N\, B^H(t),\ t \in \mathbb{R}^N\Big\}, \tag{1.4}$$

where $A = (a_{ij})$ is the $N \times N$ diagonal matrix with $a_{ii} = 1/H_i$ for all $1 \le i \le N$ and $a_{ij} = 0$ if $i \neq j$, and $X \stackrel{d}{=} Y$ means that the two processes have the same finite dimensional distributions. Note that if $N = 1$, then $B^H$ is a fractional Brownian motion in $\mathbb{R}^d$ with Hurst index $H_1 \in (0, 1)$; if $N > 1$ and $H = \langle 1/2 \rangle$, then $B^H$ is the $(N, d)$-Brownian sheet, denoted by $B$. Hence $B^H$ can be regarded as a natural generalization of one parameter fractional Brownian motion in $\mathbb{R}^d$, as well as a generalization of the Brownian sheet.

Several authors have studied various properties of fractional Brownian sheets. For example, Ayache, et al. [2] provided a moving average representation for $B_0^H$ and studied its sample path continuity as well as its continuity in $H$. Dunker [14], Mason and Shi [32], Belinski and Linde [6], Kühn and Linde [30] studied the small ball probabilities of $B_0^H$. Mason and Shi [32] also computed the Hausdorff dimension of some exceptional sets related to the oscillation of the sample paths of $B_0^H$. Ayache and Taqqu [3] derived an optimal wavelet series expansion for the fractional Brownian sheet $B_0^H$; see also Kühn and Linde [30], Dzhaparidze and van Zanten [15] for other optimal series expansions for $B_0^H$. Xiao and Zhang [49] studied the existence of local times of an $(N, d)$-fractional Brownian sheet $B^H$ and proved a sufficient condition for the joint continuity of the local times. Kamont [26] and Ayache [1] studied the box and Hausdorff dimensions of the graph set of an $(N, 1)$-fractional Brownian sheet.

Recently, Ayache and Xiao [5] investigated the uniform and local asymptotic properties of $B^H$ by using wavelet methods, and determined the Hausdorff dimensions of the image $B^H([0, 1]^N)$, the graph $\mathrm{Gr}B^H([0, 1]^N)$ and the level set $L_x = \{t \in (0, \infty)^N : \ B^H(t) = x\}$.

The main objective of this paper is to investigate the geometric and Fourier analytic properties of the image $B^H(E)$ of an arbitrary Borel set $E \subset (0,\infty)^N$. In particular, we compute the Hausdorff and Fourier dimensions of $B^H(E)$, and provide a sufficient condition for $B^H(E)$ to have interior points. Such problems for fractional Brownian motion and the Brownian sheet have been investigated by Kahane [22], Pitt [39], Mountford [35], Khoshnevisan and Xiao [29], Shieh and Xiao [41], Khoshnevisan, Wu and Xiao [28]. Our present study of fractional Brownian sheets is different from those of the previous authors in two aspects.

Firstly, unlike the well-known cases of fractional Brownian motion and the Brownian sheet, the Hausdorff dimension of $B^H(E)$ can not be determined by $\dim_{\mathcal{H}} E$ and the index $H$ alone due to the anisotropic nature of $B^H$ [see Example 1]. We solve this problem by introducing a new concept of "dimension" [we call it *Hausdorff dimension contour*] for finite Borel measures and Borel sets. We prove that the Hausdorff and Fourier dimensions of $B^H(E)$ can be represented in terms of the Hausdorff dimension contour of $E$ and the Hurst index $H$. We believe that the concept of Hausdorff dimension contour is of independent interest because it carries more information about the geometric properties of Borel measures and sets than Hausdorff dimension does. It is an appropriate notion for studying the image sets and the local times of fractional Brownian sheets and other anisotropic random fields, as shown by the results in this paper and in Wu and Xiao [43]. It can be shown that, for all sets $E \subset \mathbb{R}^N$, the Hausdorff dimension contour $E$ is related to the Hausdorff dimension of $E$ with respect to an "anisotropic metric"; see Remark 2.

Secondly, the dependence structure of $B^H$ is significantly different from those of fractional Brownian motion and the Brownian sheet, namely, $B^H$ is not locally nondeterministic and does not have the property of independent increments. We overcome this difficulty by showing that $B^H$ satisfies a type of "sectorial local nondeterminism" [see Theorem 1]. This is motivated by a result of Khoshnevisan and Xiao [29] for the Brownian sheet, but our approach here is different and relies on the harmonizable representation (2.1). The property of sectorial local nondeterminism not only plays an important role in this paper, but also in studying other problems such as local times of fractional Brownian sheets [see Ayache, Wu and Xiao [4]].

The rest of this paper is organized as follows. In Section 2, we prove that fractional Brownian sheets satisfy the property of sectorial local nondeterminism. In Section 3, we establish an explicit formula for the Hausdorff dimension of the image $B^H(E)$ in terms of the Hausdorff dimension contour of $E$ and the Hurst index $H$. Moreover, when $H = \langle \alpha \rangle$, we prove the following uniform Hausdorff dimension result for its images: If $N \le \alpha d$, then with probability one,

$$\dim_{\mathcal{H}} B^{\langle\alpha\rangle}(E) = \frac{1}{\alpha}\dim_{\mathcal{H}} E \quad \text{ for all Borel sets } \ E \subset (0,\infty)^N. \tag{1.5}$$

This extends the results of Mountford [35] and Khoshnevisan, Wu and Xiao [28] for the Brownian sheet. Our proof is based on the sectorial local nondeterminism of $B^{\langle\alpha\rangle}$ and is similar to that of Khoshnevisan, Wu and Xiao [28].

Let $\mu$ be a probability measure carried by $E$ and let $\nu = \mu_{_{B^H}}$ be the image measure of $\mu$ under the mapping $t \mapsto B^H(t)$. In Section 4, we study the asymptotic properties of the Fourier transform $\widehat{\nu}(\xi)$ of $\nu$ as $\xi \to \infty$. In particular, we show that the image $B^H(E)$ is a Salem set whenever $s(H,E) \le d$, see Section 3 for the definition of $s(H,E)$. These results extend those of Kahane [22][23] and Khoshnevisan, Wu and Xiao [28] for fractional Brownian motion and the Brownian sheet, respectively.

In Section 5, we prove a sufficient condition for $B^H(E)$ to have interior points. This problem is closely related to the existence of a continuous local time of $B^H$ on $E$ [cf. Pitt [39], Geman and Horowitz [19], Kahane [22][23]]. Our Theorem 6 extends and improves the previous result of Khoshnevisan and Xiao [29] for the Brownian sheet.

Throughout this paper, the underlying parameter space is $\mathbb{R}^N$ or $\mathbb{R}^N_+ = [0,\infty)^N$. A typical parameter, $t \in \mathbb{R}^N$ is written as $t = (t_1, \ldots, t_N)$, or as $\langle c\rangle$, if $t_1 = \cdots = t_N = c$. For any $s, t \in \mathbb{R}^N$ such that $s_j < t_j$ $(j = 1, \ldots, N)$, $[s,t] = \prod_{j=1}^N [s_j, t_j]$ is called a closed interval (or a rectangle). The inner product in $\mathbb{R}^N$ is denoted by $\langle \cdot, \cdot \rangle$.

We will use $c$ to denote an unspecified positive and finite constant which may not be the same in each occurrence. More specific constants in Section $i$ are numbered as $c_{_{i,1}}, c_{_{i,2}}, \ldots$.

Finally, we refer to Kahane [22], Falconer [17] or Mattila [33] for all the notions of dimensions appeared in this paper.

## 2. Sectorial local nondeterminism

One of the main difficulties in studying sample path properties of fractional Brownian sheets is the complexity of their dependence structure. Unlike fractional Brownian motion which is locally nondeterministic [see Pitt [39]] or the Brownian sheet which has independent increments, a fractional Brownian sheet has neither of these properties. The main technical tool which we will apply to study fractional Brownian sheets is the property of "sectorial local nondeterminism" [SLND].

It should be mentioned that the concept of local nondeterminism was first introduced by Berman [9] to unify and extend his methods for studying local times of real-valued Gaussian processes, and then extended by Pitt [39] to Gaussian random fields. The notion of strong local nondeterminism was later developed to investigate the regularity of local times, small ball probabilities and other sample path properties of Gaussian processes and Gaussian random fields. We refer to Xiao [46][47] for more information on the history and applications of local nondeterminism.

For Gaussian random fields, the aforementioned properties of local nondeterminism can only be satisfied by those with approximate isotropy. It is well-known that the Brownian sheet does not satisfy these locally nondeterministic properties. Despite this, Khoshnevisan and Xiao [29] have recently proved that the Brownian sheet satisfies a type of sectorial local nondeterminism and applied this property to study geometric properties of the Brownian sheet; see also Khoshnevisan, Wu and Xiao [28].

The property of sectorial local nondeterminism for fractional Brownian sheets is an extension of that in Khoshnevisan and Xiao [29]. While the argument of Khoshnevisan and Xiao [29] relies on the property of independent increments of the Brownian sheet, our proof is based on a Fourier analytic argument in Kahane ([22], Chapter 18) and the following harmonizable representation of $B_0^H$ essentially due to Herbin [20]:

$$B_0^H(t) = K_{_H}^{-1} \int_{\mathbb{R}^N} \psi_t(\lambda)\, \widehat{W}(d\lambda), \tag{2.1}$$

where $\widehat{W}$ is the Fourier transform of the white noise in $\mathbb{R}^N$,

$$\psi_t(\lambda) = \prod_{j=1}^N \frac{e^{it_j\lambda_j} - 1}{|\lambda_j|^{H_j + \frac{1}{2}}},$$

and where $K_{_H}$ is a normalizing constant so that $\mathbb{E}\big[(B_0^H(t))^2\big] = \prod_{j=1}^N \big|t_j\big|^{2H_j}$ for all $t \in \mathbb{R}^N$.

***Theorem 1.*** *[SLND] For any fixed positive number $\varepsilon \in (0,1)$, there exists a positive constant $c_{_{2,1}}$, depending on $\varepsilon, H$ and $N$ only, such that for all positive integers $n \geq 1$, and all $u, t^1, \ldots, t^n \in [\varepsilon, \infty)^N$, we have*

$$\mathrm{Var}\left(B_0^H(u) \,\middle|\, B_0^H(t^1), \ldots, B_0^H(t^n)\right) \geq c_{_{2,1}} \sum_{j=1}^N \min_{0\leq k\leq n} \big|u_j - t_j^k\big|^{2H_j}, \tag{2.2}$$

*where $t^0 = 0$.*

***Proof.*** Let $\ell \in \{1, \ldots, N\}$ be fixed and denote $r_{_\ell} \equiv \min_{0\leq k\leq n} \big|u_\ell - t_\ell^k\big|$. Firstly, we prove that there exists a positive constant $c_{_\ell}$ such that the following inequality holds:

$$\mathrm{Var}\left(B_0^H(u) \,\middle|\, B_0^H(t^1), \ldots, B_0^H(t^n)\right) \geq c_{_\ell}\, r_{_\ell}^{2H_\ell}. \tag{2.3}$$

Summing over $\ell$ from 1 to $N$ in (2.3), we obtain (2.2).

In order to prove (2.3), we only need to consider those $\ell$ with $r_\ell > 0$ [otherwise, the inequality holds automatically]. Working in the Hilbert space setting, we write the conditional variance in (2.3) as the square of the $L^2(\mathbb{P})$-distance of $B_0^H(u)$ from the subspace generated by $\{B_0^H(t^1), \ldots, B_0^H(t^n)\}$. Hence it suffices to show that for all $a_k \in \mathbb{R}$,

$$\mathbb{E}\biggl(B_0^H(u) - \sum_{k=1}^n a_k B_0^H(t^k)\biggr)^2 \ge c_\ell r_\ell^{2H_\ell}. \tag{2.4}$$

It follows from the harmonizable representation (2.1) of $B_0^H$ that

$$\begin{aligned}\mathbb{E}\biggl(B_0^H(u) - \sum_{k=1}^n a_k B_0^H(t^k)\biggr)^2 &= K_H^{-2}\int_{\mathbb{R}^N}\biggl|\psi_u(\lambda) - \sum_{k=1}^n a_k \psi_{t^k}(\lambda)\biggr|^2 d\lambda \\ = K_H^{-2}\int_{\mathbb{R}^N}\biggl|\prod_{j=1}^N (e^{iu_j\lambda_j}-1) &- \sum_{k=1}^n a_k \prod_{j=1}^N (e^{it_j^k\lambda_j}-1)\biggr|^2 f_H(\lambda)d\lambda,\end{aligned} \tag{2.5}$$

where

$$f_H(\lambda) = \prod_{j=1}^N |\lambda_j|^{-2H_j-1}.$$

Now for every $j = 1, \ldots, N$, we choose a bump function $\delta_j(\cdot) \in C^\infty(\mathbb{R})$ with values in $[0,1]$ such that $\delta_j(0) = 1$ and strictly decreasing in $|\cdot|$ near 0 [e.g., on $(-\varepsilon, \varepsilon)$] and vanishes outside the open interval $(-1,1)$. Let $\widehat{\delta}_j$ be the Fourier transform of $\delta_j$. It can be verified that $\widehat{\delta}_j(\lambda_j)$ is also in $C^\infty(\mathbb{R})$ and decays rapidly as $\lambda_j \to \infty$. Also, the Fourier inversion formula gives

$$\delta_j(s_j) = (2\pi)^{-1}\int_{\mathbb{R}} e^{-is_j\lambda_j}\widehat{\delta}_j(\lambda_j)d\lambda_j. \tag{2.6}$$

Let $\delta_\ell^{r_\ell}(s_\ell) = r_\ell^{-1}\delta_\ell(\frac{s_\ell}{r_\ell})$, then by (2.6) and a change of variables, we have

$$\delta_\ell^{r_\ell}(s_\ell) = (2\pi)^{-1}\int_{\mathbb{R}} e^{-is_\ell\lambda_\ell}\widehat{\delta}_\ell(r_\ell\lambda_\ell)d\lambda_\ell. \tag{2.7}$$

By the definition of $r_\ell$, we have $\delta_\ell^{r_\ell}(u_\ell) = 0$ and $\delta_\ell^{r_\ell}(u_\ell - t_\ell^k) = 0$ for all

$k = 1, \ldots, n$. Hence it follows from (2.6) and (2.7) that

$$
\begin{aligned}
I &\equiv \int_{\mathbb{R}^N} \bigg( \prod_{j=1}^N (e^{iu_j\lambda_j} - 1) - \sum_{k=1}^n a_k \prod_{j=1}^N (e^{it_j^k \lambda_j} - 1) \bigg) \\
&\qquad\qquad \times \bigg( \prod_{j=1}^N e^{-iu_j\lambda_j} \bigg) \bigg( \prod_{j\neq\ell}^N \widehat{\delta}_j(\lambda_j) \bigg) \widehat{\delta}_\ell(r_\ell \lambda_\ell)\, d\lambda \\
&= (2\pi)^N \bigg[ \bigg( \prod_{j\neq\ell} \big(\delta_j(0) - \delta_j(u_j)\big) \bigg) \Big( \delta_\ell^{r_\ell}(0) - \delta_\ell^{r_\ell}(u_\ell) \Big) \bigg] \\
&\quad - (2\pi)^N \bigg[ \sum_{k=1}^n a_k \bigg( \prod_{j\neq\ell} \big(\delta_j(u_j - t_j^k) - \delta_j(u_j)\big) \bigg) \Big( \delta_\ell^{r_\ell}(u_\ell - t_\ell^k) - \delta_\ell^{r_\ell}(u_\ell) \Big) \bigg] \\
&\geq (2\pi)^N \prod_{j\neq\ell} \big(1 - \delta_j(\varepsilon)\big)\, r_\ell^{-1}.
\end{aligned} \tag{2.8}
$$

On the other hand, by the Cauchy-Schwarz inequality, (2.5) and (2.8), we have

$$
\begin{aligned}
I^2 &\leq \int_{\mathbb{R}^N} \bigg| \prod_{j=1}^N \big(e^{iu_j\lambda_j} - 1\big) - \sum_{k=1}^n a_k \prod_{j=1}^N \big(e^{it_j^k\lambda_j} - 1\big) \bigg|^2 f_H(\lambda) d\lambda \\
&\qquad\qquad \times \int_{\mathbb{R}^N} \frac{1}{f_H(\lambda)} \bigg| \bigg( \prod_{j\neq\ell}^N \widehat{\delta}_j(\lambda_j) \bigg) \widehat{\delta}_\ell(r_\ell\lambda_\ell) \bigg|^2 d\lambda \\
&= K_H^2\, \mathbb{E} \bigg( B_0^H(u) - \sum_{k=1}^n a_k B_0^H(t^k) \bigg)^2 \cdot r_\ell^{-2H_\ell - 2} \int_{\mathbb{R}^N} \frac{1}{f_H(\lambda)} \prod_{j=1}^N \big|\widehat{\delta}_j(\lambda_j)\big|^2 d\lambda \\
&= c_{2,2}\, r_\ell^{-2H_\ell - 2}\, \mathbb{E} \bigg( B_0^H(u) - \sum_{k=1}^n a_k B_0^H(t^k) \bigg)^2.
\end{aligned} \tag{2.9}
$$

Combining (2.8) and (2.9) yields (2.3). This finishes the proof of the theorem. □

Given jointly Gaussian random variables $Z_1, \ldots, Z_n$, we denote by $\mathrm{detCov}\big(Z_1, \ldots, Z_n\big)$ the determinant of their covariance matrix. If $\mathrm{detCov}\big(Z_1, \ldots, Z_n\big) > 0$, then we have the identity

$$
\frac{(2\pi)^{n/2}}{\mathrm{detCov}\big(Z_1, \ldots, Z_n\big)} = \int_{\mathbb{R}^n} \mathbb{E} \exp\bigg( -i \sum_{k=1}^n u_k Z_k \bigg) du_1 \cdots du_n. \tag{2.10}
$$

By using the fact that, for every $k$, the conditional distribution of $Z_k$ given $Z_1, \ldots, Z_{k-1}$ is still Gaussian with mean $\mathbb{E}(Z_k | Z_1, \ldots, Z_{k-1})$ and variance

$\mathrm{Var}(Z_k|Z_1,\ldots,Z_{k-1})$, one can evaluate the integral in the right-hand side of (2.10) and thus verify the following formula:

$$\mathrm{detCov}(Z_1,\ldots,Z_n)=\mathrm{Var}(Z_1)\prod_{k=2}^{n}\mathrm{Var}\big(Z_k\big|Z_1,\ldots,Z_{k-1}\big). \tag{2.11}$$

A little thought reveals that (2.11) still holds when $\mathrm{detCov}\big(Z_1,\ldots,Z_n\big)=0$.

Combined with (2.2), the identity (2.11) can be applied to estimate the joint distribution of the Gaussian random variables $B_0^H(t^1),\ldots,B_0^H(t^n)$. It is for this reason why sectorial local nondeterminism is essential in this paper and in studying local times of fractional Brownian sheets.

The following simple result will be needed in Section 5.

***Lemma 1.*** *Let $n\ge 1$ be a fixed integer. Then for all $t^1,\ldots,t^n\in[\varepsilon,\infty)^N$ such that $t_j^1,\ldots,t_j^n$ are all distinct for some $j\in\{1,\ldots,N\}$, the Gaussian random variables $B_0^H(t^1),\ldots,B_0^H(t^n)$ are linearly independent.*

***Proof.*** Let $t^1,\ldots,t^n\in[\varepsilon,\infty)^N$ be given as above. Then it follows from Theorem 1 and (2.11) that $\mathrm{detCov}\big(B_0^H(t^1),\ldots,B_0^H(t^n)\big)>0$. This proves the lemma. □

## 3. Hausdorff dimension results for the images

In this section, we study the Hausdorff dimension of the image set $B^H(E)$ of an arbitrary Borel set $E\subset(0,\infty)^N$. When $E=[0,1]^N$ or any Borel set with positive Lebesgue measure, this problem has been solved by Ayache and Xiao [5]. However, when $E\subset(0,\infty)^N$ is a fractal set, the Hausdorff dimension of $B^H(E)$ can not be determined by $\dim_{\mathcal H}E$ and the Hurst index $H$ alone, as shown by Example 1 below. This is in contrast with the cases of fractional Brownian motion or the Brownian sheet.

To solve this problem, we will introduce a new notion of dimension, namely, *Hausdorff dimension contour*, for finite Borel measures and Borel sets. It turns out that the Hausdorff dimension contour of $E$ is the natural object in determining the Hausdorff dimension and other geometric properties of $B^H(E)$ for all Borel sets $E$.

We start with the following proposition which determines $\dim_{\mathcal H}B^H(E)$ when $E$ belongs to a special class of Borel sets in $\mathbb{R}_+^N$.

***Proposition 1.*** *Let $B^H=\{B^H(t),t\in\mathbb{R}^N\}$ be an $(N,d)$-fractional Brownian sheet with index $H=(H_1,\ldots,H_N)$. Assume that $E_j$ $(j=1,\ldots,N)$ are Borel sets in $(0,\infty)$ satisfying the following property: $\exists\{j_1,\ldots,j_{N-1}\}\subset\{1,\ldots,N\}$ such that $\dim_{\mathcal H}E_{j_k}=\dim_{\mathrm P}E_{j_k}$ for $k=1,\ldots,N-1$. Let $E=$*

$E_1 \times \cdots \times E_N \subset (0,\infty)^N$, *then we have*

$$\dim_{\mathcal{H}} B^H(E) = \min\left\{d; \quad \sum_{j=1}^{N} \frac{\dim_{\mathcal{H}} E_j}{H_j}\right\}, \quad \text{a.s.} \tag{3.1}$$

In the above, $\dim_{\rm P} F$ denotes the packing dimension of $F$ which is defined as follows. For any $\varepsilon > 0$ and any bounded set $F \subseteq \mathbb{R}^d$, let $N(F,\varepsilon)$ be the smallest number of balls of radius $\varepsilon$ needed to cover $F$. Then the *upper box-counting dimension* of $F$ is defined as

$$\overline{\dim}_{\rm B} F = \limsup_{\varepsilon\to 0} \frac{\log N(F,\varepsilon)}{-\log \varepsilon}. \tag{3.2}$$

The packing dimension of $F$ can be defined by

$$\dim_{\rm P} F = \inf\left\{\sup_n \overline{\dim}_{\rm B} F_n : \quad F \subseteq \bigcup_{n=1}^{\infty} F_n\right\}. \tag{3.3}$$

Further information on packing dimension can be found in Tricot [42], Falconer [17] and Mattila [33].

For proving Proposition 1, we need the next two lemmas that are due to Ayache and Xiao [5].

***Lemma 2.*** *Let $B^H = \{B^H(t), t \in \mathbb{R}^N\}$ be an $(N,d)$-fractional Brownian sheet with index $H = (H_1,\ldots,H_N)$. For all $T > 0$, there exist a random variable $A_1 = A_1(\omega) > 0$ of finite moments of any order and an event $\Omega_1^*$ of probability 1 such that for every $\omega \in \Omega_1^*$,*

$$\sup_{s,t\in[0,T]^N} \frac{\left|B^H(s,\omega) - B^H(t,\omega)\right|}{\sum_{j=1}^N |s_j - t_j|^{H_j} \sqrt{\log\left(3 + |s_j - t_j|^{-1}\right)}} \le A_1(\omega). \tag{3.4}$$

***Lemma 3.*** *Let $B_0^H = \{B_0^H(t), t \in \mathbb{R}^N\}$ be an $(N,1)$-fractional Brownian sheet with index $H = (H_1,\ldots,H_N)$, then for any $0 < \varepsilon < T$, there exist positive and finite constants $c_{3,1}$ and $c_{3,2}$ such that for all $s,t \in [\varepsilon,T]^N$,*

$$c_{3,1} \sum_{j=1}^{N} |s_j - t_j|^{2H_j} \le \mathbb{E}\Big[\big(B_0^H(s) - B_0^H(t)\big)^2\Big] \le c_{3,2} \sum_{j=1}^{N} |s_j - t_j|^{2H_j}. \tag{3.5}$$

Now we prove Proposition 1.

***Proof of Proposition 1.*** As usual, the proof of (3.1) is divided into proving the upper and lower bounds separately. We will show that the upper bound in (3.1) follows from the modulus of continuity of $B^H$ and a covering argument, and the lower bound follows from Frostman's theorem [see e.g., Kahane ([22], Chapter 10)] and Lemma 3.

For simplicity of notation, we will only consider the case $N = 2$ and $\dim_{\mathcal{H}} E_1 = \dim_{\mathrm{P}} E_1$. The proof for the general case is similar.

**Upper bound:** By the $\sigma$-stability of $\dim_{\mathcal{H}}$ and (3.3), it is sufficient to prove that for every Borel set $E = E_1 \times E_2$,

$$\dim_{\mathcal{H}} B^H(E) \le \min\left\{d;\quad \frac{\overline{\dim}_{\mathrm{B}} E_1}{H_1} + \frac{\dim_{\mathcal{H}} E_2}{H_2}\right\}, \quad \text{a.s.} \tag{3.6}$$

For any $\gamma_1 > \overline{\dim}_{\mathrm{B}} E_1,\ \gamma_2 > \dim_{\mathcal{H}} E_2$, we choose and fix $\gamma_2' \in (\dim_{\mathcal{H}} E_2, \gamma_2)$. Then there exists a constant $r_0 > 0$ such that, for all $r \le r_0$, $E_1$ can be covered by $N(E_1, r) \le r^{-\gamma_1}$ many small intervals of length $r$; and there exists a covering $\{U_n, n \ge 1\}$ of $E_2$ such that $r_n := |U_n| \le r_0$ and

$$\sum_{n=1}^{\infty} r_n^{\gamma_2'} \le 1. \tag{3.7}$$

For every $n \ge 1$ and any constant $\delta \in (0,1)$ small enough, let $\{V_{n,m} : 1 \le m \le N_n\}$ be $N_n := N(E_1, r_n^{(H_2-\delta)/(H_1-\delta)})$ intervals of length $r_n^{(H_2-\delta)/(H_1-\delta)}$ which cover $E_1$. Then the rectangles $\{V_{n,m} \times U_n :\, n \ge 1, 1 \le m \le N_n\}$ form a covering of $E_1 \times E_2$, that is,

$$E \subset \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{N_n} V_{n,m} \times U_n,$$

and thus

$$B^H(E) \subset \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{N_n} B^H(V_{n,m} \times U_n). \tag{3.8}$$

It follows from Lemma 2 that, almost surely, $B^H(V_{n,m} \times U_n)$ can be covered by a ball of radius $c\, r_n^{H_2-\delta}$. By this and (3.8), we have covered $B^H(E)$ a.s. by balls of radius $c\, r_n^{H_2-\delta}$ $(n = 1, 2, \ldots)$. Moreover, recalling that $N_n \le r_n^{-\gamma_1 (H_2-\delta)/(H_1-\delta)}$, we have

$$\begin{aligned}\sum_{n=1}^{\infty} \sum_{m=1}^{N_n} \Big(r_n^{H_2-\delta}\Big)^{\frac{\gamma_1}{H_1} + \frac{\gamma_2}{H_2}} &\le \sum_{n=1}^{\infty} r_n^{-\gamma_1 \frac{H_2-\delta}{H_1-\delta}} \cdot r_n^{(H_2-\delta)\left(\frac{\gamma_1}{H_1} + \frac{\gamma_2}{H_2}\right)} \\ &= \sum_{n=1}^{\infty} r_n^{\gamma_2 - \gamma_1\left(\frac{H_2-\delta}{H_1-\delta} - \frac{H_2-\delta}{H_1}\right) - \frac{\delta\gamma_2}{H_2}}.\end{aligned} \tag{3.9}$$

Now we choose $\delta > 0$ small enough so that

$$\gamma_2 - \gamma_1\Big(\frac{H_2-\delta}{H_1-\delta} - \frac{H_2-\delta}{H_1}\Big) - \frac{\delta\gamma_2}{H_2} > \gamma_2'.$$

Then (3.7) and (3.9) imply that

$$\sum_{n=1}^{\infty}\sum_{m=1}^{N_n}\left(r_n^{H_2-\delta}\right)^{\frac{\gamma_1}{H_1}+\frac{\gamma_2}{H_2}}\le 1. \tag{3.10}$$

It should be clear that (3.6) follows from (3.10).

**Lower bound:** Recall that for any Borel measure $\mu$ in $\mathbb{R}^p$ and a constant $\gamma>0$, the $\gamma$-energy of $\mu$ is defined by

$$I_\gamma(\mu)=\int\int\frac{\mu(dx)\mu(dy)}{|x-y|^\gamma}. \tag{3.11}$$

The connection between the Hausdorff dimension of a Borel set $A$ and energy of Borel measures on $A$ is described by Frostman's theorem, which provides a powerful way to determine $\dim_{\mathcal{H}}A$; see Kahane [22], Falconer [17] or Mattila [33].

To use this method, we choose $\gamma_1,\gamma_2$ such that $0<\gamma_1<\dim_{\mathcal{H}}E_1$, $0<\gamma_2<\dim_{\mathcal{H}}E_2$ and $\frac{\gamma_1}{H_1}+\frac{\gamma_2}{H_2}<d$. It follows from Frostman's theorem that there exist probability measures $\sigma_1$ on $E_1$ and $\sigma_2$ on $E_2$ such that

$$\int_{E_1}\int_{E_1}\frac{\sigma_1(ds_1)\sigma_1(dt_1)}{|s_1-t_1|^{\gamma_1}}<\infty\quad\text{and}\quad\int_{E_2}\int_{E_2}\frac{\sigma_2(ds_2)\sigma_2(dt_2)}{|s_2-t_2|^{\gamma_2}}<\infty. \tag{3.12}$$

Let $\sigma=\sigma_1\times\sigma_2$. Then $\sigma$ is a probability measure on $E$. By Lemma 3 and the fact that $\frac{\gamma_1}{H_1}+\frac{\gamma_2}{H_2}<d$, we have

$$\begin{aligned}
&\mathbb{E}\int_E\int_E\frac{\sigma(ds)\sigma(dt)}{|B^H(s)-B^H(t)|^{\frac{\gamma_1}{H_1}+\frac{\gamma_2}{H_2}}}\\
&\le c\int_E\int_E\frac{\sigma_1(ds_1)\sigma_1(dt_1)\sigma_2(ds_2)\sigma_2(dt_2)}{\left(|s_1-t_1|^{2H_1}+|s_2-t_2|^{2H_2}\right)^{\frac12(\frac{\gamma_1}{H_1}+\frac{\gamma_2}{H_2})}}\\
&\le c\int_{E_1}\int_{E_1}\sigma_1(ds_1)\sigma_1(dt_1)\int_{E_2}\int_{E_2}\frac{\sigma_2(ds_2)\sigma_2(dt_2)}{\left(|s_1-t_1|^{H_1}+|s_2-t_2|^{H_2}\right)^{\frac{\gamma_1}{H_1}+\frac{\gamma_2}{H_2}}}.
\end{aligned} \tag{3.13}$$

By an inequality for the weighted arithmetic mean and geometric mean with $\beta_1=\frac{H_2\gamma_1}{H_2\gamma_1+H_1\gamma_2}$ and $\beta_2=1-\beta_1=\frac{H_1\gamma_2}{H_2\gamma_1+H_1\gamma_2}$, we have

$$\begin{aligned}
|s_1-t_1|^{H_1}+|s_2-t_2|^{H_2}&\ge\beta_1|s_1-t_1|^{H_1}+\beta_2|s_2-t_2|^{H_2}\\
&\ge\left(|s_1-t_1|^{H_1}\right)^{\beta_1}\left(|s_2-t_2|^{H_2}\right)^{\beta_2}\\
&=|s_1-t_1|^{\frac{H_1H_2\gamma_1}{H_2\gamma_1+H_1\gamma_2}}\,|s_2-t_2|^{\frac{H_1H_2\gamma_2}{H_2\gamma_1+H_1\gamma_2}}.
\end{aligned} \tag{3.14}$$

Therefore, the last denominator in (3.13) can be bounded from below by $|s_1-t_1|^{\gamma_1}|s_2-t_2|^{\gamma_2}$. It follows from this and (3.12), (3.13) that

$$\mathbb{E}\int_E\int_E\frac{\sigma(ds)\sigma(dt)}{|B^H(s)-B^H(t)|^{\frac{\gamma_1}{H_1}+\frac{\gamma_2}{H_2}}}<\infty.$$

This yields $\dim_{\mathcal{H}} B^H(E) \geq \frac{\gamma_1}{H_1} + \frac{\gamma_2}{H_2}$ a.s., and the lower bound in (3.1) follows. $\square$

The following simple example illustrates that, in general, $\dim_{\mathcal{H}} E$ alone is not enough to determine the Hausdorff dimension of $B^H(E)$.

***Example 1.*** Let $B^H = \{B^H(t), t \in \mathbb{R}^2\}$ be a $(2, d)$-fractional Brownian sheet with index $H = (H_1, H_2)$ and $H_1 < H_2$. Let $E = E_1 \times E_2$ and $F = E_2 \times E_1$, where $E_1 \subset (0, \infty)$ satisfies $\dim_{\mathcal{H}} E_1 = \dim_{\mathrm{P}} E_1$ and $E_2 \subset (0, \infty)$ is arbitrary. It is well known that

$$\dim_{\mathcal{H}} E = \dim_{\mathcal{H}} E_1 + \dim_{\mathcal{H}} E_2 = \dim_{\mathcal{H}} F,$$

cf. Falconer ([17], p.94). However, by Proposition 1 we have

$$\dim_{\mathcal{H}} B^H(E) = \min\left\{d; \quad \frac{\dim_{\mathcal{H}} E_1}{H_1} + \frac{\dim_{\mathcal{H}} E_2}{H_2}\right\}$$

and

$$\dim_{\mathcal{H}} B^H(F) = \min\left\{d; \quad \frac{\dim_{\mathcal{H}} E_2}{H_1} + \frac{\dim_{\mathcal{H}} E_1}{H_2}\right\}.$$

We see that $\dim_{\mathcal{H}} B^H(E) \neq \dim_{\mathcal{H}} B^H(F)$ unless $\dim_{\mathcal{H}} E_1 = \dim_{\mathcal{H}} E_2$.

Example 1 shows that in order to determine $\dim_{\mathcal{H}} B^H(E)$, we need to have more information about the geometry of $E$ than its Hausdorff dimension. We have found that it is more convenient to work with Borel measures carried by $E$.

Recall that the Hausdorff dimension of a Borel measure $\mu$ on $\mathbb{R}^N$ (or lower Hausdorff dimension as it is sometimes called) is defined by

$$\dim_{\mathcal{H}} \mu = \inf\big\{\dim_{\mathcal{H}} F : \ \mu(F) > 0 \text{ and } F \subseteq \mathbb{R}^N \text{ is a Borel set}\big\}. \tag{3.15}$$

The packing dimension of $\mu$, denoted by $\dim_{\mathrm{P}} \mu$, is defined by replacing $\dim_{\mathcal{H}} F$ in (3.15) by $\dim_{\mathrm{P}} F$.

Hu and Taylor [21] proved the following characterization of $\dim_{\mathcal{H}} \mu$: if $\mu$ is a finite Borel measure on $\mathbb{R}^N$ then

$$\dim_{\mathcal{H}} \mu = \sup\left\{\gamma \geq 0 : \ \limsup_{r \to 0^+} \frac{\mu\big(U(t,r)\big)}{r^\gamma} = 0 \ \text{ for } \mu\text{-a.e. } \ t \in \mathbb{R}^N\right\}, \tag{3.16}$$

where $U(t, r) = \{s \in \mathbb{R}^N : |s - t| \leq r\}$. It can be verified that for every Borel set $E \subset \mathbb{R}^N$, we have

$$\dim_{\mathcal{H}} E = \sup\big\{\dim_{\mathcal{H}} \mu : \ \mu \in \mathcal{M}_c^+(E)\big\}, \tag{3.17}$$

where $\mathcal{M}_c^+(E)$ denotes the family of finite Borel measures on $E$ with compact support in $E$.

From (3.16), we note that $\dim_{\mathcal{H}}\mu$ only describes the local behavior of $\mu$ in an isotropic way and is not quite informative if $\mu$ is highly anisotropic as what we are dealing with in this paper. To overcome this difficulty, we introduce the following new notion of "dimension" for $E \subset (0, \infty)^N$ that is natural for studying $B^H(E)$.

***Definition 1.*** Given a Borel probability measure $\mu$ on $\mathbb{R}^N$, we define the set $\Lambda_\mu \subseteq \mathbb{R}^N_+$ by

$$\Lambda_\mu = \left\{ \lambda = (\lambda_1, \ldots, \lambda_N) \in \mathbb{R}^N_+ : \limsup_{r\to 0^+} \frac{\mu\left(R(t,r)\right)}{r^{\langle \lambda, H^{-1}\rangle}} = 0 \ \text{ for } \mu\text{-a.e. } t \in \mathbb{R}^N \right\}, \tag{3.18}$$

where $R(t,r) = \prod_{j=1}^N [t_j - r^{1/H_j}, t_j + r^{1/H_j}]$ and $H^{-1} = (\frac{1}{H_1}, \ldots, \frac{1}{H_N})$.

Some basic properties of $\Lambda_\mu$ are summarized in the following lemma. For simplicity of notation, without loss of generality, we will assume $H_1 = \min\{H_j : 1 \le j \le N\}$.

***Lemma 4.*** *$\Lambda_\mu$ has the following properties:*

*(i)* *The set $\Lambda_\mu$ is bounded:*

$$\Lambda_\mu \subseteq \left\{ \lambda = (\lambda_1, \ldots, \lambda_N) \in \mathbb{R}^N_+ : \langle \lambda, H^{-1}\rangle \le \frac{N}{H_1} \right\}. \tag{3.19}$$

*(ii)* *For all $\beta \in (0,1]^N$ and $\lambda \in \Lambda_\mu$, the Hadamard product of $\beta$ and $\lambda$, $\beta \circ \lambda = (\beta_1\lambda_1, \ldots, \beta_N\lambda_N) \in \Lambda_\mu$.*

*(iii)* *$\Lambda_\mu$ is convex, i.e. $\forall \lambda, \eta \in \Lambda_\mu$ and $0 < b < 1$, $b\lambda + (1-b)\eta \in \Lambda_\mu$.*

*(iv)* *For every $a \in (0,\infty)^N$, $\sup_{\lambda\in\Lambda_\mu} \langle \lambda, a\rangle$ is achieved on the boundary of $\Lambda_\mu$.*

Because of (iv) and its importance in this paper, we call the boundary of $\Lambda_\mu$, denoted by $\partial\Lambda_\mu$, the *Hausdorff dimension contour* of $\mu$.

***Proof.*** Suppose $\lambda = (\lambda_1, \ldots, \lambda_N) \in \Lambda_\mu$. Then

$$\limsup_{r\to 0^+} \frac{\mu\left(R(t,r)\right)}{r^{\langle \lambda, H^{-1}\rangle}} = 0 \qquad \text{for } \mu\text{-a.e. } t \in \mathbb{R}^N. \tag{3.20}$$

Fix a $t \in \mathbb{R}^N$ such that (3.20) holds. Since for any $a > 0$, the ball $U(t,a)$ centered at $t$ with radius $a$ can be covered by $R(t, a^{H_1})$, it follows from (3.20) that

$$\limsup_{r\to 0^+} \frac{\mu\left(U(t,a)\right)}{a^{H_1\langle \lambda, H^{-1}\rangle}} = 0 \qquad \text{for } \mu\text{-a.e. } t \in \mathbb{R}^N. \tag{3.21}$$

It follows from (3.16) and (3.21) that $\dim_{\mathcal{H}}\mu \ge H_1\langle \lambda, H^{-1}\rangle$. Hence we have $\langle \lambda, H^{-1}\rangle \le N/H_1$. This proves (i).

Statements (ii) and (iii) follow directly from the definition of $\Lambda_\mu$. To prove (iv), we note that for every $a \in (0,\infty)^N$, Property (i) implies that $\sup_{\lambda\in\Lambda_\mu} \langle a, \lambda\rangle < \infty$. On the other hand, the function $\lambda \mapsto \langle \lambda, a\rangle$ is increasing in each $\lambda_j$. Hence $\sup_{\lambda\in\Lambda_\mu} \langle \lambda, a\rangle$ must be achieved on the boundary of $\Lambda_\mu$. □

As examples, we mention that if $m$ is the Lebesgue measure on $\mathbb{R}^N_+$, then

$$\Lambda_m = \left\{(\lambda_1, \ldots, \lambda_N) \in \mathbb{R}^N_+ : \sum_{j=1}^N \frac{\lambda_j}{H_j} < \sum_{j=1}^N \frac{1}{H_j}\right\} \tag{3.22}$$

and $\sup_{\lambda\in\Lambda_m} \langle H^{-1}, \lambda\rangle = \sum_{j=1}^N \frac{1}{H_j}$. More generally we can verify that, if $\mu = \sigma_1 \times \cdots \times \sigma_N$, where $\sigma_j$ $(j = 1, \ldots, N)$ are Borel probability measures on $\mathbb{R}$ such that $\dim_{\mathcal{H}}\sigma_{j_k} = \dim_{\mathrm{P}}\sigma_{j_k}$ for some $\{j_1, \ldots, j_{N-1}\} \subset \{1, \ldots, N\}$, then

$$\Lambda_\mu = \left\{(\lambda_1, \ldots, \lambda_N) \in \mathbb{R}^N_+ : \sum_{j=1}^N \frac{\lambda_j}{H_j} < \sum_{j=1}^N \frac{\beta_j}{H_j}\right\},$$

where $\beta_j = \dim_{\mathcal{H}}\sigma_j$ for $j = 1, \ldots, N$. In the special case of $H_1 = \cdots = H_N = \alpha \in (0, 1)$, we derive from (3.16) that for every Borel measure $\mu$,

$$\Lambda_\mu = \left\{(\lambda_1, \ldots, \lambda_N) \in \mathbb{R}^N_+ : \sum_{j=1}^N \lambda_j < \dim_{\mathcal{H}}\mu\right\}. \tag{3.23}$$

For any Borel measure $\mu$ on $\mathbb{R}^N_+$, its image measure under the mapping $t \mapsto B^H(t)$ is defined by

$$\mu_{_{B^H}}(F) = \mu\Big\{t \in \mathbb{R}^N_+ : B^H(t) \in F\Big\} \quad \text{for all Borel sets } F \subset \mathbb{R}^d.$$

We will make use of the following result. Note that the exceptional null probability event does not depend on $\mu$.

***Proposition 2.*** *Let $B^H = \{B^H(t), t \in \mathbb{R}^N\}$ be an $(N, d)$-fractional Brownian sheet with index $H \in (0, 1)^N$. Then almost surely,*

$$\dim_{\mathcal{H}}\mu_{_{B^H}} \le s_\mu(H) \wedge d \quad \textit{for all } \mu \in \mathcal{M}^+_c(\mathbb{R}^N_+), \tag{3.24}$$

*where $s_\mu(H) = \sup_{\lambda\in\Lambda_\mu} \langle \lambda, H^{-1}\rangle$.*

***Proof.*** To prove the upper bound in (3.24), note that, without loss of generality, we may and will assume the support of $\mu$ is contained in $[\varepsilon, T]^N$ for some $0 < \varepsilon < T < \infty$. Furthermore, if $s_\mu(H) \ge d$, then $\dim_{\mathcal{H}}\mu_{_{B^H}} \le s_\mu(H) \wedge d$ holds trivially. Therefore, we will also assume that $s_\mu(H) < d$.

Note that $\forall\,\delta_j \in (0, H_j)$ $(j = 1, \ldots, N)$, Lemma 2 implies that for almost all $\omega \in \Omega$,

$$\big|B^H(s) - B^H(t)\big| \le c \sum_{j=1}^{N} |s_j - t_j|^{H_j-\delta_j} \quad \forall s,\, t \in [\varepsilon, T]^N. \tag{3.25}$$

We choose $\delta_1, \ldots, \delta_N$ in the following way: $\forall\delta \in (0,\, H_1)$,

$$\delta_1 = \delta \quad \text{and} \quad \delta_j = \frac{H_j}{H_1}\delta, \quad \text{for } j = 2, \ldots, N. \tag{3.26}$$

and fix an $\omega \in \Omega$ such that (3.25) holds. For any $0 < \gamma < \dim_{\mathcal{H}} \mu_{_{B^H}}$, by (3.16) we have

$$\limsup_{\rho\to 0} \frac{\mu_{_{B^H}}\big(U(u, \rho)\big)}{\rho^\gamma} = 0 \quad \text{for } \ \mu_{_{B^H}}\text{-a.e. } u \in \mathbb{R}^d. \tag{3.27}$$

This is equivalent to

$$\limsup_{\rho\to 0} \frac{1}{\rho^\gamma} \int_{[\varepsilon,T]^N} \mathbb{1}_{\big\{|B^H(s)-B^H(t)|\le\rho\big\}}\, \mu(ds) = 0 \quad \text{for } \mu\text{-a.e. } t \in \mathbb{R}^N_+. \tag{3.28}$$

It follows from (3.28) and (3.25) that almost surely

$$\limsup_{\rho\to 0} \frac{1}{\rho^\gamma}\, \mu\Big(\widetilde{R}\big(t, \rho\big)\Big) = 0 \quad \text{for } \mu\text{-a.e. } t \in \mathbb{R}^N_+, \tag{3.29}$$

where $\widetilde{R}\big(t, \rho\big) = \prod_{j=1}^N \big[t_j - \rho^{\frac{1}{H_j-\delta_j}},\, t_j + \rho^{\frac{1}{H_j-\delta_j}}\big]$. Equation (3.29), together with (3.26), implies

$$\limsup_{\rho\to 0} \frac{1}{\rho^\gamma} \mu\Big(R\big(t, \rho^{\frac{H_1}{H_1-\delta}}\big)\Big) = 0 \quad \text{for } \mu\text{-a.e. } t \in \mathbb{R}^N_+. \tag{3.30}$$

We claim that $\gamma \le s_\mu(H)$. In fact, if $\gamma > s_\mu(H)$, then there exists $\beta \notin \Lambda_\mu$ such that $\langle\beta,\, H^{-1}\rangle < \gamma$. Since $\beta \notin \Lambda_\mu$, there is a set $A \subset [\varepsilon, T]^N$ with positive $\mu$-measure such that

$$\limsup_{r\to 0} \frac{\mu\big(R(t, r)\big)}{r^{\langle\beta, H^{-1}\rangle}} > 0 \quad \text{for every } \ t \in A. \tag{3.31}$$

Now we choose $\delta > 0$ small enough such that

$$\gamma - \frac{H_1}{H_1 - \delta}\langle\beta,\, H^{-1}\rangle > 0. \tag{3.32}$$

Then (3.31) and (3.32) imply that for every $t \in A$,

$$\begin{aligned}
&\limsup_{\rho\to 0} \frac{1}{\rho^{\gamma}} \mu\Big(R\big(t, \rho^{\frac{H_1}{H_1-\delta}}\big)\Big) \\
&= \limsup_{\rho\to 0} \frac{1}{\rho^{\gamma - \frac{H_1}{H_1-\delta}\langle \beta, H^{-1}\rangle}} \times \frac{\mu\Big(R\big(t, \rho^{\frac{H_1}{H_1-\delta}}\big)\Big)}{\rho^{\frac{H_1}{H_1-\delta}\langle \beta, H^{-1}\rangle}} = \infty.
\end{aligned} \tag{3.33}$$

This contradicts (3.30). Therefore, we have proved that $\gamma \le s_\mu(H)$. Since $\gamma < \dim_{\mathcal{H}} \mu_{{}_{B^H}}$ is arbitrary, we have $\dim_{\mathcal{H}} \mu_{{}_{B^H}} \le s_\mu(H) \wedge d$. This finishes the proof of Proposition 2. $\square$

The following corollary follows directly from Proposition 2 and (3.22). It is related to Theorem 3.1 in Ayache and Xiao [5].

***Corollary 1.*** *Let $m$ be the Lebesgue measure on $\mathbb{R}^N_+$, then $\dim_{\mathcal{H}} m_{{}_{B^H}} \le d \wedge \sum_{j=1}^N H_j^{-1}$ a.s.*

***Remark 1.*** Applying a moment argument [see, e.g., Xiao [44]] and the sectorial local nondeterminism of $B^H$, we can also prove that $\dim_{\mathcal{H}} m_{{}_{B^H}} \ge d \wedge \sum_{j=1}^N H_j^{-1}$ a.s. Since this result is not needed in this paper and its proof is rather long, we omit it.

For any Borel set $E \subset (0,\, \infty)^N$, we define

$$\Lambda(E) = \bigcup_{\mu \in \mathcal{M}_c^+(E)} \Lambda_\mu. \tag{3.34}$$

Recall that $\mathcal{M}_c^+(E)$ is the family of finite Borel measures with compact support in $E$. Similar to the case for measures, we call the set $\cup_{\mu \in \mathcal{M}_c^+(E)} \partial \Lambda_\mu$ the Hausdorff dimension contour of $E$. It follows from Lemma 4 that, for every $a \in (0,\infty)^N$, the supermum $\sup_{\lambda \in \Lambda(E)} \langle \lambda, a\rangle$ is determined by the Hausdorff dimension contour of $E$.

The following is the main result of this section.

***Theorem 2.*** *Let $B^H$ be an $(N,d)$-fractional Brownian sheet with index $H \in (0,1)^N$. Then, for any Borel set $E \subset (0,\infty)^N$,*

$$\dim_{\mathcal{H}} B^H(E) = s(H,E) \wedge d \qquad a.s., \tag{3.35}$$

*where $s(H,E) = \sup_{\lambda \in \Lambda(E)} \langle \lambda, H^{-1}\rangle = \sup_{\mu \in \mathcal{M}_c^+(E)} s_\mu(E)$.*

***Remark 2.*** Given the importance of $s(H,E)$ in this paper, it is of interest to determine its value more explicitly. Xiao [48] has shown that it is the Hausdorff dimension of $E$ with respect to an “anisotropic metric”.

We need the following lemmas to prove Theorem 2.

***Lemma 5.*** *Let $E \subset \mathbb{R}^N$ be an analytic set and let $f : \mathbb{R}^N \to \mathbb{R}^d$ be a continuous function. If $0 \le \tau < \dim_{\mathcal{H}} f(E)$, then there exists a compact set $E_0 \subseteq E$ such that $\tau < \dim_{\mathcal{H}} f(E_0)$.*

***Proof.*** The proof is the same as that of Lemma 4.1 in Xiao [45], with packing dimension replaced by Hausdorff dimension. □

***Lemma 6.*** *Let $E \subset (0,\infty)^N$ be an analytic set. Then for any continuous function $f : \mathbb{R}^N \to \mathbb{R}^d$,*

$$\dim_{\mathcal{H}} f(E) = \sup\big\{\dim_{\mathcal{H}} \mu_f : \mu \in \mathcal{M}_c^+(E)\big\}. \tag{3.36}$$

***Proof.*** For any $\mu \in \mathcal{M}_c^+(E)$, we have $\mu_f \in \mathcal{M}_c^+(f(E))$. By (3.17), we have

$$\dim_{\mathcal{H}} f(E) = \sup\big\{\dim_{\mathcal{H}} \nu : \nu \in \mathcal{M}_c^+(f(E))\big\}, \tag{3.37}$$

which implies that

$$\dim_{\mathcal{H}} f(E) \ge \sup\big\{\dim_{\mathcal{H}} \mu_f : \mu \in \mathcal{M}_c^+(E)\big\}. \tag{3.38}$$

To prove the reverse inequality, let $\tau < \dim_{\mathcal{H}} f(E)$. By Lemma 5, there exists a compact set $E_0 \subset E$ such that $\dim_{\mathcal{H}} f(E_0) > \tau$. Hence, by (3.37), there exists a finite Borel measure $\nu \in \mathcal{M}_c^+(f(E_0))$ such that $\dim_{\mathcal{H}} \nu > \tau$. It follows from Theorem 1.20 in Mattila [33] that there exists $\mu \in \mathcal{M}_c^+(E_0)$ such that $\nu = \mu_f$, which implies $\sup\{\dim_{\mathcal{H}} \mu_f : \mu \in \mathcal{M}_c^+(E)\} > \tau$. Since $\tau < \dim_{\mathcal{H}} f(E)$ is arbitrary, we have

$$\dim_{\mathcal{H}} f(E) \le \sup\{\dim_{\mathcal{H}} \mu_f : \mu \in \mathcal{M}_c^+(E)\}. \tag{3.39}$$

Equation (3.36) now follows from (3.38) and (3.39). □

***Proof of Theorem 2.*** First we prove the lower bound:

$$\dim_{\mathcal{H}} B^H(E) \ge s(H,E) \wedge d \qquad \text{a.s.} \tag{3.40}$$

For any $0 < \gamma < s(H,E) \wedge d$, there exists a Borel measure $\mu$ with compact support in $E$ such that $\gamma < s_\mu(H) \wedge d$. Hence we can find $\lambda' = (\lambda'_1, \ldots, \lambda'_N) \in \Lambda_\mu$ such that $\gamma < \sum_{j=1}^N \frac{\lambda'_j}{H_j} \wedge d$ and

$$\limsup_{r \to 0} \frac{\mu(R(t,r))}{r^{\langle \lambda', H^{-1}\rangle}} = 0 \qquad \text{for } \mu\text{-a.e. } t \in \mathbb{R}^N_+. \tag{3.41}$$

For $\varepsilon > 0$ we define

$$E_\varepsilon = \Big\{t \in E : \mu\big(R(t,r)\big) \le r^{\langle \lambda', H^{-1}\rangle} \quad \text{for all } \ 0 < r \le \varepsilon\Big\}. \tag{3.42}$$

Then (3.41) implies that $\mu(E_\varepsilon) > 0$ if $\varepsilon$ is small enough. In order to prove (3.40), it suffices to show $\dim_{\mathcal{H}} B^H(E_\varepsilon) \ge \gamma$ a.s.

The proof of the latter using Frostman's theorem is standard: we only need to show

$$\mathbb{E}\int_{E_\varepsilon}\int_{E_\varepsilon}\frac{\mu(ds)\mu(dt)}{|B^H(t)-B^H(s)|^\gamma}<\infty. \tag{3.43}$$

By Lemma 3 and the fact that $\gamma < d$, we have

$$\mathbb{E}\int_{E_\varepsilon}\int_{E_\varepsilon}\frac{\mu(ds)\mu(dt)}{|B^H(t)-B^H(s)|^\gamma}\le c\int_{E_\varepsilon}\int_{E_\varepsilon}\frac{\mu(ds)\mu(dt)}{\big(\sum_{j=1}^N|s_j-t_j|^{2H_j}\big)^{\gamma/2}}. \tag{3.44}$$

Let $t \in E_\varepsilon$ be fixed and let $n_0$ be the smallest integer $n$ such that $2^{-n} \le \varepsilon$. For every $n \ge n_0$, denote

$$D_n=\big\{s\in\mathbb{R}^N_+:\,2^{-(n+1)/H_j}<|s_j-t_j|\le 2^{-n/H_j}\ \text{ for all } 1\le j\le N\big\}.$$

Then by (3.42) we have

$$\begin{aligned}\int_{E_\varepsilon}\frac{\mu(ds)}{\big(\sum_{j=1}^N|s_j-t_j|^{2H_j}\big)^{\gamma/2}}&\le c+\sum_{n=n_0}^{\infty}\int_{D_n}\frac{\mu(ds)}{\big(\sum_{j=1}^N|s_j-t_j|^{2H_j}\big)^{\gamma/2}}\\&\le c+c\sum_{n=n_0}^{\infty}2^{-n(\sum_{j=1}^N\frac{\lambda_j'}{H_j}-\gamma)}\\&\le c_{3,10}.\end{aligned} \tag{3.45}$$

This and (3.44) yield (3.43). So we have proved (3.40).

Now we prove the upper bound in (3.35). It follows from Proposition 2 that almost surely for all $\mu \in \mathcal{M}_c^+(E)$ we have

$$\dim_{\mathcal{H}}\mu_{B^H}\le s_\mu(H)\wedge d. \tag{3.46}$$

Hence by (3.34) and (3.36) we derive

$$\dim_{\mathcal{H}}B^H(E)\le s(H,E)\wedge d\quad \text{a.s.} \tag{3.47}$$

Combining (3.40) and (3.47) finishes the proof. □

In the special case of $H = \langle\alpha\rangle$, Theorem 2 implies that for every Borel set $E \subset (0,\infty)^N$,

$$\dim_{\mathcal{H}}B^{\langle\alpha\rangle}(E)=\min\left\{d,\frac{1}{\alpha}\dim_{\mathcal{H}}E\right\}\quad \text{a.s.} \tag{3.48}$$

The following theorem gives us a uniform version of (3.48). It extends the results of Mountford [35] and Khoshnevisan, Wu and Xiao [28] for the Brownian sheet.

***Theorem 3.*** *If $N \le \alpha d$, then with probability 1*

$$\dim_{\mathcal{H}}B^{\langle\alpha\rangle}(E)=\frac{1}{\alpha}\dim_{\mathcal{H}}E \textit{ for all Borel sets } E\subset(0,\infty)^N. \tag{3.49}$$

Our proof of Theorem 3 is reminiscent to that of Khoshnevisan, Wu and Xiao [28] for the Brownian sheet. The key step is provided by the following lemma, which will be proved by using the sectorial local nondeterminism of $B^{\langle\alpha\rangle}$.

***Lemma 7.*** *Assume $N \le \alpha d$ and let $\delta > 0$ and $0 < 2\alpha - \delta < \beta < 2\alpha$ be given constants. Then with probability 1, for all integers n large enough, there do not exist more than $2^{n\delta d}$ distinct points of the form $t^j = 4^{-n}\, k^j$, where $k^j \in \{1, 2, \ldots, 4^n\}^N$, such that*

$$\left|B^{\langle\alpha\rangle}(t^i) - B^{\langle\alpha\rangle}(t^j)\right| < 3 \cdot 2^{-n\beta} \qquad \textit{for } i \neq j. \tag{3.50}$$

***Proof.*** Let $A_n$ be the event that there do exist more than $2^{n\delta d}$ distinct points of the form $4^{-n}k^j$ such that (3.50) holds. Let $N_n$ be the number of $n$-tuples of distinct $t^1, \ldots, t^n$ such that (3.50) holds. Then

$$A_n \subseteq \left\{ N_n \ge \binom{[2^{n\delta d} + 1]}{n} \right\}.$$

Consequently,

$$\mathbb{P}(A_n) \le \frac{\mathbb{E}(N_n)}{\binom{[2^{n\delta d} + 1]}{n}}. \tag{3.51}$$

In order to estimate $\mathbb{E}(N_n)$, we write it as

$$\begin{aligned}
\mathbb{E}(N_n) &= \mathbb{E}\Bigg[ \underbrace{\sum_{t^1} \sum_{t^2} \cdots \sum_{t^n}}_{\text{distinct}} \mathbf{1}_{\big\{(3.50) \text{ holds}\big\}} \Bigg] \\
&= \underbrace{\sum_{t^1} \sum_{t^2} \cdots \sum_{t^n}}_{\text{distinct}} \mathbb{P}\Big\{ \left|B^{\langle\alpha\rangle}(t^i) - B^{\langle\alpha\rangle}(t^j)\right| < 3 \cdot 2^{-n\beta}, \quad \forall i \neq j \le n \Big\}.
\end{aligned} \tag{3.52}$$

Now we fix $n-1$ distinct points $t^1, \ldots, t^{n-1}$ and estimate the following sum first:

$$\sum_{t^n} \mathbb{P}\Big\{ \left|B^{\langle\alpha\rangle}(t^i) - B^{\langle\alpha\rangle}(t^j)\right| < 3 \cdot 2^{-n\beta}, \quad \forall i \neq j \le n \Big\}. \tag{3.53}$$

Note that for fixed $t^1 = k^1 4^{-n}, \ldots, t^{n-1} = k^{n-1} 4^{-n}$, there are at most $(n-1)^N$ points $\tau^u = (\tau^u_1, \ldots, \tau^u_N)$ defined by

$$\tau^u_\ell = t^j_\ell \qquad \text{for some } j = 1, \ldots, n-1.$$

We denote the collection of $\tau^u$'s by $\Gamma_n = \{\tau^u\}$. Clearly, $t^1, \ldots, t^{n-1}$ are all included in $\Gamma_n$.

It follows from Theorem 1 that, for every $t^n \notin \Gamma_n$, there exists $\tau^{u_n} \in \Gamma_n$ such that

$$\operatorname{Var}\left(B_0^{\langle\alpha\rangle}(t^n)|B_0^{\langle\alpha\rangle}(t^1), \ldots, B_0^{\langle\alpha\rangle}(t^{n-1})\right) \geq c_{3,5}\ |t^n - \tau^{u_n}|^{2\alpha}. \tag{3.54}$$

In this case, since $B_1^{\langle\alpha\rangle}, \ldots, B_d^{\langle\alpha\rangle}$ are the independent copies of $B_0^{\langle\alpha\rangle}$, a standard conditioning argument and (3.54) yield

$$\begin{aligned} &\mathbb{P}\left\{ \left|B^{\langle\alpha\rangle}(t^i) - B^{\langle\alpha\rangle}(t^j)\right| < 3 \cdot 2^{-n\beta}, \quad \forall i \neq j \leq n\right\} \\ &\leq \mathbb{P}\left\{ \left|B^{\langle\alpha\rangle}(t^i) - B^{\langle\alpha\rangle}(t^j)\right| < 3 \cdot 2^{-n\beta}, \quad \forall i \neq j \leq n-1\right\} \\ &\qquad \times \left(\frac{3 \cdot 2^{-n\beta}}{c_{3,5}^{1/2}\ |t^n - \tau^{u_n}|^{\alpha}}\right)^d. \end{aligned} \tag{3.55}$$

If $t^n \in \Gamma_n$, then we use the trivial bound

$$\begin{aligned} &\mathbb{P}\left\{ \left|B^{\langle\alpha\rangle}(t^i) - B^{\langle\alpha\rangle}(t^j)\right| < 3 \cdot 2^{-n\beta}, \quad \forall i \neq j \leq n\right\} \\ &\leq \mathbb{P}\left\{ \left|B^{\langle\alpha\rangle}(t^i) - B^{\langle\alpha\rangle}(t^j)\right| < 3 \cdot 2^{-n\beta}, \quad \forall i \neq j \leq n-1\right\}. \end{aligned} \tag{3.56}$$

Hence, by combining (3.55) and (3.56), we obtain

$$\begin{aligned} &\sum_{t^n} \mathbb{P}\left\{ \left|B^{\langle\alpha\rangle}(t^i) - B^{\langle\alpha\rangle}(t^j)\right| < 3 \cdot 2^{-n\beta}, \quad \forall i \neq j \leq n\right\} \\ &\leq \mathbb{P}\left\{ \left|B^{\langle\alpha\rangle}(t^i) - B^{\langle\alpha\rangle}(t^j)\right| < 3 \cdot 2^{-n\beta}, \quad \forall i \neq j \leq n-1\right\} \\ &\qquad \times \left[\sum_{t^n \notin \Gamma_n} c_{3,6} \left(\frac{3 \cdot 2^{-n\beta}}{|t^n - \tau^{u_n}|^{\alpha}}\right)^d + (n-1)^N\right]. \end{aligned} \tag{3.57}$$

Note that

$$\begin{aligned} \sum_{t^n \notin \Gamma_n} \left(\frac{3 \cdot 2^{-n\beta}}{|t^n - \tau^{u_n}|^{\alpha}}\right)^d &\leq \sum_{\tau^u \in \Gamma_n} \sum_{t^n \neq \tau^u} \left(\frac{3 \cdot 2^{-n\beta}}{|t^n - \tau^u|^{\alpha}}\right)^d \\ &\leq \sum_{\tau^u \in \Gamma_n} 3^d \cdot 2^{-n\beta d} \sum_{t^n \neq \tau^u} \frac{1}{|t^n - \tau^u|^{\alpha d}} \\ &\leq c_{3,7}\, (n-1)^{N+1}\, 2^{n(2\alpha-\beta)d}. \end{aligned} \tag{3.58}$$

In deriving the last inequality, we have used the fact that if $N \leq \alpha d$ then for all fixed $\tau^u$,

$$\sum_{t^n \neq \tau^u} \frac{1}{|t^n - \tau^u|^{\alpha d}} \leq c \cdot 2^{2\alpha n d}\, n.$$

Plug (3.58) into (3.57), we get

$$\begin{aligned}\sum_{t^n} \mathbb{P}\Big\{ \Big|B^{\langle\alpha\rangle}(t^i) - B^{\langle\alpha\rangle}(t^j)\Big| < 3 \cdot 2^{-n\beta}, \quad \forall i \neq j \leq n \Big\} \\ \leq \mathbb{P}\Big\{ \Big|B^{\langle\alpha\rangle}(t^i) - B^{\langle\alpha\rangle}(t^j)\Big| < 3 \cdot 2^{-n\beta}, \quad \forall i \neq j \leq n-1 \Big\} \\ \times\, c_{3,8}\, (n-1)^{N+1}\, 2^{n(2\alpha-\beta)d}.\end{aligned} \tag{3.59}$$

Therefore, by iteration, we obtain

$$\begin{aligned}\underbrace{\sum_{t^1}\sum_{t^2}\cdots\sum_{t^n}}_{\text{distinct}} \mathbb{P}\Big\{ \Big|B^{\langle\alpha\rangle}(t^i) - B^{\langle\alpha\rangle}(t^j)\Big| < 3 \cdot 2^{-n\beta}, \quad \forall i \neq j \leq n \Big\} \\ \leq c_{3,9}^n\, \big[(n-1)!\big]^{N+1}\, 2^{n^2(2\alpha-\beta)d},\end{aligned} \tag{3.60}$$

which implies

$$\mathbb{E}\big(N_n\big) \leq c_{3,9}^n\, (n-1)^{n(N+1)}\, 2^{n^2(2\alpha-\beta)d}. \tag{3.61}$$

By (3.51), (3.61) and the elementary inequality

$$\binom{[2^{n\delta d}+1]}{n} \geq \left(\frac{2^{n\delta d}+1}{n}\right)^n \geq \frac{2^{n^2\delta d}}{n^n},$$

we obtain

$$\mathbb{P}\big(A_n\big) \leq c_{3,10}^n\, (n-1)^{n(N+2)}\, 2^{n^2(2\alpha-\beta-\delta)d}. \tag{3.62}$$

Since $0 < 2\alpha - \beta < \delta$, by (3.62), we get $\sum_n \mathbb{P}(A_n) < \infty$. Hence the Borel-Cantelli Lemma implies that $\mathbb{P}\left(\overline{\lim}_n A_n\right) = 0$. This completes the proof of our lemma. □

For $n = 1, 2, \ldots$ and $k = (k_1, \ldots, k_N)$, where each $k_i \in \{1, 2, \ldots, 4^n\}$, define

$$I_k^n = \big\{t \in [0,1]^N : (k_i - 1)4^{-n} \leq t_i \leq k_i 4^{-n} \quad \text{for all } i = 1, \ldots, N\big\}. \tag{3.63}$$

The following lemma is a consequence of Lemmas 2 and 7.

***Lemma 8.*** *Let $\delta > 0$ and $0 < 2\alpha - \delta < \beta < 2\alpha$. Then with probability 1, for all large enough $n$, there exists no ball $O \subset \mathbb{R}^d$ of radius $2^{-n\beta}$ for which $B^{-1}(O)$ intersects more than $2^{n\delta d}$ cubes $I_k^n$.*

Now we are ready to prove Theorem 3.

***Proof of Theorem 3.*** For simplicity, we shall only prove (3.49) for all Borel sets $E \subseteq (0,1]^N$. By Lemma 2, we know that almost surely $B^{\langle\alpha\rangle}(t)$ satisfies a uniform Hölder condition on $[0,1]^N$ of any order smaller than $\alpha$. This and Theorem 6 in Kahane [22] [or Proposition 2.3 in Falconer [17]] imply that $\mathbb{P}\big\{\dim_{\mathcal{H}} B(E) \le \frac{1}{\alpha}\dim_{\mathcal{H}} E \quad \text{for every Borel set} \quad E \subset [0,1]^N\big\} = 1$.

To prove the lower bound we need only to show that almost surely for every compact set $F \subseteq \mathbb{R}^d$,

$$\dim_{\mathcal{H}}\big\{t \in (0,1]^N : B^{\langle\alpha\rangle}(t) \in F\big\} \le \alpha \dim_{\mathcal{H}} F. \tag{3.64}$$

This follows from Lemma 8 and a simple covering argument as in Khoshnevisan, Wu and Xiao [28]. Therefore, the proof of Theorem 3 is finished. □

## 4. Salem sets

Fourier transforms of deterministic and random measures have been studied extensively in harmonic analysis [see, e.g., Mattila [33] and the reference therein]. Of special interest to fractal geometry is the following relationship between energy and the Fourier transform [see Kahane ([22], Ch. 10) or Mattila ([33], Ch. 12)]: Let $\nu$ be a finite Borel measure on $\mathbb{R}^d$. Then for any $\gamma \in (0,d)$,

$$I_\gamma(\nu) = \int_{\mathbb{R}^d} \big|\widehat{\nu}(\xi)\big|^2 \big|\xi\big|^{\gamma-d}\, d\xi, \tag{4.1}$$

where $I_\gamma(\nu)$ and $\widehat{\nu}$ are the $\gamma$-energy [see Equation (3.11) for a definition] and the Fourier transform of $\nu$, respectively.

Let us recall from Kahane [22][23] the definitions of Fourier dimension and Salem set. Given a constant $\beta \ge 0$, a Borel set $F \subset \mathbb{R}^d$ is said to be an $M_\beta$-set if there exists a probability measure $\nu$ on $F$ such that

$$\big|\widehat{\nu}(\xi)\big| = o(|\xi|^{-\beta}) \qquad \text{as} \quad |\xi| \to \infty. \tag{4.2}$$

The asymptotic behavior of $\widehat{\nu}(\xi)$ at infinity carries some information about the geometry of $F$. It can be verified that (i) if $\beta > d/2$ in (4.2), then $\widehat{\nu} \in L^2(\mathbb{R}^d)$ and, consequently, $F$ has positive $d$-dimensional Lebesgue measure; (ii) if $\beta > d$, then $\widehat{\nu} \in L^1(\mathbb{R}^d)$. Hence $\nu$ has a continuous density function which implies that $F$ has interior points.

For any Borel set $F \subset \mathbb{R}^d$, the Fourier dimension of $F$, denoted by $\dim_{\mathcal{F}} F$, is defined as

$$\dim_{\mathcal{F}} F = \sup\big\{\gamma \in [0,d] : \ F \text{ is an } \ M_{\gamma/2}\text{-set } \big\}. \tag{4.3}$$

It follows from Frostman's theorem and (4.1) that $\dim_{\mathcal{F}} F \le \dim_{\mathcal{H}} F$ for all Borel sets $F \subset \mathbb{R}^d$. The strict inequality may hold. For example, the Fourier dimension of triadic Cantor set is 0, but its Hausdorff dimension is $\log 2/\log 3$. It has been known that the Hausdorff dimension $\dim_{\mathcal{H}} F$ describes a metric property of $F$, whereas the Fourier dimension measures an arithmetic property of $F$. As a further example of this aspect, we mentioned that every set $F \subset \mathbb{R}^d$ with positive Fourier dimension generates $\mathbb{R}^d$ as a group.

We say that a Borel set $F \subset \mathbb{R}^d$ is a *Salem set* if $\dim_{\mathcal{F}} F = \dim_{\mathcal{H}} F$. Such sets are of importance in studying the problem of uniqueness and multiplicity for trigonometric series [cf. Zygmund ([50], Chapter 9) and Kahane and Salem [25]] and the restriction problem for the Fourier transforms [cf. Mockenhaupt [34]].

Many random sets have been proved to be Salem sets. Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a centered Gaussian random field with values in $\mathbb{R}^d$. When $X$ is an $(N,d)$-fractional Brownian motion of index $\gamma \in (0,1)$, Kahane [22][23] studied the asymptotic properties of the Fourier transforms of the image measures of $X$ and proved that, for every Borel set $E \subset \mathbb{R}^N$ with $\dim_{\mathcal{H}} E \le \gamma d$, $X(E)$ is a Salem set almost surely. Kahane [24] further raised the question of studying the Fourier dimensions of other random sets. Recently, Shieh and Xiao [41] extended Kahane's results to a large class of Gaussian random fields with stationary increments and Khoshnevisan, Wu and Xiao [28] proved similar results for the Brownian sheet. However, all the Gaussian random fields considered so far are at least approximately isotropic.

In this section, we study the asymptotic properties of the Fourier transforms of the image measures of the $(N,d)$-fractional Brownian sheet $B^H$. The main result of this section is Theorem 4 below, whose proof depends crucially on the ideas of sectorial local nondeterminism and Hausdorff dimension contour. Moreover, by combining Theorems 2 and 4 we show that, for every Borel set $E \subset (0,\infty)^N$, $B^H(E)$ is almost surely a Salem set whenever $s(H,E) \le d$. Recall that $s(H,E)$ is defined in Theorem 2.

Let $0 < \varepsilon < T$ be fixed. For all $n \ge 2$, $t^1, \ldots, t^n, s^1, \ldots, s^n \in E \subset [\varepsilon, T]^N$, denote $\mathbf{s} = (s^1, \ldots, s^n)$, $\mathbf{t} = (t^1, \ldots, t^n)$ and

$$\Psi(\mathbf{s},\mathbf{t}) = \mathbb{E}\Big[\sum_{k=1}^{n} \big(B_0^H(t^k) - B_0^H(s^k)\big)\Big]^2. \tag{4.4}$$

For $\mathbf{s} \in E^n$ and $r > 0$, we define

$$F(\mathbf{s}, r) = \bigcup_{i_1=1}^{n} \cdots \bigcup_{i_N=1}^{n} \bigcap_{j=1}^{N} \Big\{u \in E : \Big|u_j - s_j^{i_j}\Big| \le r^{1/H_j}\Big\}.$$

This is a union of at most $n^N$ rectangles of side-lengths $2r^{1/H_1}, \ldots, 2r^{1/H_N}$, centered at $(s_1^{i_1}, \ldots, s_N^{i_k})$. Let

$$G(\mathbf{s}, r) = \big\{\mathbf{t} = (t^1, \ldots, t^n) : t^k \in F(\mathbf{s}, r) \text{ for } 1 \le k \le n\big\}. \tag{4.5}$$

The following lemma is essential for the proof of Theorem 4.

***Lemma 9.*** *There exists a positive constant $c_{4,1}$, depending on $\varepsilon, T, H$ and $N$ only, such that for all $r \in (0, \varepsilon]$ and all $\mathbf{s}, \mathbf{t} \in E^n$ with $\mathbf{t} \notin G(\mathbf{s}, r)$, we have $\Psi(\mathbf{s}, \mathbf{t}) \geq c_{4,1}\, r^2$.*

***Proof.*** Since $\mathbf{t} \notin G(\mathbf{s}, r)$, there exist indices $k_0 \in \{1, \ldots, n\}$ and $j_0 \in \{1, \ldots, N\}$ such that $\left|t_{j_0}^{k_0} - s_{j_0}^k\right| > r^{1/H_{j_0}}$ for all $k = 1, \ldots, n$. It follows from (2.1) that

$$\Psi(\mathbf{s}, \mathbf{t}) = K_H^{-2} \int_{\mathbb{R}^N} \left| \sum_{k=1}^n \bigg( \prod_{j=1}^N \big(e^{it_j^k \lambda_j} - 1\big) - \prod_{j=1}^N \big(e^{is_j^k \lambda_j} - 1\big) \bigg) \right|^2 f_H(\lambda)\, d\lambda, \tag{4.6}$$

where $f_H(\lambda) = \prod_{j=1}^N |\lambda_j|^{-2H_j - 1}$. Let $\delta_j(\cdot) \in C^\infty(\mathbb{R})$ $(1 \leq j \leq N)$ be the bump functions in the proof of Theorem 1. We define $\delta_{j_0}^r(u_{j_0}) = r^{-1/H_{j_0}} \delta_{j_0}(r^{-1/H_{j_0}} u_{j_0})$ and $\delta_j^\varepsilon(u_j) = \varepsilon^{-1} \delta_j(\varepsilon^{-1} u_j)$, if $j \neq j_0$. Then, by using the Fourier inversion formula again, we have

$$\delta_{j_0}^r(u_{j_0}) = (2\pi)^{-1} \int_{\mathbb{R}} e^{-iu_{j_0}\lambda_{j_0}} \widehat{\delta}_{j_0}(r^{1/H_{j_0}} \lambda_{j_0}) d\lambda_{j_0}$$

and similar identities holds for $\delta_j^\varepsilon(u_j)$ with $j \neq j_0$.

Since $r \in (0, \varepsilon)$, we have $\delta_{j_0}^r(t_{j_0}^{k_0}) = 0$ and $\delta_j^\varepsilon(t_j^{k_0}) = 0$ for all $j \neq j_0$. Similarly, $\delta_{j_0}^r(t_{j_0}^{k_0} - s_{j_0}^k) = 0$ for all $k = 1, \ldots, n$. Hence,

$$\begin{aligned}
J &\equiv \int_{\mathbb{R}^N} \left[ \sum_{k=1}^n \bigg( \prod_{j=1}^N (e^{it_j^k \lambda_j} - 1) - \prod_{j=1}^N (e^{is_j^k \lambda_j} - 1) \bigg) \right] \\
&\qquad \times \bigg( \prod_{j=1}^N e^{-it_j^{k_0} \lambda_j} \bigg) \bigg( \prod_{j \neq j_0}^N \widehat{\delta}_j(\varepsilon \lambda_j) \bigg) \widehat{\delta}_{j_0}\big(r^{1/H_{j_0}} \lambda_{j_0}\big)\, d\lambda \\
&= (2\pi)^N \sum_{k=1}^n \bigg( \prod_{j \neq j_0}^N \big(\delta_j^\varepsilon(t_j^{k_0} - t_j^k) - \delta_j^\varepsilon(t_j^{k_0})\big) \bigg) \Big( \delta_{j_0}^r(t_{j_0}^{k_0} - t_{j_0}^k) - \delta_{j_0}^r(t_{j_0}^{k_0}) \Big) \\
&\quad - (2\pi)^N \sum_{k=1}^n \bigg( \prod_{j \neq j_0}^N \big(\delta_j^\varepsilon(t_j^{k_0} - s_j^k) - \delta_j^\varepsilon(t_j^{k_0})\big) \bigg) \Big( \delta_{j_0}^r(t_{j_0}^{k_0} - s_{j_0}^k) - \delta_{j_0}^r(t_{j_0}^{k_0}) \Big) \\
&\geq (2\pi)^N \varepsilon^{-(N-1)} r^{-1/H_{j_0}}.
\end{aligned} \tag{4.7}$$

In the above, all the terms in the first sum are non-negative and the second sum equals 0.

On the other hand, by the Cauchy-Schwarz inequality, (4.6) and (4.7), we get

$$\begin{aligned} J^2 &\le K_H^2 \Psi(\mathbf{t},\mathbf{s}) \int_{\mathbb{R}^N} \frac{1}{f_H(\lambda)} \prod_{j\neq j_0}^N \left|\widehat{\delta}_j(\varepsilon\lambda_j)\right|^2 \left|\widehat{\delta}_{j_0}\big(r^{1/H_{j_0}}\lambda_{j_0}\big)\right|^2 d\lambda \\ &= K_H^2 \Psi(\mathbf{t},\mathbf{s})\, \varepsilon^{-2(N-1)-2\sum_{j\neq j_0} H_j}\, r^{-2-2/H_{j_0}} \prod_{j=1}^N \int_{\mathbb{R}} |\lambda_j|^{2H_j+1} \left|\widehat{\delta}_j(\lambda_j)\right|^2 d\lambda_j \\ &= c_{4,2} r^{-2-2/H_{j_0}} \Psi(\mathbf{t},\mathbf{s}). \end{aligned} \tag{4.8}$$

Square the both sides of (4.7) and combine it with (4.8), the lemma follows. □

For any Borel probability measure $\mu$ on $\mathbb{R}_+^N$, let $\nu = \mu_{B^H}$ be the image measure of $\mu$ under $B^H$. The Fourier transform of $\nu$ can be written as

$$\widehat{\nu}(\xi) = \int_{\mathbb{R}_+^N} e^{i\langle \xi, B^H(t)\rangle}\, \mu(dt). \tag{4.9}$$

The following theorem describes the asymptotic behavior of $\widehat{\nu}(\xi)$ as $\xi \to \infty$. Contrast to the results for fractional Brownian motion and the Brownian sheet, the behavior of $\widehat{\nu}(\xi)$ is anisotropic.

***Theorem 4.*** *Let $B^H = \{B^H(t), t \in \mathbb{R}^N\}$ be an $(N,d)$-fractional Brownian sheet. Assume that, for every $j = 1, \ldots, N$, the function $\tau_j : \mathbb{R}_+ \to \mathbb{R}_+$ is non-decreasing such that $\tau_j(0) = 0$ and $\tau_j(2r) \le c_{4,3}\, \tau_j(r)$ for all $r \ge 0$ [i.e., $\tau_j$ satisfies the doubling property]. If $\mu$ is a Borel probability measure on $[\varepsilon, T]^N$ such that*

$$\mu\big(R(t,r)\big) \le c_{4,4} \prod_{j=1}^N \tau_j(r^{1/H_j}), \quad \forall\ t \in \mathbb{R}_+^N, \tag{4.10}$$

*where $R(t,r) = \prod_{j=1}^N [t_j - r^{1/H_j}, t_j + r^{1/H_j}]$. Then there exists a constant $\varrho > 0$ such that the Fourier transform of $\nu$ satisfies*

$$\limsup_{|\xi|\to\infty} \frac{\big|\widehat{\nu}(\xi)\big|}{\sqrt{\Big(\prod_{j=1}^N \tau_j\big(|\xi|^{-\frac{1}{H_j}}\big)\Big) \log^{\varrho} |\xi|}} < \infty, \qquad a.s. \tag{4.11}$$

***Proof.*** First note that by considering the restriction of $\mu$ on subsets of its support and the linearity of the Fourier transform, we see that, without loss of generality, we may and will assume that $\mu$ is supported on a Borel set $E \subset [\varepsilon, T]^N$ with $\mathrm{diam} E < \varepsilon^{1/H_1}$ [we have assumed that $H_1 = \min\{H_j : 1 \le j \le N\}$]. The reason for this reduction will become clear below.

The argument is similar to that of Kahane [22]. For any integer $n \geq 1$, (4.9) yields

$$\begin{aligned} \mathbb{E}\big(\big|\widehat{\nu}(\xi)\big|^{2n}\big) &= \mathbb{E}\int_{\mathbb{R}_+^{nN}}\int_{\mathbb{R}_+^{nN}} e^{i\langle \xi, \sum_{k=1}^n (B^H(t^k)-B^H(s^k))\rangle}\, \mu^n(d\mathbf{s})\mu^n(d\mathbf{t}) \\ &= \int_{\mathbb{R}_+^{nN}}\int_{\mathbb{R}_+^{nN}} \exp\Big(-\frac{1}{2}|\xi|^2\Psi(\mathbf{s},\mathbf{t})\Big)\, \mu^n(d\mathbf{s})\mu^n(d\mathbf{t}), \end{aligned} \tag{4.12}$$

where $\mu^n(d\mathbf{s}) = \mu(ds^1)\cdots\mu(ds^n)$.

Let $\mathbf{s} \in [\varepsilon, T]^{nN}$ be fixed and we write

$$\begin{aligned} &\int_{\mathbb{R}_+^{nN}} \exp\Big(-\frac{1}{2}|\xi|^2\Psi(\mathbf{s},\mathbf{t})\Big)\mu^n(d\mathbf{t}) \\ &\quad= \int_{G(\mathbf{s},r)} \exp\Big(-\frac{1}{2}|\xi|^2\Psi(\mathbf{s},\mathbf{t})\Big)\, \mu^n(d\mathbf{t}) \\ &\qquad+ \sum_{m=1}^{\infty}\int_{G(\mathbf{s},r2^m)\backslash G(\mathbf{s},r2^{m-1})} \exp\Big(-\frac{1}{2}|\xi|^2\Psi(\mathbf{s},\mathbf{t})\Big)\, \mu^n(d\mathbf{t}). \end{aligned} \tag{4.13}$$

Since $\mu$ is supported on $E$ with $\operatorname{diam} E < \varepsilon^{1/H_1}$, the above summation is taken over the integers $m$ such that $r2^m \leq \varepsilon$. Hence we can apply Lemma 9 to estimate the integrands.

By (4.10), we always have

$$\int_{G(\mathbf{s},r)} \exp\Big(-\frac{1}{2}|\xi|^2\Psi(\mathbf{s},\mathbf{t})\Big)\, \mu^n(d\mathbf{t}) \leq \Big(c_{4,4}\, n^N \prod_{j=1}^N \tau_j(r^{\frac{1}{H_j}})\Big)^n. \tag{4.14}$$

Given $\xi \in \mathbb{R}^d\backslash\{0\}$, we take $r = |\xi|^{-1}$. It follows from Lemma 9, the doubling property of functions $\tau_j$ and (4.10) that

$$\begin{aligned} &\int_{G(\mathbf{s},r2^m)\backslash G(\mathbf{s},r2^{m-1})} \exp\Big(-\frac{1}{2}|\xi|^2\Psi(\mathbf{s},\mathbf{t})\Big)\, \mu^n(d\mathbf{t}) \\ &\quad\leq \exp\Big(-\frac{1}{2}\, c_{4,1}\, |\xi|^2 (r2^{m-1})^2\Big)\cdot\Big(c_{4,4}\, n^N \prod_{j=1}^N \tau_j(2^m r^{\frac{1}{H_j}})\Big)^n \\ &\quad\leq \Big(c_{4,4}\, n^N \prod_{j=1}^N \tau_j(r^{1/H_j})\Big)^n \exp\Big(-c_{4,5}\, 2^{2m}\Big)\cdot c_{4,3}^{Nmn}. \end{aligned} \tag{4.15}$$

Note that

$$1 + \sum_{m=1}^{\infty} \exp\Big(-c_{4,5}\, 2^{2m}\Big)\cdot c_{4,3}^{Nmn} \leq c_{4,6}^n n^{\rho n}, \tag{4.16}$$

where $\rho = N/(2\log c_{4,3})$.

Combining (4.14), (4.15) and (4.16), we derive an upper bound for the integral in (4.13):

$$\int_{\mathbb{R}_+^{nN}} \exp\Big(-\frac{1}{2}|\xi|^2\Psi(\mathbf{s},\mathbf{t})\Big)\,\mu^n(d\mathbf{t}) \le c_{4,7}^n\, n^{(N+\rho)n}\cdot\bigg(\prod_{j=1}^N \tau_j\big(|\xi|^{-\frac{1}{H_j}}\big)\bigg)^n. \tag{4.17}$$

Integrating the both sides of (4.17) in $\mu^n(d\mathbf{s})$, we get

$$\mathbb{E}\big(\big|\widehat{\nu}(\xi)\big|^{2n}\big) \le c_{4,7}^n\cdot n^{\varrho n}\cdot\bigg(\prod_{j=1}^N \tau_j\big(|\xi|^{-\frac{1}{H_j}}\big)\bigg)^n, \tag{4.18}$$

where $\varrho = N+\rho$ is a constant.

The same argument as in Kahane ([22], pp. 254–255) using (4.18) and the Borel-Cantelli lemma implies that almost surely

$$\limsup_{z\in\mathbb{Z}^d, |z|\to\infty} \frac{\big|\widehat{\nu}(z)\big|}{\sqrt{\Big(\prod_{j=1}^N \tau_j\big(|z|^{-\frac{1}{H_j}}\big)\Big)\log^{\varrho}|z|}} < \infty. \tag{4.19}$$

Therefore (4.11) follows from (4.19) and Lemma 1 of Kahane ([22], p.252). This finishes the proof of Theorem 4. □

***Theorem 5.*** *Let $B^H = \{B^H(t), t\in\mathbb{R}^N\}$ be an $(N,d)$-fractional Brownian sheet with Hurst index $H\in(0,1)^N$. Then for every Borel set $E\subset(0,\infty)^N$ with $s(H,E)\le d$, $B^H(E)$ is almost surely a Salem set with Fourier dimension $s(H,E)$.*

***Proof.*** By using (3.35) and the fact that $\dim_{\mathcal{F}} F \le \dim_{\mathcal{H}} F$ for all Borel sets $F\subset\mathbb{R}^d$, we see that for every Borel set $E\subset(0,\infty)^N$ satisfying $s(H,E)\le d$, we have $\dim_{\mathcal{F}} B^H(E) \le \dim_{\mathcal{H}} B^H(E) = s(H,E)$ almost surely.

To prove the reverse inequality, it suffices to show that if $s(H,E)\le d$ then for all $\gamma\in(0,s(H,E))$ we have $\dim_{\mathcal{F}} B^H \ge \gamma$ a.s.

Note that for any $0<\gamma<s(H,E)$, there exists a Borel probability measure $\mu$ with compact support in $E$ such that $\gamma < s_\mu(H)$. Hence we can find $\lambda' = (\lambda'_1,\ldots,\lambda'_N)\in\Lambda_\mu$ such that $\gamma < \sum_{j=1}^N \frac{\lambda'_j}{H_j}$ and (3.41) holds. Let $\mu_\varepsilon$ be the restriction of $\mu$ to the set $E_\varepsilon$ defined by (3.42). Then $\mu_\varepsilon$ satisfies the condition (4.10) with $\tau_j(r) = r^{\lambda'_j}$ $(j=1,\ldots,N)$.

Let $\nu$ be the image measure of $\mu_\varepsilon$ under $B^H$. Then by Theorem 4 we have almost surely,

$$\big|\widehat{\nu}(\xi)\big| = O\bigg(\sqrt{|\xi|^{-\sum_{j=1}^N \frac{\lambda'_j}{H_j}}\log^{\varrho}|\xi|}\bigg), \qquad \text{as } |\xi|\to\infty. \tag{4.20}$$

This and (4.3) imply that $\dim_{\mathcal{F}} B^H(E) \geq \sum_{j=1}^N \frac{\lambda'_j}{H_j}$ a.s., which yields

$$\dim_{\mathcal{F}} B^H(E) \geq \gamma \quad \text{a.s.}$$

Since $\gamma \in (0,\, s(H,E))$ is arbitrary, we have proved that $\dim_{\mathcal{F}} B^H(E) = \dim_{\mathcal{H}} B^H(E)$ a.s. Therefore, $B^H(E)$ is a Salem set. □

Applying Theorems 4 and 5 to $B^{\langle\alpha\rangle}$, we have the following result.

***Corollary 2.*** *Let $B^{\langle\alpha\rangle} = \{B^{\langle\alpha\rangle}(t), t \in \mathbb{R}^N\}$ be an $(N,d)$-fractional Brownian sheet, and let $\tau : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-decreasing function satisfying $\tau(0) = 0$ and the doubling property. If $\mu$ is a probability measure on $[\varepsilon, T]^N$ such that*

$$\mu(B(x,r)) \leq c_{4,8}\, \tau(2r), \qquad \forall x \in \mathbb{R}_+^N,\ r \geq 0, \tag{4.21}$$

*then there exists a positive and finite constant $\varrho$ such that*

$$\limsup_{|\xi|\to\infty} \frac{|\widehat{\nu}(\xi)|}{\sqrt{\tau\big(|\xi|^{-\frac{1}{\alpha}}\big) \log^{\varrho} |\xi|}} < \infty. \tag{4.22}$$

*Moreover, for every Borel set $E \subset (0,\infty)^N$ with $\dim_{\mathcal{H}} E \leq \alpha d$, $B^{\langle\alpha\rangle}(E)$ is almost surely a Salem set with Fourier dimension $\dim_{\mathcal{H}} E/\alpha$.*

## 5. Interior points

Given a Borel set $E \subset (0,\infty)^N$ with $s(H,E) \geq d$, Theorem 2 implies that $\dim_{\mathcal{H}} B^H(E) = d$ a.s. Two natural questions arise: (1) Does $B^H(E)$ have positive $d$-dimensional Lebesgue measure a.s. ? (2) Does $B^H(E)$ have interior points a.s.?

Question (1) can be studied by using the Fourier analytic argument of Kahane [22]. One can show the following: If a Borel set $E \subset (0,\infty)^N$ carries a probability measure $\mu$ such that

$$\int_E \int_E \frac{1}{\big(\sum_{j=1}^N |s_j - t_j|^{2H_j}\big)^{d/2}}\, \mu(ds)\mu(dt) < \infty, \tag{5.1}$$

then almost surely, $B^H(E)$ has positive $d$-dimensional Lebesgue measure. In particular, it follows from the proof of Theorem 2 that, if $E \subset (0,\infty)^N$ satisfies $s(H,E) > d$, then the condition (5.1) is satisfied, and $B^H(E)$ has positive $d$-dimensional Lebesgue measure. Question (2) is more difficult to answer. This question for Brownian motion was first considered by Kaufman [27], and then extended by Pitt [39] and Kahane [22][23] to fractional Brownian motion and by Khoshnevisan and Xiao [29] to the Brownian sheet.

Recently, Shieh and Xiao [41] proved similar results under more general conditions for a large class of strongly locally nondeterministic, approximately isotropic Gaussian random fields.

In the following, we prove that a condition similar to that in Shieh and Xiao [41] is sufficient for $B^H(E)$ to have interior points almost surely. This theorem extends and improves the result of Khoshnevisan and Xiao [29] mentioned above.

***Theorem 6.*** *Let $B^H = \{B^H(t), t \in \mathbb{R}^N\}$ be an $(N,d)$-fractional Brownian sheet with index $H \in (0,1)^N$. If a Borel set $E \subset (0,\infty)^N$ carries a probability measure $\mu$ such that for some finite constants $c_{5,1} > 0$ and $\gamma > N$ we have*

$$\int_{\mathbb{R}^N_+} \frac{1}{\big(\sum_{j=1}^N |s_j - t_j|^{2H_j}\big)^{d/2}} \log_+^{(N+1)\gamma}\left(\frac{1}{\sum_{j=1}^N |s_j - t_j|^{2H_j}}\right) \mu(ds) \le c_{5,1} \tag{5.2}$$

*for all $t \in \mathbb{R}^N_+$, where $\log_+ x = \max\{1, \log x\}$. Then $B^H(E)$ has interior points almost surely.*

From Theorem 6 we derive the following corollaries.

***Corollary 3.*** *If $E \subset (0,\infty)^N$ is a Borel set with $s(H,E) > d$, then $B^H(E)$ a.s. has interior points.*

***Proof.*** It follows from the proof of Theorem 2 that, if $s(H,E) > d$, then there is a Borel probability measure $\mu$ on $E$ satisfying (5.2). Hence the conclusion follows from Theorem 6. □

***Corollary 4.*** *Let $B^{\langle\alpha\rangle} = \{B^{\langle\alpha\rangle}(t), t \in \mathbb{R}^N\}$ be an $(N,d)$-fractional Brownian sheet with Hurst index $H = \langle\alpha\rangle$. If a Borel set $E \subset (0,\infty)^N$ carries a probability measure $\mu$ such that*

$$\sup_{t\in\mathbb{R}^N_+} \int_{\mathbb{R}^N_+} \frac{1}{|s-t|^{\alpha d}} \log_+^{(N+1)\gamma}\left(\frac{1}{|s-t|}\right) \mu(ds) \le c_{5,2} \tag{5.3}$$

*for some finite constants $c_{5,2} > 0$ and $\gamma > N$. Then $B^{\langle\alpha\rangle}(E)$ has interior points almost surely.*

The existence of interior points in $B^H(E)$ is related to the regularity of the local times of $B^H$ on $E$. Let us recall briefly the definition of local time of $B^H$ on $E$. For any Borel probability measure $\mu$ on $E$, let $\mu_{B^H}$ be the image measure of $\mu$ under $B^H$. If $\mu_{B^H}$ is absolutely continuous with respect to the Lebesgue measure $m_d$ in $\mathbb{R}^d$, then $B^H$ is said to have a local time on $E$. The local time $l_\mu(x)$ is defined to be the Radon–Nikodým derivative $d\mu_{B^H}/dm_d(x)$ and it satisfies the following occupation density formula: For all Borel measurable functions $f : \mathbb{R}^d \to \mathbb{R}_+$,

$$\int_E f\big(B^H(s)\big)\, \mu(ds) = \int_{\mathbb{R}^d} f(x) l_\mu(x)\, dx. \tag{5.4}$$

It is known from Geman and Horowitz [19] and Kahane [22] that, when (5.1) holds, then $l_\mu(x) \in L^2(\mathbb{R}^d)$ a.s.

In order to prove Theorem 6, we will make use of the following continuity lemma of Garsia [18].

***Lemma 10.*** *Assume that $p(u)$ and $\Psi(u)$ are two positive increasing functions on $[0, \infty)$ such that $p(u) \downarrow 0$ as $u \downarrow 0$, $\Psi(u)$ is convex and $\Psi(u) \uparrow \infty$ as $u \uparrow \infty$. Let $D$ denote an open hypercube in $\mathbb{R}^d$. If the function $f(x) : D \to \mathbb{R}$ is measurable and*

$$A := A(D, f) = \int_D \int_D \Psi\Big(\frac{|f(x) - f(y)|}{p(|x-y|/\sqrt{d})}\Big) dxdy < \infty, \tag{5.5}$$

*then after modifying $f(x)$ on a set of Lebesgue measure 0, we have*

$$|f(x) - f(y)| \le 8 \int_0^{|x-y|} \Psi^{-1}\Big(\frac{A}{u^{2d}}\Big)\, dp(u) \quad \textit{for all } x, y \in D. \tag{5.6}$$

We take the function $p(u)$ in Garsia's lemma as follows: Let $\gamma$ be the constant in (5.2) and define

$$p(u) = \begin{cases} 0, & \text{if } u = 0, \\ \log^{-\gamma}\left(\frac{e}{u}\right), & \text{if } 0 < u \le 1, \\ \gamma u - \gamma + 1, & \text{if } u > 1. \end{cases} \tag{5.7}$$

Clearly, the function $p(u)$ is strictly increasing on $[0, \infty)$ and $p(u) \downarrow 0$ as $u \downarrow 0$.

Now we proceed to prove Theorem 6.

***Proof of Theorem 6.*** First note that, since $\mu$ is a Borel probability measure on $E$, without loss of generality, we can assume that $E$ is compact. Hence there are constants $0 < \varepsilon < T < \infty$ such that $E \subseteq [\varepsilon, T]^N$. Since $B^H(E)$ is a compact subset of $\mathbb{R}^d$, (5.4) implies that $\{x : l_\mu(x) > 0\}$ is a subset of $B^H(E)$. Hence, in order to prove our theorem, it is sufficient to prove that the local time $l_\mu(x)$ has a version which is continuous in $x$; see Pitt ([39], p.324) or Geman and Horowitz ([19], p.12). This will be proved by deriving moment estimates for the local time $l_\mu$ and by applying Garsia's continuity lemma.

Secondly, as in Khoshnevisan and Xiao [29], we may and will assume that the Borel probability measure $\mu$ in (5.2) has the following property: For any constant $c > 0$ and $\ell = 1, \ldots, N$,

$$\mu\{t = (t_1, \ldots, t_N) \in E : \ t_\ell = c\} = 0. \tag{5.8}$$

Otherwise, we can replace $B^H$ by an $(N-1, d)$-fractional Brownian sheet $\widetilde{B}^H$ and prove the desired conclusion for $\widetilde{B}^H(E_\ell)$, where $E_\ell$ is the set in (5.8) with positive $\mu$-measure.

Consider the set

$$\widetilde{E}_n = \left\{ \bar{\mathbf{t}} = (t^1, \ldots, t^n) \in E^n : \ t^j_\ell = t^i_\ell \text{ for some } i \neq j \ \text{ and } \ 1 \leq \ell \leq N \right\}. \tag{5.9}$$

It follows from (5.8) and the Fubini–Tonelli theorem that $\mu^n(\widetilde{E}_n) = 0$.

The following lemma provides estimates on high moments of the local time, which is the key for finishing the proof of Theorem 6.

***Lemma 11.*** *Let $\mu$ be a Borel probability measure on $E \subset [\varepsilon, T]^N$ satisfying (5.2) and (5.8) and let $p(u)$ be defined by (5.7). Then for every hypercube $D \subset \mathbb{R}^d$ there exists a finite constant $c_{5,3} > 0$, depending on $N$, $d$, $\gamma$, $\mu$ and $D$ only, such that for all even integers $n \geq 2$,*

$$\mathbb{E} \int_D \int_D \left( \frac{l_\mu(x) - l_\mu(y)}{p(|x-y|/\sqrt{d})} \right)^n dxdy \leq c_{5,3}^n \, (n!)^N \, \log^{n(N+1)\gamma} n. \tag{5.10}$$

We now continue with the proof of Theorem 6 and defer the proof of Lemma 11 to the end of this section.

Let $\Psi(u) = u \exp(u^\theta)$, where $\theta \in \left(\frac{1}{\gamma}, \frac{1}{N}\right)$ is a constant. Then $\Psi$ is increasing and convex on $(0, \infty)$. It follows from Jensen's inequality, the Fubini–Tonelli theorem and Lemma 11 that for all closed hypercubes $D \subset \mathbb{R}^d$ and all integers $n$ with $\theta + 1/n < 1$,

$$\begin{aligned} &\mathbb{E} \int_D \int_D \left( \frac{|l_\mu(x) - l_\mu(y)|}{p(|x-y|/\sqrt{d})} \right)^{n\theta+1} dxdy \\ &\quad \leq \big(m_d(D)\big)^{2-\theta-1/n} \left\{ \mathbb{E} \int_D \int_D \left( \frac{|l_\mu(x) - l_\mu(y)|}{p(|x-y|/\sqrt{d})} \right)^n dxdy \right\}^{\theta+1/n} \\ &\quad \leq c^n \, (n!)^{N(\theta+1/n)} \, (\log n)^{n(N+1)\gamma(\theta+1/n)} \\ &\quad \leq c_{5,4}^n \, (n!)^{N\theta} \, \log^{n(N+1)\gamma\theta} n, \end{aligned} \tag{5.11}$$

where $c_{5,4}$ is a finite constant depending on $N, d, \theta$, $D$ and $c_{5,3}$ only.

Expanding $\Psi(u)$ into a power series and applying the inequality (5.11), we derive

$$\begin{aligned} &\mathbb{E} \int_D \int_D \Psi\left( \frac{|l_\mu(x) - l_\mu(y)|}{p(|x-y|/\sqrt{d})} \right) dxdy \\ &= \sum_{n=0}^\infty \frac{1}{n!} \, \mathbb{E} \int_D \int_D \left( \frac{|l_\mu(x) - l_\mu(y)|}{p(|x-y|/\sqrt{d})} \right)^{n\theta+1} dxdy \\ &\leq c_{5,5} < \infty, \end{aligned} \tag{5.12}$$

where the last inequality follows from the fact that $N\theta < 1$. Hence Garsia's lemma implies that there are positive and finite random variables $A_1$ and

$A_2$ such that for almost all $x, y \in D$ with $|x-y| \le e^{-1}$,

$$\begin{aligned} |l_\mu(x) - l_\mu(y)| &\le \int_0^{|x-y|} \Psi^{-1}\Big(\frac{A_1}{u^{2d}}\Big)\, dp(u) \\ &\le A_2\Big[\log\big(1/|x-y|\big)\Big]^{-(\gamma-1/\theta)}. \end{aligned}$$

Note that, by our choice of $\theta$, we have $\theta > 1/\gamma$ and hence $B^H$ has almost surely a local time $l_\mu(x)$ on $E$ that is continuous for all $x \in D$. Finally, by taking a sequence of closed hypercubes $\{D_n,\ n \ge 1\}$ such that $\mathbb{R}^d = \cup_{n=1}^\infty D_n$, we have proved that almost surely $l_\mu(x)$ is continuous for all $x \in \mathbb{R}^d$. This completes the proof of Theorem 6. □

It remains to prove Lemma 11. Our proof relies on the sectorial local nondeterminism of $B^H$ and on an argument which improves those in Khoshnevisan and Xiao [29] and Shieh and Xiao [41].

We will need several lemmas. Lemma 12 is essentially due to Cuzick and DuPreez [12], where the extra condition on $g$ is dropped in Khoshnevisan and Xiao [29]. Lemma 13 is a slight modification of Lemma 4 in Cuzick and DuPreez [12].

**Lemma 12.** *Let $Z_k\,(k = 1, \ldots, n)$ be centered, jointly Gaussian random variables which are linearly independent. If $g : \mathbb{R} \to \mathbb{R}_+$ is a Borel measurable function, then*

$$\int_{\mathbb{R}^n} g(v_1)\, e^{-\frac{1}{2}\mathrm{Var}(\langle v, Z\rangle)}\, dv = \frac{(2\pi)^{(n-1)/2}}{Q^{1/2}} \int_{-\infty}^{\infty} g(z/\sigma_1)\, e^{-\frac{1}{2}z^2}\, dz, \qquad (5.13)$$

*where $\sigma_1^2 = \mathrm{Var}(Z_1 \,|\, Z_2, \ldots, Z_n)$ and $Q = \det \mathrm{Cov}(Z_1, \ldots, Z_n)$ is the determinant of the covariance matrix of $Z_1, \ldots, Z_n$.*

**Lemma 13.** *If $\alpha \ge e^2/2$, then*

$$\int_1^\infty \log^\alpha x \, \exp\Big(-\frac{x^2}{2}\Big)\, dx \le \sqrt{\pi}\, \log^\alpha \alpha. \qquad (5.14)$$

Consider the non-decreasing function $\Lambda(u) = 2\min\{1, u\}$ on $[0, \infty)$. Later we will make use of the elementary inequality

$$|e^{iu} - 1| \le \Lambda(|u|), \qquad \forall u \in \mathbb{R}. \qquad (5.15)$$

**Lemma 14.** *Assume $h(y)$ is any positive and non-decreasing function on $[0, \infty)$ such that $h(0) = 0$, $y^n/h^n(y)$ is non-decreasing on $[0, 1]$, and $\int_1^\infty h^{-2}(y)dy < \infty$. Then there exists a constant $c_{5,6}$ such that for all integers $n \ge 1$ and $v \in (0, \infty)$,*

$$\int_0^\infty \frac{\Lambda^n(vy)}{h^n(y)}\, dy \le c_{5,6}^n\, h_-^{-n}\Big(\frac{1}{v}\Big), \qquad (5.16)$$

*where* $h_-(y) = \min\{1, h(y)\}$ *so that* $h_-^{-n}(y) = \max\{1, h^{-n}(y)\}$.

***Proof.*** The proof is the same as that of Lemma 3 in Cuzick and DuPreez [12]. □

The following result is about the function $p(u)$ defined by (5.7).

***Lemma 15.*** *Let* $p(u)$ *be defined as in (5.7). Then for all* $\sigma > 0$ *and integers* $n \geq 1$,

$$\int_0^\infty p_-^{-n}\Big(\frac{\sigma}{v}\Big) \exp\Big(-\frac{v^2}{2}\Big) dv \leq c_{5,7}^n \left[\log^{n\gamma} n + \log_+^{n\gamma}\left(\frac{e}{\sigma}\right)\right], \tag{5.17}$$

*where* $\log_+ x = \max\{1, \log x\}$.

***Proof.*** Since

$$p_-^{-n}(x) = \begin{cases} \log^{n\gamma}\left(\frac{e}{x}\right), & \text{if } 0 < x < 1, \\ 1 & \text{if } x \geq 1 \end{cases}$$

and $\log_+^\alpha(xy) \leq 2^\alpha(\log_+^\alpha x + \log_+^\alpha y)$ for all $\alpha \geq 0$, we deduce that the integral in (5.17) is at most

$$\begin{aligned} &\int_{\sigma/v \geq 1} \exp\Big(-\frac{v^2}{2}\Big)\, dv + 2^{n\gamma} \int_{\sigma/v<1} \log_+^{n\gamma}(v) \exp\Big(-\frac{v^2}{2}\Big)\, dv \\ &\qquad + 2^{n\gamma} \int_{\sigma/v<1} \log_+^{n\gamma}\Big(\frac{e}{\sigma}\Big) \exp\Big(-\frac{v^2}{2}\Big)\, dv. \end{aligned} \tag{5.18}$$

It follows from (5.18) and Lemma 13 that

$$\begin{aligned} \int_0^\infty p_-^{-n}\Big(\frac{\sigma}{v}\Big) \exp\Big(-\frac{v^2}{2}\Big) dv &\leq c^n \Big[\log^{n\gamma}(n\gamma) + \log_+^{n\gamma}\Big(\frac{e}{\sigma}\Big)\Big] \\ &\leq c_{5,7}^n \Big[\log^{n\gamma} n + \log_+^{n\gamma}\Big(\frac{e}{\sigma}\Big)\Big]. \end{aligned} \tag{5.19}$$

This completes the proof of Lemma 15. □

Finally, we are in a position to prove Lemma 11.

***Proof of Lemma 11.*** By (25.7) in Geman and Horowitz [19], we have that for every $x$, $y \in \mathbb{R}^d$ and all even integers $n \geq 2$,

$$\begin{aligned} \mathbb{E}\left[(l_\mu(x) - l_\mu(y))^n\right] = (2\pi)^{-nd} \int_{E^n} \int_{\mathbb{R}^{nd}} \prod_{k=1}^n \Big[e^{-i\langle u^k, x\rangle} - e^{-i\langle u^k, y\rangle}\Big] \\ \times \exp\bigg[-\frac{1}{2}\mathrm{Var}\Big(\sum_{k=1}^n \langle u^k, B^H(t^k)\rangle\Big)\bigg]\, d\mathbf{u}\, \mu^n(d\mathbf{t}). \end{aligned} \tag{5.20}$$

In the above, $\mathbf{u} = (u^1, \ldots, u^n)$, $u^k \in \mathbb{R}^d$ for each $k = 1, \ldots, n$ and we will write it coordinate-wise as $u^k = (u^k_1, \ldots, u^k_d)$.

Note that for $u^1, \ldots, u^n,\ y \in \mathbb{R}^d$, the triangle inequality implies

$$\begin{aligned}
&\prod_{k=1}^{n} \Big| \exp(-i\langle u^k,\, y\rangle) - 1 \Big| \\
&\quad \le \prod_{k=1}^{n} \Big| \sum_{j=1}^{d} \Big[ \exp\Big( -i \sum_{\ell=0}^{j} u^k_\ell\, y_\ell \Big) - \exp\Big( -i \sum_{\ell=0}^{j-1} u^k_\ell\, y_\ell \Big) \Big] \Big| \\
&\quad \le \prod_{k=1}^{n} \Big[ \sum_{j=1}^{d} \Big| \exp(-iu^k_j\, y_j) - 1 \Big| \Big] \\
&\quad = \sum{}' \prod_{k=1}^{n} \Big| \exp(-iu^k_{j_k}\, y_{j_k}) - 1 \Big|,
\end{aligned} \tag{5.21}$$

where $y_0 = u^j_0 = 0$ in the first inequality and the last summation $\sum'$ is taken over all sequences $(j_1, \cdots, j_n) \in \{1, \ldots, d\}^n$.

It follows from (5.20), (5.21), (5.15) and the Fubini–Tonelli theorem that for any fixed hypercube $D \subset \mathbb{R}^d$ and any even integer $n \ge 2$, we have

$$\begin{aligned}
&\mathbb{E} \int_D \int_D \bigg( \frac{l_\mu(x) - l_\mu(y)}{p(|x-y|/\sqrt{d})} \bigg)^n dx dy \\
&\quad \le \sum{}' \int_D \int_D \int_{E^n} \int_{\mathbb{R}^{nd}} \prod_{k=1}^{n} \frac{\big| \exp(iu^k_{j_k}(y_{j_k} - x_{j_k})) - 1 \big|}{p(|y-x|/\sqrt{d})} \\
&\qquad\qquad \times \exp\bigg[ -\frac{1}{2} \mathrm{Var}\bigg( \sum_{k=1}^{n} \langle u^k,\, B^H(t^k)\rangle \bigg) \bigg]\, d\mathbf{u}\, \mu^n(d\mathbf{t})\, dx\, dy \\
&\quad \le m_d(D) \sum{}' \int_{D\ominus D} \int_{E^n} \int_{\mathbb{R}^{nd}} \prod_{k=1}^{n} \frac{\Lambda(|u^k_{j_k} y_{j_k}|)}{p(|y|/\sqrt{d})} \\
&\qquad\qquad \times \exp\bigg[ -\frac{1}{2} \mathrm{Var}\bigg( \sum_{k=1}^{n} \langle u^k,\ B^H(t^k)\rangle \bigg) \bigg]\, d\mathbf{u}\, \mu^n(d\mathbf{t})\, dy.
\end{aligned} \tag{5.22}$$

In the above, we have made a change of variables and $D \ominus D = \{x - y : x, y \in D\}$. By our assumptions on $\mu$, we see that the integral in (5.22) with respect to $\mu^n$ can be taken over the set $E^n \backslash \widetilde{E}_n$, where $\widetilde{E}_n$ is defined by (5.9).

Now we fix $\mathbf{t} \in E^n \backslash \widetilde{E}_n$, a sequence $\mathbf{j} = (j_1, \ldots, j_n) \in \{1, \ldots, d\}^n$ and define $\mathcal{M}_n(\mathbf{t}) \equiv \mathcal{M}_n(\mathbf{j}, \mathbf{t})$ by

$$\mathcal{M}_n(\mathbf{t}) = \int_{D\ominus D} \int_{\mathbb{R}^{nd}} \prod_{k=1}^{n} \frac{\Lambda\big(|u^k_{j_k} y_{j_k}|\big)}{p(|y|/\sqrt{d})}\ \exp\bigg[ -\frac{1}{2} \mathrm{Var}\bigg( \sum_{k=1}^{n} \langle u^k,\ B^H(t^k)\rangle \bigg) \bigg] d\mathbf{u} dy. \tag{5.23}$$

Then the last integral in (5.22) corresponding to the sequence $\mathbf{j} = (j_1, \ldots, j_n)$ can be written as

$$\mathcal{N}_{\mathbf{j}} \equiv \int_{E^n \setminus \widetilde{E}_n} \mathcal{M}_n(\mathbf{t})\, \mu^n(d\mathbf{t}). \tag{5.24}$$

We will estimate the above integral by integrating in the order $\mu(dt^n)$, $\mu(dt^{n-1})$, $\ldots, \mu(dt^1)$. For this purpose, we need to derive an upper bound for $\mathcal{M}_n(\mathbf{t})$. Observe that for any positive numbers $\beta_1, \ldots, \beta_n$ satisfying $\sum_{k=1}^n \beta_k = n$, we can write

$$\begin{aligned}\mathcal{M}_n(\mathbf{t}) = \int_{D\ominus D}\int_{\mathbb{R}^{nd}} \prod_{k=1}^n \frac{\Lambda\big(|u^k_{j_k} y_{j_k}|\big)}{p^{\beta_k}(|y|/\sqrt{d})}& \\ \times \exp\bigg[-\frac{1}{2}\mathrm{Var}\bigg(\sum_{k=1}^n \langle u^k,\ B^H(t^k)\rangle\bigg)\bigg]&\, d\mathbf{u}\, dy.\end{aligned} \tag{5.25}$$

Later it will be clear that the flexibility in choosing $\beta_k$ in (5.25) is essential to our proof. More precisely, by choosing the constants $\beta_k\,(1 \le k \le n)$ appropriately, we minimize the effect of "bad points" [see (5.37) below].

For any $n$ points $t^1, \ldots, t^n \in E^n \setminus \widetilde{E}_n$, Lemma 1 implies that the Gaussian random variables $B^H_j(t^k)$ $(j = 1, \ldots, d,\ k = 1, \ldots, n)$ are linearly independent. By applying the generalized Hölder's inequality, Lemma 12 and a change of variables, we see that $\mathcal{M}_n(\mathbf{t})$ is bounded by

$$\begin{aligned}&\prod_{k=1}^n \bigg\{\int_{D\ominus D}\int_{\mathbb{R}^{nd}} \frac{\Lambda^n\big(|u^k_{j_k} y_{j_k}|\big)}{p^{n\beta_k}(|y|/\sqrt{d})}\, \exp\bigg[-\frac{1}{2}\mathrm{Var}\bigg(\sum_{k=1}^n\sum_{\ell=1}^d u^k_\ell B^H_\ell(t^k)\bigg)\bigg]\, d\mathbf{u}\, dy\bigg\}^{1/n} \\ &= \frac{c_{5,8}^{nd}}{[\det\mathrm{Cov}(B^H_0(t^1), \ldots, B^H_0(t^n))]^{d/2}} \\ &\qquad \times \prod_{k=1}^n \bigg\{\int_{D\ominus D}\int_{\mathbb{R}} \frac{\Lambda^n\big(|u^k_{j_k} y_{j_k}|/\sigma_k\big)}{p^{n\beta_k}(|y|/\sqrt{d})}\, \exp\bigg(-\frac{(u^k_{j_k})^2}{2}\bigg)\, du^k_{j_k} dy\bigg\}^{1/n},\end{aligned} \tag{5.26}$$

where, for every $1 \le k \le n$, $\sigma_k^2 \equiv \sigma_k^2(\mathbf{t})$ is the conditional variance of $B^H_{j_k}(t^k)$ given $B^H_\ell(t^m)$ ($\ell \ne j_k$ and $1 \le m \le n$, or $\ell = j_k$ and $m \ne k$).

Denote the $n$ integrals in the last product of (5.26) by $\mathfrak{J}_1, \ldots, \mathfrak{J}_n$, respectively. In order to estimate them, we will apply the sectorial local nondeterminism of $B^H_0$. Since $B^H_1, \ldots, B^H_d$ are independent copies of $B^H_0$, we have

$$\sigma_k^2(\mathbf{t}) = \mathrm{Var}\left(B^H_0(t^k)\ \Big|\ \big\{B^H_0(t^m)\big\}_{m\ne k}\right). \tag{5.27}$$

It follows from Theorem 1 that for every $1 \le k \le n$,

$$\sigma_k^2(\mathbf{t}) \ge c_{2,1} \sum_{\ell=1}^N \min_{m\ne k} \left|t^m_\ell - t^k_\ell\right|^{2H_\ell}. \tag{5.28}$$

In order to estimate the sum in (5.28) as a function of $t^n$, we introduce $N$ permutations $\Gamma_1, \ldots, \Gamma_N$ of $\{1, \ldots, n-1\}$ such that for every $\ell = 1, \ldots, N$,

$$t_\ell^{\Gamma_\ell(1)} < t_\ell^{\Gamma_\ell(2)} < \ldots < t_\ell^{\Gamma_\ell(n-1)}. \tag{5.29}$$

This is possible since $t_\ell^k$ ($1 \le k \le n-1$, $1 \le \ell \le N$) are distinct. For convenience, we denote $t_\ell^{\Gamma_\ell(0)} = \varepsilon$ and $t_\ell^{\Gamma_\ell(n)} = T$ for all $1 \le \ell \le N$.

For every sequence $(i_1, \ldots, i_N) \in \{1, \ldots, n-1\}^N$, let $\tau_{i_1,\cdots,i_N} = (t_1^{\Gamma_1(i_1)}, \ldots, t_N^{\Gamma_N(i_N)})$ be the "center" of the rectangle

$$I_{i_1,\cdots,i_N} = \prod_{\ell=1}^N \Big[t_\ell^{\Gamma_\ell(i_\ell)} - \frac{1}{2}\big(t_\ell^{\Gamma_\ell(i_\ell)} - t_\ell^{\Gamma_\ell(i_\ell-1)}\big),\, t_\ell^{\Gamma_\ell(i_\ell)} + \frac{1}{2}\big(t_\ell^{\Gamma_\ell(i_\ell+1)} - t_\ell^{\Gamma_\ell(i_\ell)}\big)\Big) \tag{5.30}$$

with the convention that the left-end point of the interval is $\varepsilon$ whenever $i_\ell = 1$; and the interval is closed and its right-end is $T$ whenever $i_\ell = n-1$. Thus the rectangles $\{I_{i_1,\cdots,i_N}\}$ form a partition of $[\varepsilon, T]^N$.

For every $t^n \in E$, let $I_{i_1,\cdots,i_N}$ be the unique rectangle containing $t^n$. Then (5.28) yields the following lower bound for $\sigma_n^2(\mathbf{t})$:

$$\sigma_n^2(\mathbf{t}) \ge c_{2,1} \sum_{\ell=1}^N \Big|t_\ell^n - t_\ell^{\Gamma_\ell(i_\ell)}\Big|^{2H_\ell}. \tag{5.31}$$

For every $k = 1, \ldots, n-1$, we say that $I_{i_1,\cdots,i_N}$ *cannot see* $t^k$ *from direction* $\ell$ if

$$t_\ell^k \notin \Big[t_\ell^{\Gamma_\ell(i_\ell)} - \frac{1}{2}\big(t_\ell^{\Gamma_\ell(i_\ell)} - t_\ell^{\Gamma_\ell(i_\ell-1)}\big),\, t_\ell^{\Gamma_\ell(i_\ell)} + \frac{1}{2}\big(t_\ell^{\Gamma_\ell(i_\ell+1)} - t_\ell^{\Gamma_\ell(i_\ell)}\big)\Big]. \tag{5.32}$$

We emphasize that if $I_{i_1,\cdots,i_N}$ cannot see $t^k$ from all $N$ directions, then

$$\big|t_\ell^k - t_\ell^n\big| \ge \frac{1}{2} \min_{m \ne k,n} \big|t_\ell^k - t_\ell^m\big| \qquad \text{for all } 1 \le \ell \le N. \tag{5.33}$$

Thus $t^n$ does not contribute to the sum in (5.28). More precisely, the latter means that

$$\sigma_k^2(\mathbf{t}) \ge c_{5,9} \sum_{\ell=1}^N \min_{m \ne k,n} \big|t_\ell^k - t_\ell^m\big|^{2H_\ell}. \tag{5.34}$$

The right hand side of (5.34) only depends on $t^1, \ldots, t^{n-1}$, which will be denoted by $\widetilde{\sigma}_k^2(\mathbf{t})$. Because of this, $t^k$ is called a "good" point for $I_{i_1,\cdots,i_N}$ [or for $t^n$] provided (5.32) holds for every $\ell = 1, \ldots, N$.

Let $1 \le k \le n-1$. If $I_{i_1,\cdots,i_N}$ sees the point $t^k$ from a direction and $t^k \ne \tau_{i_1,\cdots,i_N}$, then it is impossible to control $\sigma_k^2(\mathbf{t})$ from below as in (5.31) or (5.34). We say that $t^k$ is a "bad" point for $I_{i_1,\cdots,i_N}$ [or for $t^n$]. It is important to note that, because of (5.29), the rectangle $I_{i_1,\cdots,i_N}$ can only have at most $N$ bad points $t^k$ $(1 \le k \le n-1)$, i.e., at most one in each direction. We denote the set of bad points for $I_{i_1,\cdots,i_N}$ by

$$\Theta^n_{i_1,\ldots,i_N} = \big\{1 \le k \le n-1 : \, t^k \ \text{ is a bad point for } \ I_{i_1,\cdots,i_N}\big\}$$

and denote its cardinality by $\#(\Theta^n_{i_1,\ldots,i_N})$. Note that by definition $n \notin \Theta^n_{i_1,\ldots,i_N}$ and $\#(\Theta^n_{i_1,\ldots,i_N}) \le N$.

Now we choose the constants $\beta_1, \ldots, \beta_n$ [they depend on the sequence $(i_1, \ldots, i_N)$] as follows: $\beta_k = 0$ if $t^k$ is a bad point for $I_{i_1,\cdots,i_N}$; $\beta_k = 1$ if $t^k$ is a good point for $I_{i_1,\cdots,i_N}$ and

$$\beta_n = 1 + \#(\Theta^n_{i_1,\ldots,i_N}).$$

Clearly, $\beta_n \le N+1$.

By Lemma 14 and Lemma 15, we have

$$\begin{aligned}
\mathfrak{I}_n &= \int_{D\ominus D}\int_{\mathbb{R}} \frac{\Lambda^n\big(|u^n_{j_n} y_{j_n}|/\sigma_n(\mathbf{t})\big)}{p^{n\beta_n}(|y|/\sqrt{d})} \exp\bigg(-\frac{(u^n_{j_n})^2}{2}\bigg)\, du^n_{j_n}\, dy \\
&\le c \int_{\mathbb{R}} \exp\bigg(-\frac{v^2}{2}\bigg)\, dv \int_0^\infty \frac{\Lambda^n(v y_{j_n}/\sigma_n(\mathbf{t}))}{p^{n\beta_n}(y_{j_n}/\sqrt{d})}\, dy_{j_n} \\
&\le c^n_{5,10} \int_{\mathbb{R}} p_-^{-n\beta_n}\bigg(\frac{\sigma_n(\mathbf{t})}{v}\bigg) \exp\bigg(-\frac{v^2}{2}\bigg)\, dv \\
&\le c^n_{5,11}\bigg[\log^{n(N+1)\gamma} n + \log_+^{n(N+1)\gamma}\bigg(\frac{e}{\sigma_n(\mathbf{t})}\bigg)\bigg].
\end{aligned} \tag{5.35}$$

In the above, we have also used the fact that $p(|y|/\sqrt{d}) \ge p(|y_j|/\sqrt{d})$ for all $j = 1, \ldots, d$.

If $t^k$ is a good point for $I_{i_1,\cdots,i_N}$, then by the monotonicity of the function $\Lambda$ we have

$$\begin{aligned}
\mathfrak{I}_k &= \int_{D\ominus D}\int_{\mathbb{R}} \frac{\Lambda^n\big(|u^k_{j_k} y_{j_k}|/\sigma_k(\mathbf{t})\big)}{p^{n\beta_k}(|y|/\sqrt{d})} \exp\bigg(-\frac{(u^k_{j_k})^2}{2}\bigg)\, du^k_{j_k}\, dy \\
&\le \int_{D\ominus D}\int_{\mathbb{R}} \frac{\Lambda^n(u^k_{j_k} y_{j_k}/\widetilde{\sigma}_k(\mathbf{t}))}{p^n(|y|/\sqrt{d})} \exp\bigg(-\frac{(u^k_{j_k})^2}{2}\bigg)\, du^k_{j_k}\, dy.
\end{aligned} \tag{5.36}$$

When $t^k$ is a bad points for $I_{i_1,\cdots,i_N}$, we use the inequality $\Lambda(u) \le 2$ to obtain

$$\mathfrak{I}_k \le 2^n \int_{D\ominus D}\int_{\mathbb{R}} \exp\bigg(-\frac{(u^k_{j_k})^2}{2}\bigg)\, du^k_{j_k}\, dy \le c^n_{5,12}. \tag{5.37}$$

Since there are at most $N$ bad points for $I_{i_1,\cdots,i_N}$, their total contribution to $\mathcal{M}_n(\mathbf{t})$ is bounded by a constant $c_{5,13}$, which depends on $D$, $d$ and $N$ only.

Combining (5.26), (5.35), (5.36) and (5.37), we derive that

$$
\begin{aligned}
\mathcal{M}_n(\mathbf{t}) \le\ & \frac{c_{5,14}^n}{[\det\mathrm{Cov}(B_0^H(t^1),\ldots,B_0^H(t^n))]^{d/2}} \\
& \qquad \times \left[\log^{(N+1)\gamma} n + \log_+^{(N+1)\gamma}\left(\frac{e}{\sigma_n(\mathbf{t})}\right)\right] \\
& \times \prod_{k\notin\Theta^n_{i_1,\ldots,i_N}} \left\{\int_{D\ominus D}\int_{\mathbb{R}} \frac{\Lambda^n(u^k_{j_k} y_{j_k}/\widetilde{\sigma}_k(\mathbf{t}))}{p^n(|y|/\sqrt{d})}\ \exp\left(-\frac{(u^k_{j_k})^2}{2}\right) du^k_{j_k}\, dy\right\}^{1/n} \\
=\ & \frac{c_{5,15}^n}{[\det\mathrm{Cov}(B_0^H(t^1),\ldots,B_0^H(t^{n-1}))]^{d/2}} \\
& \times \prod_{k\notin\Theta^n_{i_1,\ldots,i_N}} \left\{\int_{D\ominus D}\int_{\mathbb{R}} \frac{\Lambda^n(u^k_{j_k} y_{j_k}/\widetilde{\sigma}_k(\mathbf{t}))}{p^n(|y|/\sqrt{d})}\ \exp\left(-\frac{(u^k_{j_k})^2}{2}\right) du^k_{j_k}\, dy\right\}^{1/n} \\
& \times \frac{1}{\sigma_n(\mathbf{t})^d}\left[\log^{(N+1)\gamma} n + \log_+^{(N+1)\gamma}\left(\frac{e}{\sigma_n(\mathbf{t})}\right)\right].
\end{aligned}
\tag{5.38}
$$

Note that Condition (5.2) implies

$$
\begin{aligned}
& \int_{I_{i_1,\ldots,i_N}} \frac{1}{\sigma_n(\mathbf{t})^d}\left[\log^{(N+1)\gamma} n + \log_+^{(N+1)\gamma}\left(\frac{e}{\sigma_n(\mathbf{t})}\right)\right]\mu(dt^n) \\
& \quad \le c_{5,16}\int_{I_{i_1,\ldots,i_N}} \frac{1}{\left(\sum_{\ell=1}^N \left|t^n_\ell - t_\ell^{\Gamma_\ell(i_\ell)}\right|^{2H_\ell}\right)^{d/2}} \\
& \qquad\quad \times\left[\log^{(N+1)\gamma} n + \log^{(N+1)\gamma}\left(\frac{1}{\sum_{\ell=1}^N \left|t^n_\ell - t_\ell^{\Gamma_\ell(i_\ell)}\right|^{2H_\ell}}\right)\right]\mu(dt^n) \\
& \quad \le c_{5,17}\ \log^{(N+1)\gamma} n.
\end{aligned}
\tag{5.39}
$$

Hence, by integrating $\mathcal{M}_n(\mathbf{t})$ as a function of $t_n$ with respect to $\mu$ on $I_{i_1,\ldots,i_N}$ and using (5.38) and (5.39), we obtain

$$
\begin{aligned}
& \int_{I_{i_1,\ldots,i_N}} \mathcal{M}_n(\mathbf{t})\,\mu(dt^n) \le \frac{c_{5,18}^n\ \log^{(N+1)\gamma} n}{[\det\mathrm{Cov}(B_0^H(t^1),\ldots,B_0^H(t^{n-1}))]^{d/2}} \\
& \times \prod_{k\notin\Theta^n_{i_1,\ldots,i_N}} \left\{\int_{D\ominus D}\int_{\mathbb{R}} \frac{\Lambda^n(u^k_{j_k} y_{j_k}/\widetilde{\sigma}_k(\mathbf{t}))}{p^n(|y|/\sqrt{d})}\ \exp\left(-\frac{(u^k_{j_k})^2}{2}\right) du^k_{j_k}\, dy\right\}^{1/n}.
\end{aligned}
\tag{5.40}
$$

It is important that the right hand side of (5.40) depends on $t^1,\ldots,t^{n-1}$ only and is similar to (5.26).

Summing (5.40) over all the sequences $(i_1, \ldots, i_N) \in \{1, \ldots, n-1\}^N$, we derive that the integral $\mathcal{N}_{\mathbf{j}}$ in (5.24) is bounded by

$$
\begin{aligned}
& c_{5,18}^n \log^{(N+1)\gamma} n \sum_{i_1,\ldots,i_N} \int_{E^{n-1}} \frac{1}{[\det \mathrm{Cov}(B_0^H(t^1), \ldots, B_0^H(t^{n-1}))]^{d/2}} \\
& \times \prod_{k \notin \Theta^n_{i_1,\ldots,i_N}} \bigg\{ \int_{D \ominus D} \int_{\mathbb{R}} \frac{\Lambda^n(u^k_{j_k} y_{j_k} / \widetilde{\sigma}_k(\mathbf{t}))}{p^n(|y|/\sqrt{d})} \\
& \qquad \times \exp\Big( - \frac{(u^k_{j_k})^2}{2} \Big) \, du^k_{j_k} \, dy \bigg\}^{1/n} \mu(dt^{n-1}) \ldots \mu(dt^1).
\end{aligned} \tag{5.41}
$$

Note that, for different sequences $(i_1, \ldots, i_N)$, the index sets $\Theta^n_{i_1,\ldots,i_N}$ may be the same. We say that a set $\Theta^n \subseteq \{1, \ldots, n-1\}$ is *admissible* if it is the set of bad points for some $I_{i_1,\ldots,i_N}$. It can be seen that every admissible set $\Theta^n$ has the following properties:

**(i)** $\#(\Theta^n) \le N$ [recall that there are at most $N$ bad points for each $I_{i_1,\ldots,i_N}$];

**(ii)** Denote by $\chi(\Theta^n)$ the number of sequences $(i_1, \ldots, i_N)$ such that $\Theta^n_{i_1,\ldots,i_N} = \Theta^n$. If $\#(\Theta^n) = p$, then $\chi(\Theta^n) \le c\, n^{N-p}$.

It follows from (i), (ii) and an elementary combinatorics argument that

$$
\sum_{\Theta^n} \chi(\Theta^n) = \sum_{p=1}^{N} \sum_{\#(\Theta^n)=p} \chi(\Theta^n) \le c_{5,19} n^N, \tag{5.42}
$$

where the first summation is taken over all admissible sets $\Theta^n \subseteq \{1, \ldots, n-1\}$.

By regrouping $\Theta^n_{i_1,\ldots,i_N}$ in (5.41), we can rewrite

$$
\begin{aligned}
\mathcal{N}_{\mathbf{j}} \le\ & c_{5,20}^n \log^{(N+1)\gamma} n \sum_{\Theta^n} \chi(\Theta^n) \int_{E^{n-1}} \frac{1}{[\det \mathrm{Cov}(B_0^H(t^1), \ldots, B_0^H(t^{n-1}))]^{d/2}} \\
& \times \prod_{k \notin \Theta^n} \bigg\{ \int_{D \ominus D} \int_{\mathbb{R}} \frac{\Lambda^n(u^k_{j_k} y_{j_k} / \widetilde{\sigma}_k(\mathbf{t}))}{p^n(|y|/\sqrt{d})} \\
& \qquad \times \exp\Big( - \frac{(u^k_{j_k})^2}{2} \Big) \, du^k_{j_k} \, dy \bigg\}^{1/n} \mu(dt^{n-1}) \ldots \mu(dt^1),
\end{aligned} \tag{5.43}
$$

where, as in (5.42), the summation is taken over all admissible sets $\Theta^n \subseteq \{1, \ldots, n-1\}$.

We now carry out the procedure iteratively. In order to simplify the computation, we will make some further reductions:

**(iii)** Since increasing the number of elements in $\Theta^n$ changes the integrals in (5.43) only by a constant factor [recall (5.37) and the fact that we have used $\#(\Theta^n) \le N$ in deriving the first inequality in (5.38)], we may just consider the admissible sets $\Theta^n$ with $\#(\Theta^n) = N$ and

**(iv)** Since $\det \mathrm{Cov}\big(B_1(t^1), \ldots, B_1(t^{n-1})\big)$ is symmetric in $t^1, \ldots, t^{n-1}$, we can further assume $\Theta^n = \{1, \ldots, N\}$.

Based on the above observations we can repeat the preceding argument and integrate $\mu(dt^{n-1})$ [we define $n-N-1$ constants $\beta_k$, $k \in \{N+1, \ldots, n-1\}$ accordingly] and then, in the same way, continue to integrate with respect to $\mu(dt^{n-2}), \ldots, \mu(dt^1)$, respectively. We obtain

$$\begin{aligned} \mathcal{N}_{\mathbf{j}} &\le c_{5,21}^n \ \log^{n(N+1)\gamma} n \sum_{\Theta^n} \cdots \sum_{\Theta^1} \chi(\Theta^n) \cdots \chi(\Theta^1) \\ &\le c_{5,22}^n \, (n!)^N \ \log^{n(N+1)\gamma} n, \end{aligned} \tag{5.44}$$

where last inequality follows from (5.42) and, moreover, the positive constant $c_{5,22}$ is independent of $\mathbf{j}$.

By combining (5.22), (5.23), (5.24) and (5.44) we derive that

$$\mathbb{E} \int_D \int_D \left( \frac{l_\mu(x) - l_\mu(y)}{p(|x-y|/\sqrt{d})} \right)^n dxdy \le c_{5,23}^n \, (n!)^N \ \log^{n(N+1)\gamma} n. \tag{5.45}$$

This finishes the proof of Lemma 11. $\square$

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