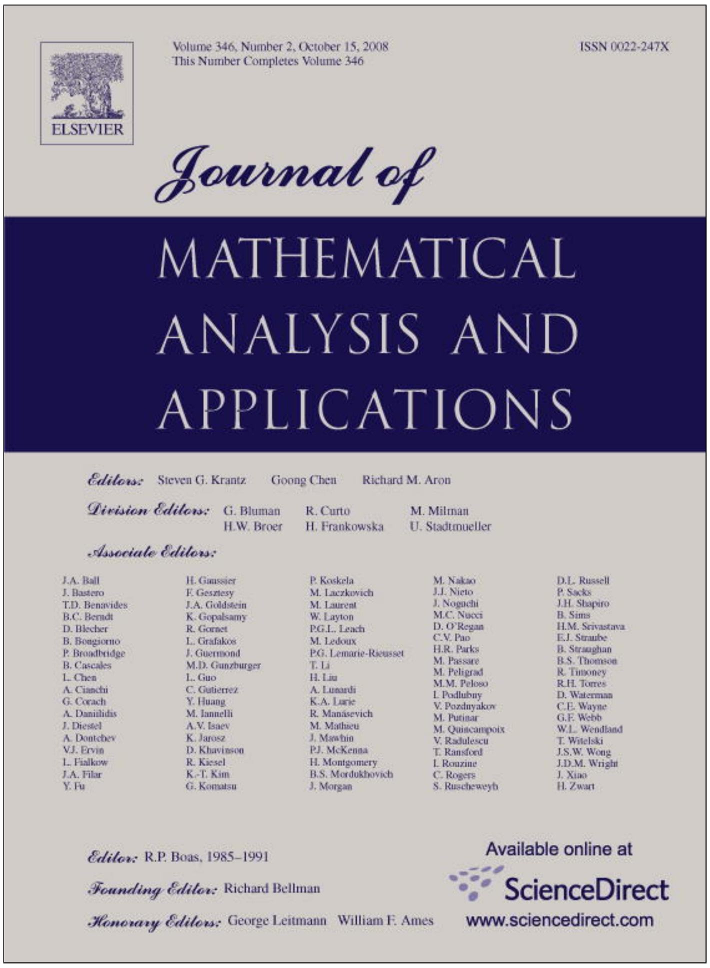
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Large deviations for local time fractional Brownian motion and applications

Mark M. Meerschaert [1], Erkan Nane *, Yimin Xiao [2]

Department of Probability and Statistics, Michigan State University, East Lansing, MI 48823, USA

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A B S T R A C T

Let $W^H = \{W^H(t),\ t \in \mathbb{R}\}$ be a fractional Brownian motion of Hurst index $H \in (0,1)$ with values in $\mathbb{R}$, and let $L = \{L_t,\ t \geqslant 0\}$ be the local time process at zero of a strictly stable Lévy process $X = \{X_t,\ t \geqslant 0\}$ of index $1 < \alpha \leqslant 2$ independent of W^H. The α-stable local time fractional Brownian motion $Z^H = \{Z^H(t),\ t \geqslant 0\}$ is defined by $Z^H(t) = W^H(L_t)$. The process Z^H is self-similar with self-similarity index $H(1-\frac{1}{\alpha})$ and is related to the scaling limit of a continuous time random walk with heavy-tailed waiting times between jumps [P. Becker-Kern, M.M. Meerschaert, H.P. Scheffler, Limit theorems for coupled continuous time random walks, Ann. Probab. 32 (2004) 730–756; M.M. Meerschaert, H.P. Scheffler, Limit theorems for continuous time random walks with infinite mean waiting times, J. Appl. Probab. 41 (2004) 623–638]. However, Z^H does not have stationary increments and is non-Gaussian. In this paper we establish large deviation results for the process Z^H. As applications we derive upper bounds for the uniform modulus of continuity and the laws of the iterated logarithm for Z^H.

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1. Introduction

Self-similar processes arise naturally in limit theorems of random walks and other stochastic processes, and they have been applied to model various phenomena in a wide range of scientific areas including telecommunications, turbulence, image processing and finance. The most important example of self-similar processes is fractional Brownian motion (fBm) which is a centered Gaussian process $W^H = \{W^H(t),\ t \in \mathbb{R}\}$ with $W^H(0) = 0$ and covariance function

$$\mathbb{E}\big(W^H(s)W^H(t)\big) = \frac{1}{2}\big(|s|^{2H} + |t|^{2H} - |s-t|^{2H}\big), \tag{1.1}$$

where $H \in (0,1)$ is a constant. By using (1.1) one can verify that W^H is self-similar with index H (i.e., for all constants $c > 0$, the processes $\{W^H(ct),\ t \in \mathbb{R}\}$ and $\{c^H W^H(t),\ t \in \mathbb{R}\}$ have the same finite-dimensional distributions) and has stationary increments. When $H = 1/2$, W^H is a two-sided Brownian motion, which will be written as W.

Many authors have constructed and investigated various classes of non-Gaussian self-similar processes. See, for example, [39] for information on self-similar stable processes with stationary increments. Burdzy [10,11] introduced the so-called *iterated Brownian motion* (IBM) by replacing the time parameter in W by an independent one-dimensional Brownian motion

* Corresponding author. Address at time of publication: Department of Mathematics and Statistics, Auburn University, 221 Parker Hall, Auburn, AL 36849, USA.

E-mail addresses: mcubed@stt.msu.edu (M.M. Meerschaert), nane@stt.msu.edu (E. Nane), xiao@stt.msu.edu (Y. Xiao).

URLs: http://www.stt.msu.edu/~mcubed/ (M.M. Meerschaert), http://www.stt.msu.edu/~nane (E. Nane), http://www.stt.msu.edu/~xiaoyimi (Y. Xiao).

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$B = \{B_t,\ t \geqslant 0\}$. His work inspired many researchers to explore the connections between IBM (or other iterated processes) and PDEs [1,2,4,35], to establish potential theoretical results [5,19,33,34,36,37] and to study its sample path properties [12,14–16,24,25,28,41].

In this paper, we consider another class of iterated self-similar processes which is related to continuous-time random walks considered in [6,31]. Let $W^H = \{W^H(t),\ t \in \mathbb{R}\}$ be a fractional Brownian motion of Hurst index $H \in (0,1)$ with values in $\mathbb{R}$. Let $X = \{X_t,\ t \geqslant 0\}$ be a real-valued, strictly stable Lévy process of index $1 < \alpha \leqslant 2$. We assume that X is independent of W^H. Let $L = \{L_t,\ t \geqslant 0\}$ be the local time process at zero of X (see Section 2 for its definition). Let $Z^H = \{Z^H(t),\ t \geqslant 0\}$ be a real-valued stochastic process defined by $Z^H(t) = W^H(L_t)$ for all $t \geqslant 0$. This iterated process will be called an *α-stable local time H-fractional Brownian motion* or simply *local time fractional Brownian motion*.

Since the sample functions of W^H and L are a.s. continuous, the local time fractional Brownian motion Z^H also has continuous sample functions. Moreover, by using the facts that W^H is self-similar with index H and L is self-similar with index $1 - 1/\alpha$, one can readily verify that Z^H is self-similar with index $H(1 - 1/\alpha)$. However, Z^H is non-Gaussian, non-Markovian and does not have stationary increments. When $H = 1/2$, we will call $Z^{1/2}$ the local time Brownian motion and denote it by Z for convenience.

The local time Brownian motion Z emerges as the scaling limit of a continuous time random walk with heavy-tailed waiting times between jumps [6,31]. Moreover, local time Brownian motion has a close connection to fractional partial differential equations. Baeumer and Meerschaert [3] showed that the process Z can be applied to provide a solution to the fractional Cauchy problem. More precisely, they proved that, if L_t is the local time at 0 of a symmetric stable Lévy process, then $u(t,x) = \mathbb{E}_x[f(W(L_t))]$ solves the following fractional in time PDE

$$\frac{\partial^\beta}{\partial t^\beta} u(t,x) = \Delta_x u(t,x), \qquad u(0,x) = f(x), \tag{1.2}$$

where $\beta = 1 - 1/\alpha$ and $\partial^\beta g(t)/\partial t^\beta$ is the Caputo fractional derivative in time, which can be defined as the inverse Laplace transform of $s^\beta \tilde{g}(s) - s^{\beta-1} g(0)$, where $\tilde{g}(s) = \int_0^\infty e^{-st} g(t)\,dt$ is the usual Laplace transform. Recently Baeumer, Meerschaert and Nane [4] further established the equivalence of the governing PDEs of $W(L_t)$ and $W(|B_t|)$ when $\alpha = 2$ and $\beta = 1/2$. Here $B = \{B_t,\ t \geqslant 0\}$ is another Brownian motion independent of W and X. The process Z has also appeared in the works of Borodin [8,9], Ikeda and Watanabe [26], Kasahara [27], and Papanicolaou et al. [38]. In [17], Csáki, Földes and Révész studied the Strassen type law of the iterated logarithm of $Z(t) = W(L_t)$ when L_t is the local time at zero of a symmetric stable Lévy process.

For all $H \in (0,1)$ and $\alpha \in (1,2]$, α-stable local time H-fractional Brownian motions form a new class of self-similar processes. It is natural to expect that they arise as scaling limit of continuous-time *correlated* random walks with heavy-tailed waiting times and, as such, they are potentially useful as stochastic models. Hence it is of interest in both theory and applications to investigate their probabilistic and analytic properties. Due to the non-Gaussian and non-Markovian nature of local time fractional Brownian motions, the existing theories on Markov and/or Gaussian processes cannot be applied to them directly and some new tools will have to be developed. The literature on iterated Brownian motion mentioned above provides an instructive guideline for studying local time fractional Brownian motions.

The objective of the present paper is to establish large deviation results for the local time fractional Brownian motion Z^H and apply them to study regularity properties of the sample paths of Z^H. We will consider the interesting problem of determining the domain of attraction of Z^H in a subsequent paper.

The following Theorems 1.1 and 1.2 are our main results.

Theorem 1.1. *Let $Z^H = \{Z^H(t),\ t \geqslant 0\}$ be an α-stable local time H-fractional Brownian motion with values in $\mathbb{R}$ and $2H < \alpha$. Then for every Borel set $D \subseteq \mathbb{R}$,*

$$\limsup_{t\to\infty} t^{-\frac{2H(\alpha-1)}{\alpha-2H}} \log \mathbb{P}\left\{t^{-\frac{2H(\alpha-1)}{\alpha-2H}} Z^H(t) \in D\right\} \leqslant -\inf_{x\in\overline{D}} \Lambda_1^*(x) \tag{1.3}$$

and

$$\liminf_{t\to\infty} t^{-\frac{2H(\alpha-1)}{\alpha-2H}} \log \mathbb{P}\left\{t^{-\frac{2H(\alpha-1)}{\alpha-2H}} Z^H(t) \in D\right\} \geqslant -\inf_{x\in D^\circ} \Lambda_1^*(x), \tag{1.4}$$

where $\overline{D}$ and D° denote respectively the closure and interior of D and

$$\Lambda_1^*(x) = \frac{\alpha+2H}{2\alpha}\left(\frac{\alpha-2H}{2\alpha B_1}\right)^{\frac{\alpha-2H}{\alpha+2H}} x^{\frac{2\alpha}{\alpha+2H}}, \quad \forall x \in \mathbb{R}. \tag{1.5}$$

In the above, $B_1 = B_1(H,\alpha,\chi,\nu)$ is the positive constant defined by

$$B_1 = \frac{\alpha-2H}{2\alpha}\left(\frac{HA_1^\alpha}{(1-\frac{1}{\alpha})^{\alpha-1}}\right)^{\frac{2H}{\alpha-2H}}, \tag{1.6}$$

where A_1 is the constant given by

$$A_1 = \frac{\Gamma(1-\frac{1}{\alpha})\Gamma(\frac{1}{\alpha})\chi^{1/\alpha}\cos(\frac{1}{\alpha}\arctan(\nu\tan(\frac{\pi\alpha}{2})))}{\pi\alpha[1+(\nu\tan(\frac{\pi\alpha}{2}))^2]^{1/(2\alpha)}} \tag{1.7}$$

and $\nu\in[-1,1]$ and $\chi>0$ are the parameters of the stable Lévy process X defined in (2.1).

In (1.5) and the sequel, for any $\gamma>0$ and $x\in\mathbb{R}$, the term $x^{2\gamma}$ is defined as $(x^2)^\gamma$. Since $2H<\alpha$, one can see that the function Λ_1^* in (1.5) is even, convex and differentiable on $\mathbb{R}$.

In the terminology of [20], Theorem 1.1 states that the pair $(t^{-\frac{2H(\alpha-1)}{\alpha-2H}}Z^H(t), t^{\frac{2H(\alpha-1)}{\alpha-2H}})$ satisfies a large deviation principle with good rate function Λ_1^*. When $H=1/2$, it yields a large deviation result for the local time Brownian motion and, moreover, the constants B_1 in (1.6) can be simplified.

Letting $D=[x,\infty)$, we derive from Theorem 1.1 and the self-similarity of Z^H the asymptotic tail probability $\mathbb{P}\{Z^H(1)\geqslant x\}$ as $x\to\infty$. The following theorem is more general because it holds for all $H\in(0,1)$ and $\alpha\in(1,2]$.

Theorem 1.2. *Let $Z^H=\{Z^H(t),\ t\geqslant 0\}$ be an α-stable local time H-fractional Brownian motion with values in $\mathbb{R}$. Then for any $0\leqslant a\leqslant b<\infty$,*

$$\lim_{x\to\infty}\frac{\log\mathbb{P}\{|Z^H(b)-Z^H(a)|>x\}}{x^{\frac{2\alpha}{\alpha+2H}}} = -B_2, \tag{1.8}$$

where $B_2=B_2(H,\alpha,\chi,\nu)$ is the positive constant defined by

$$B_2 = \frac{\alpha+2H}{2\alpha}\left(\frac{HA_1^\alpha}{(1-\frac{1}{\alpha})^{\alpha-1}}\right)^{-\frac{2H}{\alpha+2H}}(b-a)^{-\frac{2H(\alpha-1)}{\alpha+2H}}. \tag{1.9}$$

In order to prove Theorems 1.1 and 1.2, we first study the analytic properties of the moment generating functions of $Z^H(t)$ and $|Z^H(b)-Z^H(a)|$. This is done by calculating the moments of $Z^H(t)$ and $|Z^H(b)-Z^H(a)|$ for $0\leqslant a\leqslant b$ directly and by using a theorem of Valiron [40]. Then Theorems 1.1 and 1.2 follow respectively from the Gärtner–Ellis theorem (cf. [20]) and a result of Davies [18].

The rest of the paper is organized as follows. In Section 2, we derive sharp estimates on the moments of the local time L_t of X, and the moments of $Z^H(b)-Z^H(a)$. These estimates are applied in Section 3 to study the analyticity of the moment generating functions of Z^H, and to derive large time behavior of the logarithmic moment generating functions $\log\mathbb{E}[\exp(\theta Z^H(t))]$ and $\log\mathbb{E}[\exp(t|Z^H(b)-Z^H(a)|^\beta)]$ for suitably chosen $\beta>0$. In Section 4, we prove Theorems 1.1 and 1.2. We will also establish similar tail estimates for the maximum $\max_{t\in[a,b]}|Z^H(t)-Z^H(a)|$. In Section 5, by combining the large deviation result with the methods in [13, Theorem 3.1], we establish local and uniform moduli of continuity for Z^H. We also obtain an upper bound in the law of the iterated logarithm for Z^H.

2. Moment estimates

A Lévy process $X=\{X_t,\ t\geqslant 0\}$ with values in $\mathbb{R}$ is called strictly stable of index $\alpha\in(0,2]$ if its characteristic function is given by

$$\mathbb{E}\big[\exp(i\xi X_t)\big] = \exp\left(-t|\xi|^\alpha\frac{1+i\nu\,\mathrm{sgn}(\xi)\tan(\frac{\pi\alpha}{2})}{\chi}\right), \tag{2.1}$$

where $-1\leqslant\nu\leqslant 1$ and $\chi>0$ are constants. In the terminology of [39, Definition 1.1.6], ν and $\chi^{-1/\alpha}$ are respectively the skewness and scale parameters of the stable random variable X_1. When $\alpha=2$ and $\chi=2$, X is Brownian motion. In general, many properties of stable Lévy processes can be characterized by the parameters α,ν and χ. For a systematic account on Lévy processes we refer to [7].

For any Borel set $I\subseteq\mathbb{R}$, the occupation measure of X on I is defined by

$$\mu_I(A)=\lambda_1\{t\in I:\ X_t\in A\} \tag{2.2}$$

for all Borel sets $A\subseteq\mathbb{R}$, where λ_1 is the one-dimensional Lebesgue measure. If μ_I is absolutely continuous with respect to the Lebesgue measure λ_1 on $\mathbb{R}$, we say that X has a local time on I and define its local time $L(x,I)$ to be the Radon–Nikodým derivative of μ_I with respect to λ_1, i.e.,

$$L(x,I)=\frac{d\mu_I}{d\lambda_1}(x),\quad \forall x\in\mathbb{R}.$$

In the above, x is the so-called *space variable*, and I is the *time variable* of the local time. If $I=[0,t]$, we will write $L(x,I)$ as $L(x,t)$. Moreover, if $x=0$ then we will simply write $L(0,t)$ as L_t.

By using a monotone class argument, one can verify that $L(x,I)$ satisfies the following *occupation density formula*: For every measurable function $f:\mathbb{R}\to\mathbb{R}_+$,

$$\int_I f\big(X(t)\big)\,dt = \int_{\mathbb{R}} f(x)L(x,I)\,dx. \tag{2.3}$$

It is well known (see, e.g. [7]) that a strictly stable Lévy process X has a local time if and only if $\alpha \in (1,2]$. In the later case, $L(x,t)$ has a version that is continuous in (x,t). Throughout this paper, we tacitly work with such a version so that the local time process $L = \{L_t,\ t \geqslant 0\}$ has continuous sample paths.

It follows from (2.3) and the self-similarity of X that $L(x,t)$ has the following scaling property: For every constant $c > 0$, $c^{1-1/\alpha} L(c^{-1/\alpha}x, c^{-1}t)$ is a version of $L(x,t)$. In particular, by letting $x = 0$ we see that L_t is self-similar with index $1 - \frac{1}{\alpha}$.

For the purpose of the present paper, it will be convenient to express the local time $L(x,t)$ as the inverse Fourier transform of $\widehat{\mu}(u,t) := \widehat{\mu_{[0,t]}}(u)$, namely

$$L(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} \exp(-iux)\widehat{\mu}(u,t)\,du = \frac{1}{2\pi}\int_0^t \int_{\mathbb{R}} \exp\big(-iux + iuX(s)\big)\,du\,ds. \tag{2.4}$$

This formal expression can be justified rigorously (see [21]). Moreover, it follows from (25.2) and (25.7) in [21] that for all $x \in \mathbb{R}$, $I \in \mathcal{B}(\mathbb{R}_+)$ and all integers $n \geqslant 1$, we have

$$\mathbb{E}\big[L(x,I)\big]^n = (2\pi)^{-n} \int_{I^n}\int_{\mathbb{R}^n} \exp\left(-i\sum_{j=1}^n u_j x\right) \mathbb{E}\exp\left(i\sum_{j=1}^n u_j X(t_j)\right) d\bar{u}\,d\bar{t}. \tag{2.5}$$

In the above and in the sequel, $d\bar{u} = du_1 \cdots du_n$ and $d\bar{t} = dt_1 \cdots dt_n$.

We start with the following moment estimates for the local time of X.

Lemma 2.1. *Let $L = \{L_t,\ t \geqslant 0\}$ be the local time at zero of a strictly stable Lévy process $X = \{X_t,\ t \geqslant 0\}$ with values in $\mathbb{R}$ and index $1 < \alpha \leqslant 2$. Then for all $0 < a \leqslant b < \infty$ and all integers $n \geqslant 1$,*

$$\left(\frac{b-a}{b}\right)^{1/\alpha} \frac{A_1^n n!(b-a)^{n(1-1/\alpha)}}{\Gamma(1-\frac{1}{\alpha})\Gamma(1+\frac{1}{\alpha}+n(1-\frac{1}{\alpha}))} \leqslant \mathbb{E}\big[|L_b - L_a|^n\big] \leqslant \left(\frac{b-a}{b}\right)^{1/\alpha} \frac{A_1^n n!(b-a)^{n(1-1/\alpha)}}{\Gamma(1+n(1-\frac{1}{\alpha}))}, \tag{2.6}$$

where $A_1 > 0$ is the constant defined by (1.7). In the case $a = 0$, we have the equality

$$\mathbb{E}\big[|L_b|^n\big] = \frac{A_1^n n!}{\Gamma(1+n(1-\frac{1}{\alpha}))} b^{n(1-1/\alpha)}. \tag{2.7}$$

Proof. Applying (2.5) with $x = 0$ and $I = (a,b]$ and making a change of variables $s_j = t_j - a$ $(j = 1,\ldots,n)$, we obtain

$$\mathbb{E}\big[|L_b - L_b|^n\big] = (2\pi)^{-n} \int_{(a,b]^n}\int_{\mathbb{R}^n} \mathbb{E}\exp\left(i\sum_{j=1}^n u_j X(t_j)\right) d\bar{u}\,d\bar{t} = \frac{n!}{(2\pi)^n}\int_{S_n}\int_{\mathbb{R}^n} \mathbb{E}\exp\left(i\sum_{j=1}^n u_j X(s_j + a)\right) d\bar{u}\,d\bar{s}, \tag{2.8}$$

where

$$S_n = \big\{(s_1,\ldots,s_n):\ 0 \leqslant s_1 \leqslant s_2 \leqslant \cdots \leqslant s_n \leqslant b-a\big\}.$$

Let $v_j = \sum_{l=j}^n u_l$ $(j = 1,\ldots,n)$, then the last sum in (2.8) can be written as

$$\sum_{j=1}^n u_j X(s_j + a) = \sum_{j=1}^n v_j\big(X(s_j + a) - X(s_{j-1} + a)\big), \tag{2.9}$$

where $s_0 := -a$ so that $s_0 + a = 0$. Denote

$$\phi(\xi) = |\xi|^\alpha \frac{1 + i\nu\,\mathrm{sgn}(\xi)\tan(\frac{\pi\alpha}{2})}{\chi}.$$

Since the process X has stationary and independent increments, we have

$$\mathbb{E}\exp\left(i\sum_{j=1}^n v_j\big(X(s_j+a) - X(s_{j-1}+a)\big)\right) = \exp\left(-\sum_{j=2}^n (s_j - s_{j-1})\phi(v_j)\right)\exp\big(-(s_1+a)\phi(v_1)\big). \tag{2.10}$$

It follows from (2.8)–(2.10) and a change of variables that

$$\mathbb{E}\big[|L_b - L_a|^n\big] = \frac{n!}{(2\pi)^n}\int_{S_n}\int_{\mathbb{R}^n} \exp\left(-\sum_{j=2}^n (s_j - s_{j-1})\phi(v_j)\right)\exp\big(-(s_1+a)\phi(v_1)\big)\,d\bar{v}\,d\bar{s}, \tag{2.11}$$

where $d\bar{v} = dv_1 \cdots dv_n$.

We know from symmetry that for any $s>0$:

$$\int_{\mathbb{R}} \exp\bigl(-s\phi(v)\bigr)\,dv = 2\int_0^\infty \exp\left(-\frac{sv^\alpha}{\chi}\right)\cos\left(-\frac{sv^\alpha \nu}{\chi}\tan\left(\frac{\alpha\pi}{2}\right)\right)dv.$$

By a change of variable $z=v^\alpha$, this equals

$$\frac{2}{\alpha}\int_0^\infty z^{1/\alpha-1}\exp\left(-\frac{sz}{\chi}\right)\cos\left(-\frac{sz\nu}{\chi}\tan\left(\frac{\alpha\pi}{2}\right)\right)dz,$$

which by Eq. 3.944(6) in Gradshteyn and Ryzhik [22] equals

$$\frac{2\chi^{1/\alpha}\Gamma(\frac{1}{\alpha})}{\alpha s^{1/\alpha}(1+[\nu\tan(\frac{\alpha\pi}{2})]^2)^{1/(2\alpha)}}\cos\left(\frac{1}{\alpha}\arctan\left(\nu\tan\left(\frac{\alpha\pi}{2}\right)\right)\right).$$

Combining the above with (2.11) we obtain

$$\mathbb{E}\bigl[|L_b-L_a|^n\bigr] = n!C(\alpha)^n\int_{S_n}(s_1+a)^{-1/\alpha}\prod_{j=2}^n (s_j-s_{j-1})^{-1/\alpha}\,d\bar{s}, \tag{2.12}$$

where $C(\alpha)$ is the constant given by

$$C(\alpha) = \frac{\chi^{1/\alpha}\Gamma(\frac{1}{\alpha})}{\pi\alpha(1+[\nu\tan(\frac{\alpha\pi}{2})]^2)^{1/(2\alpha)}}\cos\left(\frac{1}{\alpha}\arctan\left(\nu\tan\left(\frac{\alpha\pi}{2}\right)\right)\right). \tag{2.13}$$

We denote the multiple integral in (2.12) by J_n. When $a=0$, it can be evaluated in terms of the Gamma function. When $a>0$ the same induction method can still be applied. We include a proof for completeness.

First we integrate over $s_n \in [s_{n-1}, b-a]$ to get

$$\int_{s_{n-1}}^{b-a}(s_n-s_{n-1})^{-1/\alpha}\,ds_n = \frac{(b-a-s_{n-1})^{1-1/\alpha}}{1-\frac{1}{\alpha}}.$$

Next we integrate over $s_{n-1}\in[s_{n-2}, b-a]$. By changing variables twice $v=s_{n-1}-s_{n-2}$ and $t=v/(b-a-s_{n-2})$, we obtain

$$\begin{aligned}
&\int_{s_{n-2}}^{b-a}(s_{n-1}-s_{n-2})^{-1/\alpha}\frac{(b-a-s_{n-1})^{1-1/\alpha}}{1-\frac{1}{\alpha}}\,ds_{n-1}\\
&\quad=\frac{1}{1-\frac{1}{\alpha}}\int_0^{b-a-s_{n-2}} v^{-1/\alpha}(b-a-s_{n-2}-v)^{1-1/\alpha}\,dv = \frac{(b-a-s_{n-2})^{2(1-1/\alpha)}}{1-\frac{1}{\alpha}}\int_0^1 t^{(1-1/\alpha)-1}(1-t)^{(2-1/\alpha)-1}\,dt\\
&\quad=\frac{(b-a-s_{n-2})^{2(1-1/\alpha)}}{1-\frac{1}{\alpha}}\cdot\frac{\Gamma(1-\frac{1}{\alpha})\Gamma(2-\frac{1}{\alpha})}{\Gamma(1+2(1-\frac{1}{\alpha}))} = \frac{\Gamma(1-\frac{1}{\alpha})^2}{\Gamma(1+2(1-\frac{1}{\alpha}))}(b-a-s_{n-2})^{2(1-1/\alpha)}.
\end{aligned}$$

Iterating this procedure, we derive

$$\begin{aligned}
J_n &= \frac{\Gamma(1-\frac{1}{\alpha})^{n-1}}{\Gamma(1+(n-1)(1-\frac{1}{\alpha}))}\int_0^{b-a}(b-a-s_1)^{(n-1)(1-1/\alpha)}(s_1+a)^{-1/\alpha}\,ds_1\\
&= \frac{\Gamma(1-\frac{1}{\alpha})^{n-1}}{\Gamma(1+(n-1)(1-\frac{1}{\alpha}))}b^{n(1-1/\alpha)}\int_0^{(b-a)/b} v^{(n-1)(1-1/\alpha)}(1-v)^{-1/\alpha}\,dv,
\end{aligned} \tag{2.14}$$

where the second equality follows from change of variables.

If $a=0$, then the last integral equals $\mathrm{Beta}(1-\frac{1}{\alpha}, 1+(n-1)(1-\frac{1}{\alpha}))$, where Beta denotes the Beta function. This and (2.14) yield

$$J_n = \frac{\Gamma(1-\frac{1}{\alpha})^n}{\Gamma(1+n(1-\frac{1}{\alpha}))}b^{n(1-1/\alpha)}. \tag{2.15}$$

It follows from (2.12) and (2.15) that for $a = 0$,

$$\mathbb{E}\big[|L_b - L_a|^n\big] = \frac{\Gamma(1-\frac{1}{\alpha})^n C(\alpha)^n n!}{\Gamma(1+n(1-\frac{1}{\alpha}))} b^{n(1-1/\alpha)}. \tag{2.16}$$

By (1.7) and (2.13), we have $A_1 = \Gamma(1-\frac{1}{\alpha})C(\alpha)$. Hence the desired result (2.7) follows from (2.16).

If $a > 0$, then last integral in (2.14) is an incomplete Beta function [22, 8.384(1) and 9.100]. Write

$$f(n,a,b) = \int_0^{(b-a)/b} v^{(n-1)(1-1/\alpha)}(1-v)^{-1/\alpha}\,dv = \left(\frac{b-a}{b}\right)^{n(1-1/\alpha)+1/\alpha} \int_0^1 v^{(n-1)(1-1/\alpha)}\left(1-\frac{b-a}{b}v\right)^{-1/\alpha} dv. \tag{2.17}$$

Then one can verify that

$$\begin{aligned} f(n,a,b) &\leqslant \left(\frac{b-a}{b}\right)^{n(1-1/\alpha)+1/\alpha} \int_0^1 v^{(n-1)(1-1/\alpha)}(1-v)^{-1/\alpha}\,dv \\ &= \left(\frac{b-a}{b}\right)^{n(1-1/\alpha)+1/\alpha} \frac{\Gamma(1-\frac{1}{\alpha})\Gamma(1+(n-1)(1-\frac{1}{\alpha}))}{\Gamma(1+n(1-\frac{1}{\alpha}))} \end{aligned} \tag{2.18}$$

and

$$f(n,a,b) \geqslant \left(\frac{b-a}{b}\right)^{n(1-1/\alpha)+1/\alpha} \int_0^1 v^{(n-1)(1-1/\alpha)}\,dv = \left(\frac{b-a}{b}\right)^{n(1-1/\alpha)+1/\alpha} \frac{1}{1+(n-1)(1-\frac{1}{\alpha})}. \tag{2.19}$$

It follows from (2.12), (2.14) and (2.18) that

$$\mathbb{E}\big[|L_b - L_a|^n\big] \leqslant \left(\frac{b-a}{b}\right)^{1/\alpha} \frac{\Gamma(1-\frac{1}{\alpha})^n C(\alpha)^n n!}{\Gamma(1+n(1-\frac{1}{\alpha}))} (b-a)^{n(1-1/\alpha)}. \tag{2.20}$$

Recalling $A_1 = \Gamma(1-\frac{1}{\alpha})C(\alpha)$, we see that (2.20) gives the upper bound in (2.6).

Similarly, we use (2.12), (2.14) and (2.19) to derive

$$\mathbb{E}\big[|L_b - L_a|^n\big] \geqslant \left(\frac{b-a}{b}\right)^{1/\alpha} \frac{\Gamma(1-\frac{1}{\alpha})^{n-1} C(\alpha)^n n!}{\Gamma(1+\frac{1}{\alpha}+n(1-\frac{1}{\alpha}))} (b-a)^{n(1-1/\alpha)}, \tag{2.21}$$

which yields the lower bound in (2.6). This proves Lemma 2.1. □

Now we consider the moments of the increment $Z^H(b) - Z^H(a)$. As we will see in Remark 4.1, the following lemma is sufficient for proving (1.8) in Theorem 1.2.

Lemma 2.2. *Let $W^H = \{W^H(t),\ t \in \mathbb{R}\}$ be a fractional Brownian motion of index H in $\mathbb{R}$, and L_t be the local time at zero of a strictly stable process $X = \{X_t,\ t \geqslant 0\}$ of index $1 < \alpha \leqslant 2$ independent of W^H. Then for all $0 < a \leqslant b < \infty$ and all positive integers n,*

$$C_1(n)(b-a)^{n(1-1/\alpha)} \leqslant \mathbb{E}\big(\big|W^H(L_b) - W^H(L_a)\big|^{n/H}\big) \leqslant C_2(n)(b-a)^{n(1-1/\alpha)}, \tag{2.22}$$

where

$$C_1(n) = \frac{1}{\sqrt{\pi}\,\Gamma(1-\frac{1}{\alpha})} \left(\frac{b-a}{b}\right)^{1/\alpha} \big(2^{1/(2H)}A_1\big)^n \frac{n!\,\Gamma(\frac{n}{2H}+\frac{1}{2})}{\Gamma(1+\frac{1}{\alpha}+n(1-\frac{1}{\alpha}))} \tag{2.23}$$

and

$$C_2(n) = \frac{1}{\sqrt{\pi}} \left(\frac{b-a}{b}\right)^{1/\alpha} \big(2^{1/(2H)}A_1\big)^n \frac{n!\,\Gamma(\frac{n}{2H}+\frac{1}{2})}{\Gamma(1+n(1-\frac{1}{\alpha}))}. \tag{2.24}$$

In the above $A_1 > 0$ is the constant defined by (1.7). Moreover, when $a = 0$ we have the equality

$$\mathbb{E}\big(\big|W^H(L_b)\big|^{n/H}\big) = \frac{1}{\sqrt{\pi}} \big(2^{1/(2H)}A_1\big)^n \frac{n!\,\Gamma(\frac{n}{2H}+\frac{1}{2})}{\Gamma(1+n(1-\frac{1}{\alpha}))} b^{n(1-1/\alpha)}. \tag{2.25}$$

Proof. Since L_t is a non-decreasing process, W^H is H-self-similar with stationary increments, and these two processes are independent, we use a conditioning argument to derive

$$\mathbb{E}\big(\big|W^H(L_b)-W^H(L_a)\big|^{n/H}\big)=\mathbb{E}\big(\big|W^H(1)\big|^{n/H}\big)\mathbb{E}\big(|L_b-L_a|^n\big). \tag{2.26}$$

On the other hand, since $W^H(1)$ has a standard normal density, one can use a change of variables to verify that

$$\mathbb{E}\big(\big|W^H(1)\big|^{n/H}\big)=\frac{1}{\sqrt{\pi}}2^{n/(2H)}\Gamma\left(\frac{n}{2H}+\frac{1}{2}\right). \tag{2.27}$$

Combining (2.26), (2.27) with (2.6) and (2.7) proves (2.22) and (2.25). □

3. Analytic results: Exponential integrability

In this section we study the exponential integrability of the random variable $Z^H(t)$ and some analytic properties of its logarithmic moment generating function. Our main results of this section are Theorems 3.1 and 3.4, which are the main ingredients for proving Theorems 1.1 and 1.2.

Theorem 3.1. *Let $Z^H=\{Z^H(t),\ t\geqslant 0\}$ be an α-stable local time H-fractional Brownian motion with values in $\mathbb{R}$ and $2H<\alpha$. Then for every $\theta\in\mathbb{R}$,*

$$\lim_{t\to\infty} t^{-\frac{2H(\alpha-1)}{\alpha-2H}}\log\mathbb{E}\exp\big(\theta Z^H(t)\big)=\Lambda_1(\theta), \tag{3.1}$$

where Λ_1 is the function on $\mathbb{R}$ defined by $\Lambda_1(\theta)=B_1\theta^{\frac{2\alpha}{\alpha-2H}}$ for all $\theta\in\mathbb{R}$. Recall from (1.6) *and* (1.7) *that the constants $B_1=B_1(H,\alpha,\chi,\nu)$ and A_1 are defined as*

$$B_1=\frac{\alpha-2H}{2\alpha}\left(\frac{HA_1^{\alpha}}{(1-\frac{1}{\alpha})^{\alpha-1}}\right)^{\frac{2H}{\alpha-2H}} \tag{3.2}$$

and

$$A_1=\frac{\Gamma(1-\frac{1}{\alpha})\Gamma(\frac{1}{\alpha})\chi^{1/\alpha}\cos(\frac{1}{\alpha}\arctan(\nu\tan(\frac{\pi\alpha}{2})))}{\pi\alpha[1+(\nu\tan(\frac{\pi\alpha}{2}))^2]^{1/(2\alpha)}}, \tag{3.3}$$

where $\nu\in[-1,1]$ and $\chi>0$ are the constants defined in (2.1)*.*

Note that for $2H<\alpha$ the above function $\Lambda_1(\cdot)$ is even, convex and differentiable on $\mathbb{R}$, and the function $\Lambda_1^*(\cdot)$ defined by (1.5) is the Fenchel–Legendre transform of Λ_1, that is, $\Lambda_1^*(x)=\sup_{\theta\in\mathbb{R}}(\theta x-\Lambda_1(\theta))$ for all $x\in\mathbb{R}$.

The proof of Theorem 3.1 relies on explicit calculation of the moments of $Z^H(t)$ and the following theorem in Valiron [40, p. 44].

Lemma 3.2. *Let $f(z)=\sum_{p=0}^{\infty}c_pz^p$ be an entire function such that $c_p\neq 0$ for infinitely many p's. For any $r>0$, let $M(r)=\sup_{|z|=r}|f(z)|$. Then a necessary and sufficient condition for*

$$\lim_{r\to\infty}\frac{\log M(r)}{r^{\rho}}=B \tag{3.4}$$

is that, for all values of ε and all sufficiently large integers p, we have

$$\frac{1}{\rho e}pc_p^{\rho/p}\leqslant B+\varepsilon, \tag{3.5}$$

and there exists a sequence of integers p_n, such that

$$\lim_{n\to\infty}\frac{p_{n+1}}{p_n}=1, \tag{3.6}$$

for which

$$\lim_{n\to\infty}\frac{1}{\rho e}p_nc_{p_n}^{\rho/p_n}=B. \tag{3.7}$$

Proof of Theorem 3.1. Similar to the proof of Lemma 2.2, we apply a conditioning argument and the formula for the moment generating function of a Gaussian random variable to derive that for all $\theta \in \mathbb{R}$,

$$\mathbb{E}\exp\big(\theta Z^H(t)\big) = \mathbb{E}\exp\left(\frac{\theta^2}{2}t^{2H(1-1/\alpha)}L_1^{2H}\right). \tag{3.8}$$

In order to prove (3.1), we show that, for $2H < \alpha$, the function $f(z) = \mathbb{E}\exp(zL_1^{2H})$ is an entire function and the coefficients of its Taylor expansion verify the conditions of Lemma 3.2.

Let us first consider the Taylor series

$$M_1(r) = \sum_{n=0}^{\infty} \frac{\mathbb{E}(L_1^{2Hn})}{n!} r^n. \tag{3.9}$$

We will make use of the following consequence of Jensen's inequality: For any constant $\gamma \geqslant 1$ and nonnegative random variable Δ,

$$\big(\mathbb{E}\big(\Delta^{\lfloor\gamma\rfloor}\big)\big)^{\gamma/\lfloor\gamma\rfloor} \leqslant \mathbb{E}\big(\Delta^{\gamma}\big) \leqslant \big(\mathbb{E}\big(\Delta^{\lfloor\gamma\rfloor+1}\big)\big)^{\gamma/(\lfloor\gamma\rfloor+1)}. \tag{3.10}$$

Here and in the sequel, $\lfloor\gamma\rfloor$ denotes the largest integer $\leqslant \gamma$.

It follows from (3.10) with $\Delta = L_1$ and $\gamma = 2Hn$, (2.7) in Lemma 2.1 and Stirling's formula that

$$\frac{\mathbb{E}(L_1^{2Hn})}{n!} \asymp \left(A_1 \frac{(2H)^{1/\alpha}}{(1-\frac{1}{\alpha})^{1-1/\alpha}}\right)^{2Hn} e^{n(1-\frac{2H}{\alpha})} n^{-n(1-\frac{2H}{\alpha})}, \tag{3.11}$$

where A_1 is the constant in (1.7). In the above, $x_n \asymp y_n$ means that, for all n large enough, x_n/y_n is bounded from below and above by constant multiple of $n^{-\eta}$. Here η is a constant depending on H and α only. The omitted factors have no influence on the limit in (3.12) below.

By (3.11), we see that the Taylor series in (3.9) represents an analytic function on $\mathbb{R}$ if and only if $2H < \alpha$. In the latter case, we choose $\rho_1 = \frac{\alpha}{\alpha - 2H}$ and derive

$$\lim_{n\to\infty} \frac{1}{\rho_1 e} n \left(\frac{\mathbb{E}(L_1^{2Hn})}{n!}\right)^{\rho_1/n} = \frac{1}{\rho_1}\left(A_1 \frac{(2H)^{1/\alpha}}{(1-\frac{1}{\alpha})^{1-1/\alpha}}\right)^{2H\rho_1}. \tag{3.12}$$

Hence, Lemma 3.2 implies that

$$\lim_{r\to\infty} \frac{\log M_1(r)}{r^{\rho_1}} = \frac{1}{\rho_1}\left(A_1 \frac{(2H)^{1/\alpha}}{(1-\frac{1}{\alpha})^{1-1/\alpha}}\right)^{2H\rho_1}. \tag{3.13}$$

It follows from (3.8) that $\mathbb{E}\exp(\theta Z^H(t)) = M_1(\frac{\theta^2}{2}t^{2H(1-1/\alpha)})$. Hence (3.1) follows from (3.13) and a simple change of variables. $\square$

The proof of Theorem 3.1 shows that, if $2H > \alpha$, then $\mathbb{E}(e^{\theta Z^H(t)}) = \infty$ for all $\theta > 0$. Hence we cannot prove a large deviation principle for $Z^H(t)$ by applying the Gärtner–Ellis theorem. However, for studying the tail probability of $Z^H(b) - Z^H(a)$, it is sufficient to consider the exponential integrability of $|Z^H(b) - Z^H(a)|^\beta$ for appropriately chosen $\beta > 0$.

Proposition 3.3. *Let $0 \leqslant a < b < \infty$ be given constants. For any $\beta > 0$ and $t \in \mathbb{R}$, let $g_\beta(t) = \mathbb{E}(e^{t|W^H(L_b) - W^H(L_a)|^\beta})$. The following statements hold*:

(i) *If $0 < \beta < \frac{2\alpha}{2H+\alpha}$, then the function $g_\beta(t)$ is analytic on $\mathbb{R}$.*
(ii) *If $\beta = \frac{2\alpha}{2H+\alpha}$, then $g_\beta(t)$ is analytic in $(-\infty, \delta_0)$ for some $\delta_0 > 0$.*
(iii) *If $\beta > \frac{2\alpha}{2H+\alpha}$, then $g_\beta(t) = \infty$ for all $t > 0$.*

Proof. Let us consider the Taylor series

$$\sum_{n=0}^{\infty} \frac{\mathbb{E}(|W^H(L_b) - W^H(L_a)|^{\beta n})}{n!} t^n.$$

As in the proof of Lemma 2.2, we have

$$\mathbb{E}\big(\big|W^H(L_b) - W^H(L_a)\big|^{\beta n}\big) = \mathbb{E}\big(\big|W^H(1)\big|^{\beta n}\big)\mathbb{E}\big(|L_b - L_a|^{\beta H n}\big) = \frac{1}{\sqrt{\pi}} 2^{(\beta n)/2} \Gamma\left(\frac{\beta n}{2} + \frac{1}{2}\right)\mathbb{E}\big(|L_b - L_a|^{\beta H n}\big). \tag{3.14}$$

Applying (3.10) to $\Delta = |L_b - L_a|$ and $\gamma = \beta H n$ and using (3.14), the moment estimates in Lemma 2.1 and Stirling's formula, we derive

$$\frac{\mathbb{E}(|W^H(L_b) - W^H(L_a)|^{\beta n})}{n!} \asymp A_2^n e^{n(1-\frac{\beta}{2}-\frac{\beta H}{\alpha})} n^{-n(1-\frac{\beta}{2}-\frac{\beta H}{\alpha})}, \tag{3.15}$$

where A_2 is a constant which can be expressed explicitly in terms of α, β, H, A_1 and $b-a$. It can be verified that (3.15) implies the conclusions in Proposition 3.3. □

It follows from Proposition 3.3 that, for $\beta > 0$, $g_\beta(z)$ ($z \in \mathbb{C}$) is an entire function if and only if $\beta < \frac{2\alpha}{2H+\alpha}$. In this case, Theorem 3.4 further proves that $g_\beta(z)$ is of *very regular growth* in the sense of Valiron [40].

Theorem 3.4. *Let W^H be a fractional Brownian motion in $\mathbb{R}$, and let L_t be the local time at zero of a strictly stable process X_t of index $1 < \alpha \leqslant 2$ independent of W^H. Then for all $0 < \beta < \frac{2\alpha}{2H+\alpha}$ and $0 \leqslant a \leqslant b < \infty$,*

$$\lim_{t\to\infty} \frac{\log \mathbb{E}[\exp(t|W^H(L_b) - W^H(L_a)|^\beta)]}{t^\rho} = B_3, \tag{3.16}$$

where $\rho = \frac{2\alpha}{2\alpha-\alpha\beta-2H\beta}$ and $B_3 = B_3(H, \alpha, \nu, \chi, \beta)$ is the constant given by

$$B_3 = \frac{1}{\rho} A_1^{\beta H\rho} (b-a)^{\beta H\rho(1-1/\alpha)} \left(\frac{\beta^{\beta/(2H)} (\beta H)^{\beta/\alpha}}{(1-\frac{1}{\alpha})^{\beta(1-\frac{1}{\alpha})}} \right)^{H\rho}, \tag{3.17}$$

where A_1 is the constant given in (1.7).

Note that, when $H = 1/2$, we can take $\beta = 1$. In this case, B_3 is reduced to

$$B_3 = \left[\frac{\Gamma(1-\frac{1}{\alpha})\Gamma(\frac{1}{\alpha})\chi^{1/\alpha} \cos(\frac{1}{\alpha}\arctan(\nu\tan(\frac{\pi\alpha}{2})))}{2\pi\alpha[1+(\nu\tan(\frac{\pi\alpha}{2}))^2]^{1/(2\alpha)}} \right]^{\frac{\alpha}{\alpha-1}}.$$

Proof of Theorem 3.4. We start with the following elementary fact: Let $H \in (0,1)$ be a constant. Then for all $x \geqslant 0$,

$$1 + \sum_{n=1}^\infty \frac{x^{\lfloor n/H\rfloor}}{\lfloor \frac{n}{H}\rfloor!} \leqslant e^x \leqslant \frac{e}{H}\left(1 + \sum_{n=1}^\infty \frac{x^{\lfloor n/H\rfloor}}{(\lfloor\frac{n-1}{H}\rfloor+1)!}\right). \tag{3.18}$$

In order to verify (3.18), first note that the first inequality holds for all $x \geqslant 0$ because $\{\lfloor\frac{n}{H}\rfloor,\ n \geqslant 0\}$ is a subsequence of $\mathbb{N}$, and the second inequality holds for all $0 \leqslant x \leqslant 1$. Hence it only remains to show the second inequality holds for all $x > 1$. This can be verified by grouping the terms in the expansion $e^x = 1 + \sum_{k=1}^\infty \frac{x^k}{k!}$ in the blocks $k \in \{\lfloor\frac{n-1}{H}\rfloor + 1, \dots, \lfloor\frac{n}{H}\rfloor\}$, and noting that the number of integers in each block is at most $1 + \frac{1}{H}$, which is smaller than $\frac{e}{H}$.

Now let $\beta \in (0, \frac{2\alpha}{2H+\alpha})$ be a constant and let $t \geqslant 0$. By taking $x = t|W^H(L_b) - W^H(L_a)|^\beta$ in (3.18), we have

$$1 + \sum_{n=1}^\infty \frac{t^{\lfloor n/H\rfloor}\mathbb{E}(|W^H(L_b) - W^H(L_a)|^{\beta\lfloor n/H\rfloor})}{\lfloor\frac{n}{H}\rfloor!} \leqslant g_\beta(t) \leqslant \frac{e}{H}\left(1 + \sum_{n=1}^\infty \frac{t^{\lfloor n/H\rfloor}\mathbb{E}(|W^H(L_b) - W^H(L_a)|^{\beta\lfloor n/H\rfloor})}{(\lfloor\frac{n-1}{H}\rfloor+1)!}\right). \tag{3.19}$$

Denote the first and last terms in (3.19) by $f_1(t)$ and $f_2(t)$, respectively. In order to prove (3.16), it suffices to show that for $i = 1, 2$,

$$\lim_{t\to\infty} \frac{\log f_i(t)}{t^\rho} = B_3, \tag{3.20}$$

where $\rho = \frac{2\alpha}{2\alpha-\alpha\beta-2H\beta}$ and $B_3 = B_3(H, \alpha, \nu, \chi, \beta)$ is given by (3.17).

This can be done by showing the coefficients of f_i satisfy the conditions of Lemma 3.2. Moreover, since the proofs for $i = 1, 2$ are almost the same, we only prove (3.20) for $i = 1$.

Note that the coefficients c_p ($p = 0, 1, \dots$) of $f_1(t)$ are given by

$$c_p = \frac{\mathbb{E}(|W^H(L_b) - W^H(L_a)|^{\beta\lfloor n/H\rfloor})}{\lfloor\frac{n}{H}\rfloor!}$$

if $p = \lfloor\frac{n}{H}\rfloor$ and $c_p = 0$ otherwise. By Lemma 3.2, it suffices to show

$$\lim_{n\to\infty} \frac{1}{\rho e}\left\lfloor\frac{n}{H}\right\rfloor \left(\frac{\mathbb{E}(|W^H(L_b) - W^H(L_a)|^{\beta\lfloor n/H\rfloor})}{\lfloor\frac{n}{H}\rfloor!}\right)^{\rho/\lfloor\frac{n}{H}\rfloor} = B_3. \tag{3.21}$$

As in the proof of Lemma 2.2, we have

$$\mathbb{E}\big(\big|W^H(L_b)-W^H(L_a)\big|^{\beta\lfloor n/H\rfloor}\big)=\frac{1}{\sqrt{\pi}}2^{\beta\lfloor n/H\rfloor/2}\Gamma\bigg(\frac{\beta\lfloor n/H\rfloor}{2}+\frac{1}{2}\bigg)\mathbb{E}\big(|L_b-L_a|^{\beta H\lfloor n/H\rfloor}\big). \tag{3.22}$$

By (3.10), the moment estimates in Lemma 2.1 and Stirling's formula, we can verify that

$$\frac{\mathbb{E}(|W^H(L_b)-W^H(L_a)|^{\beta\lfloor n/H\rfloor})}{\lfloor\frac{n}{H}\rfloor!}\asymp A_3^{\lfloor n/H\rfloor}e^{n(\frac{1}{H}-\frac{\beta}{2H}-\frac{\beta}{\alpha})}n^{-n(\frac{1}{H}-\frac{\beta}{2H}-\frac{\beta}{\alpha})}, \tag{3.23}$$

where A_3 is a constant defined by

$$A_3=A_1^{\beta H}(b-a)^{\beta H(1-\frac{1}{\alpha})}\bigg(\frac{\beta^{\frac{\beta}{2H}+\frac{\beta}{\alpha}}H^{\frac{1}{H}-\frac{\beta}{2H}}}{(1-\frac{1}{\alpha})^{\beta(1-\frac{1}{\alpha})}}\bigg)^H. \tag{3.24}$$

Since $(\frac{1}{H}-\frac{\beta}{2H}-\frac{\beta}{\alpha})H\rho=1$, we see that (3.23) implies

$$\lim_{n\to\infty}\frac{1}{\rho e}\bigg\lfloor\frac{n}{H}\bigg\rfloor\bigg(\frac{\mathbb{E}(|W^H(L_b)-W^H(L_a)|^{\beta\lfloor n/H\rfloor})}{\lfloor\frac{n}{H}\rfloor!}\bigg)^{\rho/\lfloor\frac{n}{H}\rfloor}=\frac{A_3^{\rho}}{\rho H}=B_3, \tag{3.25}$$

where B_3 is given by (3.17). This proves (3.21) and hence Theorem 3.4. □

4. Large deviations results: Proofs of Theorems 1.1 and 1.2

In this section, we first prove Theorems 1.1 and 1.2. Then we apply a maximal inequality due to Móricz et al. [32] to derive upper bounds for the tail probabilities of the maxima $\max_{t\in[0,1]}Z^H(t)$ and $\max_{t\in[a,b]}|Z^H(t)-Z^H(a)|$.

Proof of Theorem 1.1. Note that the function $\Lambda_1(\theta)=B_1\theta^{\frac{2\alpha}{\alpha-2H}}$ in Theorem 3.1 is essentially smooth and continuous on $\mathbb{R}$. It follows from the Gärtner–Ellis theorem (cf. [20, Theorem 2.3.6]) that the pair $(t^{-\frac{2H(\alpha-1)}{\alpha-2H}}Z^H(t),t^{\frac{2H(\alpha-1)}{\alpha-2H}})$ satisfies a large deviation principle with the good rate function

$$\Lambda_1^*(x)=\sup_{\theta\in\mathbb{R}}\big(\theta x-\Lambda_1(\theta)\big),$$

which is the Fenchel–Legendre transform of Λ_1. It is elementary to verify that $\Lambda_1^*(x)$ coincides with (1.5). This proves Theorem 1.1. □

Proof of Theorem 1.2. Let $0<\beta<\frac{2\alpha}{2H+\alpha}$ be fixed. It follows from (3.16) in Theorem 3.4 and Davies' Theorem 1 in [18] that

$$\lim_{u\to\infty}\frac{\log\mathbb{P}\{|W^H(L_b)-W^H(L_a)|^{\beta}\geqslant u\}}{u^{\rho/(\rho-1)}}=-\big(1-\rho^{-1}\big)(\rho B_3)^{-1/(\rho-1)}. \tag{4.1}$$

Here $\rho=\frac{2\alpha}{2\alpha-\alpha\beta-2H\beta}$. Letting $x=u^{1/\beta}$ and simplifying the right-hand side of (4.1), we obtain

$$\lim_{x\to\infty}\frac{\log\mathbb{P}\{|W^H(L_b)-W^H(L_a)|\geqslant x\}}{x^{\frac{2\alpha}{\alpha+2H}}}=-\frac{\alpha+2H}{2\alpha}\bigg(\frac{HA_1^{\alpha}}{(1-\frac{1}{\alpha})^{\alpha-1}}\bigg)^{-\frac{2H}{\alpha+2H}}(b-a)^{-\frac{2H(\alpha-1)}{\alpha+2H}}. \tag{4.2}$$

This finishes the proof of Theorem 1.2. □

Remark 4.1. Theorem 1.2 can also be proved by using Lemma 2.3 in König and Mörters [29] (note that their assumption $p\in\mathbb{N}$ can be replaced by $p>0$) and the moment results in Lemma 2.2. The proof of Lemma 2.3 in [29] is based on a change-of-measure technique in large deviations. We remark that Lemma 2.3 in [29] is equivalent to Corollary 2 in Davies [18], hence it can also be proved by using an analytic method.

Similar to the proof of Theorems 3.1 and 1.1, we obtain the following theorem for the local time L_t at zero of X. They are in complement to the results of Hawkes [23] and Lacey [30] on the tail asymptotics for the local time L_t and the maximum local time $\max_{x\in\mathbb{R}}L(x,t)$, respectively.

Theorem 4.2. *Let $L = \{L_t,\ t \geqslant 0\}$ be the local time at zero of a real-valued strictly stable Lévy process $X = \{X_t,\ t \geqslant 0\}$ of index $1 < \alpha \leqslant 2$. Then the following two statements hold*:

(i) *For all $0 \leqslant a \leqslant b < \infty$,*

$$\lim_{t\to\infty} \frac{\log \mathbb{E}[\exp(t(L_b - L_a))]}{t^{\frac{\alpha}{\alpha-1}}} = (b-a)C(\alpha,\nu,\chi), \tag{4.3}$$

where $C(\alpha,\nu,\chi)$ is the constant defined by

$$C(\alpha,\nu,\chi) = \left[\frac{\Gamma(1-\frac{1}{\alpha})\Gamma(\frac{1}{\alpha})\chi^{1/\alpha}\cos(\frac{1}{\alpha}\arctan(\nu\tan(\frac{\pi\alpha}{2})))}{\pi\alpha[1+(\nu\tan(\frac{\pi\alpha}{2}))^2]^{1/2\alpha}}\right]^{\frac{\alpha}{\alpha-1}}.$$

(ii) *The pair $(t^{-1/(\alpha-1)}(L_b - L_a), t^{\alpha/(\alpha-1)})$ satisfies LDP with good rate function $\Lambda_2^*(x) = \frac{x^\alpha}{\alpha}[(\frac{\alpha}{\alpha-1})(b-a)C(\alpha,\nu,\chi)]^{-(\alpha-1)}$ if $x > 0$ and $\Lambda_2^*(x) = \infty$ if $x \leqslant 0$. That is, for every Borel set $F \subseteq \mathbb{R}$,*

$$\limsup_{t\to\infty} t^{-\frac{\alpha}{\alpha-1}} \log \mathbb{P}\left\{t^{-\frac{1}{\alpha-1}}(L_b - L_a) \in F\right\} \leqslant -\inf_{x\in\overline{F}} \Lambda_2^*(x) \tag{4.4}$$

and

$$\liminf_{t\to\infty} t^{-\frac{\alpha}{\alpha-1}} \log \mathbb{P}\left\{t^{-\frac{1}{\alpha-1}}(L_b - L_a) \in F\right\} \geqslant -\inf_{x\in F^\circ} \Lambda_2^*(x). \tag{4.5}$$

Proof. Similar to the proof of Theorem 3.1, Eq. (4.3) follows from Lemma 3.2 and the moment estimates in Lemma 2.1.

It follows from (i) that for all $\theta > 0$,

$$\lim_{t\to\infty} \frac{\log \mathbb{E}[\exp(\theta t(L_b - L_a))]}{t^{\frac{\alpha}{\alpha-1}}} = (b-a)C(\alpha,\nu,\chi)\theta^{\alpha/(\alpha-1)}. \tag{4.6}$$

Denote

$$\Lambda_2(\theta) = \begin{cases} (b-a)C(\alpha,\nu,\chi)\theta^{\alpha/(\alpha-1)} & \text{if } \theta > 0, \\ 0 & \text{if } \theta \leqslant 0. \end{cases} \tag{4.7}$$

Then, in the terminology of [20], Λ_2 is an essentially smooth, continuous function on $\mathbb{R}$ and its Fenchel–Legendre transform is given by

$$\Lambda_2^*(x) = \begin{cases} \frac{x^\alpha}{\alpha}[(\frac{\alpha}{\alpha-1})(b-a)C(\alpha,\nu,\chi)]^{-(\alpha-1)} & \text{if } x > 0, \\ \infty & \text{if } x \leqslant 0. \end{cases} \tag{4.8}$$

Therefore, as in the proof of Theorem 1.1, part (ii) follows from (4.6) and the Gärtner–Ellis theorem. □

Remark 4.3. Let $F = [1,\infty)$ in (4.4) and (4.5), we obtain the tail probability

$$\lim_{x\to\infty} \frac{\log \mathbb{P}\{L_b - L_a > x\}}{x^\alpha} = -\frac{1}{\alpha}\left[\left(\frac{\alpha}{\alpha-1}\right)(b-a)C(\alpha,\nu,\chi)\right]^{-(\alpha-1)}. \tag{4.9}$$

In case $a = 0$, the limit in (4.9) is weaker than the best known. By using the connection between L_t and a stable subordinator of index $1-1/\alpha$, one can derive more precise result (i.e., without the logarithm) on limiting behavior of $\mathbb{P}\{L_b > x\}$ as $x \to \infty$. See Hawkes [23].

Theorem 1.2 is concerned with asymptotic behavior of the tail probability $\mathbb{P}\{|Z^H(b) - Z^H(a)| > x\}$ as $x \to \infty$. In many applications, however, it is useful to have sharp bounds on $\mathbb{P}\{|Z^H(b) - Z^H(a)| > x\}$ and $\mathbb{P}\{\max_{a\leqslant t\leqslant b}|Z^H(t) - Z^H(a)| > x\}$ that hold for *all* $x > 0$. In the rest of this section, we consider these questions and in the next section we use them to derive upper bounds for the local and uniform moduli of continuity for Z^H.

Lemma 4.4. *There exists a finite constant $A_4 > 0$, depending on H, α, ν and χ only, such that for all $0 \leqslant a < b < \infty$ and all $x > 0$,*

$$\mathbb{P}\left\{\left|Z^H(b) - Z^H(a)\right| > x\right\} \leqslant \exp\left(-A_4 \frac{x^{2\alpha/(\alpha+2H)}}{(b-a)^{2H(\alpha-1)/(\alpha+2H)}}\right). \tag{4.10}$$

Proof. We consider the random variable

$$\Lambda = \frac{|Z^H(b) - Z^H(a)|}{(b-a)^{H(\alpha-1)/\alpha}}.$$

As in the proof of Lemma 2.2, we apply a conditioning argument and Lemma 2.1 to show that for all integers $n \geqslant 1$,

$$\mathbb{E}\big(\Lambda^n\big) \leqslant A_5^n n^{n(\alpha+2H)/(2\alpha)}, \tag{4.11}$$

where $A_5 > 0$ is a constant depending on H, α, ν and χ only.

For any constant $A_6 > 0$, the Markov inequality and (4.11) imply that for all $u > 0$,

$$\mathbb{P}\{\Lambda > A_6 u\} \leqslant \frac{A_5^n n^{n(\alpha+2H)/(2\alpha)}}{A_6^n u^n} = \left(\frac{A_5}{A_6}\right)^n \left(\frac{n^{(\alpha+2H)/(2\alpha)}}{u}\right)^n. \tag{4.12}$$

By taking the constant $A_6 \geqslant eA_5$ and $n = \lfloor u^{2\alpha/(\alpha+2H)} \rfloor$, we obtain

$$\mathbb{P}\big\{\big|Z^H(b) - Z^H(a)\big| > A_6 (b-a)^{H(\alpha-1)/\alpha} u\big\} \leqslant \exp\big(-u^{2\alpha/(\alpha+2H)}\big). \tag{4.13}$$

It is clear that (4.10) follows from (4.13) by letting $x = A_6(b-a)^{H(\alpha-1)/\alpha} u$. $\square$

Next we apply Lemma 4.4 and a result of Móricz et al. [32] to prove the following theorem.

Theorem 4.5. *There exist positive and finite constants A_7 and A_8, depending on H, α, ν and χ only, such that for all $0 \leqslant a < b < \infty$ and all $x > 0$,*

$$\mathbb{P}\Big\{\max_{a\leqslant t\leqslant b} \big|Z^H(t) - Z^H(a)\big| > x\Big\} \leqslant A_7 \exp\left(-A_8 \frac{x^{2\alpha/(\alpha+2H)}}{(b-a)^{2H(\alpha-1)/(\alpha+2H)}}\right). \tag{4.14}$$

Proof. For any integer $n \geqslant 2$, we divide the interval $[a, b]$ into n subintervals of length $(b-a)/n$. Let $t_{n,i} = a + \frac{i(b-a)}{n}$ ($i \in \{0, 1, \ldots, n\}$) be the end-points of these subintervals. By the sample path continuity of Z^H, it suffices to show that

$$\mathbb{P}\Big\{\max_{1\leqslant i\leqslant n} \big|Z^H(t_{n,i}) - Z^H(a)\big| > x\Big\} \leqslant A_7 \exp\left(-A_8 \frac{x^{2\alpha/(\alpha+2H)}}{(b-a)^{2H(\alpha-1)/(\alpha+2H)}}\right) \tag{4.15}$$

for all integers $n \geqslant 2$.

To this end, we define the random variables $\xi_i = Z^H(t_{n,i+1}) - Z^H(t_{n,i})$ for $i \in \{0, 1, \ldots, n-1\}$. Then for all integers $0 \leqslant j < k \leqslant n$, we have

$$Z^H(t_{n,k}) - Z^H(t_{n,j}) = \sum_{i=j}^{k-1} \xi_i := S(j,k). \tag{4.16}$$

Applying Lemma 4.4, we see that for all integers $j < k$,

$$\mathbb{P}\big\{\big|S(j,k)\big| > x\big\} \leqslant \exp\left(-A_4 \left(\frac{n}{(b-a)(k-j)}\right)^{2H(\alpha-1)/(\alpha+2H)} x^{2\alpha/(\alpha+2H)}\right). \tag{4.17}$$

Using the notation in [32], we denote $\phi(x) = x^{2\alpha/(\alpha+2H)}$ and

$$g(j,k) = A_4 \left(\frac{(b-a)(k-j)}{n}\right)^{2H(\alpha-1)/(\alpha+2H)}.$$

For simplicity denote $r = 2H(\alpha-1)/(\alpha+2H)$. Since $r \in (0,1)$, the concavity of the function $t \mapsto t^r$ implies that for all integers $1 \leqslant i \leqslant j < k \leqslant n$,

$$g(i,j) + g(j+1,k) \leqslant 2^{1-r} g(i,k). \tag{4.18}$$

Hence the function g satisfies the property of *quasi-superadditivity* with index $Q = 2^{1-r}$ in [32]. Moreover, the functions ϕ and g satisfy all the other conditions of Theorem 2.2 in Móricz et al. [32]. Consequently, the latter implies the existence of positive and finite constants A_7 and A_8 (depending on H, α, ν and χ only) such that (4.15) holds. This proves Theorem 4.5. $\square$

Remark 4.6. Note that, when $H = 1/2$, one can apply the reflection principle of Brownian motion and conditioning to prove Theorem 4.5. Our method is much more general.

5. Applications

Applying the large deviation results in the previous section, we establish uniform and local moduli of continuity for Z^H.

Theorem 5.1. *Let $Z^H = \{Z^H(t),\ t \geqslant 0\}$ be an α-stable local time H-fractional Brownian motion with values in $\mathbb{R}$. Then there exists a finite constant $A_9 > 0$ such that for all constants $0 \leqslant a < b < \infty$, we have*

$$\limsup_{h\downarrow 0} \sup_{a\leqslant t\leqslant b-h} \sup_{0\leqslant s\leqslant h} \frac{|Z^H(t+s) - Z^H(t)|}{h^{H(\alpha-1)/\alpha}(\log 1/h)^{(\alpha+2H)/(2\alpha)}} \leqslant A_9 \quad \text{a.s.} \tag{5.1}$$

Proof. For every $t \geqslant 0$ and $h > 0$, it follows from (4.13) that

$$\mathbb{P}\bigl\{\bigl|Z^H(t+h) - Z^H(h)\bigr| > A_6 h^{H(\alpha-1)/\alpha} u\bigr\} \leqslant \exp\bigl(-u^{2\alpha/(\alpha+2H)}\bigr). \tag{5.2}$$

Hence $Z^H = \{Z^H(t),\ t \geqslant 0\}$ satisfies the conditions of Lemmas 2.1 and 2.2 in [13] with $\sigma(h) = h^{H(\alpha-1)/\alpha}$ and $\beta = \frac{2\alpha}{\alpha+2H}$. Consequently (5.1) follows directly from Theorem 3.1 in [13]. □

Csáki, Földes and Révész [17] obtained a Strassen type law of the iterated logarithm (LIL) for $Z(t) = W(L_t)$ when L_t is the local time at zero of a *symmetric* stable Lévy process (see Theorem 2.4 in [17]). Part (i) of the following theorem extends partially their result to Z^H and part (ii) describes the local oscillation of Z^H in the neighborhood of any fixed point.

Theorem 5.2. *Let $Z^H = \{Z^H(t),\ t \geqslant 0\}$ be an α-stable local time H-fractional Brownian motion with values in $\mathbb{R}$. The following statements hold*:

(i) *Almost surely,*

$$\limsup_{t\to\infty} \frac{\max_{0\leqslant s\leqslant t}|Z^H(s)|}{t^{H(\alpha-1)/\alpha}(\log\log t)^{(\alpha+2H)/(2\alpha)}} \leqslant A_8^{-(\alpha+2H)/(2\alpha)}. \tag{5.3}$$

(ii) *For every $t > 0$, almost surely*

$$\limsup_{h\to 0} \frac{\max_{|s|\leqslant h}|Z^H(t+s) - Z^H(t)|}{h^{H(\alpha-1)/\alpha}(\log\log 1/h)^{(\alpha+2H)/(2\alpha)}} \leqslant A_8^{-(\alpha+2H)/(2\alpha)}. \tag{5.4}$$

In the above, A_8 is the constant in (4.14).

Proof. Since both (5.3) and (5.4) follow from Theorem 4.5 and a standard Borel–Cantelli argument, we only prove (5.3).

Fix two arbitrary constants $\gamma > A_8^{-1}$ and $\rho > 1$. For every integer $n \geqslant 1$, let $T_n = \rho^n$ and consider the event

$$E_n = \Bigl\{\omega\colon \max_{0\leqslant t\leqslant T_n}\bigl|Z^H(s)\bigr| > T_n^{H(\alpha-1)/\alpha} U(T_n)\Bigr\},$$

where $U(t) = (\gamma \log\log t)^{(\alpha+2H)/(2\alpha)}$. It follows from Theorem 4.5 that

$$\mathbb{P}(E_n) \leqslant A_7 \exp\biggl(-A_8 \frac{(T_n^{H(\alpha-1)/\alpha} U(T_n))^{2\alpha/(\alpha+2H)}}{T_n^{2H(\alpha-1)/(\alpha+2H)}}\biggr) \leqslant A_{10} n^{-A_8\gamma}. \tag{5.5}$$

Since $A_8\gamma > 1$, we have $\sum_{n=1}^{\infty} \mathbb{P}(E_n) < \infty$. The Borel–Cantelli lemma implies that

$$\limsup_{n\to\infty} \frac{\max_{0\leqslant s\leqslant T_n}|Z^H(s)|}{T_n^{H(\alpha-1)/\alpha} U(T_n)} \leqslant 1 \quad \text{a.s.} \tag{5.6}$$

Note that $T_{n+1}/T_n = \rho$ for every $n \geqslant 1$. We use the monotonicity to derive that for all $t \in [T_n, T_{n+1}]$,

$$\frac{\max_{0\leqslant s\leqslant t}|Z^H(s)|}{t^{H(\alpha-1)/\alpha} U(t)} \leqslant \rho^{H(\alpha-1)/\alpha} \frac{\max_{0\leqslant s\leqslant T_{n+1}}|Z^H(s)|}{T_{n+1}^{H(\alpha-1)/\alpha} U(T_{n+1})} \frac{U(T_{n+1})}{U(T_n)}. \tag{5.7}$$

Eqs. (5.6) and (5.7) imply

$$\limsup_{t\to\infty} \frac{\max_{0\leqslant s\leqslant t}|Z^H(s)|}{t^{H(\alpha-1)/\alpha}(\log\log t)^{(\alpha+2H)/(2\alpha)}} \leqslant \rho^{H(\alpha-1)/\alpha} \gamma^{(\alpha+2H)/(2\alpha)} \quad \text{a.s.} \tag{5.8}$$

We obtain (5.3) from (5.8) by letting $\gamma \downarrow A_8^{-1}$ and $\rho \downarrow 1$ along rational numbers. □

Remark 5.3. We believe that, up to a constant factor, both uniform and local moduli of continuity of $Z^H(t)$ are sharp. However, we have not been able to prove this due to the lack of information on the dependence structure of the process Z^H.

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