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Regularity of Intersection Local Times of Fractional Brownian Motions

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Abstract Let B^{α_i} be an (N_i, d)-fractional Brownian motion with Hurst index α_i $(i = 1, 2)$, and let B^{α_1} and B^{α_2} be independent. We prove that, if $\frac{N_1}{\alpha_1} + \frac{N_2}{\alpha_2} > d$, then the intersection local times of B^{α_1} and B^{α_2} exist, and have a continuous version. We also establish Hölder conditions for the intersection local times and determine the Hausdorff and packing dimensions of the sets of intersection times and intersection points.

One of the main motivations of this paper is from the results of Nualart and Ortiz-Latorre (J. Theor. Probab. 20:759–767, 2007), where the existence of the intersection local times of two independent $(1, d)$-fractional Brownian motions with the same Hurst index was studied by using a different method. Our results show that anisotropy brings subtle differences into the analytic properties of the intersection local times as well as rich geometric structures into the sets of intersection times and intersection points.

Keywords Intersection local time · Fractional Brownian motion · Joint continuity · Hölder condition · Hausdorff dimension · Packing dimension

Mathematics Subject Classification (2000) 60G15 · 60J55 · 60G18 · 60F25 · 28A80

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1 Introduction

Let $B_0^\gamma = \{B_0^\gamma(u), u \in \mathbb{R}^p\}$ be a p-parameter fractional Brownian motion in $\mathbb{R}$ with Hurst index $\gamma \in (0,1)$, i.e., a centered, real-valued Gaussian random field with covariance function

$$\mathbb{E}\big[B_0^\gamma(u_1)B_0^\gamma(u_2)\big] = \frac{1}{2}\big(|u_1|^{2\gamma} + |u_2|^{2\gamma} - |u_1 - u_2|^{2\gamma}\big). \tag{1.1}$$

It follows from (1.1) that $\mathbb{E}[(B_0^\gamma(u_1) - B_0^\gamma(u_2))^2] = |u_1 - u_2|^{2\gamma}$ and B_0^γ is γ-self-similar with stationary increments.

We associate with B_0^γ a Gaussian random field $B^\gamma = \{B^\gamma(u), u \in \mathbb{R}^p\}$ in $\mathbb{R}^q$ by

$$B^\gamma(u) = \big(B_1^\gamma(u), \dots, B_q^\gamma(u)\big), \quad u \in \mathbb{R}^p, \tag{1.2}$$

where $B_1^\gamma, \dots, B_q^\gamma$ are independent copies of B_0^γ. B^γ is called a (p,q)-fractional Brownian motion of index γ.

Fractional Brownian motion has been intensively studied in recent years and, because of its interesting properties such as short/long range dependence and self-similarity, has been widely applied in many areas such as finance, hydrology and telecommunication engineering.

Let $B^{\alpha_1} = \{B^{\alpha_1}(s), s \in \mathbb{R}^{N_1}\}$ and $B^{\alpha_2} = \{B^{\alpha_2}(t), t \in \mathbb{R}^{N_2}\}$ be two independent fractional Brownian motions in $\mathbb{R}^d$ with Hurst indices $\alpha_1, \alpha_2 \in (0,1)$, respectively. This paper is concerned with the regularity of the intersection local times of B^{α_1} and B^{α_2}, as well as the fractal properties of the sets of intersection times and intersection points. Without loss of generality, we further assume $\alpha_1 \le \alpha_2$ throughout this paper. For $N_1 = N_2 = 1$ and $\alpha_1 = \alpha_2 = \frac{1}{2}$, the processes are classical d-dimensional Brownian motions. The intersection local times of independent Brownian motions have been studied by several authors (see [7, 17]) and are closely related to the self-intersections (or multiple points) of Brownian motion. The approach of these papers relies on the fact that the intersection local times of independent Brownian motions can be seen as the local times at zero of some Gaussian random field. For the applications of the intersection local time theory for Brownian motions, we refer to [10, 18], among others.

The self-intersection local times of fractional Brownian motion were studied by Rosen [16] for the planar case, and by Hu and Nualart [8] for the multidimensional case. Very recently, Nualart and Ortiz-Latorre [12] proved an existence result for the intersection local times of two independent d-dimensional fractional Brownian motions with the same Hurst index.

The aim of this paper is to show that the existence of the intersection local times for two independent fractional Brownian motions B^{α_1} and B^{α_2} in $\mathbb{R}^d$ can be studied by using a Fourier analytic method and, moreover, this latter method can be applied to establish the joint continuity and sharp Hölder conditions for the intersection local times. Besides their own interest, these results are useful for studying fractal properties of the set of intersection times as well as the set of intersection points.

Let $X = \{X(s,t), (s,t) \in \mathbb{R}^N\}$ be an (N,d)-Gaussian random field, where $N = N_1 + N_2$, defined by

$$X(s,t) \equiv B^{\alpha_1}(s) - B^{\alpha_2}(t), \quad s \in \mathbb{R}^{N_1}, t \in \mathbb{R}^{N_2}. \tag{1.3}$$

We will follow the same idea as Wolpert [17] and Geman, et al. [7] and treat the intersection local times of B^{α_1} and B^{α_2} as the local times at 0 of X, with an intension to establish sharp Hölder conditions. The main ingredients for proving our results are the strong local nondeterminism of fractional Brownian motions, occupation density theory (cf. [6]), and newly developed techniques for anisotropic Gaussian random fields (cf. [2, 23]).

For later use, we mention that, by the self-similarity and stationarity of the increments of B^{α_1} and B^{α_2}, the Gaussian random field X defined by (1.3) has stationary increments and satisfies the following operator-scaling property: For every constant $c > 0$,

$$\left\{X\left(c^A(s,t)\right), (s,t) \in \mathbb{R}^N\right\} \stackrel{d}{=} \left\{cX(s,t), (s,t) \in \mathbb{R}^N\right\}, \tag{1.4}$$

where $A = (a_{ij})$ is an $N \times N$ diagonal matrix such that $a_{ii} = 1/\alpha_1$ if $1 \le i \le N_1$ and $a_{ii} = 1/\alpha_2$ if $N_1 + 1 \le i \le N$. In the above, $\stackrel{d}{=}$ denotes equality of all finite dimensional distributions and c^A is the linear operator on $\mathbb{R}^N$ defined by $c^A = \sum_{n=0}^{\infty} \frac{(\ln c)^n A^n}{n!}$.

This paper is organized as follows. In Sect. 2, we give several lemmas which will be used to prove our main results in the following sections. In Sect. 3, we study the existence and the joint continuity of the intersection local times of two independent d-dimensional fractional Brownian motions. We prove that the necessary and sufficient condition for the existence of an intersection local times in $L^2(\mathbb{P} \times \lambda_d)$ actually implies the joint continuity. We devote Sect. 4 to the study of the exponential integrability and Hölder conditions for the intersection local times. The later results imply information about the exact Hausdorff measure of the set of intersection times of B^{α_1} and B^{α_2}. Finally, in Sect. 5, we determine the Hausdorff and packing dimensions of the set of intersection points of B^{α_1} and B^{α_2}.

Throughout this paper, we use $\langle \cdot, \cdot \rangle$ and $|\cdot|$ to denote the ordinary scalar product and the Euclidean norm in $\mathbb{R}^p$, respectively, no matter what the value of the integer p is, and we use λ_p to denote the Lebesgue measure in $\mathbb{R}^p$. We denote by $O_p(u,r)$ a p-dimensional ball centered at u with radius r, and $O_{p_1,p_2}(u,r) := O_{p_1}(u_1,r) \times O_{p_2}(u_2,r)$, where $u = (u_1, u_2)$ with $u_1 \in \mathbb{R}^{p_1}$ and $u_2 \in \mathbb{R}^{p_2}$. In Sect. i, unspecified positive and finite constants will be numbered as $c_{i,1}, c_{i,2}, \ldots$.

2 Preliminaries

In this section, we provide necessary preparations for the proofs of our main results in the later sections.

It follows from Lemma 7.1 of [13] that, for any $\gamma \in (0,1)$, the real-valued fractional Brownian motion $B_0^\gamma = \{B_0^\gamma(u), u \in \mathbb{R}^p\}$ has the following important property of *strong local nondeterminism*: There exists a constant $0 < c_{2,1} < \infty$ such that for all integers $n \ge 1$ and all $u, u_1, \ldots, u_n \in \mathbb{R}^p$,

$$\mathrm{Var}\left(B_0^\gamma(u) \middle| B_0^\gamma(u_1), \ldots, B_0^\gamma(u_n)\right) \ge c_{2,1} \min_{0 \le k \le n} |u - u_k|^{2\gamma}, \tag{2.1}$$

where $\mathrm{Var}(B_0^\gamma(u) | B_0^\gamma(u_1), \ldots, B_0^\gamma(u_n))$ denotes the conditional variance of $B_0^\gamma(u)$ given $B_0^\gamma(u_1), \ldots, B_0^\gamma(u_n)$, and where $u_0 \equiv 0$. The strong local nondeterminism has played important rôles in studying various sample path properties of fractional Brownian motion. See [19, 21, 22] and the references therein for further information. It will be the main technical tool of this paper as well.

We consider the real-valued Gaussian random field $X_0 = \{X_0(s,t), (s,t) \in \mathbb{R}^N\}$ defined by $X_0(s,t) := B_0^{\alpha_1}(s) - B_0^{\alpha_2}(t)$ for $s \in \mathbb{R}^{N_1}$ and $t \in \mathbb{R}^{N_2}$. Then the coordinate processes of X defined by (1.3) are independent copies of X_0.

The following Lemma 2.1 is a consequence of the property of strong local nondeterminism of fractional Brownian motion, and will be useful in our approach.

Lemma 2.1 *There exists a constant* $0 < c_{2,2} < \infty$ *such that for all integers* $n \ge 1$ *and all* $(v,w), (s_1,t_1), \ldots, (s_n,t_n) \in \mathbb{R}^N$, *we have*

$$\begin{aligned} &\mathrm{Var}\big(X_0(v,w)\big|X_0(s_1,t_1),\ldots,X_0(s_n,t_n)\big) \\ &\quad \ge c_{2,2}\Big(\min_{0\le k\le n}|v-s_k|^{2\alpha_1} + \min_{0\le k\le n}|w-t_k|^{2\alpha_2}\Big), \end{aligned} \tag{2.2}$$

where $s_0 = t_0 = 0$.

Proof By definition we can write

$$\begin{aligned} &\mathrm{Var}\big(X_0(v,w)\big|X_0(s_1,t_1),\ldots,X_0(s_n,t_n)\big) \\ &\quad = \inf_{a_i\in\mathbb{R},1\le i\le n} \mathbb{E}\bigg[\bigg(X_0(v,w) - \sum_{i=1}^n a_i X_0(s_i,t_i)\bigg)^2\bigg]. \end{aligned} \tag{2.3}$$

Since $B_0^{\alpha_1}$ and $B_0^{\alpha_2}$ are independent, we have

$$\begin{aligned} &\mathrm{Var}\big(X_0(v,w)\big|X_0(s_1,t_1),\ldots,X_0(s_n,t_n)\big) \\ &\quad = \inf_{a_i\in\mathbb{R},1\le i\le n}\bigg\{\mathbb{E}\bigg[\bigg(B_0^{\alpha_1}(v) - \sum_{i=1}^n a_i B_0^{\alpha_1}(s_i)\bigg)^2\bigg] \\ &\qquad + \mathbb{E}\bigg[\bigg(B_0^{\alpha_2}(w) - \sum_{i=1}^n a_i B_0^{\alpha_2}(t_i)\bigg)^2\bigg]\bigg\} \\ &\quad \ge \inf_{a_i\in\mathbb{R},1\le i\le n} \mathbb{E}\bigg[\bigg(B_0^{\alpha_1}(v) - \sum_{i=1}^n a_i B_0^{\alpha_1}(s_i)\bigg)^2\bigg] \\ &\qquad + \inf_{b_i\in\mathbb{R},1\le i\le n} \mathbb{E}\bigg[\bigg(B_0^{\alpha_2}(w) - \sum_{i=1}^n b_i B_0^{\alpha_2}(t_i)\bigg)^2\bigg] \\ &\quad = \mathrm{Var}\big(B_0^{\alpha_1}(v)\big|B_0^{\alpha_1}(s_1),\ldots,B_0^{\alpha_1}(s_n)\big) \\ &\qquad + \mathrm{Var}\big(B_0^{\alpha_2}(w)\big|B_0^{\alpha_2}(t_1),\ldots,B_0^{\alpha_2}(t_n)\big). \end{aligned} \tag{2.4}$$

Hence (2.2) follows from (2.4) and (2.1). □

Combining Lemma 2.1 with the following well-known fact, which will be used repeatedly throughout the paper, that

$$\det\mathrm{Cov}(Z_1,\ldots,Z_n)=\mathrm{Var}(Z_1)\prod_{k=2}^{n}\mathrm{Var}(Z_k|Z_1,\ldots,Z_{k-1}) \tag{2.5}$$

for any Gaussian random vector $(Z_1,\ldots,Z_n)$, we have that, for any $(s_1,t_1),\ldots,(s_n,t_n)\in\mathbb{R}^N_+$,

$$\begin{aligned}&\det\mathrm{Cov}(X_0(s_1,t_1),\ldots,X_0(s_n,t_n))\\ &\quad\ge\prod_{j=1}^{n}\Big[\mathrm{Var}\big(B_0^{\alpha_1}(s_j)\big|B_0^{\alpha_1}(s_1),\ldots,B_0^{\alpha_1}(s_{j-1})\big)\\ &\qquad+\mathrm{Var}\big(B_0^{\alpha_2}(t_j)\big|B_0^{\alpha_2}(t_1),\ldots,B_0^{\alpha_2}(t_{j-1})\big)\Big]\\ &\quad\ge c_{2,2}^{n}\prod_{j=1}^{n}\Big(\min_{0\le k\le j-1}|s_j-s_k|^{2\alpha_1}+\min_{0\le k\le j-1}|t_j-t_k|^{2\alpha_2}\Big).\end{aligned} \tag{2.6}$$

To prove the existence and continuity of the intersection local times of B^{α_1} and B^{α_2}, we will make use of the following lemmas. Lemma 2.2 is similar to Lemma 8.6 in [23] whose proof is elementary. Lemmas 2.3 and 2.4 are extensions of Lemma 2.3 in [19] and will be useful for dealing with anisotropy of the Gaussian random field X_0. Lemma 2.5, due to [4], is a technical lemma.

Lemma 2.2 *Let β, γ and p be positive constants, then for all $A\in(0,1)$*

$$\int_0^1\frac{r^{p-1}}{(A+r^\gamma)^\beta}\,dr\asymp\begin{cases}A^{\frac{p}{\gamma}-\beta} & \text{if } \beta\gamma>p,\\ \log\big(1+A^{-1/\gamma}\big) & \text{if } \beta\gamma=p,\\ 1 & \text{if } \beta\gamma<p.\end{cases} \tag{2.7}$$

In the above, $f(A)\asymp g(A)$ means that the ratio $f(A)/g(A)$ is bounded from below and above by positive constants that do not depend on $A\in(0,1)$.

Proof This can be verified directly and we omit the details. □

Lemma 2.3 *Let β, γ and p be positive constants such that $\gamma\beta\ge p$.*

(i) *If $\gamma\beta>p$, then there exists a constant $c_{2,3}>0$ whose value depends on γ, β and p only such that for all $A\in(0,1)$, $r>0$, $u^*\in\mathbb{R}^p$, all integers $n\ge1$ and all distinct $u_1,\ldots,u_n\in O_p(u^*,r)$ we have*

$$\int_{O_p(u^*,r)}\frac{du}{[A+\min\{|u-u_j|^\gamma,\ j=1,\ldots,n\}]^\beta}\le c_{2,3}\,nA^{\frac{p}{\gamma}-\beta}. \tag{2.8}$$

(ii) *If $\gamma\beta=p$, then for any $\kappa\in(0,1)$ there exists a constant $c_{2,4}>0$ whose value depends on γ,β,κ and p only such that for all $A\in(0,1)$, $r>0$, $u^*\in\mathbb{R}^p$, all* *integers* $n \geq 1$ *and all distinct* $u_1, \ldots, u_n \in O_p(u^*, r)$ *we have*

$$\int_{O_p(u^*,r)} \frac{du}{[A + \min\{|u - u_j|^\gamma, j = 1, \ldots, n\}]^\beta} \leq c_{2,4} n \log\left[e + \left(A^{-1/\gamma} \frac{r}{n^{1/p}}\right)^\kappa\right]. \tag{2.9}$$

Proof The idea of proof is similar to that of Lemma 2.3 in [19]. Let

$$\Gamma_i = \big\{u \in O_p(u^*, r) : |u - u_i| = \min\big\{|u - u_j|,\ j = 1, \ldots, n\big\}\big\}.$$

Then

$$O_p(u^*, r) = \bigcup_{i=1}^n \Gamma_i \quad \text{and} \quad \lambda_p\big(O_p(u^*, r)\big) = \sum_{i=1}^n \lambda_p(\Gamma_i). \tag{2.10}$$

For every $u \in \Gamma_i$, we write $u = u_i + \rho\theta$, where $0 \leq \rho \leq \rho_i(\theta)$ and $\theta \in S_{p-1}$, the unit sphere in $\mathbb{R}^p$. Then

$$\begin{aligned} \lambda_p(\Gamma_i) &= C_p \int_{S_{p-1}} \nu(d\theta) \int_0^{\rho_i(\theta)} \rho^{p-1}\, d\rho \\ &= \frac{C_p}{p} \int_{S_{p-1}} \rho_i(\theta)^p \nu(d\theta), \end{aligned} \tag{2.11}$$

where ν is the normalized surface area in S_{p-1} and C_p is a positive constant depending on p only.

Denote the integral in (2.8) and (2.9) by I_1. We first consider the case of $\gamma\beta > p$. By (2.10), a change of variables and Lemma 2.2, we can write I_1 as

$$\begin{aligned} I_1 &= \sum_{i=1}^n \int_{\Gamma_i} \frac{du}{[A + \min\{|u - u_j|^\gamma, j = 1, \ldots, n\}]^\beta} \\ &= \sum_{i=1}^n C_p \int_{S_{p-1}} \nu(d\theta) \int_0^{\rho_i(\theta)} \frac{\rho^{p-1}}{(A + \rho^\gamma)^\beta}\, d\rho \\ &= \sum_{i=1}^n C_p A^{\frac{p}{\gamma} - \beta} \int_{S_{p-1}} \nu(d\theta) \int_0^{A^{-1/\gamma}\rho_i(\theta)} \frac{\rho^{p-1}}{(1 + \rho^\gamma)^\beta}\, d\rho \\ &\leq c_{2,3} \sum_{i=1}^n A^{\frac{p}{\gamma} - \beta} \int_{S_{p-1}} \nu(d\theta) \\ &= c_{2,3} n A^{\frac{p}{\gamma} - \beta}. \end{aligned} \tag{2.12}$$

This proves inequality (2.8).

Now we assume $\gamma\beta = p$. As above, we use (2.10) and a change of variables to get

$$I_1 = \sum_{i=1}^{n} C_p \int_{S_{p-1}} \nu(d\theta) \int_0^{A^{-1/\gamma}\rho_i(\theta)} \frac{\rho^{p-1}}{(1+\rho^\gamma)^\beta}\, d\rho$$
$$\le \frac{2C_p}{\kappa} \sum_{i=1}^{n} \int_{S_{p-1}} \log\big[e + \big(A^{-1/\gamma}\rho_i(\theta)\big)^\kappa\big]\nu(d\theta). \tag{2.13}$$

In the above, we have used the fact that if $\gamma\beta = p$ and $\kappa \in (0, 1)$, then for all $x \ge 0$

$$\int_0^x \frac{\rho^{p-1}}{(1+\rho^\gamma)^\beta}\, d\rho \le \frac{2}{\kappa} \log(e + x^\kappa).$$

Since the function $\psi_1(x) = \log(e + x^{\kappa/p})$ is concave on $(0, \infty)$, we apply (2.11) and Jensen's inequality twice to derive

$$I_1 \le c_{2,5} n \sum_{i=1}^{n} \frac{1}{n} \psi_1\big(A^{-p/\gamma}\lambda_p(\Gamma_i)\big)$$
$$\le c_{2,4} n \log\left[e + \left(A^{-1/\gamma} \frac{r}{n^{1/p}}\right)^\kappa\right]. \tag{2.14}$$

This finishes the proof of (2.9). □

Lemma 2.4 *Let $\beta > 0$ be a constant and let $p \ge 1$ be an integer such that $\beta < p$. Then the following statements hold*:

(i) *For all $r > 0$, $u^* \in \mathbb{R}^p$, all integers $n \ge 1$, and all distinct $u_1, \dots, u_n \in O_p(u^*, r)$, we have*

$$\int_{O_p(u^*,r)} \frac{du}{\min\{|u - u_j|^\beta, j = 1, \dots, n\}} \le c_{2,6} n^{\frac{\beta}{p}} r^{p-\beta}, \tag{2.15}$$

where $c_{2,6} > 0$ is a constant whose value depends on β and p only.

(ii) *For all constants $r > 0$ and $K > 0$, all $u^* \in \mathbb{R}^p$, integers $n \ge 1$, and all distinct $u_1, \dots, u_n \in O_p(u^*, r)$, we have*

$$\int_{O_p(u^*,r)} \log\big[e + K\big(\min\{|u - u_j|, j = 1, \dots, n\}\big)^{-\beta}\big]\, du$$
$$\le c_{2,7} r^p \log\left[e + K\left(\frac{r}{n^{1/p}}\right)^{-\beta}\right], \tag{2.16}$$

where $c_{2,7} > 0$ is a constant whose value depends on β and p only.

Proof Part (i) is a special case of Lemma 2.3 in [19]. Hence, it only remains to prove Part (ii). Denote the integral in (2.16) by I_2. As in the proof of Lemma 2.3, we have

$$\begin{aligned} I_2 &= \sum_{i=1}^{n} \int_{\Gamma_i} \log\big[e + K\big(\min\{|u-u_j|, j=1,\ldots,n\}\big)^{-\beta}\big]\,du \\ &= \sum_{i=1}^{n} C_p \int_{S_{p-1}} \nu(d\theta) \int_0^{\rho_i(\theta)} \rho^{p-1} \log\big(e + K\rho^{-\beta}\big)\,d\rho \\ &= \sum_{i=1}^{n} C_p \int_{S_{p-1}} \rho_i(\theta)^p \nu(d\theta) \int_0^1 \rho^{p-1} \log\big(e + K\rho_i(\theta)^{-\beta}\rho^{-\beta}\big)\,d\rho \\ &\le c_{2,8} \sum_{i=1}^{n} \int_{S_{p-1}} \rho_i(\theta)^p \log\big(e + K\rho_i(\theta)^{-\beta}\big)\nu(d\theta). \end{aligned} \tag{2.17}$$

In deriving the last inequality, we have use the fact that $\log(e+xy) \le \log(e+x) + \log(e+y)$ for all $x, y \ge 0$. Since $\beta < p$, we can verify that the function $\psi_2(x) = x\log(e + Kx^{-\beta/p})$ is concave on $(0,\infty)$. By using Jensen's inequality twice, we obtain

$$\begin{aligned} I_2 &\le c_{2,8} \sum_{i=1}^{n} \psi_2\left(\int_{S_{p-1}} \rho_i(\theta)^p \nu(d\theta)\right) \\ &\le c_{2,8}\, n\psi_2\left(\frac{1}{n}\sum_{i=1}^{n} \lambda_p(\Gamma_\ell)\right) \le c_{2,7}\, n\psi_2\left(\frac{r^p}{n}\right). \end{aligned} \tag{2.18}$$

This finishes the proof of Lemma 2.4. □

Lemma 2.5 *Let $Z_1, \ldots, Z_n$ be the mean zero Gaussian random variables which are linearly independent and assume that*

$$\int_{-\infty}^{\infty} g(v) e^{-\varepsilon v^2}\,dv < \infty$$

for all $\varepsilon > 0$. Then

$$\begin{aligned} &\int_{\mathbb{R}^n} g(v_1) \exp\left[-\frac{1}{2}\mathrm{Var}\left(\sum_{j=1}^{n} v_j Z_j\right)\right] dv_1 \cdots dv_n \\ &\quad = \frac{(2\pi)^{(n-1)/2}}{(\det\mathrm{Cov}(Z_1,\ldots,Z_n))^{1/2}} \int_{-\infty}^{\infty} g\left(\frac{v}{\sigma_1}\right) e^{-v^2/2}\,dv, \end{aligned} \tag{2.19}$$

where $\sigma_1^2 = \mathrm{Var}(Z_1|Z_2,\ldots,Z_n)$ is the conditional variance of Z_1 given $Z_2, \ldots, Z_n$.

3 Intersection Local Times and Their Joint Continuity

In this section, we briefly recall the definition of local time as occupation density (cf. [6]) and then study the existence and joint continuity of the intersection local times of B^{α_1} and B^{α_2}.

Let $Y(t)$ be a [random] Borel vector field on $\mathbb{R}^p$ with values in $\mathbb{R}^q$. For any Borel set $E \subseteq \mathbb{R}^p$, the occupation measure of Y on E is defined as the following measure on $\mathbb{R}^q$:

$$\mu_E(\bullet) = \lambda_p\big\{t \in E : Y(t) \in \bullet\big\}.$$

If μ_E is absolutely continuous with respect to the Lebesgue measure λ_q, we say that $Y(t)$ has *local time* on E, and define its local time, $L(\bullet, E)$, as the Radon–Nikodým derivative of μ_E with respect to λ_q, i.e.,

$$L(x, E) = \frac{d\mu_E}{d\lambda_q}(x), \quad \forall x \in \mathbb{R}^q.$$

In the above, x is the so-called *space variable*, and E is the *time variable*. Note that if Y has local times on E then for every Borel set $F \subseteq E$, $L(x, F)$ also exists.

It follows from Theorem 6.4 in [6] that the local time has a measurable modification that satisfies the following *occupation density formula*: For every Borel set $E \subseteq \mathbb{R}^p$, and for every measurable function $f : \mathbb{R}^q \to \mathbb{R}_+$,

$$\int_E f\big(Y(t)\big)\, dt = \int_{\mathbb{R}^q} f(x) L(x, E)\, dx. \tag{3.1}$$

Suppose we fix a rectangle $E = [a, a+h] \subseteq \mathbb{R}^p$, where $a \in \mathbb{R}^p$ and $h \in \mathbb{R}^p_+$. If we can choose a version of the local time, still denoted by $L(x, [a, a+t])$, such that it is a continuous function of $(x, t) \in \mathbb{R}^q \times [0, h]$, Y is said to have a *jointly continuous local time* on E. When a local time is jointly continuous, $L(x, \cdot)$ can be extended to be a finite Borel measure supported on the level set

$$Y_E^{-1}(x) = \big\{t \in E : Y(t) = x\big\}; \tag{3.2}$$

see Theorem 8.6.1 in [1] for details. This makes local times, besides of interest on their own right, a useful tool in studying fractal properties of Y.

It follows from (25.5) and (25.7) in [6] that, for all $x, y \in \mathbb{R}^q$, $E \subseteq \mathbb{R}^p$ a closed interval and all integers $n \geq 1$,

$$\begin{aligned}\mathbb{E}\big[L(x, E)^n\big] = (2\pi)^{-nq} \int_{E^n} \int_{\mathbb{R}^{nq}} \exp\left(-i \sum_{j=1}^n \langle u_j, x\rangle\right) \\ \times \mathbb{E}\exp\left(i \sum_{j=1}^n \big\langle u_j, Y(t_j)\big\rangle\right) d\bar{u}\, d\bar{t}\end{aligned} \tag{3.3}$$

and, for all even integers $n \geq 2$,

$$\mathbb{E}\big[\big(L(x,E)-L(y,E)\big)^n\big] = (2\pi)^{-nq} \int_{E^n} \int_{\mathbb{R}^{nq}} \prod_{j=1}^{n} \big[e^{-i\langle u_j,x\rangle} - e^{-i\langle u_j,y\rangle}\big] \times \mathbb{E}\exp\Bigg(i\sum_{j=1}^{n}\langle u_j, Y(t_j)\rangle\Bigg) d\overline{u}\, d\overline{t}, \tag{3.4}$$

where $\overline{u} = (u_1,\dots,u_n)$, $\overline{t} = (t_1,\dots,t_n)$, and each $u_j \in \mathbb{R}^q$, $t_j \in E$. In the coordinate notation we then write $u_j = (u_{j,1},\dots,u_{j,q})$.

The main results of this section are the following Theorems 3.1 and 3.3 for the existence and the joint continuity of the intersection local times of two independent fractional Brownian motions in $\mathbb{R}^d$.

Theorem 3.1 *Let $B^{\alpha_1} = \{B^{\alpha_1}(s), s \in \mathbb{R}^{N_1}\}$ and $B^{\alpha_2} = \{B^{\alpha_2}(t), t \in \mathbb{R}^{N_2}\}$ be two independent fractional Brownian motions with values in $\mathbb{R}^d$ and Hurst indices α_1 and α_2, respectively. Let $X = \{X(s,t), (s,t) \in \mathbb{R}^N\}$ be the (N,d)-Gaussian random field defined by (1.3). Then, for any given constant $R > 0$, X has a local time $L(x, O_{N_1,N_2}(0,R)) \in L^2(\mathbb{P}\times\lambda_d)$ if and only if $\frac{N_1}{\alpha_1} + \frac{N_2}{\alpha_2} > d$. Furthermore, if it exists, the local time of X admits the following L^2-representation*

$$L\big(x, O_{N_1,N_2}(0,R)\big) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-i\langle y,x\rangle} \int_{O^2_{N_1,N_2}(0,R)} e^{i\langle y, B^{\alpha_1}(s) - B^{\alpha_2}(t)\rangle}\, ds\, dt\, dy, \tag{3.5}$$

and the local time L can be chosen as a kernel $L(\cdot,\cdot)$ on $\mathbb{R}^d \times \mathcal{B}(O_{N_1,N_2}(0,R))$. In particular, if $\frac{N_1}{\alpha_1} + \frac{N_2}{\alpha_2} > d$, then B^{α_1} and B^{α_2} have an intersection local time which can be defined as $L^{\alpha_1,\alpha_2}(O_{N_1,N_2}(0,R)) := L(0, O_{N_1,N_2}(0,R))$.

Some remarks about Theorem 3.1 are in order.

Remark 3.2 (i) When $N_1 = N_2 = 1$, $\alpha_1 = \alpha_2 = H$ and $Hd < 2$, the existence of the intersection local time was proved by Nualart and Oritz-Latorre [12] as the L^2-limit of

$$I_\varepsilon(B^H, \widetilde{B}^H) \equiv \int_0^R \int_0^R p_\varepsilon\big(B^H(s) - \widetilde{B}^H(t)\big)\, ds\, dt, \quad \text{as } \varepsilon \to 0, \tag{3.6}$$

where $p_\varepsilon(x) = (2\pi\varepsilon)^{d/2}\exp(-|x|^2/(2\varepsilon))$. They also proved that if $Hd \geq 2$, then

$$\lim_{\varepsilon\downarrow 0} \mathbb{E}\big[I_\varepsilon\big(B^H, \widetilde{B}^H\big)\big] = \infty \quad \text{and} \quad \lim_{\varepsilon\downarrow 0} \mathrm{Var}\big[I_\varepsilon\big(B^H, \widetilde{B}^H\big)\big] = \infty.$$

In the above, $B^H = \{B^H(t), t \geq 0\}$ and $\widetilde{B}^H = \{\widetilde{B}^H(t), t \geq 0\}$ are two independent fractional Brownian motions with values in $\mathbb{R}^d$ and index $H \in (0,1)$. Similar method can be applied to show that the intersection local time $L^{\alpha_1,\alpha_2}(O_{N_1,N_2}(0,R))$ in Theorem 3.1 can be chosen as the L^2-limit of the following approximating functionals

$$I_\varepsilon\big(B^{\alpha_1}, B^{\alpha_2}\big) \equiv \int_{O_{N_1,N_2}(0,R)} p_\varepsilon\big(B^{\alpha_1}(s) - B^{\alpha_2}(t)\big)\, ds\, dt, \quad \text{as } \varepsilon \to 0. \tag{3.7}$$

Moreover, we are able to show that, if $\frac{N_1}{\alpha_1}+\frac{N_2}{\alpha_2}=d$, then

$$\mathbb{E}\big[I_\varepsilon\big(B^{\alpha_1},B^{\alpha_2}\big)\big]\sim c(\alpha_1,\alpha_2,N_1,N_2)\ln\bigg(\frac{1}{\varepsilon}\bigg),\quad \text{as } \varepsilon\to 0, \tag{3.8}$$

where $c(\alpha_1,\alpha_2,N_1,N_2)>0$ is a constant depending on α_1,α_2 and N_1,N_2 only. This raises an interesting question whether I_ε can be renormalized to converge to a non-trivial limiting process. This and other related questions will be dealt with elsewhere since they require different methods.

(ii) It follows from the operator-scaling property (1.4) of X and (3.5) that the intersection local time $L^{\alpha_1,\alpha_2}(O_{N_1,N_2}(0,R))$ has the following scaling property: For any constant $c>0$,

$$\big\{L^{\alpha_1,\alpha_2}\big(c^A O_{N_1,N_2}(0,R)\big),\,R>0\big\}\overset{d}{=}\Big\{c^{\frac{N_1}{\alpha_1}+\frac{N_2}{\alpha_2}-d}L^{\alpha_1,\alpha_2}\big(O_{N_1,N_2}(0,R)\big),\,R>0\Big\}. \tag{3.9}$$

Here A is the $N\times N$ diagonal matrix as in (1.4).

(iii) We say that the sample functions of B^{α_1} and B^{α_2} intersect if there exist $s\in\mathbb{R}^{N_1}$ and $t\in\mathbb{R}^{N_2}$ such that $B^{\alpha_1}(s)=B^{\alpha_2}(t)$. It is also of interest to study the geometric properties of the set of intersection times

$$M_2=\big\{(s,t)\in\mathbb{R}^N: B^{\alpha_1}(s)=B^{\alpha_2}(t)\big\}$$

and the set of intersection points

$$D_2=\big\{x\in\mathbb{R}^d: x=B^{\alpha_1}(s)=B^{\alpha_2}(t) \text{ for some } (s,t)\in\mathbb{R}^N\big\},$$

because they are often random fractals. The existence of the intersection local time and its properties are closely related to the existence of intersections of the sample functions of B^{α_1} and B^{α_2} and the geometric properties of M_2 and D_2. Similar to Theorem 7.1 in [23], we can prove that if $\frac{N_1}{\alpha_1}+\frac{N_2}{\alpha_2}>d$ then $M_2\neq\emptyset$ with positive probability. On the other hand, Theorem 3.2 in [20] proved that if $\frac{N_1}{\alpha_1}+\frac{N_2}{\alpha_2}\le d$ then $M_2=\emptyset$ almost surely. In Sect. 4, we will give more information on the Hausdorff and packing dimensions of M_2, as well as a lower bound for the exact Hausdorff measure of M_2. The Hausdorff and packing dimensions of D_2 are determined in Sect. 5.

Proof of Theorem 3.1 Note that the Fourier transform of the occupation measure $\mu_{O_{N_1,N_2}(0,R)}$ of X is

$$\widehat{\mu}_{O_{N_1,N_2}(0,R)}(\xi)=\int_{O_{N_1,N_2}(0,R)}e^{i\langle\xi,X(s,t)\rangle}\,ds\,dt.$$

It follows from the Plancherel Theorem that X has a local time $L(x,O_{N_1,N_2}(0,R))\in L^2(\mathbb{P}\times\lambda_d)$ with a representation (3.5) if and only if

$$\mathcal{J}:=\int_{O^2_{N_1,N_2}(0,R)}ds\,dt\,dv\,dw\int_{\mathbb{R}^d}\big|\mathbb{E}\exp\big(i\langle y,X(s,t)-X(v,w)\rangle\big)\big|dy<\infty. \tag{3.10}$$

See Theorem 21.9 of [6]. Hence, it suffices to prove that (3.10) holds if and only if $\frac{N_1}{\alpha_1} + \frac{N_2}{\alpha_2} > d$. For this purpose, we use the independence of the coordinate processes of X, (1.3) and (1.1) to deduce that

$$\begin{aligned}\mathcal{J} &= \int_{O^2_{N_1,N_2}(0,R)} \frac{ds\,dt\,dv\,dw}{[\mathbb{E}(X_0(s,t) - X_0(v,w))^2]^{d/2}} \\ &= \int_{O^2_{N_1,N_2}(0,R)} \frac{ds\,dt\,dv\,dw}{[|s-v|^{2\alpha_1} + |t-w|^{2\alpha_2}]^{d/2}}. \end{aligned} \tag{3.11}$$

By using spherical variable substitutions and Lemma 2.2, it is elementary to verify that the last integral in (3.11) is finite if and only if $\frac{N_1}{\alpha_1} + \frac{N_2}{\alpha_2} > d$.

When the later holds, one can apply Theorem 6.3 in [6] to choose a version of the local time of X, still denoted by L, such that it is a kernel in the following sense: For every $x \in \mathbb{R}^d$, $L(x, \cdot)$ is a finite measure on $\mathcal{B}(O_{N_1,N_2}(0,R))$ and, for every Borel set $E \in \mathcal{B}(O_{N_1,N_2}(0,R))$, $x \mapsto L(x,E)$ is a measurable function. This proves the main conclusion of Theorem 3.1. Finally, by taking $x = 0$ we prove the last conclusion of Theorem 3.1. □

Theorem 3.3 *Let B^{α_1} and B^{α_2} be defined as that in Theorem 3.1. If $\frac{N_1}{\alpha_1} + \frac{N_2}{\alpha_2} > d$, then B^{α_1} and B^{α_2} have almost surely a continuous intersection local time on $\mathbb{R}^{N_1+N_2}$.*

As in the proof of Theorem 3.1, we will prove a stronger result that X has almost surely a jointly continuous local time on $\mathbb{R}^{N_1+N_2}$. The proof is based on the following Lemmas 3.4 and 3.6. They will also play an essential rôle in Sect. 4 for establishing Hölder conditions for the intersection local times.

Under the condition $\frac{N_1}{\alpha_1} + \frac{N_2}{\alpha_2} > d$, define

$$\tau = \begin{cases} 1 & \text{if } \frac{N_1}{\alpha_1} > d, \\ 2 & \text{if } \frac{N_1}{\alpha_1} \le d < \frac{N_1}{\alpha_1} + \frac{N_2}{\alpha_2} \end{cases} \tag{3.12}$$

and

$$\beta_\tau = \begin{cases} N - \alpha_1 d & \text{if } \tau = 1, \\ N_2 + \frac{\alpha_2}{\alpha_1} N_1 - \alpha_2 d & \text{if } \tau = 2. \end{cases} \tag{3.13}$$

(Recall that we assumed $\alpha_1 \le \alpha_2$ throughout the paper, and $N = N_1 + N_2$.) We will also make use of the following notation:

$$\eta_\tau = \begin{cases} \frac{\alpha_1 d}{N_1} & \text{if } \tau = 1, \\ \frac{\alpha_2 d}{N_2} + 1 - \frac{\alpha_2 N_1}{\alpha_1 N_2} & \text{if } \tau = 2. \end{cases} \tag{3.14}$$

Note that, if $N_1 = \alpha_1 d$, then $\beta_\tau = N_2$ and $\eta_\tau = 1$. To emphasize the importance of β_τ and η_τ, we point out that β_τ is the Hausdorff dimension of the set M_2 of intersection times and η_τ is useful for determining the exact Hausdorff measure of M_2. See Sect. 4 for more information.

Lemma 3.4 *Suppose the assumptions of Theorem 3.3 hold. Then, there exist positive and finite constants $\varepsilon \in (0, 1/e)$ and $c_{3,1}$, which depend on $\alpha_1, \alpha_2, N_1, N_2$ and d only, such that for all $r \in (0, \varepsilon)$, $D := O_{N_1,N_2}(u, r)$, where $u = (u_1, u_2) \in \mathbb{R}^N$, all $x \in \mathbb{R}^d$ and all integers $n \geq 1$, we have*

$$\mathbb{E}\big[L(x, D)^n\big] \leq \begin{cases} c_{3,1}^n (n!)^{\eta_1} r^{n\beta_1} & \text{if } \frac{N_1}{\alpha_1} > d, \\ c_{3,1}^n n! r^{nN_2} \prod_{j=1}^n \log\big(e + \frac{j^{(\frac{\alpha_2}{N_2} - \frac{\alpha_1}{N_1})^+}}{r^{\alpha_2-\alpha_1}}\big) & \text{if } \frac{N_1}{\alpha_1} = d, \\ c_{3,1}^n (n!)^{\eta_2} r^{n\beta_2} & \text{if } \frac{N_1}{\alpha_1} < d < \frac{N_1}{\alpha_1} + \frac{N_2}{\alpha_2}. \end{cases} \tag{3.15}$$

In the above, $y^+ = \max\{y, 0\}$ for every $y \in \mathbb{R}$.

Remark 3.5 From (3.13) and (3.14), it can be verified that

$$\frac{N - \beta_\tau}{N} \leq \eta_\tau \leq N - \beta_\tau. \tag{3.16}$$

We observe that the power of $n!$ in (3.15) becomes $(N - \beta_\tau)/N$ when X is an isotropic Gaussian field as in [19] and is $N - \beta_\tau$ when X is anisotropic in every coordinate (with the same scaling or Hölder index) as in [2]. These seem to be the extreme cases. In the present paper, if we assume $N_1 \neq N_2$ and $\alpha_1 \neq \alpha_2$, then strict inequalities in (3.16) may hold and if, in addition, $N_1 = \alpha_1 d$, then extra logarithmic factors appear in the estimate (3.15). Lemma 3.4 suggests that the local time $L(x, \cdot)$ may satisfy a law of the iterated logarithm which is different from those for the local times of an (N, d)-fractional Brownian motion or an (N, d)-fractional Brownian sheet with index $(\alpha, \ldots, \alpha)$; see (4.10), (4.12) and (4.13) below. This leads us to expect that the exact Hausdorff measure function for M_2 may be different from those for the level sets of fractional Brownian motion and fractional Brownian sheets, respectively. It would be interesting to investigate these problems.

Proof of Lemma 3.4 Even though the proof of Lemma 3.4 follows the same spirit of the proofs of Lemma 2.5 in [19] and Lemma 3.7 in [2], there are some subtle differences (see the remark above). Hence we give a complete proof. In particular, we provide a direct way to estimate the last integral in (3.17) below. We believe that this method will be useful elsewhere.

It follows from (3.3) and the fact that $X_1, \ldots, X_d$ are independent copies of X_0 that, for all integers $n \geq 1$,

$$\begin{aligned} \mathbb{E}\big[L(x, D)^n\big] &\leq (2\pi)^{-nd} \int_{D^n} \prod_{k=1}^d \bigg\{ \int_{\mathbb{R}^n} \exp\bigg[-\frac{1}{2}\mathrm{Var}\bigg(\sum_{j=1}^n u_{j,k} X_0(s_j, t_j)\bigg)\bigg] d\overline{u}_k \bigg\} d\bar{\mathbf{t}} \\ &= (2\pi)^{-nd/2} \int_{D^n} \big[\det\mathrm{Cov}\big(X_0(s_1, t_1), \ldots, X_0(s_n, t_n)\big)\big]^{-\frac{d}{2}} d\bar{\mathbf{t}}, \end{aligned} \tag{3.17}$$

where $\overline{u}_k = (u_{1,k}, \dots, u_{n,k}) \in \mathbb{R}^n$, $\bar{\mathbf{t}} = (s_1, t_1, \dots, s_n, t_n)$ and the equality follows from the fact that for any positive definite $n \times n$ matrix Γ,

$$\int_{\mathbb{R}^n} \frac{[\det(\Gamma)]^{1/2}}{(2\pi)^{n/2}} \exp\Big(-\frac{1}{2}x'\Gamma x\Big)\, dx = 1. \tag{3.18}$$

In order to prove (3.15), we consider the three cases separately: $\frac{N_1}{\alpha_1} > d$, $\frac{N_1}{\alpha_1} < d < \frac{N_1}{\alpha_1} + \frac{N_2}{\alpha_2}$, and $\frac{N_1}{\alpha_1} = d$.

In the case $\frac{N_1}{\alpha_1} > d$, thanks to (2.6), we have

$$\begin{aligned}
&\mathbb{E}\big[L(x, D)^n\big] \\
&\quad\le c_{3,2}^n \int_{D^n} \prod_{j=1}^n \frac{1}{\min\{|s_j - s_i|^{\alpha_1 d}, 0 \le i \le j-1\}}\, d\bar{s}\, d\bar{t} \\
&\quad= c_{3,2}^n \int_{O^n_{N_2}(u_2,r)} \bigg(\int_{O^n_{N_1}(u_1,r)} \prod_{j=1}^n \frac{1}{(\min\{|s_j - s_i|^{\alpha_1}, 0 \le i \le j-1\})^d}\, d\bar{s}\bigg)\, d\bar{t} \\
&\quad= c_{3,3}^n r^{nN_2} \int_{O^n_{N_1}(u_1,r)} \prod_{j=1}^n \frac{1}{\min\{|s_j - s_i|^{\alpha_1 d}, 0 \le i \le j-1\}}\, d\bar{s},
\end{aligned} \tag{3.19}$$

where $s_0 := 0$, $\bar{s} = (s_1, \dots, s_n)$ and $\bar{t} = (t_1, \dots, t_n)$.

Since $N_1 > \alpha_1 d$, we integrate the last integral in (3.19) in the order $ds_n, \dots, ds_1$ and apply Part (i) of Lemma 2.4 iteratively. This yields

$$\mathbb{E}\big[L(x, D)^n\big] \le c_{3,1}^n (n!)^{\frac{\alpha_1 d}{N_1}} r^{n(N_1 - \alpha_1 d)} \times r^{nN_2} = c_{3,1}^n (n!)^{\eta_1} r^{n\beta_1}, \tag{3.20}$$

which proves (3.15) for the case $\frac{N_1}{\alpha_1} > d$ [i.e., $\tau = 1$].

In the second and third cases [i.e., $\frac{N_1}{\alpha_1} \le d < \frac{N_1}{\alpha_1} + \frac{N_2}{\alpha_2}$] we use (3.17) and (2.6) to obtain

$$\begin{aligned}
&\mathbb{E}\big[L(x, D)^n\big] \\
&\le c_{3,4}^n \int_{D^n} \prod_{j=1}^n \frac{1}{(\min_{0 \le k \le j-1} |s_j - s_k|^{\alpha_1} + \min_{0 \le k \le j-1} |t_j - t_k|^{\alpha_2})^d}\, d\bar{s}\, d\bar{t}.
\end{aligned} \tag{3.21}$$

To estimate the last integral in (3.21), we will integrate in the order of $ds_n, dt_n, \dots, ds_1, dt_1$. In the case of $\frac{N_1}{\alpha_1} < d < \frac{N_1}{\alpha_1} + \frac{N_2}{\alpha_2}$, we apply Part (i) of Lemma 2.3 with $A = \min_{0 \le k \le n-1} |t_n - t_k|^{\alpha_2}$ to derive

$$\begin{aligned}
&\int_{O_{N_1}(u_1,r)} \frac{ds_n}{(\min_{0 \le k \le n-1} |s_n - s_k|^{\alpha_1} + \min_{0 \le k \le n-1} |t_n - t_k|^{\alpha_2})^d} \\
&\qquad\le \frac{c_{2,3}\, n}{[\min_{0 \le k \le n-1} |t_n - t_k|^{\alpha_2}]^{d - \frac{N_1}{\alpha_1}}}.
\end{aligned} \tag{3.22}$$

Since $\alpha_2(d-\frac{N_1}{\alpha_1})<N_2$, it follows from (3.22) and Part (i) of Lemma 2.4 that

$$\begin{aligned}\int_D &\frac{ds_n dt_n}{(\min_{0\le k\le n-1}|s_n-s_k|^{\alpha_1}+\min_{0\le k\le n-1}|t_n-t_k|^{\alpha_2})^d}\\ &\le c_{2,3}n\int_{O_{N_2}(u_2,r)}\frac{dt_n}{(\min_{0\le k\le n-1}|t_n-t_k|^{\alpha_2})^{d-\frac{N_1}{\alpha_1}}}\\ &\le c_{3,5}n^{1+\frac{\alpha_2(\alpha_1 d-N_1)}{\alpha_1 N_2}}r^{N_2-\alpha_2(d-\frac{N_1}{\alpha_1})}=c_{3,5}n^{\eta_2}r^{\beta_2}.\end{aligned}\tag{3.23}$$

Repeating the above procedure yields (3.15) for the case of $\frac{N_1}{\alpha_1}<d<\frac{N_1}{\alpha_1}+\frac{N_2}{\alpha_2}$.

Finally, we consider the case of $\frac{N_1}{\alpha_1}=d$. Let $\kappa\in(0,1)$ be a constant such that $\kappa\alpha_2/\alpha_1<N_2$. Applying Part (ii) of Lemma 2.3 with $A=\min_{0\le k\le n-1}|t_n-t_k|^{\alpha_2}$, we have

$$\begin{aligned}\int_{O_{N_1}(u_1,r)}&\frac{ds_n}{(\min_{0\le k\le n-1}|s_n-s_k|^{\alpha_1}+\min_{0\le k\le n-1}|t_n-t_k|^{\alpha_2})^d}\\ &\le c_{3,6}n\log\Bigg[e+\bigg(\Big(\min_{0\le k\le n-1}|t_n-t_k|^{\alpha_2}\Big)^{-1/\alpha_1}\frac{r}{n^{1/N_1}}\bigg)^{\kappa}\Bigg].\end{aligned}\tag{3.24}$$

It follows from (3.24) and Part (ii) of Lemma 2.4 (with $\beta=\kappa\alpha_2/\alpha_1$ and $K=(rn^{-1/N_1})^{\kappa}$) that

$$\begin{aligned}\int_D &\frac{ds_n dt_n}{(\min_{0\le k\le n-1}|s_n-s_k|^{\alpha_1}+\min_{0\le k\le n-1}|t_n-t_k|^{\alpha_2})^d}\\ &\le c_{3,6}n\int_{O_{N_2}(u_2,r)}\log\Bigg[e+\Big(\min_{0\le k\le n-1}|t_n-t_k|\Big)^{-\kappa\alpha_2/\alpha_1}\Big(\frac{r}{n^{1/N_1}}\Big)^{\kappa}\Bigg]dt_n\\ &\le c_{3,7}nr^{N_2}\log\Bigg[e+\Big(\frac{r}{n^{1/N_2}}\Big)^{-\kappa\alpha_2/\alpha_1}\Big(\frac{r}{n^{1/N_1}}\Big)^{\kappa}\Bigg]\\ &=c_{3,7}nr^{N_2}\log\Bigg[e+\bigg(\frac{n^{\frac{\alpha_2}{N_2}-\frac{\alpha_1}{N_1}}}{r^{\alpha_2-\alpha_1}}\bigg)^{\kappa/\alpha_1}\Bigg]\\ &\le c_{3,8}nr^{N_2}\log\bigg(e+\frac{n^{(\frac{\alpha_2}{N_2}-\frac{\alpha_1}{N_1})^+}}{r^{\alpha_2-\alpha_1}}\bigg).\end{aligned}\tag{3.25}$$

(Recall that $y^+=\max\{y,0\}$.)

By iterating the procedure and integrating $ds_{n-1},dt_{n-1},\dots,ds_1,dt_1$, we obtain

$$\mathbb{E}\big[L(x,D)^n\big]\le c_{3,1}^n(n!)^{\eta_2}r^{n\beta_2}\prod_{j=1}^n\log\bigg(e+\frac{j^{(\frac{\alpha_2}{N_2}-\frac{\alpha_1}{N_1})^+}}{r^{\alpha_2-\alpha_1}}\bigg).\tag{3.26}$$

This finishes the proof of the moment estimate (3.15). □

The following lemma estimates the higher moments of the increments of the local times of X. Combined with Kolmogorov's continuity theorem, it immediately implies the existence of a continuous version of $x \mapsto L(x, D)$.

Lemma 3.6 *Suppose the assumptions of Theorem 3.3 hold. Then, there exist positive constants $c_{3,9}$ and κ_1, depending on $\varepsilon, \alpha_1, \alpha_2, N_1, N_2$ and d only, such that, for any $r > 0$, $D := O_{N_1,N_2}(u, r)$ for $u = (u_1, u_2) \in \mathbb{R}^N$, all $x, y \in \mathbb{R}^d$ with $|x - y| \le 1$, all even integers $n \ge 1$, and all $\gamma \in (0, 1)$ small enough, we have*

$$\mathbb{E}\big[\big(L(x, D) - L(y, D)\big)^n\big] \le c_{3,9}^n (n!)^{\eta_\tau + \kappa_1\gamma} |x - y|^{n\gamma} r^{n(\beta_\tau - \kappa_1\gamma)}. \tag{3.27}$$

Proof Let $\gamma \in (0, 1)$ be a small constant whose value will be determined later. Note that by the elementary inequalities

$$|e^{iu} - 1| \le 2^{1-\gamma} |u|^\gamma \quad \text{for all } u \in \mathbb{R} \tag{3.28}$$

and $|u + v|^\gamma \le |u|^\gamma + |v|^\gamma$, we see that for all $u_1, \dots, u_n,\ x,\ y \in \mathbb{R}^d$,

$$\prod_{j=1}^n \big|e^{-i\langle u_j, x\rangle} - e^{-i\langle u_j, y\rangle}\big| \le 2^{(1-\gamma)n}\ |x - y|^{n\gamma} \sum{}' \prod_{j=1}^n |u_{j,k_j}|^\gamma, \tag{3.29}$$

where the summation $\sum'$ is taken over all the sequences $(k_1, \dots, k_n) \in \{1, \dots, d\}^n$.

It follows from (3.4) and (3.29) that for every even integer $n \ge 2$,

$$\begin{aligned} &\mathbb{E}\big[\big(L(x, D) - L(y, D)\big)^n\big] \\ &\quad \le (2\pi)^{-nd} 2^{(1-\gamma)n} |x - y|^{n\gamma} \\ &\qquad \times \sum{}' \int_{D^n} \int_{\mathbb{R}^{nd}} \prod_{m=1}^n |u_{m,k_m}|^\gamma\, \mathbb{E} \exp\Bigg(-i \sum_{j=1}^n \langle u_j, X(s_j, t_j)\rangle\Bigg)\, d\overline{u}\, d\overline{\mathbf{t}} \\ &\quad \le c_{3,10}^n |x - y|^{n\gamma} \sum{}' \int_{D^n} d\overline{\mathbf{t}} \\ &\qquad \times \prod_{m=1}^n \Bigg\{\int_{\mathbb{R}^{nd}} |u_{m,k_m}|^{n\gamma} \exp\Bigg[-\frac{1}{2}\mathrm{Var}\Bigg(\sum_{j=1}^n \langle u_j, X(s_j, t_j)\rangle\Bigg)\Bigg]\, d\overline{u}\Bigg\}^{1/n}, \end{aligned} \tag{3.30}$$

where the last inequality follows from the generalized Hölder inequality.

Now we fix a vector $\overline{k} = (k_1, k_2, \dots, k_n) \in \{1, \dots, d\}^n$ and n points $(s_1, t_1), \dots, (s_n, t_n) \in D\backslash\{0\}$ such that $s_1, t_1, \dots, s_n, t_n$ are all distinct [the set of such points has full nN-dimensional Lebesgue measure]. Let $\mathcal{M} = \mathcal{M}(\overline{k}, \overline{\mathbf{t}}, \gamma)$ be defined by

$$\mathcal{M} = \prod_{m=1}^n \Bigg\{\int_{\mathbb{R}^{nd}} |u_{m,k_m}|^{n\gamma} \exp\Bigg[-\frac{1}{2}\mathrm{Var}\Bigg(\sum_{j=1}^n \langle u_j, X(s_j, t_j)\rangle\Bigg)\Bigg]\, d\overline{u}\Bigg\}^{1/n}. \tag{3.31}$$

Note that X_ℓ $(1 \le \ell \le d)$ are independent copies of X_0. By the strong local nondeterminism of fractional Brownian motions $B_0^{\alpha_1}$ and $B_0^{\alpha_2}$ and (2.6), the random variables

$X_\ell(s_j, t_j)$ $(1 \le \ell \le d, 1 \le j \le n)$ are linearly independent. Hence Lemma 2.5 gives

$$\begin{aligned}
&\int_{\mathbb{R}^{nd}} \left|u_{k_m}^m\right|^{n\gamma} \exp\left[-\frac{1}{2}\mathrm{Var}\left(\sum_{j=1}^n \langle u_j, X(s_j,t_j)\rangle\right)\right] d\overline{u} \\
&\quad = \frac{(2\pi)^{(nd-1)/2}}{[\det\mathrm{Cov}(X_0(s_1,t_1),\dots,X_0(s_n,t_n))]^{d/2}} \int_{\mathbb{R}} \left(\frac{v}{\sigma_m}\right)^{n\gamma} e^{-\frac{v^2}{2}}\, dv \\
&\quad \le \frac{c_{3,11}^n (n!)^\gamma}{[\det\mathrm{Cov}(X_0(s_1,t_1),\dots,X_0(s_n,t_n))]^{d/2}} \frac{1}{\sigma_m^{n\gamma}},
\end{aligned} \tag{3.32}$$

where σ_m^2 is the conditional variance of $X_{k_m}(s_m, t_m)$ given $X_i(s_j, t_j)$ ($i \ne k_m$ or $i = k_m$ but $j \ne m$), and the last inequality follows from Stirling's formula.

Combining (3.31) and (3.32) we obtain

$$\mathcal{M} \le \frac{c_{3,11}^n (n!)^\gamma}{[\det\mathrm{Cov}(X_0(s_1,t_1),\dots,X_0(s_n,t_n))]^{d/2}} \prod_{m=1}^n \frac{1}{\sigma_m^\gamma}. \tag{3.33}$$

The second product in (3.33) is a "perturbation" factor and will be shown to be small when integrated. For this purpose, we use again the independence of the coordinate processes of X, (2.2) and (2.1) to derive

$$\begin{aligned}
\sigma_m^2 &= \mathrm{Var}\big(X_{k_m}(s_m,t_m) \,\big|\, X_{k_m}(s_j,t_j),\ j \ne m\big) \\
&\ge \mathrm{Var}\big(B_{k_m}^{\alpha_1}(s_m) \,\big|\, B_{k_m}^{\alpha_1}(s_j),\ j \ne m\big) + \mathrm{Var}\big(B_{k_m}^{\alpha_2}(t_m) \,\big|\, B_{k_m}^{\alpha_2}(t_j),\ j \ne m\big) \\
&\ge c_{3,12}^2\big(\min\big\{|s_m - s_j|^{2\alpha_1} : j \ne m\big\} + \min\big\{|t_m - t_j|^{2\alpha_2} : j \ne m\big\}\big).
\end{aligned} \tag{3.34}$$

As in the proof of (3.15), we will prove (3.27) by cases.

If $\frac{N_1}{\alpha_1} > d$, then we take $\gamma \in (0, \frac{1}{2}(\frac{N_1}{\alpha_1} - d))$ so that

$$\alpha_1(d + 2\gamma) < N_1. \tag{3.35}$$

For any n points $(s_1, t_1), \dots, (s_n, t_n) \in D\backslash\{0\}$, we define a permutation π_s of $\{1, 2, \dots, n\}$ such that

$$\begin{aligned}
&|s_{\pi_s(1)}| = \min\big\{|s_i|, i = 1, \dots, n\big\}, \\
&|s_{\pi_s(j)} - s_{\pi_s(j-1)}| = \min\big\{|s_i - s_{\pi_s(j-1)}|, i \in \{1,\dots,n\}\big\backslash\{\pi_s(1),\dots,\pi_s(j-1)\}\big\}.
\end{aligned}$$

Then, by (3.34), we have

$$\begin{aligned}
\prod_{m=1}^n \frac{1}{\sigma_m^\gamma} &\le \prod_{m=1}^n \frac{1}{c_{3,12}[|s_{\pi_s(m)} - s_{\pi_s(m-1)}|^{\alpha_1} \wedge |s_{\pi_s(m+1)} - s_{\pi_s(m)}|^{\alpha_1}]^\gamma} \\
&\le c_{3,12}^{-n} \prod_{m=1}^n \frac{1}{|s_{\pi_s(m)} - s_{\pi_s(m-1)}|^{2\alpha_1\gamma}} \\
&\le c_{3,12}^{-n} \frac{1}{[\det\mathrm{Cov}(B_0^{\alpha_1}(s_1),\dots,B_0^{\alpha_1}(s_n))]^\gamma}.
\end{aligned} \tag{3.36}$$

It follows from (3.33), (2.6) and (3.36) that

$$\begin{aligned}\mathcal{M} &\le \frac{c_{3,11}^n (n!)^\gamma}{[\det\mathrm{Cov}(B_0^{\alpha_1}(s_1), \dots, B_0^{\alpha_1}(s_n))]^{d/2}} \prod_{m=1}^n \frac{1}{\sigma_m^\gamma} \\ &\le c_{3,13}^n (n!)^\gamma \frac{1}{[\det\mathrm{Cov}(B_0^{\alpha_1}(s_1), \dots, B_0^{\alpha_1}(s_n))]^{(d+2\gamma)/2}} \\ &\le c_{3,14}^n (n!)^\gamma \prod_{j=1}^n \frac{1}{\min\{|s_j - s_i|^{\alpha_1(d+2\gamma)}, 0 \le i \le j-1\}}. \end{aligned} \tag{3.37}$$

Therefore, by (3.35) and Lemma 2.4, we have

$$\begin{aligned}\int_{D^n} \mathcal{M}\big(\bar{k}, \bar{\mathbf{t}}, \gamma\big)\, d\bar{\mathbf{t}} &\le c_{3,14}^n (n!)^\gamma \int_{D^n} \prod_{j=1}^n \frac{1}{\min\{|s_j - s_i|^{\alpha_1(d+2\gamma)}, 0 \le i \le j-1\}}\, d\bar{\mathbf{t}} \\ &\le c_{3,15}^n (n!)^{\frac{\alpha_1(d+2\gamma)}{N_1}+\gamma} r^{n\left(N_1 - \alpha_1(d+2\gamma)\right)} \times r^{nN_2} \\ &= c_{3,15}^n (n!)^{\eta_1 + \left(\frac{2\alpha_1}{N_1}+1\right)\gamma} r^{n(\beta_1 - 2\alpha_1\gamma)}. \end{aligned} \tag{3.38}$$

We combine (3.30) and (3.38) to obtain

$$\mathbb{E}\Big[\big(L(x, D) - L(y, D)\big)^n\Big] \le c_{3,16}^n (n!)^{\eta_1 + \left(\frac{2\alpha_1}{N_1}+1\right)\gamma} |x-y|^{n\gamma} r^{n(\beta_1 - 2\alpha_1\gamma)}. \tag{3.39}$$

By choosing the constant $\kappa_1 \ge \max\{\frac{2\alpha_1}{N_1} + 1, 2\alpha_1\}$, we prove (3.27) for the case $\frac{N_1}{\alpha_1} > d$ (i.e., $\tau = 1$).

Now we prove (3.27) for the case of $\frac{N_1}{\alpha_1} \le d < \frac{N_1}{\alpha_1} + \frac{N_2}{\alpha_2}$. Inspired by Lemma 3.4 in [2], we choose

$$\begin{aligned}\gamma &\in \left(0, \frac{1}{4}\left(\frac{N_1}{\alpha_1} + \frac{N_2}{\alpha_2} - d\right)\right), \\ \delta &= \frac{1}{2}\min\left\{1, \alpha_1\left(\frac{N_1}{\alpha_1} + \frac{N_2}{\alpha_2} - d\right), \alpha_1\gamma\right\}\end{aligned} \tag{3.40}$$

and set

$$\frac{1}{p_1} = \frac{N_1 - \delta}{\alpha_1 d}, \qquad \frac{1}{p_2} = 1 - \frac{1}{p_1}. \tag{3.41}$$

Clearly, we have

$$p_1 > 1, \qquad p_2 > 1, \qquad \frac{1}{p_1} + \frac{1}{p_2} = 1 \tag{3.42}$$

and

$$\begin{aligned} \frac{\alpha_1 d}{p_1} &= N_1 - \delta < N_1, \\ \frac{\alpha_2 d}{p_2} &= \alpha_2\left(d - \frac{N_1}{\alpha_1} + \frac{\delta}{\alpha_1}\right) < N_2, \end{aligned} \tag{3.43}$$

where the last inequality follows from the fact that $\delta \le \frac{\alpha_1}{2}(\frac{N_1}{\alpha_1} + \frac{N_2}{\alpha_2} - d)$. By a simple computation, we also have

$$\frac{\alpha_1 d}{p_1} + \frac{\alpha_2 d}{p_2} = N_1 + \alpha_2 d - \frac{\alpha_2}{\alpha_1} N_1 + \left(\frac{\alpha_2}{\alpha_1} - 1\right)\delta = N - \beta_2 + \left(\frac{\alpha_2}{\alpha_1} - 1\right)\delta \tag{3.44}$$

and

$$\frac{\alpha_1 d}{N_1 p_1} + \frac{\alpha_2 d}{N_2 p_2} = 1 + \frac{\alpha_2 d}{N_2} - \frac{\alpha_2 N_1}{\alpha_1 N_2} + \left(\frac{\alpha_2}{\alpha_1 N_2} - \frac{1}{N_1}\right)\delta = \eta_2 + \left(\frac{\alpha_2}{\alpha_1 N_2} - \frac{1}{N_1}\right)\delta. \tag{3.45}$$

Furthermore, from the way we define γ, δ and p_2, we know

$$\frac{\alpha_2 d}{p_2} + 2\alpha_2 \gamma < N_2. \tag{3.46}$$

For any n points $(s_1, t_1), \dots, (s_n, t_n) \in D \backslash \{0\}$, we define a permutation π_t of $\{1, 2, \dots, n\}$ such that

$$\begin{aligned} &|t_{\pi_t(1)}| = \min\bigl\{|t_i|, i = 1, \dots, n\bigr\}, \\ &|t_{\pi_t(j)} - t_{\pi_t(j-1)}| = \min\bigl\{|t_i - t_{\pi_t(j-1)}|, i \in \{1, \dots, n\} \backslash \bigl\{\pi_t(1), \dots, \pi_t(j-1)\bigr\}\bigr\}. \end{aligned}$$

Then, by (3.34), we have

$$\begin{aligned} \prod_{m=1}^{n} \frac{1}{\sigma_m^{\gamma}} &\le \prod_{m=1}^{n} \frac{1}{c_{3,12}[|t_{\pi_t(m)} - t_{\pi_t(m-1)}|^{\alpha_2} \wedge |t_{\pi_t(m+1)} - t_{\pi_t(m)}|^{\alpha_2}]^{\gamma}} \\ &\le c_{3,12}^{-n} \frac{1}{[\det\mathrm{Cov}(B_0^{\alpha_2}(t_1), \dots, B_0^{\alpha_2}(t_n))]^{\gamma}}. \end{aligned} \tag{3.47}$$

Recall from (2.6) that

$$\begin{aligned} &\det\mathrm{Cov}\bigl(X_0(s_1, t_1), \dots, X_0(s_n, t_n)\bigr) \\ &\quad \ge \det\mathrm{Cov}\bigl(B_0^{\alpha_1}(s_1), \dots, B_0^{\alpha_1}(s_n)\bigr) + \det\mathrm{Cov}\bigl(B_0^{\alpha_2}(t_1), \dots, B_0^{\alpha_2}(t_n)\bigr). \end{aligned} \tag{3.48}$$

Hence,

$$\begin{aligned} &\bigl[\det\mathrm{Cov}\bigl(X_0(s_1, t_1), \dots, X_0(s_n, t_n)\bigr)\bigr]^{-\frac{1}{2}} \\ &\quad \le \bigl[\det\mathrm{Cov}\bigl(B_0^{\alpha_1}(s_1), \dots, B_0^{\alpha_1}(s_n)\bigr)\bigr]^{-\frac{1}{2p_1}} \bigl[\det\mathrm{Cov}\bigl(B_0^{\alpha_2}(t_1), \dots, B_0^{\alpha_2}(t_n)\bigr)\bigr]^{-\frac{1}{2p_2}}. \end{aligned} \tag{3.49}$$

It follows from (3.33), (3.47) and (3.49) that

$$\mathcal{M} \le \frac{c_{3,11}^n (n!)^\gamma}{[\det\mathrm{Cov}(X_0(s_1,t_1),\ldots,X_0(s_n,t_n))]^{d/2}} \prod_{m=1}^n \frac{1}{\sigma_m^\gamma}$$
$$\le \frac{c_{3,17}^n (n!)^\gamma}{[\det\mathrm{Cov}(B_0^{\alpha_1}(s_1),\ldots,B_0^{\alpha_1}(s_n))]^{\frac{d}{2p_1}}[\det\mathrm{Cov}(B_0^{\alpha_2}(t_1),\ldots,B_0^{\alpha_2}(t_n))]^{\frac{d}{2p_2}+\gamma}}. \tag{3.50}$$

Combining (3.50), (3.43), (3.46) and Lemma 2.4, we obtain

$$\int_{D^n} \mathcal{M}\big(\bar{k},\bar{\mathbf{t}},\gamma\big)\,d\bar{\mathbf{t}} \le c_{3,17}^n (n!)^\gamma \int_{O_{N_1}^n(u_1,r)} \frac{d\bar{s}}{[\det\mathrm{Cov}(B_0^{\alpha_1}(s_1),\ldots,B_0^{\alpha_1}(s_n))]^{\frac{d}{2p_1}}}$$
$$\times \int_{O_{N_2}^n(u_2,r)} \frac{d\bar{t}}{[\det\mathrm{Cov}(B_0^{\alpha_2}(t_1),\ldots,B_0^{\alpha_2}(t_n))]^{\frac{d}{2p_2}+\gamma}}$$
$$\le c_{3,18}^n (n!)^{\sum_{\ell=1}^2 \frac{\alpha_\ell d}{N_\ell p_\ell}+(1+\frac{2\alpha_2}{N_2})\gamma} r^{n(N-\sum_{\ell=1}^2\frac{\alpha_\ell d}{p_\ell}-\alpha_2\gamma)}$$
$$\le c_{3,18}^n (n!)^{\eta_2+\kappa_1\gamma} r^{n(\beta_2-\kappa_1\gamma)}. \tag{3.51}$$

In the above, the constant $\kappa_1 > 0$ is chosen appropriately by taking into account (3.44), (3.45) and (3.39). The value of κ_1 depends on $\alpha_1, \alpha_2, N_1, N_2$ and d only.

We combine (3.30) and (3.51) to obtain

$$\mathbb{E}\big[\big(L(x,D)-L(y,D)\big)^n\big] \le c_{3,9}^n (n!)^{\eta_2+\kappa_1\gamma} |x-y|^{n\gamma} r^{n(\beta_2-\kappa_1\gamma)}. \tag{3.52}$$

This proves (3.27) for the case of $\frac{N_1}{\alpha_1} \le d < \frac{N_1}{\alpha_1} + \frac{N_2}{\alpha_2}$ (i.e., $\tau = 2$). The proof of Lemma 3.6 is complete. □

Now we are ready to prove Theorem 3.3.

Proof of Theorem 3.3 The proof of the joint continuity of the local time of X is similar to that of Theorem 3.1 in [2] (see also the proof of Theorem 8.2 in [23]). Hence we only give a sketch of it.

It suffices to show that for any fixed $u = (u_1, u_2) \in \mathbb{R}^N$ and $R > 0$, the local time $L(x,(s,t)) := L(x, [u_1, u_1+s] \times [u_2, u_2+t])$ has a version which is continuous in $(x,s,t) \in \mathbb{R}^d \times [0,R]^N$ almost surely. For simplicity of notation, we assume $u = (0,0)$. Observe that for all $x, y \in \mathbb{R}^d$, $(s,t), (v,w) \in [0,R]^N$ and all even integers $n \ge 1$, we have

$$\mathbb{E}\big[\big(L\big(x,[0,s]\times[0,t]\big)-L\big(y,[0,v]\times[0,w]\big)\big)^n\big]$$
$$\le 2^{n-1}\big\{\mathbb{E}\big[\big(L\big(x,[0,s]\times[0,t]\big)-L\big(x,[0,v]\times[0,w]\big)\big)^n\big]$$
$$+\mathbb{E}\big[\big(L\big(x,[0,v]\times[0,w]\big)-L\big(y,[0,v]\times[0,w]\big)\big)^n\big]\big\}. \tag{3.53}$$

Since $L(x,\cdot)$ is a finite Borel measure, the difference $L(x,[0,s]\times[0,t]) - L(x,[0,v]\times[0,w])$ can be bounded by a sum of finite number of terms of the form $L(x,D_j)$, where each D_j is a closed subset of $[0,R]^N$ of the form $O_{N_1}(\cdot,r)\times O_{N_2}(\cdot,r)$ with the radius $r\le \frac{1}{2}|(s,t)-(v,w)| := \frac{1}{2}\sqrt{|s-v|^2+|t-w|^2}$. We can use (3.15) to bound the first term in (3.53). On the other hand, the second term in (3.53) can be dealt with by using (3.27). Consequently, there exist some constants $\gamma\in(0,1)$ and n_0 such that for all $x,y\in\mathbb{R}^d$, $(s,t),(v,w)\in O_{N_1,N_2}(0,R)\cap\mathbb{R}^N_+$ and all even integers $n\ge n_0$,

$$\begin{aligned}&\mathbb{E}\big[\big(L\big(x,[0,s]\times[0,t]\big)-L\big(y,[0,v]\times[0,w]\big)\big)^n\big]\\&\quad\le c_{3,19}^n\big(|x-y|+\big|(s,t)-(v,w)\big|\big)^{n\gamma}.\end{aligned}\tag{3.54}$$

It follows from (3.54) and the multiparameter version of Kolmogorov's continuity theorem (cf. [9]) that there exists a modification of the local times of X, still denoted by $L(x,(s,t))$, such that it is continuous for $x\in\mathbb{R}^d$, $(s,t)\in[0,R]^N$. This finishes the proof of Theorem 3.3. □

4 Exponential Integrability and Hölder Conditions for the Intersection Local Times

In this section, we investigate the exponential integrability and asymptotic behavior of the local time $L(x,\cdot)$ of X. As applications of the later result, we obtain a lower bound for the exact Hausdorff measure of the set M_2 of the intersection times of B^{α_1} and B^{α_2}.

The following two technical lemmas will play essential rôles in our derivation.

Lemma 4.1 *Under the conditions of Theorem 3.3, there exist positive and finite constants $\varepsilon\in(0,1/e)$, $c_{4,1}$ and $c_{4,2}$, depending on α_1,α_2, N_1, N_2, and d only, such that the following hold*:

(i) *For all $(a_1,a_2)\in\mathbb{R}^N$ and $D=O_{N_1,N_2}((a_1,a_2),r)$ with radius $r\in(0,\varepsilon)$, $x\in\mathbb{R}^d$ and all integers $n\ge 1$,*

$$\mathbb{E}\big[L\big(x+X(a_1,a_2),D\big)^n\big]\le\begin{cases}c_{4,1}^n(n!)^{\eta_\tau}r^{n\beta_\tau} & \text{if } \frac{N_1}{\alpha_1}\ne d,\\ c_{4,1}^n n! r^{nN_2}\prod_{j=1}^n\log\big(e+\frac{j^{(\frac{\alpha_2}{N_2}-\frac{\alpha_1}{N_1})^+}}{r^{\alpha_2-\alpha_1}}\big) & \text{if } \frac{N_1}{\alpha_1}=d.\end{cases}\tag{4.1}$$

(ii) *For all $(a_1,a_2)\in\mathbb{R}^N$ and $D=O_{N_1,N_2}((a_1,a_2),r)$ with radius $r>0$, $x,y\in\mathbb{R}^d$ with $|x-y|\le 1$, all even integers $n\ge 1$ and all $\gamma\in(0,1)$ small,*

$$\begin{aligned}&\mathbb{E}\big[\big(L\big(x+X(a_1,a_2),D\big)-L\big(y+X(a_1,a_2),D\big)\big)^n\big]\\&\quad\le c_{4,2}^n(n!)^{\eta_\tau+\kappa_1\gamma}|x-y|^{n\gamma}r^{n(\beta_\tau-\kappa_1\gamma)}.\end{aligned}\tag{4.2}$$

In the above, $\kappa_1>0$ is the same constant as in Lemma 3.6.

Proof For any fixed $(a_1, a_2) \in \mathbb{R}^N$, we define the Gaussian random field $Y = \{Y(s,t),\ (s,t) \in \mathbb{R}^N\}$ with values in $\mathbb{R}^d$ by $Y(s,t) = X(s,t) - X(a_1,a_2)$. It follows from (3.1) that if X has a local time $L(x,S)$ on any Borel set S, then Y also has a local time $\tilde{L}(x,S)$ on S and, moreover, $L(x + X(a_1,a_2), S) = \tilde{L}(x,S)$. Since X has stationary increments, both Lemmas 3.4 and 3.6 hold for the Gaussian field Y. This proves (4.1) and (4.2). □

The following lemma is a consequence of Lemma 4.1 and Chebyshev's inequality.

Lemma 4.2 *Assume the conditions of Theorem 3.3 hold. For any $b > 0$, there exist positive and finite constants $\varepsilon \in (0, 1/e)$, $c_{4,3}, c_{4,4}, c_{4,5}$ (depending on $\alpha_1, \alpha_2, N_1, N_2$ and d only), such that for all $(a_1,a_2) \in \mathbb{R}^N$, $D = O_{N_1,N_2}((a_1,a_2), r)$ with $r \in (0,\varepsilon)$, $x \in \mathbb{R}^d$ and $u > 1$ large enough, the following inequalities hold:*

(i) *If $N_1 \neq \alpha_1 d$, then*

$$\mathbb{P}\big\{L\big(x + X(a_1,a_2), D\big) \geq c_{4,3} r^{\beta_\tau} u^{\eta_\tau}\big\} \leq \exp(-bu). \tag{4.3}$$

(ii) *If $N_1 = \alpha_1 d$, then*

$$\mathbb{P}\left\{L\big(x + X(a_1,a_2), D\big) \geq c_{4,4} r^{N_2} u \log\left(e + \frac{u^{(\frac{\alpha_2}{N_2} - \frac{\alpha_1}{N_1})^+}}{r^{\alpha_2 - \alpha_1}}\right)\right\} \leq \exp(-bu). \tag{4.4}$$

(iii) *For $x, y \in \mathbb{R}^d$ with $|x - y| \leq 1$ and $\gamma > 0$ small,*

$$\begin{aligned} &\mathbb{P}\big\{\big|L\big(x + X(a_1,a_2), D\big) - L\big(y + X(a_1,a_2), D\big)\big| \\ &\quad \geq c_{4,5} |x-y|^\gamma r^{\beta_\tau - \kappa\gamma} u^{\eta_\tau + \kappa\gamma}\big\} \leq \exp(-bu). \end{aligned} \tag{4.5}$$

Proof The proofs of Parts (i) and (iii) based on Lemma 4.1 and Chebyshev's inequality are standard, hence omitted. In the following we prove (ii). Define the random variable $\Lambda = L(x + X(a_1,a_2), D)/r^{N_2}$. For $u > 0$ large, let $n = \lfloor u \rfloor$, the largest positive integer no bigger than u. We apply Chebyshev's inequality and Lemma 4.1 to obtain

$$\begin{aligned} &\mathbb{P}\left\{\Lambda \geq cu \log\left(e + \frac{u^{(\frac{\alpha_2}{N_2} - \frac{\alpha_1}{N_1})^+}}{r^{\alpha_2 - \alpha_1}}\right)\right\} \\ &\quad \leq \left(\frac{c_{4,1}}{ec}\right)^n \frac{\prod_{j=1}^n \log\big(e + \frac{j^{(\frac{\alpha_2}{N_2} - \frac{\alpha_1}{N_1})^+}}{r^{\alpha_2 - \alpha_1}}\big)}{\log^n\big(e + \frac{n^{(\frac{\alpha_2}{N_2} - \frac{\alpha_1}{N_1})^+}}{r^{\alpha_2 - \alpha_1}}\big)} \leq \left(\frac{c_{4,1}}{ec}\right)^n, \end{aligned} \tag{4.6}$$

where $c > 0$ is a constant whose value will be determined later, and where we have used the fact that for $j \in \{1, 2, \ldots, n\}$,

$$\log\left(e + \frac{j^{(\frac{\alpha_2}{N_2} - \frac{\alpha_1}{N_1})^+}}{r^{\alpha_2 - \alpha_1}}\right) \leq \log\left(e + \frac{n^{(\frac{\alpha_2}{N_2} - \frac{\alpha_1}{N_1})^+}}{r^{\alpha_2 - \alpha_1}}\right).$$

By taking $c = c_{4,4}$ large so that $\log(c_{4,1}/(ec)) \leq -b$, we obtain (4.4). □

The following result about the exponential integrability of $L(x, D)$ is a direct consequence of Lemma 4.2. We omit its proof.

Theorem 4.3 *Assume that the conditions of Theorem 3.3 hold and let* $D_1 := O_{N_1,N_2}(0, 1)$. *Then there exists a constant* $\delta > 0$, *depending on* $\alpha_1, \alpha_2, N_1, N_2$ *and* d *only, such that the following hold*:

(i) *If* $N_1 \neq \alpha_1 d$, *then for every* $x \in \mathbb{R}^d$

$$\mathbb{E}\big(e^{\delta L(x,D_1)^{\eta_\tau}}\big) \leq 1, \tag{4.7}$$

where η_τ *is the constant given in* (3.14).

(ii) *If* $N_1 = \alpha_1 d$, *then*

$$\mathbb{E}\big(e^{\delta \psi_3(L(x,D_1))}\big) \leq 1, \tag{4.8}$$

where $\psi_3(y) = y/\log(e + y)$ *for all* $y > 0$.

Now we study the local Hölder condition of the intersection local time $L^{\alpha_1,\alpha_2}(\cdot)$ and its connection to fractal properties of the set of intersection times M_2 of B^{α_1} and B^{α_2}.

Since M_2 is the zero-set of X, namely, $M_2 = X^{-1}(0)$, and the Gaussian random field X satisfies the conditions in [23]. It follows from Theorem 7.1 in [23] that

$$\dim_{\mathrm{H}} M_2 = \dim_{\mathrm{P}} M_2 = \beta_\tau \tag{4.9}$$

with positive probability. In the above, $\dim_{\mathrm{H}}$ and $\dim_{\mathrm{P}}$ denote Hausdorff and packing dimension, respectively; see [5] for more information. In Corollary 4.6 below, we will show that (4.9) holds with probability 1.

It is an interesting problem to determine the exact Hausdorff and packing measure functions for M_2. For this purpose, the limsup and liminf type laws of the iterated logarithm need to be established, respectively, for the intersection local time $L^{\alpha_1,\alpha_2}(\cdot)$.

In the following, we consider the limsup laws of the iterated logarithm for the local time $L(x, \cdot)$ of X. By applying Lemma 4.2 (with $(a_1, a_2) = (0, 0)$) and the Borel–Cantelli lemma, one can easily derive the following result: There exists a positive constant $c_{4,6}$ such that for every $x \in \mathbb{R}^d$ and $(s, t) \in \mathbb{R}^N$,

$$\limsup_{r\to 0} \frac{L(x, O_{N_1,N_2}((s,t), r))}{\varphi_1(r)} \leq c_{4,6}, \quad \text{a.s.}, \tag{4.10}$$

where

$$\varphi_1(r) = \begin{cases} r^{\beta_\tau}(\log\log(1/r))^{\eta_\tau} & \text{if } N_1 \neq \alpha_1 d, \\ r^{N_2}(\log\log(1/r))\log\Big(e + \frac{(\log\log(1/r))^{(\frac{\alpha_2}{N_2}-\frac{\alpha_1}{N_1})^+}}{r^{\alpha_2-\alpha_1}}\Big) & \text{if } N_1 = \alpha_1 d. \end{cases} \tag{4.11}$$

It is worthwhile to compare (4.10) with the corresponding results for (N, d) fractional Brownian motion of index α in [19] and the (N, d) fractional Brownian sheets with index $(\alpha, \ldots, \alpha) \in (0,1)^N$ in [2]. In the former case, X is isotropic and its local time $L(x, \cdot)$ satisfies

$$\limsup_{r\to 0} \frac{L(x, O_N(t,r))}{r^{N-\alpha d}(\log\log 1/r)^{\alpha d/N}} \le c_{4,7}, \quad \text{a.s.}, \tag{4.12}$$

while the local time of the (N, d) fractional Brownian sheet with index $(\alpha, \ldots, \alpha) \in (0,1)^N$ satisfies

$$\limsup_{r\to 0} \frac{L(x, O_N(t,r))}{r^{N-\alpha d}(\log\log 1/r)^{\alpha d}} \le c_{4,8}, \quad \text{a.s.} \tag{4.13}$$

Note that, the anisotropy of the fractional Brownian sheet only increases the power of the correction factor $\log\log 1/r$. For the Gaussian random field X defined by (1.3) with $N_1 = \alpha_1 d$, (4.10) suggests that the asymptotic properties of the local times of X may be significantly different from those in (4.12) and (4.13). In fact, when $N_1 = \alpha_1 d$ and as $r \downarrow 0$, we have

$$\varphi_1(r) \sim \begin{cases} r^{N_2} \log\log(1/r) \log\log\log(1/r) & \text{if } \alpha_2 = \alpha_1, \frac{\alpha_2}{N_2} > \frac{\alpha_1}{N_1}, \\ r^{N_2} \log\log(1/r) & \text{if } \alpha_2 = \alpha_1, \frac{\alpha_2}{N_2} \le \frac{\alpha_1}{N_1}, \\ r^{N_2} \log(1/r) \log\log(1/r) & \text{if } \alpha_2 > \alpha_1. \end{cases}$$

However, in this later case, it is unclear to us whether the logarithmic correction factor in (4.11) is sharp. It would be interesting to study this problem and establish sharp laws of the iterated logarithm for the local times of X. For such a result for the local times of a one-parameter fractional Brownian motion, see [3].

As a consequence of (4.10) we have for the intersection local time of B^{α_1} and B^{α_2} that, for every $(s,t) \in \mathbb{R}^N$,

$$\limsup_{r\to 0} \frac{L^{\alpha_1,\alpha_2}(O_{N_1,N_2}((s,t),r))}{\varphi_1(r)} \le c_{4,6}, \quad \text{a.s.} \tag{4.14}$$

It follows from Fubini's theorem that, with probability one, (4.14) holds for λ_N-almost all $(s,t) \in \mathbb{R}^N$. Now we prove a stronger version of this result, which is useful in determining the exact Hausdorff measure of M_2.

Theorem 4.4 *Assume that* $\frac{N_1}{\alpha_1} + \frac{N_2}{\alpha_2} > d$. *Let* $\tau \in \{1,2\}$ *be the integer defined in* (3.12) *and let* $D = O_{N_1,N_2}(0,R)$ *be fixed. Let* $L^{\alpha_1,\alpha_2}(\cdot)$ *be the intersection local time of* B^{α_1} *and* B^{α_2}, *which is a random measure supported on the set* M_2. *Then there exists a positive and finite constant* $c_{4,9}$ *such that with probability* 1,

$$\limsup_{r\to 0} \frac{L^{\alpha_1,\alpha_2}(O_{N_1,N_2}((s,t),r))}{\varphi_1(r)} \le c_{4,9} \tag{4.15}$$

holds for $L^{\alpha_1,\alpha_2}(\cdot)$*-almost all* $(s,t) \in D$, *where* $\varphi_1(r)$ *is defined in* (4.11).

Proof Again we work on the random field X defined by (1.3). For every integer $k > 0$, we consider the random measure $L_k(x, \bullet)$ on the Borel subsets C of

$O_{N_1,N_2}(0,R)$ defined by

$$L_k(x,C)=\int_C (2\pi k)^{d/2}\exp\biggl(-\frac{k|X(s,t)-x|^2}{2}\biggr)\,ds\,dt$$
$$=\int_C\int_{\mathbb{R}^d}\exp\biggl(-\frac{|\xi|^2}{2k}+i\langle\xi,X(s,t)-x\rangle\biggr)\,d\xi\,ds\,dt. \tag{4.16}$$

Then, by the occupation density formula (3.1) and the continuity of the function $y\mapsto L(y,C)$, one can verify that almost surely $L_k(x,C)\to L(x,C)$ as $k\to\infty$ for every Borel set $C\subseteq O_{N_1,N_2}(0,R)$.

For every integer $m\ge 1$, denote $f_m(s,t)=L(x,O_{N_1,N_2}((s,t),2^{-m}))$. From the proof of Theorem 3.3 we can see that almost surely the functions $f_m(s,t)$ are continuous and bounded. Hence we have almost surely, for all integers $m,n\ge 1$,

$$\int_{O_{N_1,N_2}(0,R)}\bigl[f_m(s,t)\bigr]^n L(x,ds\,dt)=\lim_{k\to\infty}\int_{O_{N_1,N_2}(0,R)}\bigl[f_m(s,t)\bigr]^n L_k(x,ds\,dt). \tag{4.17}$$

It follows from (4.17), (4.16) and the proof of Proposition 3.1 of [13] that, for every positive integer $n\ge 1$,

$$\mathbb{E}\int_{O_{N_1,N_2}(0,R)}\bigl[f_m(s,t)\bigr]^n L(x,ds\,dt)$$
$$=\biggl(\frac{1}{2\pi}\biggr)^{(n+1)d}\int_{O_{N_1,N_2}(0,R)}\int_{O_{N_1,N_2}((s,t),2^{-m})^n}\int_{\mathbb{R}^{(n+1)d}}\exp\biggl(-i\sum_{j=1}^{n+1}\langle x,u^j\rangle\biggr)$$
$$\times\,\mathbb{E}\exp\biggl(i\sum_{j=1}^{n+1}\langle u^j,X(s_j,t_j)\rangle\biggr)\,d\overline{u}\,d\overline{\mathbf{t}}, \tag{4.18}$$

where $\overline{u}=(u^1,\dots,u^{n+1})\in\mathbb{R}^{(n+1)d}$ and $\overline{\mathbf{t}}=(s,t,s_1,t_1,\dots,s_n,t_n)$. Similar to the proof of (3.15) we have that the right hand side of (4.18) is at most

$$\int_{O_{N_1,N_2}(0,R)}\int_{O_{N_1,N_2}((s,t),2^{-m})^n}\frac{c_{4,10}^n\,d\overline{\mathbf{t}}}{[\det\mathrm{Cov}(X_0(s,t),X_0(s_1,t_1),\dots,X_0(s_n,t_n))]^{d/2}}$$
$$\le\begin{cases}c_{4,11}^n(n!)^{\eta_\tau}2^{-mn\beta_\tau}, & \text{if } N_1\ne\alpha_1 d,\\ c_{4,11}^n n!2^{-nmN_2}\prod_{j=1}^n\log(e+j^{(\frac{\alpha_2}{N_2}-\frac{\alpha_1}{N_1})^+}2^{(\alpha_2-\alpha_1)m}) & \text{if } N_1=\alpha_1 d,\end{cases} \tag{4.19}$$

where $c_{4,11}$ is a positive finite constant depending on $\alpha_1,\alpha_2,N_1,N_2,d$ and R only.

Let $\rho>0$ be a constant whose value will be determined later. We consider the random set

$$D_m(\omega)=\bigl\{(s,t)\in O_{N_1,N_2}(0,R):f_m(s,t)\ge\rho\varphi_1(2^{-m})\bigr\}.$$

Denote by μ_ω the restriction of the random measure $L(x,\cdot)$ on $O_{N_1,N_2}(0,R)$, that is, $\mu_\omega(E)=L(x,E\cap O_{N_1,N_2}(0,R))$ for every Borel set $E\subseteq\mathbb{R}^N$. Now we take

$n = \lfloor \log m \rfloor$. Then, by applying (4.19) and by Stirling's formula, we have

$$\mathbb{E}\mu_\omega(D_m) \le \frac{\mathbb{E}\int_{O_{N_1,N_2}(0,R)} [f_m(s,t)]^n L(x, ds\, dt)}{[\rho\varphi_1(2^{-m})]^n} \le m^{-2}, \tag{4.20}$$

provided $\rho > 0$ is chosen large enough, say, $\rho \ge c_{4,11} e^2 := c_{4,9}$. This implies that

$$\mathbb{E}\left(\sum_{m=1}^{\infty} \mu_\omega(D_m)\right) < \infty.$$

Therefore, with probability 1 for μ_ω almost all $(s,t) \in O_{N_1,N_2}(0,R)$, we derive

$$\limsup_{m\to\infty} \frac{L(x, O_{N_1,N_2}((s,t), 2^{-m}))}{\varphi_1(2^{-m})} \le c_{4,9}. \tag{4.21}$$

Finally, for any $r > 0$ small enough, there exists an integer m such that $2^{-m} \le r < 2^{-m+1}$ and (4.21) is applicable. Since $\varphi_1(r)$ is increasing near $r = 0$, (4.15) follows from (4.21). □

As an application of Theorem 4.4, we derive a lower bound for the exact Hausdorff measure of the set M_2 of intersection times. The corresponding problem for the upper bound remains open.

Theorem 4.5 *Let $B^{\alpha_1} = \{B^{\alpha_1}(s), s \in \mathbb{R}^{N_1}\}$ and $B^{\alpha_2} = \{B^{\alpha_2}(t), t \in \mathbb{R}^{N_2}\}$ be two independent fractional Brownian motions with values in $\mathbb{R}^d$ and Hurst indices α_1 and α_2, respectively. Assume that $\frac{N_1}{\alpha_1} + \frac{N_2}{\alpha_2} > d$. Then, for every $R > 0$, there exists a positive constant $c_{4,12}$ such that with probability* 1,

$$\varphi_1\text{-}m\big(M_2 \cap O_{N_1,N_2}(0,R)\big) \ge c_{4,12} L\big(x, O_{N_1,N_2}(0,R)\big), \tag{4.22}$$

where φ_1-m denotes the φ_1-Hausdorff measure.

Proof As in the proof of Theorem 4.1 in [19], (4.22) follows from Theorem 4.4 and the upper density theorem of [15]. We omit the details. □

As a corollary of Theorem 4.5, we have the following result which is stronger than (4.9).

Corollary 4.6 *Let B^{α_1} and B^{α_2} be defined as that in Theorem 4.5. If $\frac{N_1}{\alpha_1} + \frac{N_2}{\alpha_2} > d$, then with probability* 1,

$$\dim_{\mathrm{H}} M_2 = \dim_{\mathrm{P}} M_2 = \begin{cases} N - \alpha_1 d & \text{if } \frac{N_1}{\alpha_1} > d, \\ N_2 + \frac{\alpha_2}{\alpha_1} N_1 - \alpha_2 d & \text{if } \frac{N_1}{\alpha_1} \le d < \frac{N_1}{\alpha_1} + \frac{N_2}{\alpha_2}. \end{cases} \tag{4.23}$$

Proof It is known from Theorem 7.1 in [23] that $\dim_{\mathrm{P}} M_2 \le \beta_\tau$ almost surely. In order to prove $\dim_{\mathrm{H}} M_2 \ge \beta_\tau$ almost surely, thanks to Theorem 4.5, it is sufficient to show that with probability 1, the intersection local time $L^{\alpha_1,\alpha_2}(O_{N_1,N_2}(0,R)) > 0$ for R large enough. We can actually prove a stronger result than this last statement. First note that, when $x = 0$, (3.17) becomes an equality. Thus, one can verify that $\mathbb{E}[L^{\alpha_1,\alpha_2}(O_{N_1,N_2}(0,1))] > 0$, which implies that $L^{\alpha_1,\alpha_2}(O_{N_1,N_2}(0,1)) > 0$ with positive probability. More precisely, there exist positive constants δ_1 and δ_2 such that $\mathbb{P}(L^{\alpha_1,\alpha_2}(O_{N_1,N_2}(0,1)) \ge \delta_1) \ge \delta_2$.

For any integer $n \ge 1$, define the event

$$A_n = \Big\{L\big(0, [0, 2^{-n/\alpha_1}]^{N_1} \times [0, 2^{-n/\alpha_2}]^{N_2}\big) \ge \delta_1 2^{-n(\frac{N_1}{\alpha_1}+\frac{N_2}{\alpha_2}-d)}\Big\}.$$

By the scaling property (3.9), we have $\mathbb{P}(A_n) \ge \delta_2$ for all $n \ge 1$. It follows from this and Fatou's lemma that $\mathbb{P}(\limsup_{n\to\infty} A_n) \ge \delta_2$. This implies that with positive probability

$$\limsup_{r\to 0} \frac{L^{\alpha_1,\alpha_2}([0, r^{1/\alpha_1}]^{N_1} \times [0, r^{1/\alpha_2}]^{N_2}))}{r^{\frac{N_1}{\alpha_1}+\frac{N_2}{\alpha_2}-d}} \ge \delta_1. \tag{4.24}$$

Finally, note that the Gaussian field X has stationary increments and satisfies the condition of Theorem 2.1 of [14], which is a zero-one law for X at 0. Hence (4.24) holds with probability 1 which, in turn, implies $L^{\alpha_1,\alpha_2}(O_{N_1,N_2}(0,R)) > 0$ for all $R > 0$. □

5 Hausdorff and Packing Dimensions of D_2

In this section, we determine the Hausdorff and packing dimensions of the set D_2 of intersection points of B^{α_1} and B^{α_2}, defined by $D_2 = \{x \in \mathbb{R}^d : x = B^{\alpha_1}(s) = B^{\alpha_2}(t)\ \text{ for some } (s,t) \in \mathbb{R}^N\}$. Note that we can rewrite D_2 as $D_2 = B^{\alpha_1}(\mathbb{R}^{N_1}) \cap B^{\alpha_2}(\mathbb{R}^{N_2})$.

Theorem 5.1 *Let B^{α_1} and B^{α_2} be defined as that in Theorem 4.5. If $\frac{N_1}{\alpha_1} + \frac{N_2}{\alpha_2} > d$, then with probability 1,*

$$\dim_{\rm H} D_2 = \dim_{\rm P} D_2 = \begin{cases} d & \text{if } N_1 > \alpha_1 d \quad \text{and } N_2 > \alpha_2 d,\\ \frac{N_2}{\alpha_2} & \text{if } N_1 > \alpha_1 d \text{ and } N_2 \le \alpha_2 d,\\ \frac{N_1}{\alpha_1} & \text{if } N_1 \le \alpha_1 d \text{ and } N_2 > \alpha_2 d,\\ \frac{N_1}{\alpha_1} + \frac{N_2}{\alpha_2} - d & \text{if } N_1 \le \alpha_1 d \text{ and } N_2 \le \alpha_2 d. \end{cases} \tag{5.1}$$

In order to prove Theorem 5.1, we will make use of the following two lemmas which are corollaries of the results in [11].

Lemma 5.2 *Let $B^\alpha = \{B^\alpha(t), t \in \mathbb{R}^p\}$ be a fractional Brownian motion with values in $\mathbb{R}^d$ and index $\alpha \in (0,1)$. If $p > \alpha d$, then almost surely $B^\alpha(\mathbb{R}^p) = \mathbb{R}^d$.*

Lemma 5.3 *Let $B^\alpha = \{B^\alpha(t), t \in \mathbb{R}^p\}$ be a fractional Brownian motion with values in $\mathbb{R}^d$ and index $\alpha \in (0,1)$. If $p \le \alpha d$, then for any constants $R \ge 1$, $\varepsilon > 0$ and $\beta > 0$*

such that $0 < \alpha - \varepsilon < \beta < \alpha$, *the following statement holds*: *With probability* 1, *for large enough n and for all balls* $U \subseteq \mathbb{R}^d$ *of radius* $2^{-n\beta}$, *the inverse image* $(B^\alpha)^{-1}(U)$ *can intersect at most* $2^{n\varepsilon d}$ *cubes* $I_{n,\bar{k}}$ *of the form*

$$I_{n,\bar{k}} = \bigl\{t \in [0, R]^p : (k_i - 1)2^{-n} \le t_i \le k_i 2^{-n},\ i = 1, 2, \dots, p\bigr\},$$

where $\bar{k} = (k_1, \dots, k_p)$ *and* $1 \le k_i \le R2^n$ *for* $i = 1, \dots, p$.

Proof of Theorem 5.1 We prove (5.1) by considering the four cases separately.

Firstly, we assume that $N_1 > \alpha_1 d$ and $N_2 > \alpha_2 d$. It follows from Lemma 5.2 that almost surely $B^{\alpha_1}(\mathbb{R}^{N_1}) = B^{\alpha_2}(\mathbb{R}^{N_2}) = \mathbb{R}^d$. Hence $D_2 = B^{\alpha_1}(\mathbb{R}^{N_1}) \cap B^{\alpha_2}(\mathbb{R}^{N_2}) = \mathbb{R}^d$ a.s., which implies that $\dim_{\mathrm{H}} D_2 = \dim_{\mathrm{P}} D_2 = d$ almost surely.

Secondly, we assume that $N_1 > \alpha_1 d$ and $N_2 \le \alpha_2 d$. Then $D_2 = B^{\alpha_1}(\mathbb{R}^{N_1}) \cap B^{\alpha_2}(\mathbb{R}^{N_2}) = B^{\alpha_2}(\mathbb{R}^{N_2})$ a.s., which yields $\dim_{\mathrm{H}} D_2 = \dim_{\mathrm{P}} D_2 = N_2/\alpha_2$ almost surely. The proof for the case $N_1 \le \alpha_1 d$ and $N_2 > \alpha_2 d$ is similar.

Finally, we consider the case of $N_1 \le \alpha_1 d$ and $N_2 \le \alpha_2 d$ (in addition to $\frac{N_1}{\alpha_1} + \frac{N_2}{\alpha_2} > d$). Let S_2 be the projection of M_2 on $\mathbb{R}^{N_2}$. Then $B^{\alpha_2}(S_2) = D_2$. Since, for every $\varepsilon > 0$, $B^{\alpha_2}(t)$ satisfies a uniform Hölder condition of order $\alpha_2 - \varepsilon$ on every compact interval of $\mathbb{R}^{N_2}$, we have

$$\dim_{\mathrm{P}} B^{\alpha_2}(S_2) \le \frac{1}{\alpha_2} \dim_{\mathrm{P}} S_2 \le \frac{N_1}{\alpha_1} + \frac{N_2}{\alpha_2} - d, \quad \text{a.s.,} \tag{5.2}$$

where the last inequality follows from (4.23).

It only remains to prove $\dim_{\mathrm{H}} D_2 \ge \frac{N_1}{\alpha_1} + \frac{N_2}{\alpha_2} - d$ almost surely. For this purpose, we denote $\ell = \frac{N_1}{\alpha_1} + \frac{N_2}{\alpha_2} - d$ and define an $(N, 2d)$-Gaussian random field $Z = \{Z(s,t), (s,t) \in \mathbb{R}^N\}$ by

$$Z(s,t) = \bigl(B^{\alpha_1}(s), B^{\alpha_2}(t)\bigr), \quad (s,t) \in \mathbb{R}^N.$$

Set $\tilde{D}_2 = \{(x,x) : x \in D_2\}$. Then

$$Z^{-1}(\tilde{D}_2) = M_2. \tag{5.3}$$

Fix an $\omega \in \Omega$ such that the conclusion of Lemma 5.3 holds. Assume that for some constant $\eta > 0$, $\dim_{\mathrm{H}} D_2(\omega) < \ell - \eta$. (We will suppress ω from now on.) Then, for any n large enough, there exists a sequence of balls $\{U_i\}$ in $\mathbb{R}^d$ with radius $\le 2^{-n}$, such that

$$D_2 \subseteq \bigcup_i U_i \quad \text{and} \quad \sum_i (\operatorname{diam} U_i)^{\ell - \eta} \le 1, \tag{5.4}$$

where $\operatorname{diam} U$ denotes the diameter of U. Choose positive constants ε, $\gamma_1 < \alpha_1$ and $\gamma_2 < \alpha_2$ such that

$$\gamma_1 < \gamma_2 \quad \text{and} \quad \varepsilon d \left(\frac{\gamma_2}{\gamma_1} + 1 \right) < \frac{\gamma_2 \eta}{2}. \tag{5.5}$$

Let m_i and n_i be integers that satisfy

$$2^{-(m_i+1)\gamma_1} \le \operatorname{diam} U_i \le 2^{-m_i\gamma_1} \quad \text{and} \quad 2^{-(n_i+1)\gamma_2} \le \operatorname{diam} U_i \le 2^{-n_i\gamma_2}. \tag{5.6}$$

By (5.3) and Lemma 5.3, we have

$$\begin{aligned} M_2 \cap [0, R]^N &\subseteq \bigcup_i Z^{-1}(U_i \times U_i) \\ &\subseteq \bigcup_i \bigl\{\text{the union of at most } 2^{(m_i+n_i)\varepsilon d} \text{ cubes } I_{m_i,\bar{j}} \times I_{n_i,\bar{k}}\bigr\}. \end{aligned} \tag{5.7}$$

Denote the cubes in the right-hand side of (5.7) by C_{ij}. Note that, since $\gamma_1 < \gamma_2$, we derive from (5.6) that $\operatorname{diam} C_{ij} \le 3 \cdot 2^{-n_i}$ for i (or n) large enough. Combining this with (5.4), (5.6) and (5.6), we derive that for all n large enough,

$$\begin{aligned} \sum_i \sum_j (\operatorname{diam} C_{ij})^{\gamma_2(\ell-\frac{\eta}{2})} &\le c_{5,1} \sum_i 2^{(m_i+n_i)\varepsilon d} (2^{-n_i})^{\gamma_2(\ell-\frac{\eta}{2})} \\ &\le c_{5,2} \sum_i 2^{n_i(1+\frac{\gamma_2}{\gamma_1})\varepsilon d} (2^{-n_i\gamma_2})^{\ell-\frac{\eta}{2}} \\ &\le c_{5,3} \sum_i (\operatorname{diam} U_i)^{\ell-\eta} \le c_{5,3}. \end{aligned} \tag{5.8}$$

It follows from (5.7) and (5.8) that

$$\dim_{\mathrm{H}}\bigl(M_2 \cap [0, R]^N\bigr) \le \gamma_2\left(\ell - \frac{\eta}{2}\right) < \alpha_2\left(\ell - \frac{\eta}{2}\right).$$

Hence we have proven that, for any $\eta > 0$,

$$\mathbb{P}\bigl\{\dim_{\mathrm{H}} D_2 \ge \ell - \eta\bigr\} \ge \mathbb{P}\left\{\dim_{\mathrm{H}}\bigl(M_2 \cap [0, R]^N\bigr) \ge \alpha_2\left(\ell - \frac{\eta}{2}\right)\right\}. \tag{5.9}$$

Letting $R \uparrow \infty$ and $\eta \downarrow 0$ along the rational numbers and by using (4.23), we obtain $\dim_{\mathrm{H}} D_2 \ge \ell$ almost surely. This finishes the proof. □

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