HAUSDORFF DIMENSION OF THE CONTOURS OF SYMMETRIC ADDITIVE LÉVY PROCESSES

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Abstract. Let $X_1, \ldots, X_N$ denote N independent, symmetric Lévy processes on $\mathbf{R}^d$. The corresponding *additive Lévy process* is defined as the following N-parameter random field on $\mathbf{R}^d$:

$$\mathfrak{X}(\boldsymbol{t}) := X_1(t_1) + \cdots + X_N(t_N) \qquad (\boldsymbol{t} \in \mathbf{R}_+^N). \tag{0.1}$$

Khoshnevisan and Xiao (2002) have found a necessary and sufficient condition for the zero-set $\mathfrak{X}^{-1}(\{0\})$ of $\mathfrak{X}$ to be non-trivial with positive probability. They also provide bounds for the Hausdorff dimension of $\mathfrak{X}^{-1}(\{0\})$ which hold with positive probability in the case that $\mathfrak{X}^{-1}(\{0\})$ can be non-void.

Here we prove that the Hausdorff dimension of $\mathfrak{X}^{-1}(\{0\})$ is a constant almost surely on the event $\{\mathfrak{X}^{-1}(\{0\}) \neq \varnothing\}$. Moreover, we derive a formula for the said constant. This portion of our work extends the well known formulas of Horowitz (1968) and Hawkes (1974) both of which hold for one-parameter Lévy processes.

More generally, we prove that for every nonrandom Borel set F in $(0, \infty)^N$, the Hausdorff dimension of $\mathfrak{X}^{-1}(\{0\}) \cap F$ is a constant almost surely on the event $\{\mathfrak{X}^{-1}(\{0\}) \cap F \neq \varnothing\}$. This constant is computed explicitly in many cases.

1. Introduction

Let $X_1, \ldots, X_N$ denote N independent symmetric Lévy processes on $\mathbf{R}^d$, all starting from 0. We construct the N-parameter random field $\mathfrak{X} := \{\mathfrak{X}(\boldsymbol{t})\}_{\boldsymbol{t} \in \mathbf{R}_+^N}$ on $\mathbf{R}^d$ as follows:

$$\mathfrak{X}(\boldsymbol{t}) := X_1(t_1) + \cdots + X_N(t_N), \tag{1.1}$$

where $\boldsymbol{t} := (t_1, \ldots, t_N)$ ranges over $\mathbf{R}_+^N$. Thus, $\mathfrak{X}$ is called a "symmetric additive Lévy process," and has found a number of applications in the study of classical Lévy processes (Khoshnevisan and Xiao, 2002; 2003; 2005; Khoshnevisan, Xiao and Zhong, 2003). Occasionally we follow the notation of these references and denote the random field $\mathfrak{X}$ also by $X_1 \oplus \cdots \oplus X_N$.

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Consider the level set at x,

$$\mathfrak{X}^{-1}(\{x\}) := \left\{ \boldsymbol{t} \in (0\,,\infty)^N : \ \mathfrak{X}(\boldsymbol{t}) = x \right\} \qquad \text{for } x \in \mathbf{R}^d. \tag{1.2}$$

By defining $\mathfrak{X}^{-1}(\{x\})$ in this way we have deliberately ruled out the points $\boldsymbol{t} \in \partial\mathbf{R}_+^N$ with $\mathfrak{X}(\boldsymbol{t}) = x$, where $\partial\mathbf{R}_+^N := \{\boldsymbol{t} \in [0,\infty)^N : t_i = 0 \text{ for some } 1 \le j \le N\}$ denotes the boundary of $\mathbf{R}_+^N$, since the problems for the latter can be reduced to the level sets of additive Lévy processes with fewer parameters.

Khoshnevisan and Xiao (2002) assert that, under a mild technical condition, $\mathfrak{X}^{-1}(\{0\}) \neq \varnothing$ with positive probability if and only if a certain function Φ is locally integrable. Moreover, the function Φ is easy to describe: It is the density function of $\mathfrak{X}(|t_1|\,, \dots, |t_N|)$ at $x = 0$.

As a by-product of their arguments, Khoshnevisan and Xiao (2002) produce bounds on the Hausdorff dimension of $\mathfrak{X}^{-1}(\{0\})$ as well. In fact, they exhibit two numbers $\gamma \le \overline{\gamma}$, both computable in terms of the Lévy exponents of $X_1, \dots, X_N$, such that

$$\gamma \le \dim_{\mathrm{H}} \mathfrak{X}^{-1}(\{0\}) \le \bar{\gamma} \qquad \text{with positive probability.} \tag{1.3}$$

Originally, the present paper was motivated by our desire to have better information on the Hausdorff dimension of $\mathfrak{X}^{-1}(\{0\})$ in the truly multiparameter setting $N \ge 2$. Recall that when $N = 1$, $\mathfrak{X}$ is a Lévy process in the classical sense, and in addition,

$$\text{either } \mathrm{P}\left\{\mathfrak{X}^{-1}(\{0\}) = \varnothing\right\} = 1 \text{ or } \mathrm{P}\left\{\mathfrak{X}^{-1}(\{0\}) \text{ is uncountable}\right\} = 1. \tag{1.4}$$

This is a consequence of the general theory of Markov processes; see Proposition 3.5 and Theorem 3.8 of Blumenthal and Getoor (1968, pp. 213 and 214). Moreover, it is known exactly when $\mathfrak{X}^{-1}(\{0\})$ is uncountable and, in general, $\mathfrak{X}^{-1}(\{0\})$ differs from the range of a subordinator in at most countably-many places. Consequently, a nice formula for $\dim_{\mathrm{H}} \mathfrak{X}^{-1}(\{0\})$ can be derived from the result of Horowitz (1968) on the Hausdorff dimension of the range of a subordinator. For a modern elegant treatment see Theorem 15 of Bertoin (1996, p. 94). See also Hawkes (1974) where $\dim_{\mathrm{H}} \mathfrak{X}^{-1}(\{0\})$ is described solely in terms of the Lévy exponent of $\mathfrak{X}$.

We were puzzled by why the extension of the said refinements to $N \ge 2$ are so much more difficult to obtain. For example, the issue of when $\{0\}$ is regular for itself—i.e., (1.4)—becomes much more delicate once $N \ge 2$. [We hope to deal with this matter elsewhere.] Thus, it is not obvious—nor does it appear to be true—that $\dim_{\mathrm{H}} \mathfrak{X}^{-1}(\{0\})$ is a.s. a constant.

In the present paper we prove that, under a mild technical condition, the Hausdorff dimension of $\mathfrak{X}^{-1}(\{0\})$ is a simple function of ω. In fact, it is a constant a.s. on the set where $\mathfrak{X}^{-1}(\{0\})$ is non-trivial.

We are even able to find a nice formula for the Hausdorff dimension of the zero set $\mathfrak{X}^{-1}(\{0\})$, a.s. on the event that it is nonempty. See Theorem 1.1 below. It can be shown that when $N = 1$ our formula agrees with the one-parameter findings of Horowitz (1968) and Hawkes (1974). Moreover, suppose $\mathfrak{X}^{-1}(\{0\})$ were replaced by the *closure* of $\mathfrak{X}^{-1}(\{0\})$ in $(0\,,\infty)^N$, then our derivations show that the same formula holds almost surely on the event that the said closure is nonempty.

The remainder of the Introduction is dedicated to developing the requisite background needed to describe our dimension formula precisely.

Let $\Psi_1, \ldots, \Psi_N$ denote the respective Lévy exponents of $X_1, \ldots, X_N$. That is, for all $1 \le j \le N$, $\xi \in \mathbf{R}^d$, and $u \ge 0$,

$$\mathrm{E}\left[e^{i\xi\cdot X_j(u)}\right] = \exp\left(-u\Psi_j(\xi)\right). \tag{1.5}$$

We recall that the functions $\Psi_1, \ldots, \Psi_N$ are real, non-negative, and symmetric. We say that $\mathfrak{X}$ is *absolutely continuous* if

$$\int_{\mathbf{R}^d} \exp\left(-u \sum_{1\le j\le N} \Psi_j(\xi)\right) d\xi < \infty \qquad \text{for all } u > 0. \tag{1.6}$$

Define for all $\boldsymbol{t} \in \mathbf{R}^N$,

$$\Phi(\boldsymbol{t}) := \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} \exp\left(-\sum_{1\le j\le N} |t_j|\Psi_j(\xi)\right) d\xi. \tag{1.7}$$

This defines Φ on $\mathbf{R}^N \backslash \{0\}$; Φ is uniformly continuous and bounded away from $\{\mathbf{0}\}$, and $\Phi(\mathbf{0}) = \infty$. As a consequence of the results of Khoshnevisan and Xiao (2002) we have

$$\begin{aligned} \mathrm{P}\left\{\mathfrak{X}^{-1}(\{0\}) \neq \varnothing\right\} > 0 \quad &\Longleftrightarrow \quad \mathrm{P}\left\{\overline{\mathfrak{X}^{-1}(\{0\})} \cap (0\,,\infty)^N \neq \varnothing\right\} > 0 \\ &\Longleftrightarrow \quad \Phi \in L^1_{loc}(\mathbf{R}^N), \end{aligned} \tag{1.8}$$

where $\bar{A}$ denotes the closure of A and $\Phi \in L^1_{loc}(\mathbf{R}^N)$ means $\Phi \in L^1([-T, T]^N)$ for every $T > 0$. Define $\|x\|$ to be the Euclidean ℓ^2 norm of x. Then the following is our main result:

Theorem 1.1. *If $X_1, \ldots, X_N$ are symmetric absolutely continuous Lévy processes in $\mathbf{R}^d$, then almost surely on $\{\mathfrak{X}^{-1}(\{0\}) \neq \varnothing\}$,*

$$\dim_{\mathrm{H}} \mathfrak{X}^{-1}(\{0\}) = \sup\left\{q > 0 : \int_{[0,1]^N} \frac{\Phi(\boldsymbol{t})}{\|\boldsymbol{t}\|^q}\, d\boldsymbol{t} < \infty\right\}. \tag{1.9}$$

Suppose, in addition, that there is a constant $K > 0$ such that

$$\Phi(\boldsymbol{t}) \le \Phi(K\|\boldsymbol{t}\|, \ldots, K\|\boldsymbol{t}\|) \quad \text{for all } \boldsymbol{t} \in (0\,,1]^N. \tag{1.10}$$

Then,

$$\dim_{\mathrm{H}} \mathfrak{X}^{-1}(\{0\}) = N - \limsup_{\boldsymbol{t}\to\boldsymbol{0}} \frac{\log \Phi(\boldsymbol{t})}{\log(1/\|\boldsymbol{t}\|)}. \tag{1.11}$$

When $N = 1$, (1.10) holds automatically, and so (1.9) and (1.11) coincide (Hawkes, 1974; Horowitz, 1968). We will show in Example 3.6 that when $N > 1$, formula (1.11) does not hold in general; an extra condition such as (1.10) is necessary.

Compared to the one-parameter case, the proof of Theorem 1.1 is considerably more complicated when $N > 1$. This is mainly due to the fact that classical covering arguments produce only (1.3) in general. Thus, we are led to a different route: We introduce a rich family of random sets with nice intersection properties, and strive to find exactly which of these random sets can intersect $\mathfrak{X}^{-1}(\{0\})$. There is a sense of symmetry about our arguments, since everything is described in terms of additive Lévy processes; the said random sets are constructed by means of introducing auxiliary additive Lévy processes. This argument allows us to establish a formula for the Hausdorff dimension of $\mathfrak{X}^{-1}(\{0\}) \cap F$ for every nonrandom Borel set $F \subset (0\,,\infty)^N$. See Theorem 3.2 and the examples in Section 3.

The idea of introducing random sets to help compute dimension seems to be due to Taylor (1966, Theorem 4). Since its original discovery, this method has been used by many others; in diverse ways, and to good effect (Barlow and Perkins, 1984; Benjamini et al., 2003; Blath and Mörters, 2005; Dalang and Nualart, 2004; Dembo et al., 2002; 1999; Khoshnevisan, 2003; Khoshnevisan et al., 2005a; 2005b; Khoshnevisan et al., 2000; Khoshnevisan and Shi, 2000; Khoshnevisan and Xiao, 2005; Klenke and Mörters, 2005; Lyons, 1992; 1990; Mörters, 2001; Peres, 1996a; 1996b; Peres and Steif, 1998).

We conclude the Introduction by introducing some notation that is used throughout and consistently.

- For every integer $m \geq 1$, and for all $x \in \mathbf{R}^m$,

$$\|x\| := \left(x_1^2 + \cdots + x_m^2\right)^{1/2}, \quad |x| := \max_{1\leq j\leq m} |x_j|, \quad \text{and} \quad [x] := |x_1| + \cdots + |x_m|. \tag{1.12}$$

  They respectively denote the ℓ^2, ℓ^∞, and ℓ^1 norms of x.
- Multidimensional "time" variables are typeset in bold letters in order to help the reader in his/her perusal.
- For all integers $m \geq 1$ and $\boldsymbol{s}, \boldsymbol{t} \in \mathbf{R}_+^m$, we write

$$\boldsymbol{s} \prec \boldsymbol{t} \quad \text{iff} \quad \boldsymbol{t} \succ \boldsymbol{s} \quad \text{iff} \quad s_i \leq t_i \ \text{ for all } \ 1 \leq i \leq m. \tag{1.13}$$

- Let $m \geq 1$ be a fixed integer and $q \geq 0$ a fixed real number. Suppose $f : \mathbf{R}^m \to \mathbf{R}_+$ is Borel measurable, and μ is a Borel probability measure on $\mathbf{R}^m$. Then,

$$I_f^{(q)}(\mu) := \iint \frac{f(x-y)}{\|x-y\|^q}\,\mu(dx)\,\mu(dy). \tag{1.14}$$

  When $f \equiv 1$ and $q > 0$, this is the *q-dimensional Bessel–Riesz energy* of μ, which will be denoted by $I^{(q)}(\mu)$.
- For any Borel set $G \subset \mathbf{R}^m$, let $\mathcal{P}(G)$ denote the collection of all Borel probability measures on G. The *q-dimensional Bessel–Riesz capacity* of G is defined by

$$\mathscr{C}_q(G) := \left[\inf_{\mu\in\mathcal{P}(G)} I^{(q)}(\mu)\right]^{-1}. \tag{1.15}$$

- If $f : \mathbf{R}^N \setminus \{0\} \to \mathbf{R}_+$, then we define the *upper index* and *lower index* of f (at $0 \in \mathbf{R}^N$) respectively as

$$\overline{\text{ind}}(f) := \limsup_{\|x\|\to 0} \frac{\log f(x)}{\log(1/\|x\|)}, \quad \underline{\text{ind}}(f) := \liminf_{\|x\|\to 0} \frac{\log f(x)}{\log(1/\|x\|)}. \tag{1.16}$$

Consequently, Theorem 1.1 asserts that if (1.10) holds then a.s. on the event that $\mathfrak{X}^{-1}(\{0\}) \neq \varnothing$,

$$\dim_{\text{H}} \mathfrak{X}^{-1}(\{0\}) = N - \overline{\text{ind}}(\Phi). \tag{1.17}$$

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2. Background on Additive Lévy Processes

2.1. **Absolute Continuity.** We follow Khoshnevisan and Xiao (2002) and call the following function Ψ the *Lévy exponent* of $\mathfrak{X}$. It is defined as follows. For $\xi \in \mathbf{R}^d$,

$$\Psi(\xi) := (\Psi_1(\xi)\,, \ldots, \Psi_N(\xi))\,. \tag{2.1}$$

In this way, we can write

$$\mathrm{E}\left[e^{i\xi\cdot\mathfrak{X}(\boldsymbol{t})}\right] = e^{-\boldsymbol{t}\cdot\Psi(\xi)} \quad \text{for } \xi \in \mathbf{R}^d \text{ and } \boldsymbol{t} \in \mathbf{R}_+^N. \tag{2.2}$$

Also, we declare $\mathfrak{X}$ to be *absolutely continuous* if the function $\xi \mapsto \exp\{-\boldsymbol{t}\cdot\Psi(\xi)\}$ is in $L^1(\mathbf{R}^d)$ for all $\boldsymbol{t} \in (0\,,\infty)^N$.

If any one of the X_j's is absolutely continuous, then so is $\mathfrak{X}$. A similar remark continues to apply if X_j is replaced by an additive process based on a proper, nonempty subset of

$\{X_1, \ldots, X_N\}$. However, it is possible to construct counter-examples and deduce that the converse to these assertions are in general false.

Here and throughout, we assume, without fail, that

$$\mathfrak{X} \text{ is absolutely continuous.} \tag{2.3}$$

It is possible to check that this is equivalent to the absolute-continuity condition (1.6) mentioned in the Introduction.

We may apply the inversion theorem and deduce that $\mathfrak{X}(\boldsymbol{t})$ has a density function $p_{\boldsymbol{t}}(\bullet)$ for all $\boldsymbol{t} \in (0\,,\infty)^N$. Moreover, for all $x \in \mathbf{R}^d$ and $\boldsymbol{t} \in (0\,,\infty)^N$,

$$p_{\boldsymbol{t}}(x) = \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} \cos(\xi \cdot x)\, e^{-\boldsymbol{t}\cdot\Psi(\xi)}\, d\xi. \tag{2.4}$$

Let $\mathbf{R}^N_{\neq}$ be the set of all $\boldsymbol{t} \in \mathbf{R}^N$ such that $(|t_1|\,, \ldots, |t_N|) \in (0\,,\infty)^N$. We abuse the notation slightly and also use $p_{\boldsymbol{t}}(x)$ to denote the density function of $\mathfrak{X}(|t_1|\,, \ldots, |t_N|)$ for all $\boldsymbol{t} \in \mathbf{R}^N_{\neq}$. Evidently, p is continuous on $\mathbf{R}^N_{\neq} \times \mathbf{R}^d$ and for all $\boldsymbol{t} \in \mathbf{R}^N_{\neq}$,

$$\sup_{x \in \mathbf{R}^d} p_{\boldsymbol{t}}(x) = p_{\boldsymbol{t}}(0) = \Phi(\boldsymbol{t}). \tag{2.5}$$

See (1.7) for the definition of Φ.

Throughout, we consider the probabilities:

$$\begin{aligned} \Phi_r(x\,;\boldsymbol{t}) &:= \frac{1}{(2r)^d} \mathrm{P}\left\{ \left|\mathfrak{X}(|t_1|\,, \ldots, |t_N|) - x\right| \le r \right\}, \\ \Phi_r(\boldsymbol{t}) &:= \Phi_r(0\,;\boldsymbol{t}), \end{aligned} \tag{2.6}$$

valid for all $r > 0$, $x \in \mathbf{R}^d$, and $\boldsymbol{t} = (t_1, \ldots, t_N) \in \mathbf{R}^N$. Evidently, for all $\boldsymbol{t} \in \mathbf{R}^N_{\neq}$,

$$\begin{aligned} \lim_{r \to 0^+} \Phi_r(\boldsymbol{t}) &= \Phi(\boldsymbol{t}), \\ \sup_{x \in \mathbf{R}^d} \Phi_r(x\,;\boldsymbol{t}) &\le \Phi(\boldsymbol{t}). \end{aligned} \tag{2.7}$$

The first statement follows from the continuity of $x \mapsto p_{\boldsymbol{t}}(x)$, and the second from (2.5). Similarly, we have

$$\lim_{r \to 0^+} \Phi_r(x\,;\boldsymbol{t}) = p_{\boldsymbol{t}}(x), \tag{2.8}$$

valid for all $\boldsymbol{t} \in \mathbf{R}^N_{\neq}$.

2.2. **Weak Unimodality.** We follow Khoshnevisan and Xiao (2002) and say that a Borel probability measure μ on $\mathbf{R}^k$ is *weakly unimodal* (*with constant* κ) if for all $r > 0$,

$$\sup_{x \in \mathbf{R}^k} \mu\big(B(x\,;r)\big) \le \kappa\, \mu\big(B(0\,;r)\big), \tag{2.9}$$

where $B(x\,;r) := \left\{y \in \mathbf{R}^k : \ |x-y| \le r\right\}$. Evidently, we can choose κ to be its optimal value,

$$\kappa := \sup_{r>0} \sup_{x\in\mathbf{R}^k} \frac{\mu\big(B(x\,;r)\big)}{\mu\big(B(0\,;r)\big)} < \infty, \tag{2.10}$$

where $0/0 := 1$.

Since $\mathfrak{X}$ is a symmetric additive Lévy process, Corollary 3.1 of Khoshnevisan and Xiao (2003) implies that the distribution of $\mathfrak{X}(\boldsymbol{t})$ is weakly unimodal with constant 16^d for all $\boldsymbol{t} \in (0\,,\infty)^N$. Equivalently, the growth of the function Φ_r of (2.6) is controlled as follows:

$$\sup_{x\in\mathbf{R}^d} \Phi_r(x\,;\boldsymbol{t}) \le 16^d \Phi_r(\boldsymbol{t}) \quad \text{for all } \boldsymbol{t} \in \mathbf{R}^N. \tag{2.11}$$

This and Lemma 2.8(i) of Khoshnevisan and Xiao (2002) together imply the following "doubling property":

$$\Phi_{2r}(\boldsymbol{t}) \le 32^d \Phi_r(\boldsymbol{t}) \quad \text{for all } \boldsymbol{t} \in \mathbf{R}^N. \tag{2.12}$$

Another important consequence of weak unimodality is that $\boldsymbol{t} \mapsto \Phi_r(\boldsymbol{t})$ is "quasi-monotone." This means that if $\boldsymbol{s} \prec \boldsymbol{t}$ and both are in $(0\,,\infty)^N$, then

$$\Phi_r(\boldsymbol{t}) \le 16^d \Phi_r(\boldsymbol{s}) \qquad \text{for all } r > 0. \tag{2.13}$$

See Lemma 2.8(ii) of Khoshnevisan and Xiao (2002).

3. **Some Key Results and Examples**

Khoshnevisan and Xiao (2002, Theorem 2.9) have proven that

$$\Phi \in L^1_{loc}(\mathbf{R}^N) \quad \text{ iff } \quad \mathrm{P}\left\{\mathfrak{X}^{-1}(\{0\}) \neq \varnothing\right\} > 0. \tag{3.1}$$

They proved also that the same is true for $\overline{\mathfrak{X}^{-1}(\{0\})} \cap (0\,,\infty)^N$. This was mentioned earlier in the Introduction of the present paper; see (1.8). In addition, Khoshnevisan and Xiao (2002) have computed bounds for the Hausdorff dimension of $\mathfrak{X}^{-1}(\{0\})$ in the case that Φ is locally integrable. The said bounds are in terms of γ and $\bar\gamma$, where

$$\begin{aligned} \gamma &:= \sup\left\{ q > 0 : \int_{[0,1]^N} \frac{\Phi(\boldsymbol{t})}{\|\boldsymbol{t}\|^q}\, d\boldsymbol{t} < \infty \right\}, \\ \bar\gamma &:= \inf\left\{ q > 0 : \liminf_{\|\boldsymbol{t}\|\to 0} \frac{\Phi(\boldsymbol{t})}{\|\boldsymbol{t}\|^{q-N}} > 0 \right\}. \end{aligned} \tag{3.2}$$

First, we offer the following.

Lemma 3.1. *It is always the case that*

$$0 \le \gamma \le \bar{\gamma} \le N - \frac{d}{2}. \tag{3.3}$$

If, in addition, (1.10) holds, then also,

$$\gamma = \inf\left\{ q > 0 : \limsup_{\|\boldsymbol{t}\|\to 0} \frac{\Phi(\boldsymbol{t})}{\|\boldsymbol{t}\|^{q-N}} > 0 \right\}. \tag{3.4}$$

Thus, in light of (1.16), we arrive at the following consequence:

$$\bar{\gamma} = N - \underline{\mathrm{ind}}(\Phi) \quad \text{whereas} \quad \gamma = N - \overline{\mathrm{ind}}(\Phi) \quad \text{if (1.10) holds.} \tag{3.5}$$

Proof of Lemma 3.1. By definition, $0 \le \gamma$. Also, if $q > \bar{\gamma}$ then there exists a positive and finite A such that $\Phi(\boldsymbol{t}) \ge A\,\|\boldsymbol{t}\|^{q-N}$ for all $\boldsymbol{t} \in [0\,,1]^N$. Consequently, $\int_{[0,1]^N} \Phi(\boldsymbol{t})\|\boldsymbol{t}\|^{-q}\,d\boldsymbol{t} \ge A\int_{[0,1]^N} \|\boldsymbol{t}\|^{-N}\,d\boldsymbol{t} = \infty$. It follows that $q \ge \gamma$. Let $q \downarrow \bar{\gamma}$ to deduce that $\bar{\gamma} \ge \gamma$.

In order to prove that $\bar{\gamma} \le N - (d/2)$, we first recall that $\Psi_j(\xi) = O(\|\xi\|^2)$ as $\|\xi\| \to \infty$ (Bochner, 1955, eq. (3.4.14), p. 67). Therefore, there exists a positive and finite constant A such that $|\boldsymbol{s}\cdot\Psi(\xi)| \le A\,\|\boldsymbol{s}\|\,(1+\|\xi\|^2)$ for all $\xi \in \mathbf{R}^d$ and $\boldsymbol{s} \in \mathbf{R}^N$. Consequently, for all $\boldsymbol{s} \in \mathbf{R}^N$,

$$\Phi(\boldsymbol{s}) \ge \int_{\mathbf{R}^d} e^{-A\,\|\boldsymbol{s}\|\,(1+\|\xi\|^2)}\,d\xi = \frac{A' e^{-A\,\|\boldsymbol{s}\|}}{\|\boldsymbol{s}\|^{d/2}}, \tag{3.6}$$

where A' depends only on d and A. This yields $\bar{\gamma} \le N - (d/2)$ readily.

It remains to verify (3.4) under condition (1.10). From now on, it is convenient to define temporarily,

$$\theta := \inf\left\{ q > 0 : \limsup_{\|\boldsymbol{t}\|\to 0} \frac{\Phi(\boldsymbol{t})}{\|\boldsymbol{t}\|^{q-N}} > 0 \right\}. \tag{3.7}$$

If $0 < q < \theta$, then $\Phi(\boldsymbol{t}) = o(\|\boldsymbol{t}\|^{q-N})$, and for all $\epsilon > 0$ and for all sufficiently large n,

$$\int_{\{2^{-n-1}<\|\boldsymbol{t}\|\le 2^{-n}\}} \frac{\Phi(\boldsymbol{t})}{\|\boldsymbol{t}\|^{q-\epsilon}}\,d\boldsymbol{t} = O(2^{-n\epsilon}) \qquad \text{as } n \to \infty. \tag{3.8}$$

Consequently, the left-most terms form a summable sequence indexed by n. In other words, for all $\epsilon > 0$, $\boldsymbol{t} \mapsto \|\boldsymbol{t}\|^{\epsilon-q}\Phi(\boldsymbol{t})$ is integrable on neighborhoods of the origin in $\mathbf{R}^N$. We have proved that $q \le \gamma + \epsilon$. Let $\epsilon \downarrow 0$ and $q \uparrow \theta$ to find that $\theta \le \gamma$. [This does not require (1.10).]

If $0 < q < \gamma$ and (1.10) holds, then

$$\infty > \int_{\{\|\boldsymbol{t}\|\le 1\}} \frac{\Phi(\boldsymbol{t})}{\|\boldsymbol{t}\|^q}\,d\boldsymbol{t} = \sum_{1\le n<\infty} \int_{\{2^{-n-1}<\|\boldsymbol{t}\|\le 2^{-n}\}} \frac{\Phi(\boldsymbol{t})}{\|\boldsymbol{t}\|^q}\,d\boldsymbol{t}. \tag{3.9}$$

Thus,

$$\lim_{n\to\infty}\int_{\{2^{-n-1}<\|\boldsymbol{t}\|\le 2^{-n}\}}\frac{\Phi(\boldsymbol{t})}{\|\boldsymbol{t}\|^q}\,d\boldsymbol{t}=0. \tag{3.10}$$

But the preceding integral is at least $2^{nq}\Phi(2^{-n},\ldots,2^{-n})$ times the volume of $\{\boldsymbol{t}\in\mathbf{R}^N_+ : 2^{-n-1}<\|\boldsymbol{t}\|\le 2^{-n}\}$. This follows from the coordinate-wise monotonicity of Φ, and proves that

$$\Phi(2^{-n},\ldots,2^{-n})=o\left(2^{-n(q-N)}\right)\quad\text{as } n\to\infty. \tag{3.11}$$

From this we conclude also that for the constant $K>0$ in (1.10),

$$\Phi(K2^{-n},\ldots,K2^{-n})=o\left(2^{-n(q-N)}\right)\quad\text{as } n\to\infty. \tag{3.12}$$

We appeal to (1.10) to deduce that

$$\frac{\Phi(K2^{-n},\ldots,K2^{-n})}{2^{-(n-1)(q-N)}}\ge A\sup_{2^{-n-1}<\|\boldsymbol{t}\|\le 2^{-n}}\frac{\Phi(\boldsymbol{t})}{\|\boldsymbol{t}\|^{q-N}}, \tag{3.13}$$

where A is positive and finite, and depends only on N. This and (3.12) prove that $q<\theta$, whence it follows that $\gamma\le\theta$. The converse bounds has already been proved. □

We are ready to present the main theorem of this section. This theorem is new even when $\mathfrak{X}$ is an ordinary Lévy process [i.e., $\mathfrak{X}:=X$ and $N=1$].

Theorem 3.2. *Let $\mathfrak{X}$ denote an N-parameter symmetric, absolutely continuous additive Lévy process on $\mathbf{R}^d$. Choose and fix a compact set $F\subset(0\,,\infty)^N$. Then, almost surely on $\{\mathfrak{X}^{-1}(\{0\})\cap F\neq\varnothing\}$,*

$$\dim_{\rm H}\left(\mathfrak{X}^{-1}(\{0\})\cap F\right)=\sup\left\{0<q<N:\ I_\Phi^{(q)}(\mu)<\infty\text{ for some }\mu\in\mathcal{P}(F)\right\}. \tag{3.14}$$

Remark 3.3. The proof of Theorem 3.2 implies that the Hausdorff dimension of $\overline{\mathfrak{X}^{-1}(\{0\})}\cap F$ has the same formula, almost surely on $\{\overline{\mathfrak{X}^{-1}(\{0\})}\cap F\neq\varnothing\}$.

In order to have a complete picture it remains to know when $\mathfrak{X}^{-1}(\{0\})\cap F$ is nonempty with positive probability. This issue is addressed by Corollary 2.13 of Khoshnevisan and Xiao (2002) as follows:

$$\begin{aligned}\mathrm{P}\left\{\mathfrak{X}^{-1}(\{0\})\cap F\neq\varnothing\right\}>0\quad&\Longleftrightarrow\quad \mathrm{P}\left\{\overline{\mathfrak{X}^{-1}(\{0\})}\cap F\neq\varnothing\right\}>0\\ &\Longleftrightarrow\quad\text{there exists }\mu\in\mathcal{P}(F)\text{ such that }I_\Phi^{(0)}(\mu)<\infty.\end{aligned} \tag{3.15}$$

[The weak unimodality assumption of Khoshnevisan and Xiao (2002, Corollary 2.13) is redundant in the present setting; see Corollary 3.1 of Khoshnevisan and Xiao (2003).] It follows from (3.15) that $\mathfrak{X}^{-1}(\{0\}) \cap F = \varnothing$ a.s. whenever $\dim_{\rm H} F < \underline{\rm ind}(\Phi)$.

The following is an immediate consequence of Theorem 3.2, used in conjunction with Frostman's theorem (Khoshnevisan, 2002, Theorem 2.2.1, p. 521). Note that, here and in the sequel, $\dim_{\rm H} E < 0$ means $E = \varnothing$.

Corollary 3.4. *If the conditions of Theorem 3.2 are met, then for all nonrandom compact sets $F \subset (0\,,\infty)^N$,*

$$\dim_{\rm H} F - \overline{\rm ind}(\Phi) \le \dim_{\rm H} \left(\mathfrak{X}^{-1}(\{0\}) \cap F\right) \le \dim_{\rm H} F - \underline{\rm ind}(\Phi), \tag{3.16}$$

almost surely on $\{\mathfrak{X}^{-1}(\{0\}) \cap F \neq \varnothing\}$.

Khoshnevisan and Xiao (2002, Theorem 2.10) have proved the following under the assumption that $\mathfrak{X}$ is absolutely continuous and symmetric:

(1) For all $C > c > 0$,

$$\mathrm{P}\left\{\gamma \le \dim_{\rm H} \left(\mathfrak{X}^{-1}(\{0\}) \cap [c\,,C]^N\right) \le \bar{\gamma}\right\} > 0. \tag{3.17}$$

(2) If there is a $K > 0$ such that $\Phi(\boldsymbol{t}) \le \Phi(K\|\boldsymbol{t}\|\,,\ldots,K\|\boldsymbol{t}\|)$, then

$$\mathrm{P}\left\{\dim_{\rm H} \left(\mathfrak{X}^{-1}(\{0\}) \cap [c\,,C]^N\right) = \gamma\right\} > 0. \tag{3.18}$$

Thus, Corollary 3.4 improves (3.17) and (3.18) in several ways.

We end this section with some examples showing applications of Theorems 1.1 and 3.2.

Example 3.5. Let $X_1, \ldots, X_N$ be N independent, identically distributed symmetric Lévy processes with stable components (Pruitt and Taylor, 1969). More precisely, let $X_1(t) = (X_{1,1}(t)\,, \ldots, X_{1,d}(t))$ for all $t \ge 0$, where the processes $X_{1,1}, \ldots, X_{1,d}$ are assumed to be independent, symmetric stable processes in $\mathbf{R}$ with respective indices $\alpha_1\,, \ldots, \alpha_d \in (0\,,2]$. Let $\mathfrak{X}$ be the associated additive Lévy process in $\mathbf{R}^d$. Then $\mathfrak{X}$ is anisotropic in the space-variable unless $\alpha_1 = \cdots = \alpha_d$.

It can be verified that $\mathfrak{X}$ satisfies the conditions of Theorem 3.2 and for all $\boldsymbol{t} \in (0\,,1]^N$,

$$\begin{aligned}\Phi(\boldsymbol{t}) &= \int_{\mathbf{R}^d} \exp\left(-\sum_{1\le j\le N} t_j \sum_{1\le k\le d} |\xi_k|^{\alpha_k}\right) d\xi \\ &\asymp \|\boldsymbol{t}\|^{-\sum_{1\le k\le d}(1/\alpha_k)}.\end{aligned} \tag{3.19}$$

In the above and sequel, "$f(\boldsymbol{t}) \asymp g(\boldsymbol{t})$ for all $\boldsymbol{t} \in T$" means that $f(\boldsymbol{t})/g(\boldsymbol{t})$ is bounded from below and above by constants that do not depend on $\boldsymbol{t} \in T$. It follows from Corollary 3.4

that for every compact set $F \subset (0\,,\infty)^N$,

$$\dim_{\mathrm{H}} \left(\mathfrak{X}^{-1}(\{0\}) \cap F\right) = \dim_{\mathrm{H}} F - \sum_{k=1}^{d} \frac{1}{\alpha_k}, \tag{3.20}$$

almost surely on $\{\mathfrak{X}^{-1}(\{0\}) \cap F \neq \varnothing\}$.

The same reasoning implies that if $\mathfrak{X}$ is an additive α-stable process in $\mathbf{R}^d$ [i.e., if $X_1, \ldots, X_N$ are symmetric α-stable Lévy processes in $\mathbf{R}^d$], then for every compact set $F \subset (0\,,\infty)^N$,

$$\dim_{\mathrm{H}} \left(\mathfrak{X}^{-1}(\{0\}) \cap F\right) = \dim_{\mathrm{H}} F - \frac{d}{\alpha}, \tag{3.21}$$

almost surely on $\{\mathfrak{X}^{-1}(\{0\}) \cap F \neq \varnothing\}$.

Next we consider additive Lévy processes which are anisotropic in the time-variable.

Example 3.6. Suppose $X_1, \ldots, X_N$ are N independent symmetric stable Lévy processes in $\mathbf{R}^d$ with indices $\alpha_1, \ldots, \alpha_N \in (0\,,2]$, respectively. Let $\mathfrak{X}$ be the additive Lévy process in $\mathbf{R}^d$ defined by $\mathfrak{X}(\boldsymbol{t}) = X_1(t_1) + \cdots + X_N(t_N)$. Because, for every $1 \le j \le N$ and fixed t_i $(i \neq j)$, the process $\mathbf{R}_+ \ni t_j \mapsto \mathfrak{X}(\boldsymbol{t})$ is [up to an independent random variable] an α_j-stable Lévy process in $\mathbf{R}^d$, $\mathfrak{X} = \{\mathfrak{X}(\boldsymbol{t})\}_{\boldsymbol{t}\in\mathbf{R}_+^N}$ is anisotropic in the time-variable.

The following result is concerned with the Hausdorff dimension of the zero set $\mathfrak{X}^{-1}(\{0\})$. For convenience, we assume

$$2 \ge \alpha_1 \ge \cdots \ge \alpha_N > 0. \tag{3.22}$$

Define

$$k(\boldsymbol{\alpha}) := \min\left\{\ell = 1\,, \ldots, N :\ \sum_{1\le j\le \ell} \alpha_j > d\right\}, \tag{3.23}$$

where $\min\varnothing := \infty$. In particular, $k(\boldsymbol{\alpha}) = \infty$ if and only if $\sum_{1\le j\le N} \alpha_j \le d$.

Theorem 3.7. *Let $\mathfrak{X} = \{\mathfrak{X}(\boldsymbol{t})\}_{\boldsymbol{t}\in\mathbf{R}_+^N}$ be the additive Lévy process defined above. Then,* $\mathrm{P}\{\mathfrak{X}^{-1}(\{0\}) \neq \varnothing\} > 0$ *if and only if $k(\boldsymbol{\alpha})$ is finite. Moreover, if $k(\boldsymbol{\alpha}) < \infty$, then almost surely on $\{\mathfrak{X}^{-1}(\{0\}) \neq \varnothing\}$,*

$$\dim_{\mathrm{H}} \mathfrak{X}^{-1}(\{0\}) = N - k(\boldsymbol{\alpha}) + \frac{\sum_{1\le j\le k(\boldsymbol{\alpha})} \alpha_j - d}{\alpha_{k(\boldsymbol{\alpha})}}. \tag{3.24}$$

First, we derive a few technical lemmas. The first is a pointwise estimate for Φ.

Lemma 3.8. *Under the preceding conditions, for all $\boldsymbol{t} \in (0\,,1]^N$,*

$$\Phi(\boldsymbol{t}) \asymp \frac{1}{\sum_{1\le j\le N} |t_j|^{d/\alpha_j}}. \tag{3.25}$$

Proof. For any fixed $\boldsymbol{t} \in (0\,,1]^N$ we let $i \in \{1\,,\ldots\,,N\}$ satisfy $|t_i|^{1/\alpha_i} = \max_{1\le j\le N} |t_j|^{1/\alpha_j}$. Because $\Phi(\boldsymbol{t}) \le \int_{\mathbf{R}^d} \exp(-|t_i|\cdot\|\xi\|^{\alpha_i})\,d\xi$, it follows that

$$\Phi(\boldsymbol{t}) \le \frac{A}{|t_i|^{d/\alpha_i}} \le \frac{A'}{\sum_{1\le j\le N} |t_j|^{d/\alpha_j}}, \tag{3.26}$$

where A and $A' < \infty$ do not depend on $\boldsymbol{t} \in (0\,,1]^N$.

For the other bound we use (3.22) to deduce the following:

$$\begin{aligned}
\Phi(\boldsymbol{t}) &= \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} \exp\left(-\sum_{1\le j\le N} \left(|t_j|^{1/\alpha_j}\|\xi\|\right)^{\alpha_j}\right) d\xi\\
&\ge \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} \exp\left(-\sum_{1\le j\le N} \left(|t_i|^{1/\alpha_i}\|\xi\|\right)^{\alpha_j}\right) d\xi\\
&\ge \frac{1}{(2\pi)^d} \int_{\{\|\xi\|\ge |t_i|^{-1/\alpha_i}\}} \exp\left(-N\,|t_i|^{\alpha_1/\alpha_i}\|\xi\|^{\alpha_1}\right)\,d\xi.
\end{aligned} \tag{3.27}$$

A change of variables then shows that

$$\Phi(\boldsymbol{t}) \ge \frac{A''}{\sum_{1\le j\le N} |t_j|^{d/\alpha_j}}, \tag{3.28}$$

where $A'' > 0$ does not depend on $\boldsymbol{t} \in (0\,,1]^N$. The lemma follows from (3.26) and (3.28). □

Our second technical lemma follows directly from Lemma 10 of Ayache and Xiao (2005) and its proof.

Lemma 3.9. *Let $a, b, c \ge 0$ be fixed. Define for all $u, v > 0$,*

$$J_{a,b,c}(u\,,v) := \int_0^1 \frac{dt}{(u+t^a)^b(v+t)^c}. \tag{3.29}$$

Define for all $u, v > 0$,

$$\bar{J}_{a,b,c}(u\,,v) := \begin{cases} u^{-b+(1/a)}v^{-c}, & \text{if } ab > 1,\\ v^{-c}\log\left(1+vu^{-1/a}\right), & \text{if } ab = 1,\\ 1+v^{-ab-c+1}, & \text{if } ab < 1 \text{ and } ab+c \ne 1. \end{cases} \tag{3.30}$$

Then, as long as $u \le v^a$, we have $J_{a,b,c}(u\,,v) \asymp \bar{J}_{a,b,c}(u\,,v)$.

Proof of Theorem 3.7. It can be verified that the additive process $\mathfrak{X}$ satisfies the symmetry and absolute continuity conditions of Theorem 1.1.

According to Lemma 3.8, we have that for all $q \geq 0$,

$$
\begin{aligned}
\int_{[0,1]^N} \frac{\Phi(\boldsymbol{t})}{\|\boldsymbol{t}\|^q}\, d\boldsymbol{t} &\asymp \int_{[0,1]^N} \frac{1}{\left(\sum_{1\leq j\leq N} t_j^{d/\alpha_j}\right) |\boldsymbol{t}|^q}\, d\boldsymbol{t} \\
&= \int_{[0,1]^{N-1}} J_{(d/\alpha_1),1,q}\left(\sum_{2\leq j\leq N} t_j^{d/\alpha_j}\,,\, \sum_{2\leq j\leq N} t_j\right) d\boldsymbol{t}.
\end{aligned}
\tag{3.31}
$$

This means that the left-most term converges if and only if the right-most one does. We apply induction on N, several times in conjunction with Lemma 3.9, to find that $\int_{[0,1]^N} \Phi(\boldsymbol{t})\, d\boldsymbol{t} = \infty$ if and only if $k(\boldsymbol{\alpha}) = \infty$. Therefore, in accord with Khoshnevisan and Xiao (2002), $k(\boldsymbol{\alpha}) < \infty$ if and only if $\mathrm{P}\{\mathfrak{X}^{-1}(\{0\}) \neq \varnothing\} > 0$. This proves the first part of Theorem 3.7

It remains to prove that γ equals to the right-hand side of (3.24). This is proved by appealing, once again, to (3.31), Lemma 3.9, and induction [on N]. The details are tedious but otherwise elementary. So we omit them. □

4. Proof of Theorem 3.2

Our proof of Theorem 3.2 is technical and long. We will carry it out in several parts.

Throughout the remainder of this section we enlarge the probability space enough that we can introduce symmetric, α-stable Lévy processes $\{S_j\}_{j=1}^{\infty}$—all taking values in $\mathbf{R}^N$—such that $S_1, S_2, \ldots$ are i.i.d., and totally independent of $X_1, \ldots, X_N$. We choose and fix an integer $M \geq 1$, and define $\mathfrak{S}$ to be the additive stable process $S_1 \oplus \cdots \oplus S_M$. That is, $\mathfrak{S}(\boldsymbol{t}) = S_1(t_1) + \cdots + S_M(t_M)$ for all $\boldsymbol{t} = (t_1, \ldots, t_M) \in \mathbf{R}_+^M$. The parameters $0 < \alpha < 2$ and $M \geq 1$ will be determined at the end of the proof of Theorem 3.2. For the sake of concreteness we normalize each S_j as follows:

$$
\mathrm{E}\left[e^{i\xi\cdot S_j(u)}\right] = \exp\left(-u\|\xi\|^{\alpha}\right) \quad \text{for } \xi \in \mathbf{R}^N \text{ and } u \geq 0. \tag{4.1}
$$

Let G be a nonrandom measurable subset of $\mathbf{R}^N$. According to Theorem 4.1.1 of Khoshnevisan (2002, p. 423), $\mathrm{P}\{G \cap \mathfrak{S}(\mathbf{R}_+^M) \neq \varnothing\} > 0$ if and only if $\mathscr{C}_{N-\alpha M}(G) > 0$. Owing to the independence of $\mathfrak{S}$ and $\mathfrak{X}$ we can apply the preceding with $G := \mathfrak{X}^{-1}(\{0\}) \cap F$ to find that $\mathrm{P}\{\mathfrak{X}^{-1}(\{0\}) \cap F \cap \mathfrak{S}(\mathbf{R}_+^M) \neq \varnothing\} > 0$ iff $\mathscr{C}_{N-\alpha M}(\mathfrak{X}^{-1}(\{0\}) \cap F) > 0$ with positive probability. The Frostman theorem of potential theory (Khoshnevisan, 2002, Theorem 2.2.1, p. 521) asserts that the Hausdorff dimension of $\mathfrak{X}^{-1}(\{0\}) \cap F$ is the critical $\beta \in (0\,, N)$ such that $\mathscr{C}_{\beta}(\mathfrak{X}^{-1}(\{0\}) \cap F) > 0$. Because α and M can be chosen as we like, the computation of $\dim_{\mathrm{H}}(\mathfrak{X}^{-1}(\{0\}) \cap F)$ is thus reduced to deciding when, and exactly when, $\mathfrak{X}^{-1}(\{0\}) \cap \mathfrak{S}(\mathbf{R}_+^M) \cap F$ is nonempty with positive probability. The main contribution of this paper is a precise analytic condition on F that is equivalent to the positivity of

$\mathrm{P}\{\mathfrak{X}^{-1}(\{0\}) \cap F \cap \mathfrak{S}(\mathbf{R}_+^M) \neq \varnothing\}$. See Propositions 4.7 and 4.8 below. Once we have this, the formula for the Hausdorff dimension of $\mathfrak{X}^{-1}(\{0\}) \cap F$ follows from the preceding arguments that involve the Frostman theorem.

For all Borel probability measures μ on $\mathbf{R}_+^N$, and for every $\epsilon > 0$, define

$$J_\epsilon(\mu) := \frac{1}{(2\epsilon)^{d+N}} \int_{\mathbf{R}_+^M} \left(\int_{\mathbf{R}_+^N} \mathbf{1}_{\{|\mathfrak{X}(\boldsymbol{s})| \le \epsilon\,,\, |\mathfrak{S}(\boldsymbol{t}) - \boldsymbol{s}| \le \epsilon\}}\, \mu(d\boldsymbol{s}) \right) e^{-[\boldsymbol{t}]}\, d\boldsymbol{t}. \tag{4.2}$$

It might help to recall that $[\boldsymbol{t}]$ denotes the ℓ^1-norm of $\boldsymbol{t}$.

4.1. **Some Moment Estimates.** For all $x \in \mathbf{R}^d$, we let P_x denote the law of $x + \mathfrak{X}$. Similarly, for all $y \in \mathbf{R}^N$, we define Q_y to be the law of $y + \mathfrak{S}$. These are actually measures on canonical "path spaces" defined in the usual way; see Khoshnevisan and Xiao (2002, Section 5.2) for details. Without loss of much generality, we can think of the underlying probability measure P as $\mathrm{P}_0 \times \mathrm{Q}_0$.

On our enlarged probability space, we view $\mathrm{P}_x \times \mathrm{Q}_y$ as the joint law of $(x + \mathfrak{X}\,, y + \mathfrak{S})$. Define $\mathcal{L}^k$ to be the Lebesgue measure on $\mathbf{R}^k$ for all integers $k \geq 1$. Then we can construct σ-finite measures,

$$\mathrm{P}_{\mathcal{L}^d}(\bullet) := \int_{\mathbf{R}^d} \mathrm{P}_x(\bullet)\, dx \quad \text{and} \quad \mathrm{Q}_{\mathcal{L}^N}(\bullet) := \int_{\mathbf{R}^N} \mathrm{Q}_y(\bullet)\, dy, \tag{4.3}$$

together with corresponding expectation operators,

$$\mathrm{E}_{\mathrm{P}}[f] := \int_{\mathbf{R}^d} f\, d\mathrm{P}_{\mathcal{L}^d} \quad \text{and} \quad \mathrm{E}_{\mathrm{Q}}[f] := \int_{\mathbf{R}^N} f\, d\mathrm{Q}_{\mathcal{L}^N}. \tag{4.4}$$

We are particularly interested in the σ-finite measure $\mathrm{P}_{\mathcal{L}^d} \times \mathrm{Q}_{\mathcal{L}^N}$ and its corresponding expectation operator $\mathrm{E}_{\mathrm{P}\times\mathrm{Q}}$.

It is an elementary computation that for all $\boldsymbol{s} \in \mathbf{R}_+^N$ and $\boldsymbol{t} \in \mathbf{R}_+^M$, the distribution of $(\mathfrak{X}(\boldsymbol{s})\,, \mathfrak{S}(\boldsymbol{t}))$ under $\mathrm{P}_{\mathcal{L}^d} \times \mathrm{Q}_{\mathcal{L}^N}$ is $\mathcal{L}^d \times \mathcal{L}^N$. In particular,

$$(\mathrm{P}_{\mathcal{L}^d} \times \mathrm{Q}_{\mathcal{L}^N}) \{|\mathfrak{X}(\boldsymbol{s})| \le \epsilon\,,\, |\mathfrak{S}(\boldsymbol{t}) - \boldsymbol{s}| \le \epsilon\} = (2\epsilon)^{d+N}. \tag{4.5}$$

Thus, we are led to the following formula: For all Borel probability measures μ on $\mathbf{R}_+^N$ and every $\epsilon > 0$,

$$\mathrm{E}_{\mathrm{P}\times\mathrm{Q}}\left[J_\epsilon(\mu)\right] = 1. \tag{4.6}$$

Next we bound the second moment of $J_\epsilon(\mu)$.

Proposition 4.1. *If $N > \alpha M$ then there exists a finite and positive constant A—depending only on (α, d, N, M)—such that for all Borel probability measures μ on $\mathbf{R}_+^N$ and all $\epsilon > 0$,*

$$\mathrm{E}_{\mathrm{P}\times\mathrm{Q}}\left[(J_\epsilon(\mu))^2 \right] \le A \iint \frac{\Phi(\boldsymbol{s}' - \boldsymbol{s})\, \mu(d\boldsymbol{s}')\, \mu(d\boldsymbol{s})}{\max\left(|\boldsymbol{s}' - \boldsymbol{s}|^{N-\alpha M} \ , \ \epsilon^{N-\alpha M} \right)}. \tag{4.7}$$

Proof. Combine Lemma 5.6 of Khoshnevisan and Xiao (2002) with (2.11) of the present paper to find that for all $\boldsymbol{s}, \boldsymbol{s}' \in \mathbf{R}_+^N$ and $\epsilon > 0$,

$$\begin{aligned} \mathrm{P}_{\mathcal{L}^d}\left\{ |\mathfrak{X}(\boldsymbol{s})| \le \epsilon \ , \ |\mathfrak{X}(\boldsymbol{s}')| \le \epsilon \right\} &\le (64\epsilon)^d \mathrm{P}\left\{ |\mathfrak{X}(\boldsymbol{s}) - \mathfrak{X}(\boldsymbol{s}')| \le \epsilon \right\} \\ &= 128^d \epsilon^{2d} \Phi_\epsilon(\boldsymbol{s}' - \boldsymbol{s}). \end{aligned} \tag{4.8}$$

The last line follows from symmetry; i.e., from the fact that $\mathfrak{X}(\boldsymbol{s}) - \mathfrak{X}(\boldsymbol{s}')$ has the same distribution as $\mathfrak{X}(\boldsymbol{r})$, where the jth coordinate of $\boldsymbol{r}$ is $|s_j - s_j'|$. Thanks to (2.7) we obtain the following:

$$\mathrm{P}_{\mathcal{L}^d}\left\{ |\mathfrak{X}(\boldsymbol{s})| \le \epsilon \ , \ |\mathfrak{X}(\boldsymbol{s}')| \le \epsilon \right\} \le 128^d \, \epsilon^{2d} \, \Phi(\boldsymbol{s} - \boldsymbol{s}'). \tag{4.9}$$

We follow the implicit portion of the proof of the preceding to find that for all $x, y \in \mathbf{R}^N$, $\boldsymbol{t}, \boldsymbol{t}' \in \mathbf{R}_+^M$ and $\epsilon > 0$,

$$\mathrm{Q}_{\mathcal{L}^N}\left\{ |\mathfrak{S}(\boldsymbol{t}) - x| \le \epsilon \ , \ |\mathfrak{S}(\boldsymbol{t}') - y| \le \epsilon \right\} = \mathrm{E}\left[\int_{\mathbf{R}^N} \mathbf{1}_{\{|z + \mathfrak{S}(\boldsymbol{t}) - x| \le \epsilon \ , \ |z + \mathfrak{S}(\boldsymbol{t}') - y| \le \epsilon\}} \, dz \right]. \tag{4.10}$$

We change the variables to find that

$$\begin{aligned} &\mathrm{Q}_{\mathcal{L}^N}\left\{ |\mathfrak{S}(\boldsymbol{t}) - x| \le \epsilon \ , \ |\mathfrak{S}(\boldsymbol{t}') - y| \le \epsilon \right\} \\ &\qquad = \int_{|z| \le \epsilon} \mathrm{P}\left\{ |z + \mathfrak{S}(\boldsymbol{t}') - \mathfrak{S}(\boldsymbol{t}) - (y - x)| \le \epsilon \right\} dz \\ &\qquad \le (2\epsilon)^N \mathrm{P}\left\{ |\mathfrak{S}(\boldsymbol{t}') - \mathfrak{S}(\boldsymbol{t}) - (y - x)| \le 2\epsilon \right\}. \end{aligned} \tag{4.11}$$

Thus,

$$\mathrm{E}_{\mathrm{P}\times\mathrm{Q}}\left[(J_\epsilon(\mu))^2 \right] \le \frac{32^d}{(2\epsilon)^N} \int_{\mathbf{R}_+^N} \int_{\mathbf{R}_+^N} \Phi(\boldsymbol{s} - \boldsymbol{s}') F_\epsilon(\boldsymbol{s} - \boldsymbol{s}')\, \mu(d\boldsymbol{s})\, \mu(d\boldsymbol{s}'), \tag{4.12}$$

where

$$F_\epsilon(x) := \int_{\mathbf{R}_+^M} \int_{\mathbf{R}_+^M} \mathrm{P}\left\{ |\mathfrak{S}(\boldsymbol{t}') - \mathfrak{S}(\boldsymbol{t}) - x| \le 2\epsilon \right\} e^{-[\boldsymbol{t}] - [\boldsymbol{t}']} \, d\boldsymbol{t} d\boldsymbol{t}', \tag{4.13}$$

for all $x \in \mathbf{R}^N$ and $\epsilon > 0$. Because $[\boldsymbol{t}] + [\boldsymbol{t}'] = [\boldsymbol{t} - \boldsymbol{t}'] + 2[\boldsymbol{t} \wedge \boldsymbol{t}']$ for all $\boldsymbol{t}, \boldsymbol{t}' \in \mathbf{R}_+^M$,

$$F_\epsilon(x) = \int_{|z - x| \le 2\epsilon} \int_{\mathbf{R}_+^M} \int_{\mathbf{R}_+^M} f_{\boldsymbol{t} - \boldsymbol{t}'}(z) e^{-[\boldsymbol{t} - \boldsymbol{t}'] - 2[\boldsymbol{t} \wedge \boldsymbol{t}']} \, d\boldsymbol{t} \, d\boldsymbol{t}' \, dz, \tag{4.14}$$

where f is the generalized "transition function,"

$$f_{\boldsymbol{u}}(z) := \frac{\mathrm{P}\left\{\mathfrak{S}(|u_1|, \ldots, |u_N|) \in dz\right\}}{dz} \quad \text{for } \boldsymbol{u} \in \mathbf{R}^M \text{ and } z \in \mathbf{R}^N. \tag{4.15}$$

A computation based on symmetry yields

$$\int_{\mathbf{R}_+^M} \int_{\mathbf{R}_+^M} f_{\boldsymbol{t}-\boldsymbol{t}'}(z) e^{-[\boldsymbol{t}-\boldsymbol{t}'] - 2[\boldsymbol{t} \wedge \boldsymbol{t}']} \, d\boldsymbol{t} \, d\boldsymbol{t}' = \int_{\mathbf{R}_+^M} f_{\boldsymbol{u}}(z) e^{-[\boldsymbol{u}]} \, d\boldsymbol{u} := \upsilon(z). \tag{4.16}$$

In order to see the first equality, we write the double integral as a sum of integrals over the 2^M regions:

$$D_\pi = \left\{(\boldsymbol{t}, \boldsymbol{t}') \in \mathbf{R}_+^M \times \mathbf{R}_+^M : t_i \le t_i' \text{ if } i \in \pi \text{ and } t_i > t_i' \text{ if } i \notin \pi\right\}, \tag{4.17}$$

where π ranges over all subset sets of $\{1, 2, \ldots, M\}$ including the empty set. It can been verified that the integral over D_π equals $2^{-M} \int_{\mathbf{R}_+^M} f_{\boldsymbol{u}}(z) e^{-[\boldsymbol{u}]} \, d\boldsymbol{u}$. Hence (4.16) follows.

The function $\upsilon(z)$ in (4.16) is the *one-potential density* of $\mathfrak{S}$ (Khoshnevisan, 2002, pp. 397 and 406). We cite two facts about υ:

(1) $\upsilon(z) > 0$ for all $z \in \mathbf{R}^N$, and is continuous away from $0 \in \mathbf{R}^N$. This is a consequence of eq. (3) of (*loc. cit.*, p. 406) and Bochner's subordination (*loc. cit.*, p. 378).

(2) If $N > \alpha M$, then for all $R > 0$ there exists a finite constants $A' > A > 0$ such that

$$\frac{A}{|z|^{N-\alpha M}} \le \upsilon(z) \le \frac{A'}{|z|^{N-\alpha M}} \quad \text{whenever } |z| \le R. \tag{4.18}$$

Moreover, A' can be chosen to be independent of $R > 0$. This follows from (1), together used with Proposition 4.1.1 of (*loc. cit.*, p. 420).

It follows from (4.14), (4.16) and (4.18) that for all $x \in \mathbf{R}^N$ and $\epsilon > 0$,

$$F_\epsilon(x) \le A' \int_{\substack{z \in \mathbf{R}^N: \\ |z-x| \le 2\epsilon}} \frac{dz}{|z|^{N-\alpha M}} \le A''(2\epsilon)^N \min\left(\frac{1}{|x|^{N-\alpha M}}, \frac{1}{\epsilon^{N-\alpha M}}\right). \tag{4.19}$$

Here, A'' is positive and finite, and depends only on (N, M, α). The proposition is a ready consequence of this and symmetry; see (4.12). □

We mention the following variant of Proposition 4.1. It is proved by the same argument, without using (4.9).

Proposition 4.2. *If $N > \alpha M$ then there exists a finite and positive constant A—depending only on (α, d, N, M)—such that for all Borel probability measures μ on $\mathbf{R}_+^N$ and all $\epsilon > 0$,*

$$\mathrm{E}_{\mathrm{P} \times \mathrm{Q}}\left[(J_\epsilon(\mu))^2\right] \le A \iint \frac{\Phi_\epsilon(\boldsymbol{s}' - \boldsymbol{s})}{\max\left(|\boldsymbol{s}' - \boldsymbol{s}|^{N-\alpha M}, \epsilon^{N-\alpha M}\right)} \, \mu(d\boldsymbol{s}') \, \mu(d\boldsymbol{s}). \tag{4.20}$$

Next we define two multi-parameter filtrations (Khoshnevisan, 2002, p. 233). First, define $\mathscr{X}_j$ to be the filtration of the Lévy process X_j, augmented in the usual way. Also, define $\mathscr{S}_k$ to be the corresponding filtration for S_k. Then, we consider

$$\mathscr{X}(\boldsymbol{s}) := \bigvee_{1\le j\le N} \mathscr{X}_j(s_j) \quad\text{and}\quad \mathscr{S}(\boldsymbol{t}) := \bigvee_{1\le k\le M} \mathscr{S}_k(t_k), \tag{4.21}$$

as $\boldsymbol{s}$ and $\boldsymbol{t}$ range respectively over $\mathbf{R}_+^N$ and $\mathbf{R}_+^M$. It follows from Theorem 2.1.1 of Khoshnevisan (2002, p. 233) that $\mathscr{X}$ is an N-parameter commuting filtration. Similarly, $\mathscr{S}$ is an M-parameter commuting filtration. Theorem 2.1.1 of the same reference (p. 233) can be invoked, yet again, to help deduce that $\mathscr{F}$ is an $(N+M)$-parameter commuting filtration, where

$$\mathscr{F}(\boldsymbol{s}\otimes\boldsymbol{t}) := \mathscr{X}(\boldsymbol{s}) \vee \mathscr{S}(\boldsymbol{t}) \qquad \text{for } \boldsymbol{s}\in\mathbf{R}_+^N \text{ and } \boldsymbol{t}\in\mathbf{R}_+^M. \tag{4.22}$$

We need only the following consequence of commutation; it is known as *Cairoli's strong* $(2,2)$*-inequality* (Khoshnevisan, 2002, Theorem 2.3.2, p. 235): For all $f\in L^2(\mathrm{P})$,

$$\mathrm{E}\left[\sup_{\boldsymbol{s}\in\mathbf{Q}_+^N,\,\boldsymbol{t}\in\mathbf{Q}_+^M} \left|\mathrm{E}\left[f \mid \mathscr{F}(\boldsymbol{s}\otimes\boldsymbol{t})\right]\right|^2\right] \le 4^{N+M}\mathrm{E}\left[f^2\right]. \tag{4.23}$$

[$\mathbf{Q}_+^k$ denotes the collection of all $x\in\mathbf{R}_+^k$ such that x_j is rational for all $1\le j\le k$.] Moreover, and this is significant, the same is true if we replace E by $\mathrm{E}_{\mathrm{P}\times\mathrm{Q}}$; i.e., for all $f\in L^2(\mathrm{P}_{\mathcal{L}^d}\times\mathrm{Q}_{\mathcal{L}^N})$,

$$\mathrm{E}_{\mathrm{P}\times\mathrm{Q}}\left[\sup_{\boldsymbol{s}\in\mathbf{Q}_+^N,\,\boldsymbol{t}\in\mathbf{Q}_+^M} \Big|\mathrm{E}\left[f \mid \mathscr{F}(\boldsymbol{s}\otimes\boldsymbol{t})\right]\Big|^2\right] \le 4^{N+M}\mathrm{E}_{\mathrm{P}\times\mathrm{Q}}\left[f^2\right]. \tag{4.24}$$

A proof is hashed out very briefly in Khoshnevisan and Xiao (2002, p. 90).

Proposition 4.3. *Suppose $R>0$ is fixed. Choose and fix $\boldsymbol{s}\in[0\,,R]^N$ and $\boldsymbol{t}\in\mathbf{R}_+^M$. Then, there exists a positive finite constant $A=A(\alpha,d,N,M,R)$ such that for all Borel probability measures μ that are supported on $[0\,,R]^N$,*

$$\mathrm{E}_{\mathrm{P}\times\mathrm{Q}}\left[J_\epsilon(\mu) \mid \mathscr{F}(\boldsymbol{s}\otimes\boldsymbol{t})\right] \ge A\,e^{-[\boldsymbol{t}]}\int_{\boldsymbol{s}'\succ\boldsymbol{s}} \frac{\Phi_\epsilon(\boldsymbol{s}'-\boldsymbol{s})}{\max\left(|\boldsymbol{s}'-\boldsymbol{s}|^{N-\alpha M}\;,\;\epsilon^{N-\alpha M}\right)}\,\mu(d\boldsymbol{s}'), \tag{4.25}$$

$(\mathrm{P}_{\mathcal{L}^d}\times\mathrm{Q}_{\mathcal{L}^N})$-almost everywhere on $\{|\mathfrak{X}(\boldsymbol{s})|\le\epsilon/2\,,|\mathfrak{S}(\boldsymbol{t})-\boldsymbol{s}|\le\epsilon/2\}$.

Proof. Define

$$\chi := (\mathrm{P}_{\mathcal{L}^d}\times\mathrm{Q}_{\mathcal{L}^N})\left(|\mathfrak{X}(\boldsymbol{s}')|\le\epsilon\;,\;|\mathfrak{S}(\boldsymbol{t}')-\boldsymbol{s}'|\le\epsilon\,\Big|\,\mathscr{F}(\boldsymbol{s}\otimes\boldsymbol{t})\right). \tag{4.26}$$

Owing to the Markov random-field property of Khoshnevisan and Xiao (2002, Proposition 5.8), whenever $\boldsymbol{s}' \succ \boldsymbol{s}$ and $\boldsymbol{t}' \succ \boldsymbol{t}$, we have

$$\begin{aligned}\chi &= (\mathrm{P}_{\mathcal{L}^d} \times \mathrm{Q}_{\mathcal{L}^N}) \left(|\mathfrak{X}(\boldsymbol{s}')| \le \epsilon \ , \ |\mathfrak{S}(\boldsymbol{t}') - \boldsymbol{s}'| \le \epsilon \,\middle|\, \mathfrak{X}(\boldsymbol{s}), \ \mathfrak{S}(\boldsymbol{t}) \right) \\ &= \mathrm{P}_{\mathcal{L}^d} \left(|\mathfrak{X}(\boldsymbol{s}')| \le \epsilon \,\middle|\, \mathfrak{X}(\boldsymbol{s}) \right) \cdot \mathrm{Q}_{\mathcal{L}^N} \left(|\mathfrak{S}(\boldsymbol{t}') - \boldsymbol{s}'| \le \epsilon \,\middle|\, \mathfrak{S}(\boldsymbol{t}) \right).\end{aligned} \tag{4.27}$$

We apply Lemma 5.5 of Khoshnevisan and Xiao (2002) to each term above to find that $(\mathrm{P}_{\mathcal{L}^d} \times \mathrm{Q}_{\mathcal{L}^N})$-almost everywhere,

$$\chi = \mathrm{P}\left\{ |\mathfrak{X}(\boldsymbol{s}') - \mathfrak{X}(\boldsymbol{s}) + z| \le \epsilon \right\} \Big\rfloor_{z=\mathfrak{X}(\boldsymbol{s})} \times \mathrm{P}\left\{ |\mathfrak{S}(\boldsymbol{t}') - \mathfrak{S}(\boldsymbol{t}) - \boldsymbol{s}' + w| \le \epsilon \right\} \Big\rfloor_{w=\mathfrak{S}(\boldsymbol{t})}. \tag{4.28}$$

Because $\boldsymbol{s}' \succ \boldsymbol{s}$ and $\boldsymbol{t}' \succ \boldsymbol{t}$, the distributions of $\mathfrak{X}(\boldsymbol{s}') - \mathfrak{X}(\boldsymbol{s})$ and $\mathfrak{S}(\boldsymbol{t}') - \mathfrak{S}(\boldsymbol{t})$ are the same as those of $\mathfrak{X}(\boldsymbol{s}' - \boldsymbol{s})$ and $\mathfrak{S}(\boldsymbol{t}' - \boldsymbol{t})$, respectively. Therefore, $(\mathrm{P}_{\mathcal{L}^d} \times \mathrm{Q}_{\mathcal{L}^N})$-a.e. on $\{ |\mathfrak{X}(\boldsymbol{s})| \le \epsilon/2 \ , \ |\mathfrak{S}(\boldsymbol{t}) - \boldsymbol{s}| \le \epsilon/2 \}$,

$$\begin{aligned}\chi &\ge \mathrm{P}\left\{ |\mathfrak{X}(\boldsymbol{s}') - \mathfrak{X}(\boldsymbol{s})| \le \epsilon/2 \right\} \cdot \mathrm{P}\left\{ |\mathfrak{S}(\boldsymbol{t}' - \boldsymbol{t}) - (\boldsymbol{s}' - \boldsymbol{s})| \le \epsilon/2 \right\} \\ &\ge \frac{1}{32^{d+N}} \, P_\epsilon(\boldsymbol{s}' - \boldsymbol{s} \, ; \boldsymbol{t}' - \boldsymbol{t}),\end{aligned} \tag{4.29}$$

where

$$P_\epsilon(\boldsymbol{s}' - \boldsymbol{s} \, ; \boldsymbol{t}' - \boldsymbol{t}) := \mathrm{P}\left\{ |\mathfrak{X}(\boldsymbol{s}') - \mathfrak{X}(\boldsymbol{s})| \le \epsilon \right\} \cdot \mathrm{P}\left\{ |\mathfrak{S}(\boldsymbol{t}' - \boldsymbol{t}) - (\boldsymbol{s}' - \boldsymbol{s})| \le \epsilon \right\}. \tag{4.30}$$

For the last inequality in (4.29), we have applied (2.12) to both processes $\mathfrak{X}$ and $\mathfrak{S}$. This implies that

$$\begin{aligned}&\mathrm{E}_{\mathrm{P}\times\mathrm{Q}} \left[J_\epsilon(\mu) \mid \mathscr{F}(\boldsymbol{s} \otimes \boldsymbol{t}) \right] \\ &\qquad \ge \frac{1}{32^{d+N} (2\epsilon)^{d+N}} \int_{\substack{\boldsymbol{t}' \in \mathbf{R}^M_+ : \\ \boldsymbol{t}' \succ \boldsymbol{t}}} \left(\int_{\boldsymbol{s}' \succ \boldsymbol{s}} P_\epsilon(\boldsymbol{s}' - \boldsymbol{s} \, ; \boldsymbol{t}' - \boldsymbol{t}) \, \mu(d\boldsymbol{s}') \right) e^{-[\boldsymbol{t}']} \, d\boldsymbol{t}',\end{aligned} \tag{4.31}$$

$(\mathrm{P}_{\mathcal{L}^d} \times \mathrm{Q}_{\mathcal{L}^N})$-almost everywhere on $\{ |\mathfrak{X}(\boldsymbol{s})| \le \epsilon/2 \ , \ |\mathfrak{S}(\boldsymbol{t}) - \boldsymbol{s}| \le \epsilon/2 \}$.

Recall from (4.16) the one-potential density υ of $\mathfrak{S}$. According to the Fubini–Tonelli theorem, for all $x \in \mathbf{R}^N$,

$$\begin{aligned}\int_{\substack{\boldsymbol{t}' \in \mathbf{R}^M_+ : \\ \boldsymbol{t}' \succ \boldsymbol{t}}} \mathrm{P}\left\{ |\mathfrak{S}(\boldsymbol{t}' - \boldsymbol{t}) - x| \le \epsilon \right\} e^{-[\boldsymbol{t}']} \, d\boldsymbol{t}' &\ge e^{-[\boldsymbol{t}]} \int_{\mathbf{R}^M_+} \mathrm{P}\left\{ |\mathfrak{S}(\boldsymbol{u}) - x| \le \epsilon \right\} e^{-[\boldsymbol{u}]} \, d\boldsymbol{u} \\ &= e^{-[\boldsymbol{t}]} \int_{\substack{z \in \mathbf{R}^N : \\ |z - x| \le \epsilon}} \upsilon(z) \, dx.\end{aligned} \tag{4.32}$$

Thanks to (1) and (2) [confer with the paragraph following (4.16)], we can find a finite constant $a > 0$—not depending on $(\epsilon\,, \boldsymbol{t})$—such that as long as $|x| \le R$,

$$\int_{\substack{\boldsymbol{t}' \in \mathbf{R}^M_+:\\ \boldsymbol{t}' \succ \boldsymbol{t}}} \mathrm{P}\left\{|\mathfrak{S}(\boldsymbol{t}' - \boldsymbol{t}) - x| \le \epsilon\right\} e^{-[\boldsymbol{t}']}\, d\boldsymbol{t}' \ge a e^{-[\boldsymbol{t}]}(2\epsilon)^N \min\left(\frac{1}{|x|^{N-\alpha M}}\ ,\ \frac{1}{\epsilon^{N-\alpha M}}\right). \tag{4.33}$$

[Compare with (4.19).] The proposition follows from (4.31) and (4.33) after a few lines of direct computation. □

We can use the earlier results of Khoshnevisan and Xiao (2002) to extend Proposition 4.3 further, which will be useful for proving Proposition 4.8. In light of the existing proof of Proposition 4.3, the said extension does not require any new ideas. Therefore, we will not offer a proof. However, we need to introduce a fair amount of notation in order to state the extension in its proper form.

Any subset π of $\{1, \ldots, N\}$ induces a partial order on $\mathbf{R}^N_+$ as follows: For all $\boldsymbol{s}, \boldsymbol{t} \in \mathbf{R}^N_+$,

$$\boldsymbol{s} \prec_\pi \boldsymbol{t} \quad \text{means that} \begin{cases} s_i \le t_i & \text{for all } i \in \pi, \text{ and}\\ s_i > t_i & \text{for all } i \notin \pi. \end{cases} \tag{4.34}$$

We identify each and every $\pi \subseteq \{1, \ldots, N\}$ with the partial order $\prec_\pi$.

For every $\pi \subseteq \{1, \ldots, N\}$, $1 \le j \le N$, and $u \ge 0$, define

$$\mathscr{X}^\pi_j(u) := \begin{cases} \sigma\left(\{X_j(v)\}_{0 \le v \le u}\right) & \text{if } j \in \pi,\\ \sigma\left(\{X_j(v)\}_{v \ge u}\right) & \text{if } j \notin \pi. \end{cases} \tag{4.35}$$

As is customary, $\sigma(\cdots)$ denotes the σ-algebra generated by the parenthesized quantities. For all $\pi \subseteq \{1, \ldots, N\}$ and $\boldsymbol{t} \in \mathbf{R}^N_+$ define

$$\mathscr{X}^\pi(\boldsymbol{t}) := \bigvee_{1 \le j \le N} \mathscr{X}^\pi_j(t_j). \tag{4.36}$$

It is not hard to check that $\mathscr{X}^\pi$ is an N-parameter filtration in the partial order $\prec_\pi$. That is, $\mathscr{X}^\pi(\boldsymbol{s}) \subseteq \mathscr{X}^\pi(\boldsymbol{t})$ whenever $\boldsymbol{s} \prec_\pi \boldsymbol{t}$.

For all $\pi \subseteq \{1, \ldots, N\}$, $\boldsymbol{s} \in \mathbf{R}^N_+$, and $\boldsymbol{t} \in \mathbf{R}^M_+$, define

$$\mathscr{F}^\pi(\boldsymbol{s} \otimes \boldsymbol{t}) := \mathscr{X}^\pi(\boldsymbol{s}) \vee \mathscr{S}(\boldsymbol{t}). \tag{4.37}$$

By Lemma 5.7 in Khoshnevisan and Xiao (2002), $\mathscr{F}^\pi$ is an $(N+M)$-parameter commuting filtration. It follows that, for all $f \in L^2(\mathrm{P})$ and $\pi \subseteq \{1, \ldots, N\}$,

$$\mathrm{E}\left[\sup_{\boldsymbol{s} \in \mathbf{Q}^N_+,\, \boldsymbol{t} \in \mathbf{Q}^M_+} \left|\mathrm{E}\left[f \mid \mathscr{F}^\pi(\boldsymbol{s} \otimes \boldsymbol{t})\right]\right|^2\right] \le 4^{N+M}\mathrm{E}\left[f^2\right]. \tag{4.38}$$

Also, for all $f \in L^2(\mathrm{P}_{\mathcal{L}^d} \times \mathrm{Q}_{\mathcal{L}^N})$ and $\pi \subseteq \{1, \ldots, N\}$,

$$\mathrm{E}_{\mathrm{P}\times\mathrm{Q}}\left[\sup_{\boldsymbol{s}\in\mathbf{Q}_+^N,\, \boldsymbol{t}\in\mathbf{Q}_+^M} \left|\mathrm{E}\left[f \mid \mathscr{F}^\pi(\boldsymbol{s}\otimes\boldsymbol{t})\right]\right|^2\right] \le 4^{N+M}\mathrm{E}_{\mathrm{P}\times\mathrm{Q}}\left[f^2\right]. \tag{4.39}$$

Note that when $\pi = \{1, \ldots, N\}$, (4.38) and (4.39) are the same as (4.23) and (4.24), respectively. However, the more general forms above have more content, as can be seen by considering other partial orders π than $\{1, \ldots, N\}$ [or $\varnothing$].

We are ready to present the asserted refinement of Proposition 4.3.

Proposition 4.4. *Suppose $R > 0$ is fixed. Choose and fix $\boldsymbol{s} \in [0\,, R]^N$ and $\boldsymbol{t} \in \mathbf{R}_+^M$. Then, there exists a positive finite constant $A = A(\alpha, d, N, M, R)$ such that for all Borel probability measures μ that are supported on $[0\,, R]^N$, and for all $\pi \subseteq \{1, \ldots, N\}$,*

$$\mathrm{E}_{\mathrm{P}\times\mathrm{Q}}\left[J_\epsilon(\mu) \mid \mathscr{F}^\pi(\boldsymbol{s}\otimes\boldsymbol{t})\right] \ge Ae^{-[\boldsymbol{t}]}\int_{\boldsymbol{s}'\succ_\pi \boldsymbol{s}} \frac{\Phi_\epsilon(\boldsymbol{s}'-\boldsymbol{s})}{\max\left(|\boldsymbol{s}'-\boldsymbol{s}|^{N-\alpha M}\ ,\ \epsilon^{N-\alpha M}\right)}\,\mu(d\boldsymbol{s}'), \tag{4.40}$$

$(\mathrm{P}_{\mathcal{L}^d} \times \mathrm{Q}_{\mathcal{L}^N})$-almost everywhere on $\{|\mathfrak{X}(\boldsymbol{s})| \le \epsilon/2\,, |\mathfrak{S}(\boldsymbol{t}) - \boldsymbol{s}| \le \epsilon/2\}$.

4.2. **More Moment Estimates.** Consider a compact set $B \subset (0\,,\infty)^M$ with nonempty interior. For any Borel probability measure μ on $\mathbf{R}_+^N$ and a real number $\epsilon > 0$, we define a random measure on $\mathbf{R}_+^N$ by

$$J_\epsilon^{B,\mu}(C) := \frac{1}{(2\epsilon)^{d+N}}\int_B\left(\int_C \mathbf{1}_{\{|\mathfrak{X}(\boldsymbol{s})|\le\epsilon\,,\,|\mathfrak{S}(\boldsymbol{t})-\boldsymbol{s}|\le\epsilon\}}\,\mu(d\boldsymbol{s})\right)d\boldsymbol{t}, \tag{4.41}$$

where $C \subseteq \mathbf{R}_+^N$ denotes an arbitrary Borel set.

The following is the analogue of (4.6) under the probability measure P.

Lemma 4.5. *Choose and fix a compact set $B \subset (0\,,\infty)^M$ with nonempty interior and a real number $R > 1$. Then, there exists a positive and finite number A such that for all Borel probability measures μ on $T := [R^{-1}, R]^N$,*

$$\liminf_{\epsilon\to 0^+} \mathrm{E}\left[J_\epsilon^{B,\mu}(T)\right] > A. \tag{4.42}$$

Proof. Thanks to the inversion formula, the density function of $\mathfrak{X}(\boldsymbol{s})$ is continuous for every $\boldsymbol{s} \in (0\,,\infty)^N$. Also, the density of $\mathfrak{S}(\boldsymbol{t})$ is uniformly continuous for each $\boldsymbol{t} \in (0\,,\infty)^M$. By Fatou's lemma,

$$\begin{aligned}\liminf_{\epsilon\to 0^+} \mathrm{E}\left[J_\epsilon^{B,\mu}(T)\right] &\ge \int_B\left(\int_T \Phi(\boldsymbol{s})f_{\boldsymbol{t}}(\boldsymbol{s})\,\mu(d\boldsymbol{s})\right)d\boldsymbol{t}\\ &\ge \mathcal{L}^N(B)\inf_{\boldsymbol{s}\in T}\Phi(\boldsymbol{s})\cdot\inf_{\boldsymbol{s}\in T}\inf_{\boldsymbol{t}\in B} f_{\boldsymbol{t}}(\boldsymbol{s}).\end{aligned} \tag{4.43}$$

Recall that $f_{\boldsymbol{t}}(\boldsymbol{s})$ is the density function of $\mathfrak{S}(\mathbf{t})$. It remains to prove that the two infima are strictly positive. The first fact follows from the monotonicity bound,

$$\inf_{\boldsymbol{s}\in T}\Phi(\boldsymbol{s})=\Phi\left(\frac{1}{R},\ldots,\frac{1}{R}\right)=\int_{\mathbf{R}^d}\exp\left(-\frac{1}{R}\sum_{1\le j\le N}\Psi_j(\xi)\right)d\xi, \tag{4.44}$$

and this is positive. The second fact follows from Bochner's subordination (Khoshnevisan, 2002, p. 378) and the fact that the cube T is a positive distance away from the axes of $\mathbf{R}^N_+$. □

The analogue of Proposition 4.1 follows next.

Proposition 4.6. *Choose and fix $R>1$ and a compact set $B\subset(0\,,\infty)^M$ with nonempty interior. Let $K:\mathbf{R}^N_+\times\mathbf{R}^N_+\to\mathbf{R}_+$ be a measurable function. If $N>\alpha M$, then there exists a finite and positive constant A—depending only on (α,d,N,M,B,R)—such that for all Borel probability measures μ on $T=[R^{-1},\,R]^N$ and all $\epsilon>0$,*

$$\mathrm{E}\left[\int_T\int_T K(\boldsymbol{s}\,,\boldsymbol{s}')\,J^{B,\mu}_\epsilon(d\boldsymbol{s})J^{B,\mu}_\epsilon(d\boldsymbol{s}')\right]\le A\int_T\int_T\frac{\Phi(\boldsymbol{s}-\boldsymbol{s}')K(\boldsymbol{s}\,,\boldsymbol{s}')}{|\boldsymbol{s}-\boldsymbol{s}'|^{N-\alpha M}}\,\mu(d\boldsymbol{s})\,\mu(d\boldsymbol{s}'). \tag{4.45}$$

In particular, we have

$$\sup_{\epsilon>0}\mathrm{E}\left[\left(J^{B,\mu}_\epsilon(T)\right)^2\right]\le A\,I^{(N-\alpha M)}_\Phi(\mu). \tag{4.46}$$

Proof. We use an argument that is similar to that of Khoshnevisan and Xiao (2002, Lemma 3.4). For all $\boldsymbol{s},\boldsymbol{s}'\in\mathbf{R}^N_+$ define $\boldsymbol{s}\wedge\boldsymbol{s}'$ to be the N-vector whose jth coordinate is $\min(s_j\,,s'_j)$. We write

$$Z_1:=\mathfrak{X}(\boldsymbol{s}\wedge\boldsymbol{s}'),\qquad Z_2:=\mathfrak{X}(\boldsymbol{s}')-\mathfrak{X}(\boldsymbol{s}\wedge\boldsymbol{s}'), \tag{4.47}$$

and

$$Z_3:=\mathfrak{X}(\boldsymbol{s})-\mathfrak{X}(\boldsymbol{s}\wedge\boldsymbol{s}'). \tag{4.48}$$

Then, it is easy to check that $(Z_1\,,Z_2\,,Z_3)$ are independent. Therefrom we find that $\mathrm{P}\{|\mathfrak{X}(\boldsymbol{s}')|\le\epsilon\,,|\mathfrak{X}(\boldsymbol{s})|\le\epsilon\}$ is equal to

$$\begin{aligned}&\mathrm{P}\left\{|Z_1+Z_2|\le\epsilon\ ,\ |Z_1+Z_3|\le\epsilon\right\}\\ &\qquad=\int_{\mathbf{R}^d}\mathrm{P}\left\{|z+Z_2|\le\epsilon\ ,\ |z+Z_3|\le\epsilon\right\}p_{\boldsymbol{s}\wedge\boldsymbol{s}'}(z)\,dz\\ &\qquad\le\Phi(\boldsymbol{s}'\wedge\boldsymbol{s})\int_{\mathbf{R}^d}\mathrm{P}\left\{|z+Z_2|\le\epsilon\ ,\ |z+Z_3|\le\epsilon\right\}\,dz.\end{aligned} \tag{4.49}$$

See also (2.7). After we apply the Fubini-Tonelli theorem and then change variables [$w := z + Z_2$], we find that $\mathrm{P}\{|\mathfrak{X}(\boldsymbol{s}')| \le \epsilon\,, |\mathfrak{X}(\boldsymbol{s})| \le \epsilon\}$ is at most

$$\begin{aligned}\Phi(\boldsymbol{s}'\wedge\boldsymbol{s})\int_{\{|w|\le\epsilon\}} \mathrm{P}\left\{|w+Z_3-Z_2|\le\epsilon\right\}\,dw &\le (2\epsilon)^d\Phi(\boldsymbol{s}'\wedge\boldsymbol{s})\mathrm{P}\left\{|Z_3-Z_2|\le 2\epsilon\right\}\\ &\le 32^d(2\epsilon)^{2d}\Phi(\boldsymbol{s}'\wedge\boldsymbol{s})\Phi(\boldsymbol{s}'-\boldsymbol{s}).\end{aligned}\tag{4.50}$$

The last inequality is a consequence of (2.11), because $Z_3 - Z_2 = \mathfrak{X}(\boldsymbol{s}') - \mathfrak{X}(\boldsymbol{s})$ has the same distribution as $\mathfrak{X}(\boldsymbol{r})$, where the jth coordinate of $\boldsymbol{r}$ is $r_j := |s_j' - s_j|$. In other words, for all $\epsilon > 0$ and $\boldsymbol{s}, \boldsymbol{s}' \in [1/R\,, R]^N$,

$$\frac{\mathrm{P}\left\{|\mathfrak{X}(\boldsymbol{s}')|\le\epsilon\ ,\ |\mathfrak{X}(\boldsymbol{s})|\le\epsilon\right\}}{(2\epsilon)^{2d}} \le A_1\,\Phi(\boldsymbol{s}'-\boldsymbol{s}),\tag{4.51}$$

where $A_1 := 32^d\,\Phi(1/R\,,\ldots,1/R)$.

Now consider $\boldsymbol{t}, \boldsymbol{t}' \in B$ and $\boldsymbol{s}, \boldsymbol{s}' \in [1/R\,, R]^N$. For all $\epsilon > 0$,

$$\begin{aligned}&\mathrm{P}\left\{|\mathfrak{S}(\boldsymbol{t}')-\boldsymbol{s}'|\le\epsilon\ ,\ |\mathfrak{S}(\boldsymbol{t})-\boldsymbol{s}|\le\epsilon\right\}\\ &\qquad= \mathrm{P}\left\{|W_1+W_2-\boldsymbol{s}'|\le\epsilon\ ,\ |W_1+W_3-\boldsymbol{s}|\le\epsilon\right\},\end{aligned}\tag{4.52}$$

where $W_1 := \mathfrak{S}(\boldsymbol{t}'\wedge\boldsymbol{t})$, $W_2 := \mathfrak{S}(\boldsymbol{t}') - W_1$, and $W_3 := \mathfrak{S}(\boldsymbol{t}) - W_1$. A little thought shows that $(W_1\,, W_2\,, W_3)$ are independent. Moreover, the density function of W_1 is $f_{\boldsymbol{t}'\wedge\boldsymbol{t}}$. Therefore,

$$\begin{aligned}&\mathrm{P}\left\{|\mathfrak{S}(\boldsymbol{t}')-\boldsymbol{s}'|\le\epsilon\ ,\ |\mathfrak{S}(\boldsymbol{t})-\boldsymbol{s}|\le\epsilon\right\}\\ &\qquad= \int_{\mathbf{R}^N} \mathrm{P}\left\{|z+W_2-\boldsymbol{s}'|\le\epsilon\ ,\ |z+W_3-\boldsymbol{s}|\le\epsilon\right\}\,f_{\boldsymbol{t}'\wedge\boldsymbol{t}}(z)\,dz.\end{aligned}\tag{4.53}$$

Because the density function $f_{\boldsymbol{t}'\wedge\boldsymbol{t}}$ is maximized at the origin,

$$\begin{aligned}&\mathrm{P}\left\{|\mathfrak{S}(\boldsymbol{t}')-\boldsymbol{s}'|\le\epsilon\ ,\ |\mathfrak{S}(\boldsymbol{t})-\boldsymbol{s}|\le\epsilon\right\}\\ &\qquad\le f_{\boldsymbol{t}'\wedge\boldsymbol{t}}(0)\int_{\mathbf{R}^N} \mathrm{P}\left\{|z+W_2-\boldsymbol{s}'|\le\epsilon\ ,\ |z+W_3-\boldsymbol{s}|\le\epsilon\right\}\ dz.\end{aligned}\tag{4.54}$$

Next we argue—as we did earlier in order to derive (4.49) and (4.50)—to deduce that

$$\begin{aligned}&\mathrm{P}\left\{|\mathfrak{S}(\boldsymbol{t}')-\boldsymbol{s}'|\le\epsilon\ ,\ |\mathfrak{S}(\boldsymbol{t})-\boldsymbol{s}|\le\epsilon\right\}\\ &\qquad\le f_{\boldsymbol{t}'\wedge\boldsymbol{t}}(0)\int_{\{|x|\le\epsilon\}} \mathrm{P}\left\{|x+W_2-W_3-(\boldsymbol{s}'-\boldsymbol{s})|\le\epsilon\right\}\ dx\\ &\qquad\le (2\epsilon)^N f_{\boldsymbol{t}'\wedge\boldsymbol{t}}(0)\ \mathrm{P}\left\{|W_2-W_3-(\boldsymbol{s}'-\boldsymbol{s})|\le 2\epsilon\right\}\\ &\qquad= (2\epsilon)^N f_{\boldsymbol{t}'\wedge\boldsymbol{t}}(0)\ \mathrm{P}\left\{|\mathfrak{S}(\boldsymbol{t}')-\mathfrak{S}(\boldsymbol{t})-(\boldsymbol{s}'-\boldsymbol{s})|\le 2\epsilon\right\}\\ &\qquad= (2\epsilon)^N f_{\boldsymbol{t}'\wedge\boldsymbol{t}}(0)\int_{\{|z-(\boldsymbol{s}'-\boldsymbol{s})|\le 2\epsilon\}} f_{\boldsymbol{t}'-\boldsymbol{t}}(z)\,dz.\end{aligned}\tag{4.55}$$

[It might help to confer with (4.15) at this point.]

Now,

$$f_{\boldsymbol{t}'\wedge\boldsymbol{t}}(0) = \frac{1}{(2\pi)^N}\int_{\mathbf{R}^N} e^{-[\boldsymbol{t}'\wedge\boldsymbol{t}]\cdot\|\xi\|^\alpha}\,d\xi = \frac{A}{[\boldsymbol{t}'\wedge\boldsymbol{t}]^{M/\alpha}}, \tag{4.56}$$

where $A := (2\pi)^{-N}\int_{\mathbf{R}^N}\exp(-\|x\|^\alpha)\,dx$ is positive and finite. Since $\boldsymbol{t},\boldsymbol{t}'\in B$ and B is strictly away from the axes of $\mathbf{R}^M_+$. Therefore, there exists a finite constant A_1—depending only on the distance between B and the axes of $\mathbf{R}^M_+$—such that

$$\begin{aligned}\int_B\int_B \mathrm{P}\left\{|\mathfrak{S}(\boldsymbol{t}')-\boldsymbol{s}'|\le\epsilon\ ,\ |\mathfrak{S}(\boldsymbol{t})-\boldsymbol{s}|\le\epsilon\right\}\,d\boldsymbol{t}'\,d\boldsymbol{t}&\\ \le A_1(2\epsilon)^N\int_{\{|z-(\boldsymbol{s}'-\boldsymbol{s})|\le 2\epsilon\}}\int_B\int_B f_{\boldsymbol{t}'-\boldsymbol{t}}(z)\,d\boldsymbol{t}'\,d\boldsymbol{t}\,dz&\\ \le A_2(2\epsilon)^N\int_{\{|z-(\boldsymbol{s}'-\boldsymbol{s})|\le 2\epsilon\}}\left(\int_B f_{\boldsymbol{t}}(z)\,d\boldsymbol{t}\right)dz,&\end{aligned} \tag{4.57}$$

where A_2 is another finite constant that depends only on: (a) the distance between B and the axes of $\mathbf{R}^M_+$; and (b) the distance between B and infinity; i.e., $\sup\{|x|:\ x\in B\}$. We can find a constant A_3—with the same dependencies as A_2—such that $\exp(-[\boldsymbol{t}])\ge A_3^{-1}$ for all $\boldsymbol{t}\in B$. This proves that $\int_B f_{\boldsymbol{t}}(z)\,d\boldsymbol{t}\le A_3 v(z)$ for all $z\in\mathbf{R}^N$. It follows that

$$\begin{aligned}\int_B\int_B \mathrm{P}\left\{|\mathfrak{S}(\boldsymbol{t}')-\boldsymbol{s}'|\le\epsilon\ ,\ |\mathfrak{S}(\boldsymbol{t})-\boldsymbol{s}|\le\epsilon\right\}\,d\boldsymbol{t}'\,d\boldsymbol{t}&\\ \le A_2A_3(2\epsilon)^N\int_{\{|z-(\boldsymbol{s}'-\boldsymbol{s})|\le 2\epsilon\}}\frac{dz}{|z|^{N-\alpha M}}.&\end{aligned} \tag{4.58}$$

See (4.18). From this and (4.19) we deduce that

$$\begin{aligned}\int_B\int_B \mathrm{P}\left\{|\mathfrak{S}(\boldsymbol{t}')-\boldsymbol{s}'|\le\epsilon\ ,\ |\mathfrak{S}(\boldsymbol{t})-\boldsymbol{s}|\le\epsilon\right\}\,d\boldsymbol{t}'\,d\boldsymbol{t}&\\ \le A''A_2A_3(2\epsilon)^{2N}\min\left(\frac{1}{|\boldsymbol{s}'-\boldsymbol{s}|^{N-\alpha M}}\ ,\ \frac{1}{\epsilon^{N-\alpha M}}\right).&\end{aligned} \tag{4.59}$$

This and (4.51) together imply that

$$\begin{aligned}\mathrm{E}\left[\int_T\int_T K(\boldsymbol{s}\,,\boldsymbol{s}')\,J_\epsilon^{B,\mu}(d\boldsymbol{s})J_\epsilon^{B,\mu}(d\boldsymbol{s}')\right]&\\ \le A'''\int_T\int_T\frac{\Phi(\boldsymbol{s}-\boldsymbol{s}')K(\boldsymbol{s}\,,\boldsymbol{s}')}{\max\left(|\boldsymbol{s}'-\boldsymbol{s}|^{N-\alpha M}\ ,\ \epsilon^{N-\alpha M}\right)}\,\mu(d\boldsymbol{s})\,\mu(d\boldsymbol{s}'),&\end{aligned} \tag{4.60}$$

where A''' depends only on (α,d,N,M,R,B). This proposition follows. □

4.3. **Proof of Theorem 3.2.** Our proof of Theorem 3.2 rests on two further results. Both are contributions to the potential theory of random fields, and determine when a given time set $F \subset \mathbf{R}^N_+$ is "polar" simultaneously for the range of $\mathfrak{S}$ and for the level-sets of $\mathfrak{X}$.

Proposition 4.7. *Choose and fix a compact set $F \subset (0\,,\infty)^N$. If $N > \alpha M$ and $I_\Phi^{(N-\alpha M)}(\mu)$ is finite for some $\mu \in \mathcal{P}(F)$, then $\mathfrak{X}^{-1}(\{0\}) \cap F \cap \mathfrak{S}(\mathbf{R}^M_+) \neq \varnothing$ with positive probability.*

Proof. Since $F \subset (0\,,\infty)^N$ is compact, there exists $R > 1$ such that $F \subseteq T = [R^{-1}, R]^N$. Suppose $I_\Phi^{(N-\alpha M)}(\mu) < \infty$ for some Borel probability measure μ on F. Then there exists a continuous function $\rho : \mathbf{R}^N \to [1\,,\infty)$ such that $\lim_{\boldsymbol{s}\to\boldsymbol{s}_0} \rho(\boldsymbol{s}) = \infty$ for every $\boldsymbol{s}_0 \in \mathbf{R}^N$ with at least one coordinate equals 0 and

$$\iint \frac{\Phi(\boldsymbol{s}-\boldsymbol{s}')\rho(\boldsymbol{s}-\boldsymbol{s}')}{|\boldsymbol{s}-\boldsymbol{s}'|^{N-\alpha M}}\,\mu(d\boldsymbol{s})\,\mu(d\boldsymbol{s}') < \infty. \tag{4.61}$$

See Khoshnevisan and Xiao (2002, p. 73) for a construction of ρ.

For a fixed compact set $B \subset (0\,,\infty)^M$ with nonempty interior, consider the random measures $\{J_\epsilon^{B,\mu}\}_{\epsilon>0}$ defined by (4.41). If $J_\epsilon^{B,\mu}(T) > 0$ then certainly $\mathfrak{X}^{-1}(U_\epsilon) \cap F \cap \mathfrak{S}(B) \neq \varnothing$, where $U_\epsilon := \{x \in \mathbf{R}^d : |x| \le \epsilon\}$.

It follows from Lemma 4.5, Proposition 4.6 and a second moment argument (Kahane, 1985, pp. 204–206) that there exists a subsequence $\{J_{\epsilon_n}^{B,\mu}\}$ which converges weakly to a random measure ν such that

$$\mathrm{P}\{\nu(T) > 0\} \ge \frac{a_1^2}{a_2} > 0, \tag{4.62}$$

where

$$a_1 := \inf_{0<\epsilon<1} \mathrm{E}\left[J_\epsilon^{B,\mu}(T)\right] > 0 \quad \text{and} \quad a_2 := \sup_{\epsilon>0} \mathrm{E}\left[\left(J_\epsilon^{B,\mu}(T)\right)^2\right] < \infty. \tag{4.63}$$

Moreover,

$$\mathrm{E}\left[\iint \rho(\boldsymbol{s}-\boldsymbol{s}')\,\nu(d\boldsymbol{s})\,\nu(d\boldsymbol{s}')\right] \le A \iint \frac{\Phi(\boldsymbol{s}-\boldsymbol{s}')\rho(\boldsymbol{s}-\boldsymbol{s}')}{|\boldsymbol{s}-\boldsymbol{s}'|^{N-\alpha M}}\,\mu(d\boldsymbol{s})\,\mu(d\boldsymbol{s}'). \tag{4.64}$$

This and (4.61) together imply that almost surely

$$\nu\{\boldsymbol{s} \in T : s_j = a \text{ for some } j\} = 0 \quad \text{for all} \quad a \in \mathbf{R}_+. \tag{4.65}$$

Therefore, we have shown that

$$\begin{aligned} \inf_{\mu\in\mathcal{P}(F)} I_\Phi^{(N-\alpha M)}(\mu) < \infty \quad &\Longrightarrow \quad \mathrm{P}\left\{\overline{\mathfrak{X}^{-1}(\{0\})} \cap F \cap \overline{\mathfrak{S}(B)} \neq \varnothing\right\} > 0 \\ &\Longrightarrow \quad \mathrm{P}\left\{\overline{\mathfrak{X}^{-1}(\{0\})} \cap F \cap \overline{\mathfrak{S}(\mathbf{R}^M_+)} \neq \varnothing\right\} > 0. \end{aligned} \tag{4.66}$$

Now we need to make use of some earlier results of Khoshnevisan and Xiao (2002; 2005) and Khoshnevisan et al. (2003) to remove the closure signs in (4.66). First, since the density function of $\mathfrak{S}(\boldsymbol{t})$ ($\boldsymbol{t} \in (0\,,\infty)^M$) is strictly positive everywhere, a slight modification of the proof of Lemma 4.1 in Khoshnevisan and Xiao (2005, eq.'s 4.9–4.11) implies that for every Borel set $\widetilde{F} \subseteq \mathbf{R}^N$,

$$\widetilde{F} \cap \overline{\mathfrak{S}(\mathbf{R}^M_+)} = \varnothing \quad \text{a.s.} \quad \Longleftrightarrow \quad \mathcal{L}^N\big(\widetilde{F} \ominus \overline{\mathfrak{S}(\mathbf{R}^M_+)}\big) = 0 \quad \text{a.s.} \tag{4.67}$$

On the other hand, Proposition 5.7 and the proof of Lemma 5.5 in Khoshnevisan et al. (2003) show that $\mathcal{L}^N\big(\widetilde{F} \ominus \overline{\mathfrak{S}(\mathbf{R}^M_+)}\big) = 0$ a.s. is equivalent to $\mathscr{C}_{N-\alpha M}(\widetilde{F}) = 0$, where $\mathscr{C}_\beta$ denotes the β-dimensional Bessel-Riesz capacity.

By applying the preceding facts to $\widetilde{F} = \overline{\mathfrak{X}^{-1}(\{0\})} \cap F$, we conclude that (4.66) implies that $\mathscr{C}_{N-\alpha M}(\overline{\mathfrak{X}^{-1}(\{0\})} \cap F) > 0$ with positive probability. This and Theorem 4.4 of Khoshnevisan and Xiao (2005) together yield,

$$\mathrm{P}\left\{\overline{\mathfrak{X}^{-1}(\{0\})} \cap F \cap \mathfrak{S}(\mathbf{R}^M_+) \neq \varnothing\right\} > 0. \tag{4.68}$$

We have proved the following:

$$\inf_{\mu \in \mathcal{P}(F)} I_\Phi^{(N-\alpha M)}(\mu) < \infty \;\Rightarrow\; \mathrm{P}\left\{\overline{\mathfrak{X}^{-1}(\{0\})} \cap F \cap \mathfrak{S}(\mathbf{R}^M_+) \neq \varnothing\right\} > 0. \tag{4.69}$$

It remains to prove that (4.69) still holds when $\overline{\mathfrak{X}^{-1}(\{0\})}$ is replaced by $\mathfrak{X}^{-1}(\{0\})$. This can be done by proving that the random measure ν is supported on $\mathfrak{X}^{-1}(\{0\}) \cap F \cap \mathfrak{S}(\mathbf{R}^M_+)$. For this purpose, it is sufficient to prove that for every $\delta > 0$, $\nu(D(\delta)) = 0$ a.s., where $D(\delta) := \{\boldsymbol{s} \in T : |\mathfrak{X}(\boldsymbol{s})| > \delta\}$. However, because of (4.65), the proof of the last statement is the same as that in Khoshnevisan and Xiao (2002, p. 76). The proof of Proposition 4.7 is finished. □

Proposition 4.8. *Choose and fix a compact set $F \subset (0\,,\infty)^N$. If $N > \alpha M$ and $I_\Phi^{(N-\alpha M)}(\mu)$ is infinite for all $\mu \in \mathcal{P}(F)$, then $\overline{\mathfrak{X}^{-1}(\{x\})} \cap F \cap \mathfrak{S}(\mathbf{R}^M_+) = \varnothing$ almost surely, for all $x \in \mathbf{R}^d$.*

Remark 4.9. It follows from this proposition and Theorem 4.4 of Khoshnevisan and Xiao (2005) [or Theorem 4.1.1 of Khoshnevisan (2002, p. 423)] that, under the above conditions, $\mathscr{C}_{N-\alpha M}\big(\overline{\mathfrak{X}^{-1}(\{x\})} \cap F\big) = 0$ a.s., for every $x \in \mathbf{R}^d$. Hence $\dim_{\mathrm{H}}\big(\overline{\mathfrak{X}^{-1}(\{x\})} \cap F\big) \leq N - \alpha M$ a.s. This is the argument for proving the upper bound in Theorem 3.2.

Proof. By compactness, $F \subseteq [1/R\,,R]^N$ for some $R > 1$ large enough. We fix this R throughout the proof. Also throughout, we assume that for all $\mu \in \mathcal{P}(F)$,

$$I_\Phi^{(N-\alpha M)}(\mu) = \infty. \tag{4.70}$$

Let us assume that the collection of all $(x\,,y)\in\mathbf{R}^d\times\mathbf{R}^N$ for which the following holds has positive $(\mathcal{L}^d\times\mathcal{L}^N)$-measure:

$$\mathrm{P}\left\{\overline{\mathfrak{X}^{-1}(\{x\})}\cap F\cap\left(y\oplus\mathfrak{S}\left([0\,,R]^M\right)\right)\neq\varnothing\right\}>0, \tag{4.71}$$

where $y\oplus E:=\{y+z:\ z\in E\}$ for all singletons y and all sets E. The major portion of this proof is concerned with proving that (4.71) contradicts the earlier assumption (4.70).

Note that (4.71) is equivalent to the statement that for all $(x\,,y)$ in a set of positive $(\mathcal{L}^d\times\mathcal{L}^N)$ measure,

$$(\mathrm{P}_{-x}\times\mathrm{Q}_y)\left\{\overline{\mathfrak{X}^{-1}(\{0\})}\cap F\cap\mathfrak{S}\left([0\,,R]^M\right)\neq\varnothing\right\}>0. \tag{4.72}$$

For all $\boldsymbol{s}\in[0\,,R]^N$, $\boldsymbol{t}\in\mathbf{R}^M_+$, and $\epsilon>0$ consider the event,

$$\mathbf{G}(\epsilon\,;\boldsymbol{s}\,,\boldsymbol{t}):=\left\{|\mathfrak{X}(\boldsymbol{s})|\le\frac{\epsilon}{2}\ ,\ |\mathfrak{S}(\boldsymbol{t})-\boldsymbol{s}|\le\frac{\epsilon}{2}\right\}. \tag{4.73}$$

According to Proposition 4.4, for all $\boldsymbol{s}\in[0\,,R]^N$, $\boldsymbol{t}\in\mathbf{R}^M_+$, $\epsilon>0$, and $\mu\in\mathcal{P}(F)$,

$$\begin{aligned}\sum_{\pi\subseteq\{1,\dots,N\}}&\mathrm{E}_{\mathrm{P}\times\mathrm{Q}}\left[J_\epsilon(\mu)\mid\mathscr{F}^\pi(\boldsymbol{s}\otimes\boldsymbol{t})\right]\\&\ge\frac{Ae^{-[\boldsymbol{t}]}}{(2\epsilon)^d}\int\frac{\mathrm{P}\left\{|\mathfrak{X}(\boldsymbol{s}')-\mathfrak{X}(\boldsymbol{s})|\le\epsilon\right\}}{\max\left(|\boldsymbol{s}'-\boldsymbol{s}|^{N-\alpha M}\ ,\ \epsilon^{N-\alpha M}\right)}\,\mu(d\boldsymbol{s}')\cdot\mathbf{1}_{\mathbf{G}(\epsilon\,;\boldsymbol{s}\,,\boldsymbol{t})}\\&=Ae^{-[\boldsymbol{t}]}\int\frac{\Phi_\epsilon(\boldsymbol{s}'-\boldsymbol{s})}{\max\left(|\boldsymbol{s}'-\boldsymbol{s}|^{N-\alpha M}\ ,\ \epsilon^{N-\alpha M}\right)}\,\mu(d\boldsymbol{s}')\cdot\mathbf{1}_{\mathbf{G}(\epsilon\,;\boldsymbol{s}\,,\boldsymbol{t})},\end{aligned} \tag{4.74}$$

$(\mathrm{P}_{\mathcal{L}^d}\times\mathrm{Q}_{\mathcal{L}^N})$-almost everywhere. [This uses only the fact that given $\boldsymbol{s}',\boldsymbol{s}\in\mathbf{R}^N_+$ we can find $\pi\subseteq\{1,\dots,N\}$ such that $\boldsymbol{s}'\succ_\pi\boldsymbol{s}$.]

Fix $\epsilon>0$. It is possible to see that on the same underlying probability space we can find extended random variables $\boldsymbol{\sigma}(\epsilon)\in(\mathbf{Q}^N_+\cap F)\cup\{\infty\}$ and $\boldsymbol{\tau}(\epsilon)\in(\mathbf{Q}^M_+\cap[0\,,R]^M)\cup\{\infty\}$, where $\mathbf{Q}^N_+\cap F$ and $\mathbf{Q}^M_+\cap[0\,,R]^M$ denote respectively dense subsets of F and $[0\,,R]^M$, that have the following properties:

$(\boldsymbol{\Sigma_1})$ $\boldsymbol{\sigma}(\epsilon)=\infty$ if and only if $\boldsymbol{\tau}(\epsilon)=\infty$. These conditions occur, in turn, if and only if

$$\bigcup_{\substack{\boldsymbol{s}\in\mathbf{Q}^N_+\cap F\\ \boldsymbol{t}\in\mathbf{Q}^M_+\cap[0\,,R]^N}}\mathbf{G}(\epsilon\,;\boldsymbol{s}\,,\boldsymbol{t})=\varnothing; \tag{4.75}$$

$(\boldsymbol{\Sigma_2})$ On $\{\boldsymbol{\sigma}(\epsilon)\neq\infty\}$,

$$|\mathfrak{X}(\boldsymbol{\sigma}(\epsilon))|\le\frac{\epsilon}{2}\quad\text{and}\quad|\mathfrak{S}(\boldsymbol{\tau}(\epsilon))-\boldsymbol{\sigma}(\epsilon)|\le\frac{\epsilon}{2}. \tag{4.76}$$

We can collect countably-many $(\mathrm{P}_{\mathcal{L}^d}\times \mathrm{Q}_{\mathcal{L}^N})$-null sets, lump them together, and then apply $(\boldsymbol{\Sigma_1})$ and $(\boldsymbol{\Sigma_2})$ together with (4.74) to find that

$$
\begin{aligned}
&\sum_{\pi\subseteq\{1,\ldots,N\}} \sup_{\substack{\boldsymbol{s}\in\mathbf{Q}_+^N\\ \boldsymbol{t}\in\mathbf{Q}_+^M}} \mathrm{E}_{\mathrm{P}\times\mathrm{Q}}\left[J_\epsilon(\mu) \mid \mathscr{F}^\pi(\boldsymbol{s}\otimes\boldsymbol{t})\right]\\
&\qquad \ge A e^{-[\boldsymbol{\tau}(\boldsymbol{\epsilon})]} \int \frac{\Phi_\epsilon(\boldsymbol{s}'-\boldsymbol{\sigma}(\epsilon))}{\max\left(|\boldsymbol{s}'-\boldsymbol{\sigma}(\epsilon)|^{N-\alpha M}\ ,\ \epsilon^{N-\alpha M}\right)}\,\mu(d\boldsymbol{s}')\cdot \mathbf{1}_{\{\boldsymbol{\sigma}(\epsilon)\neq\infty\}}\\
&\qquad \ge A e^{-MR} \int \frac{\Phi_\epsilon(\boldsymbol{s}'-\boldsymbol{\sigma}(\epsilon))}{\max\left(|\boldsymbol{s}'-\boldsymbol{\sigma}(\epsilon)|^{N-\alpha M}\ ,\ \epsilon^{N-\alpha M}\right)}\,\mu(d\boldsymbol{s}')\cdot \mathbf{1}_{\{\boldsymbol{\sigma}(\epsilon)\neq\infty\}}
\end{aligned}
\tag{4.77}
$$

$(\mathrm{P}_{\mathcal{L}^d}\times \mathrm{Q}_{\mathcal{L}^N})$-almost everywhere. This holds for all $\mu\in\mathcal{P}(F)$. Now we make the special choice of μ, and replace it with $\mu_{\epsilon,k}$, which we define shortly.

First of all, we note that for all $\epsilon>0$ and $k>1$,

$$
0 < \mathrm{P}_{\mathcal{L}^d}\left\{|\mathfrak{X}(\mathbf{0})|\le k\right\} = (2k)^d < \infty. \tag{4.78}
$$

At the same time, thanks to (4.72), there exists $k_0>1$ large enough so that for all $k>k_0$,

$$
\begin{aligned}
&(\mathrm{P}_{\mathcal{L}^d}\times \mathrm{Q}_{\mathcal{L}^N})\left\{\boldsymbol{\sigma}(\epsilon)\neq\infty\ ,\ |\mathfrak{X}(\mathbf{0})|\le k\right\}\\
&\qquad \ge (\mathrm{P}_{\mathcal{L}^d}\times \mathrm{Q}_{\mathcal{L}^N})\left\{\overline{\mathfrak{X}^{-1}(\{0\})}\cap F\cap \mathfrak{S}\left([0\,,R]^M\right)\neq\varnothing\ ,\ |\mathfrak{X}(\mathbf{0})|\le k\right\}\\
&\qquad > 0.
\end{aligned}
\tag{4.79}
$$

The preceding two displays together prove that for all $\epsilon>0$ and $k>k_0$, $\mu_{\epsilon,k}\in\mathcal{P}(F)$, where

$$
\mu_{\epsilon,k}(\Gamma) := \frac{(\mathrm{P}_{\mathcal{L}^d}\times \mathrm{Q}_{\mathcal{L}^N})\left\{\boldsymbol{\sigma}(\epsilon)\in\Gamma\ ,\ \boldsymbol{\sigma}(\epsilon)\neq\infty\ ,\ |\mathfrak{X}(\mathbf{0})|\le k\right\}}{(\mathrm{P}_{\mathcal{L}^d}\times \mathrm{Q}_{\mathcal{L}^N})\left\{\boldsymbol{\sigma}(\epsilon)\neq\infty\ ,\ |\mathfrak{X}(\mathbf{0})|\le k\right\}}, \tag{4.80}
$$

for all Borel sets $\Gamma\subseteq\mathbf{R}_+^N$. Apply (4.77) with μ replaced by $\mu_{\epsilon,k}$, for $k>k_0$ and $\epsilon>0$ fixed, to find that

$$
\Xi^2 \ge A'\left(\int \frac{\Phi_\epsilon(\boldsymbol{s}'-\boldsymbol{\sigma}(\epsilon))}{\max\left(|\boldsymbol{s}'-\boldsymbol{s}|^{N-\alpha M}\ ,\ \epsilon^{N-\alpha M}\right)}\,\mu_{\epsilon,k}(d\boldsymbol{s}')\right)^2 \times \mathbf{1}_{\{\boldsymbol{\sigma}(\epsilon)\neq\infty\ ,\ |\mathfrak{X}(\mathbf{0})|\le k\}} \tag{4.81}
$$

$(\mathrm{P}_{\mathcal{L}^d}\times \mathrm{Q}_{\mathcal{L}^N})$-almost everywhere, where

$$
\Xi := \sum_{\pi\subseteq\{1,\ldots,N\}} \sup_{\substack{\boldsymbol{s}\in\mathbf{Q}_+^N\\ \boldsymbol{t}\in\mathbf{Q}_+^M}} \mathrm{E}_{\mathrm{P}\times\mathrm{Q}}\left[J_\epsilon(\mu_{\epsilon,k}) \mid \mathscr{F}^\pi(\boldsymbol{s}\otimes\boldsymbol{t})\right]. \tag{4.82}
$$

For any sequence $\{a_\pi,\,\pi\subseteq\{1,\ldots,N\}\}$ of real numbers,

$$
\left(\sum_{\pi\subseteq\{1,\ldots,N\}} a_\pi\right)^2 \le 2^N \sum_{\pi\subseteq\{1,\ldots,N\}} a_\pi^2. \tag{4.83}
$$

Therefore,

$$\mathrm{E}_{\mathrm{P}\times\mathrm{Q}}\left[\Xi^2\right] \le 2^N \sum_{\pi\subseteq\{1,\ldots,N\}} \mathrm{E}_{\mathrm{P}\times\mathrm{Q}}\left[\left(\sup_{\substack{\boldsymbol{s}\in\mathbf{Q}_+^N\\ \boldsymbol{t}\in\mathbf{Q}_+^M}} \mathrm{E}_{\mathrm{P}\times\mathrm{Q}}\left[J_\epsilon(\mu_{\epsilon,k}) \mid \mathscr{F}^\pi(\boldsymbol{s}\otimes\boldsymbol{t})\right]\right)^2\right]. \tag{4.84}$$

We first apply the Cauchy–Schwarz inequality to the σ-finite measure $\mathrm{P}_{\mathcal{L}^d}\times\mathrm{Q}_{\mathcal{L}^N}$, and then use (4.24) to obtain the following:

$$\begin{aligned}\mathrm{E}_{\mathrm{P}\times\mathrm{Q}}\left[\Xi^2\right] &\le 2^N \sum_{\pi\subseteq\{1,\ldots,N\}} \mathrm{E}_{\mathrm{P}\times\mathrm{Q}}\left[\sup_{\substack{\boldsymbol{s}\in\mathbf{Q}_+^N\\ \boldsymbol{t}\in\mathbf{Q}_+^M}} \mathrm{E}_{\mathrm{P}\times\mathrm{Q}}\left[\left(J_\epsilon(\mu_{\epsilon,k})\right)^2 \,\middle|\, \mathscr{F}^\pi(\boldsymbol{s}\otimes\boldsymbol{t})\right]\right]\\ &\le 8^{N+M}\mathrm{E}_{\mathrm{P}\times Q}\left[\left(J_\epsilon(\mu_{\epsilon,k})\right)^2\right].\end{aligned} \tag{4.85}$$

Consequently, Proposition 4.2 implies the existence of a constant A—not depending on $(k\,,\epsilon)$ nor on $\mu_{\epsilon,k}$—such that

$$\mathrm{E}_{\mathrm{P}\times\mathrm{Q}}\left[\Xi^2\right] \le A\,W(\epsilon\,,k), \tag{4.86}$$

where

$$W(\epsilon\,,k) := \iint \frac{\Phi_\epsilon(\boldsymbol{s}'-\boldsymbol{s})}{\max\left(|\boldsymbol{s}'-\boldsymbol{s}|^{N-\alpha M}\,,\ \epsilon^{N-\alpha M}\right)}\,\mu_{\epsilon,k}(d\boldsymbol{s}')\,\mu_{\epsilon,k}(d\boldsymbol{s}). \tag{4.87}$$

This estimates the left-hand side of (4.81).

As for the right-hand side, let us write

$$\mathbf{A}_{\epsilon,k} := \left\{\boldsymbol{\sigma}(\epsilon)\neq\infty\ ,\ |\mathfrak{X}(\mathbf{0})|\le k\right\}, \tag{4.88}$$

for the sake of brevity. Then, we have

$$\begin{aligned}&\mathrm{E}_{\mathrm{P}\times\mathrm{Q}}\left[\left(\int \frac{\Phi_\epsilon(\boldsymbol{s}'-\boldsymbol{\sigma}(\epsilon))}{\max\left(|\boldsymbol{s}'-\boldsymbol{s}|^{N-\alpha M}\,,\ \epsilon^{N-\alpha M}\right)}\,\mu_{\epsilon,k}(d\boldsymbol{s}')\right)^2\ ;\ \mathbf{A}_{\epsilon,k}\right]\\ &= \int\left(\int \frac{\Phi_\epsilon(\boldsymbol{s}'-\boldsymbol{s})}{\max\left(|\boldsymbol{s}'-\boldsymbol{s}|^{N-\alpha M}\,,\ \epsilon^{N-\alpha M}\right)}\,\mu_{\epsilon,k}(d\boldsymbol{s}')\right)^2\,\mu_{\epsilon,k}(d\boldsymbol{s})\times\left(\mathrm{P}_{\mathcal{L}^d}\times\mathrm{Q}_{\mathcal{L}^N}\right)(\mathbf{A}_{\epsilon,k})\\ &\ge \left(W(\epsilon\,,k)\right)^2\times\left(\mathrm{P}_{\mathcal{L}^d}\times\mathrm{Q}_{\mathcal{L}^N}\right)(\mathbf{A}_{\epsilon,k}),\end{aligned} \tag{4.89}$$

thanks to the Cauchy–Schwarz inequality. Thus, (4.81), (4.86), and (4.89) together imply that

$$\left(\mathrm{P}_{\mathcal{L}^d}\times\mathrm{Q}_{\mathcal{L}^N}\right)\left\{\boldsymbol{\sigma}(\epsilon)\neq\infty\ ,\ |\mathfrak{X}(\mathbf{0})|\le k\right\} \le \frac{A'}{W(\epsilon\,,k)}, \tag{4.90}$$

where A' does not depend on $(k\,,\epsilon)$, nor on the particular choice of $\mu_{\epsilon,k}$. Now, $\{\mu_{\epsilon,k}\}_{\epsilon>0\,,k>k_0}$ is a collection of probability measures on F. According to Prohorov's theorem we can extract a weakly convergent subsequence and a weak limit $\mu_0 \in \mathcal{P}(F)$, as $k \to \infty$ and $\epsilon \to 0^+$. Without loss of too much generality we denote the implied subsequences by k and ϵ as well. [No great harm will come from this, but it is notationally simpler.] We can combine Fatou's lemma, (2.7), (4.70) and (4.90) in order to deduce that

$$\lim_{\substack{k\to\infty\\ \epsilon\to 0}} (\mathrm{P}_{\mathcal{L}^d} \times \mathrm{Q}_{\mathcal{L}^N}) \left\{ \boldsymbol{\sigma}(\epsilon) \neq \infty \ , \ |\mathfrak{X}(\mathbf{0})| \leq k \right\} = 0. \tag{4.91}$$

Thanks to the monotone convergence theorem [applied to the σ-finite measure $\mathrm{P}_{\mathcal{L}^d} \times \mathrm{Q}_{\mathcal{L}^N}$] the left-hand side is precisely

$$\int_{\mathbf{R}^d} \int_{\mathbf{R}^N_+} (\mathrm{P}_{-x} \times \mathrm{Q}_y) \left\{ \overline{\mathfrak{X}^{-1}(\{0\})} \cap F \cap \overline{\mathfrak{S}\left([0\,,R]^M\right)} \neq \varnothing \right\} \, dx \, dy, \tag{4.92}$$

which, we just proved, is zero. This implies also that

$$\int_{\mathbf{R}^d} \int_{\mathbf{R}^N_+} (\mathrm{P}_{-x} \times \mathrm{Q}_y) \left\{ \overline{\mathfrak{X}^{-1}(\{0\})} \cap F \cap \mathfrak{S}\left([0\,,R]^M\right) \neq \varnothing \right\} \, dx \, dy = 0. \tag{4.93}$$

This contradicts (4.71). That is, we have proved that the condition (4.70) implies that (4.71) fails to hold. It is the case that if (4.71) fails for some $y \in \mathbf{R}^N$ [with x held fixed] then it fails for all $y \in \mathbf{R}^N$ (Khoshnevisan et al., 2003, Proposition 6.2). This yields the following: For all $y \in \mathbf{R}^N$,

$$\int_{\mathbf{R}^d} (\mathrm{P}_{-x} \times \mathrm{Q}_y) \left\{ \overline{\mathfrak{X}^{-1}(\{0\})} \cap F \cap \mathfrak{S}\left([0\,,R]^M\right) \neq \varnothing \right\} \, dx = 0. \tag{4.94}$$

Let $R \uparrow \infty$ to find, via the monotone convergence theorem, that for all $y \in \mathbf{R}^N$,

$$\int_{\mathbf{R}^d} (\mathrm{P}_{-x} \times \mathrm{Q}_y) \left\{ \overline{\mathfrak{X}^{-1}(\{0\})} \cap F \cap \mathfrak{S}\left(\mathbf{R}^M_+\right) \neq \varnothing \right\} \, dx = 0. \tag{4.95}$$

Recall that F is a compact subset of $[1/R\,,R]^N$. Fix and choose an arbitrary $\boldsymbol{y} \in (0\,,1/R)^N$, and note that

$$\begin{aligned}(\mathrm{P}_0 \times \mathrm{Q}_{\boldsymbol{y}}) &\left\{ \overline{\mathfrak{X}^{-1}(\{0\})} \cap \mathfrak{S}(\mathbf{R}^M_+) \cap (F \ominus \boldsymbol{y}) \neq \varnothing \right\} \\ &= \mathrm{P}\left\{ ^{\exists}\boldsymbol{r} \in F \ominus \boldsymbol{y} : \ \boldsymbol{r} \in \mathfrak{S}(\mathbf{R}^M_+) \ominus \boldsymbol{y} \ , \ 0 \in \langle \mathfrak{X}(\boldsymbol{r}) \rangle \right\},\end{aligned} \tag{4.96}$$

where $A - y := \{a - y : \ a \in A\}$ for all sets A and points y, and

$$\langle \mathfrak{X}(\boldsymbol{r}) \rangle = \left\{ \sum_{1 \leq j \leq N} X_j(r_j\theta) : \ \theta \in \{+\,,-\} \right\}. \tag{4.97}$$

For example, when $N = 1$, $\mathfrak{X}$ is an ordinary Lévy process, and $\langle \mathfrak{X}(r) \rangle$ has at most two elements: $\mathfrak{X}(r)$ and $\mathfrak{X}(r-)$ [they could be equal]. Or, when $N = 2$, then the set $\langle \mathfrak{X}(\boldsymbol{r}) \rangle$ contains up to four elements: $X_1(r_1) + X_2(r_2)$, $X_1(r_1-) + X_2(r_2)$, $X_1(r_1) + X_2(r_2-)$, and

$X_1(r_1-) + X_2(r_2-)$. [Some of them are equal a.s.] In general, $\langle \mathfrak{X}(\boldsymbol{r}) \rangle$ contains up to 2^N elements.

Note that $\boldsymbol{s} \succ \boldsymbol{y}$ for all $\boldsymbol{s} \in F$. This is so only because $F \subseteq [1/R\,, R]^N$ and $\boldsymbol{y} \in (0\,, 1/R)^N$. Therefore, we can apply the Markov property of X_j at y_j to find that for all $x \in \mathbf{R}^d$,

$$\begin{aligned}(\mathrm{P}_0 \times \mathrm{Q}_{\boldsymbol{y}}) &\left\{ \overline{\mathfrak{X}^{-1}(\{x\})} \cap \mathfrak{S}(\mathbf{R}^M_+) \cap (F \ominus \boldsymbol{y}) \neq \varnothing \right\} \\ &= \mathrm{P}\left\{ ^\exists \boldsymbol{s} \in F : \ \boldsymbol{s} \in \mathfrak{S}(\mathbf{R}^M_+)\ ,\ x \in \langle \mathfrak{X}(\boldsymbol{s} - \boldsymbol{y}) \rangle \right\} \\ &= \int_{\mathbf{R}^d} \mathrm{P}\left\{ ^\exists \boldsymbol{s} \in F : \ \boldsymbol{s} \in \mathfrak{S}(\mathbf{R}^M_+)\ ,\ x + z \in \langle \mathfrak{X}(\boldsymbol{s}) \rangle \right\} p_{\boldsymbol{y}}(z)\, dz \\ &= \int_{\mathbf{R}^d} \mathrm{P}\left\{ \overline{\mathfrak{X}^{-1}(\{x+z\})} \cap \mathfrak{S}(\mathbf{R}^M_+) \cap F \neq \varnothing \right\} p_{\boldsymbol{y}}(z)\, dz \\ &= \int_{\mathbf{R}^d} (\mathrm{P}_{-w} \times \mathrm{Q}_0) \left\{ \overline{\mathfrak{X}^{-1}(\{0\})} \cap \mathfrak{S}(\mathbf{R}^M_+) \cap F \neq \varnothing \right\} p_{\boldsymbol{y}}(w - x)\, dw.\end{aligned} \tag{4.98}$$

[It might help to recall that $p_{\boldsymbol{y}}$ is the density function of $\mathfrak{X}(\boldsymbol{y})$.] We have used the fact that with probability one, $X_j(y_j) = X_j(y_j-)$ for all $1 \le j \le N$, for any fixed $\boldsymbol{y} \in (0\,, 1/R)^N$. The preceding, together with (4.95), proves the following: (4.70) implies that for all $\boldsymbol{y} \in (0\,, 1/R)^N$ and $x \in \mathbf{R}^d$,

$$(\mathrm{P}_0 \times \mathrm{Q}_{\boldsymbol{y}}) \left\{ \overline{\mathfrak{X}^{-1}(\{x\})} \cap \mathfrak{S}(\mathbf{R}^M_+) \cap (F \ominus \boldsymbol{y}) \neq \varnothing \right\} = 0. \tag{4.99}$$

Note that the "energy form" $\mu \mapsto I_\Phi^{(q)}(\mu)$ is translation invariant. That is, $I_\Phi^{(q)}(\mu) = I_\Phi^{(q)}(\mu \circ \tau_a)$ for all $a \in \mathbf{R}^N$, where $(\mu \circ \tau_a)(A) := \mu(A \ominus a)$. Therefore, for all fixed $\boldsymbol{y} \in (0\,, 1/R)^N$, (4.70) is equivalent to the following:

$$I_\Phi^{(N-\alpha M)}(\mu) = \infty \quad \text{for all } \mu \in \mathcal{P}(F \oplus \boldsymbol{y}), \tag{4.100}$$

where $A \oplus y := \{a + y : \ a \in A\}$ for all sets A and points y. Equation (4.99) is therefore implying that for all $\boldsymbol{y} \in (0\,, 1/R)^N$ and $x \in \mathbf{R}^d$,

$$(\mathrm{P}_0 \times \mathrm{Q}_{\boldsymbol{y}}) \left\{ \overline{\mathfrak{X}^{-1}(\{x\})} \cap \mathfrak{S}(\mathbf{R}^M_+) \cap F \neq \varnothing \right\} = 0. \tag{4.101}$$

Khoshnevisan et al. (2003, Proposition 6.2) implies then that the preceding holds for all $\boldsymbol{y} \in \mathbf{R}^M_+$. Apply this with $\boldsymbol{y} \equiv \mathbf{0}$ to finish. □

We are ready to prove Theorem 3.2.

Proof of Theorem 3.2. We can assume without loss in generality that

$$\mathrm{P}\left\{ \mathfrak{X}^{-1}(\{0\}) \cap F \neq \varnothing \right\} > 0. \tag{4.102}$$

For there is nothing to prove otherwise. We recall that (4.102) is equivalent to the analytic condition that there exists $\mu \in \mathcal{P}(F)$ such that the "energy integral" $\iint \Phi(\boldsymbol{s}'-\boldsymbol{s})\,\mu(d\boldsymbol{s})\,\mu(d\boldsymbol{s}')$ is finite (Khoshnevisan and Xiao, 2002).

Let $\mathfrak{S}^1, \mathfrak{S}^2, \ldots$ be i.i.d. copies of $\mathfrak{S}$, and define

$$\mathbf{K} := \bigcup_{1\le j<\infty} \mathfrak{S}^j(\mathbf{R}_+^M). \tag{4.103}$$

On one hand, according to the Borel–Cantelli lemma, the following is valid for every nonrandom Borel set $G \subset \mathbf{R}^N$:

$$\mathrm{P}\{\mathbf{K}\cap G \neq \varnothing\} = \begin{cases} 1 & \text{if } \mathrm{P}\{\mathfrak{S}(\mathbf{R}_+^M)\cap G \neq \varnothing\} > 0, \\ 0 & \text{if } \mathrm{P}\{\mathfrak{S}(\mathbf{R}_+^M)\cap G \neq \varnothing\} = 0. \end{cases} \tag{4.104}$$

On the other hand, whenever $G \neq \varnothing$,

$$\mathrm{P}\left\{\mathfrak{S}(\mathbf{R}_+^M)\cap G \neq \varnothing\right\} > 0 \qquad \text{iff} \qquad \mathscr{C}_{N-\alpha M}(G) < \infty, \tag{4.105}$$

where $\mathscr{C}_q$ denotes the q-dimensional Bessel–Riesz capacity of (1.15) (Khoshnevisan, 2002, Theorem 4.1.1, p. 423). Therefore,

$$\mathrm{P}\left(\mathfrak{X}^{-1}(\{0\})\cap F\cap\mathbf{K} \neq \varnothing \;\middle|\; \mathfrak{X}^{-1}(\{0\})\cap F \neq \varnothing\right) = \mathrm{P}\left(\boldsymbol{\Lambda} \;\middle|\; \mathfrak{X}^{-1}(\{0\})\cap F \neq \varnothing\right), \tag{4.106}$$

where $\boldsymbol{\Lambda}$ denotes the event that there exists some $\sigma \in \mathcal{P}(\mathfrak{X}^{-1}(\{0\})\cap F)$ such that

$$\iint \frac{\sigma(dx)\,\sigma(dy)}{\|x-y\|^{N-\alpha M}} < \infty. \tag{4.107}$$

It follows from Propositions 4.7 and 4.8 that a.s. on the event $\{\mathfrak{X}^{-1}(\{0\})\cap F \neq \varnothing\}$,

$$\mathscr{C}_{N-\alpha M}\left(\mathfrak{X}^{-1}(\{0\})\cap F\right) > 0 \quad \text{iff} \quad \inf_{\mu\in\mathcal{P}(F)} I_\Phi^{(N-\alpha M)}(\mu) < \infty. \tag{4.108}$$

This is a statement only about the random field $\mathfrak{X}$, and does not concern $\mathfrak{S}$. Therefore, the preceding holds for all integers $M \ge 1$, and all reals $0<\alpha<2$. By adjusting the parameters α and M, we can ensure that $q := N-\alpha M$ is any pre-described number in $(0\,, N)$. Therefore, outside a single null set

$$\mathscr{C}_q\left(\mathfrak{X}^{-1}(\{0\})\cap F\right) > 0 \quad \text{iff} \quad \inf_{\mu\in\mathcal{P}(F)} I_\Phi^{(q)}(\mu) < \infty, \tag{4.109}$$

for all rational numbers $q \in (0\,, N)$, a.s. on $\{\mathfrak{X}^{-1}(\{0\})\cap F \neq \varnothing\}$. By monotonicity, the preceding holds for *all* $q \in (0\,, N)$, off a single null set. Frostman's theorem (Khoshnevisan, 2002, Theorem 2.2.1, p. 521) then completes our proof. □

5. **Proof of Theorem 1.1**

Before commencing with our proof, we first develop a real-variable, technical lemma. We will say that $\Gamma \subset \mathbf{R}^n$ is a *cube* if and only if there exist $a \prec b$, both in $\mathbf{R}^n$, such that

$$\Gamma := [a_1\,, b_1] \times \cdots \times [a_n\,, b_n]. \tag{5.1}$$

Lemma 5.1. *Let $f : \mathbf{R}^n \mapsto [0\,,\infty]$ be continuous and finite on $\mathbf{R}^n \setminus \{0\}$, and assume that f is "quasi-monotone" in the following sense: There exists $0 < \theta \le 1$ such that $f(x) \ge \theta f(y)$ whenever $0 \prec x \prec y$. Suppose, in addition, that $f(x)$ depends on $x = (x_1, \ldots, x_n)$ only through $|x_1|\,, \ldots, |x_n|$. Then, for all cubes $\Gamma \subset (0\,,\infty)^n$,*

$$\mathcal{L}^n(\Gamma) \inf_{y\in\Gamma} \int_\Gamma f(x-y)\,dx \ge \left(\frac{\theta}{2}\right)^n \int_\Gamma \int_\Gamma f(x-z)\,dx\,dz. \tag{5.2}$$

Remark 5.2. Lemma 5.1 is a result about *symmetrization* because it is equivalent to the assertion that if U and V are i.i.d., both distributed uniformly on Γ, then

$$\inf_{y\in\Gamma} \mathrm{E}\left[f(U-y)\right] \ge \left(\frac{\theta}{2}\right)^n \mathrm{E}\left[f(U-V)\right]. \tag{5.3}$$

Our proof will make it plain that the inequality is sharp in the sense that

$$\sup_{y\in\Gamma} \mathrm{E}\left[f(U-y)\right] \le 2^n \mathrm{E}\left[f(U-V)\right]. \tag{5.4}$$

This portion does not require f to be quasi-monotone.

Proof. First we suppose that $n = 1$, and $\Gamma = [a\,,b]$, where $0 < a < b$. For all $a \le y \le b$,

$$\int_a^b f(x-y)\,dx = \int_0^{y-a} f(z)\,dz + \int_0^{b-y} f(z)\,dz. \tag{5.5}$$

[This is so because $f(x-y) = f(|x-y|)$.] Now we use the quasi-monotonicity of f to find that

$$\int_0^{b-y} f(z)\,dz \ge \theta \int_0^{b-y} f(z+y-a)\,dz = \theta \int_{y-a}^{b-a} f(z)\,dz. \tag{5.6}$$

According to (5.5) then, for all $a \le y \le b$,

$$\int_a^b f(x-y)\,dx \ge \theta \int_0^{b-a} f(z)\,dz. \tag{5.7}$$

But an argument based on symmetry shows readily that

$$\begin{aligned}\int_a^b\int_a^b f(x-z)\,dx\,dz &\le 2(b-a)\int_0^{b-a} f(z)\,dz\\ &\le \frac{2(b-a)}{\theta}\int_a^b f(x-y)\,dx,\end{aligned}\tag{5.8}$$

for all $a \le y \le b$. Thus, the lemma follows in the case that $n = 1$. The remainder follows by induction on n, using the self-evident fact that a cube in $\mathbf{R}^n$ has the form $\Gamma \times [a\,,b]$ where Γ is a cube in $\mathbf{R}^{n-1}$. □

Proof of Theorem 1.1. We only need to prove (1.9), since (1.11) follows from it and Lemma 3.1.

As before, we introduce $\mathfrak{S}$ to be an M-parameter additive stable process in $\mathbf{R}^N$, where $N > \alpha M$. Later, we will choose α and M such that $N - \alpha M \searrow \gamma$.

Choose and fix $R > 1$. According to Proposition 4.4, there exists a finite constant $A > 0$ such that for all $\boldsymbol{s} \in [0\,,R]^N$, $\boldsymbol{t} \in [0\,,R]^M$, $\epsilon > 0$, and every cube $\Gamma \subset [0\,,R]^N$,

$$\begin{aligned}\sum_{\pi\subseteq\{1,\ldots,N\}} &\mathrm{E}_{\mathrm{P}\times\mathrm{Q}}\left[J_\epsilon(\mu_{\Gamma})\mid \mathscr{F}^\pi(\boldsymbol{s}\otimes\boldsymbol{t})\right]\\ &\ge A\int\frac{\Phi_\epsilon(\boldsymbol{s}'-\boldsymbol{s})}{\max\left(|\boldsymbol{s}'-\boldsymbol{s}|^{N-\alpha M}\ ,\ \epsilon^{N-\alpha M}\right)}\,\mu_{\Gamma}(d\boldsymbol{s}')\cdot\mathbf{1}_{\mathbf{G}(\epsilon\,;\boldsymbol{s}\,,\boldsymbol{t})},\end{aligned}\tag{5.9}$$

$(\mathrm{P}_{\mathcal{L}^d}\times\mathrm{Q}_{\mathcal{L}^N})$-almost everywhere. [Recall that $\mathbf{G}(\epsilon\,;\boldsymbol{s}\,,\boldsymbol{t})$ is defined in (4.73).] Here, μ_{Γ} denotes the restriction of the Lebesgue measure $\mathcal{L}^N$ to Γ, normalized to have mass one. See also (4.74).

Define for all $x \in \mathbf{R}^N$,

$$f(x) := \frac{\Phi_\epsilon(x)}{\max\left(|x|^{N-\alpha M}\ ,\ \epsilon^{N-\alpha M}\right)}.\tag{5.10}$$

Evidently, $f(x)$ depends on $x \in \mathbf{R}^N$ only through $|x_1|,\ldots,|x_N|$. Because $N > \alpha M$, (2.13) implies that f is quasi-monotone with $\theta = 16^{-d}$. Thus, Lemma 5.1 can be used to deduce that there exists A' such that

$$\sum_{\pi\subseteq\{1,\ldots,N\}} \mathrm{E}_{\mathrm{P}\times\mathrm{Q}}\left[J_\epsilon(\mu_{\Gamma})\mid \mathscr{F}^\pi(\boldsymbol{s}\otimes\boldsymbol{t})\right] \ge A'\mathcal{I}_{\Gamma}(\epsilon)\cdot\mathbf{1}_{\mathbf{G}(\epsilon\,;\boldsymbol{s}\,,\boldsymbol{t})},\tag{5.11}$$

$(\mathrm{P}_{\mathcal{L}^d}\times\mathrm{Q}_{\mathcal{L}^N})$-almost everywhere, where

$$\mathcal{I}_{\Gamma}(\epsilon) := \iint\frac{\Phi_\epsilon(\boldsymbol{s}'-\boldsymbol{s}'')}{\max\left(|\boldsymbol{s}'-\boldsymbol{s}''|^{N-\alpha M}\ ,\ \epsilon^{N-\alpha M}\right)}\,\mu_{\Gamma}(d\boldsymbol{s}')\,\mu_{\Gamma}(d\boldsymbol{s}'').\tag{5.12}$$

We emphasize that A' does not depend on $\epsilon > 0$, $\boldsymbol{s} \in [0\,,R]^N$, or $\boldsymbol{t} \in [0\,,R]^M$. The regularity of the paths of $\mathfrak{X}$ and $\mathfrak{S}$ implies that

$$\sup_{\substack{\boldsymbol{s}\in[0,R]^N\cap\mathbf{Q}^N\\ \boldsymbol{t}\in[0,R]^M\cap\mathbf{Q}^M}} \mathbf{1}_{\mathbf{G}(\epsilon\,;\boldsymbol{s},\boldsymbol{t})} \geq \mathbf{1}_{\left\{\overline{\mathfrak{X}^{-1}(\{0\})}\cap\Gamma\cap\overline{\mathfrak{S}([0\,,R]^M)}\neq\varnothing\right\}}. \tag{5.13}$$

Therefore,

$$\begin{aligned}\sum_{\pi\subseteq\{1,\dots,N\}} \sup_{\substack{\boldsymbol{s}\in[0,R]^N\cap\mathbf{Q}^N\\ \boldsymbol{t}\in[0,R]^M\cap\mathbf{Q}^M}} \mathrm{E}_{\mathrm{P}\times\mathrm{Q}}\left[J_\epsilon(\mu_{\Gamma})\mid \mathscr{F}^\pi(\boldsymbol{s}\otimes\boldsymbol{t})\right]& \\ \geq A'\mathcal{I}_{\Gamma}(\epsilon)\cdot \mathbf{1}_{\left\{\overline{\mathfrak{X}^{-1}(\{0\})}\cap\Gamma\cap\overline{\mathfrak{S}([0\,,R]^M)}\neq\varnothing\right\}}&,\end{aligned} \tag{5.14}$$

$(\mathrm{P}_{\mathcal{L}^d}\times\mathrm{Q}_{\mathcal{L}^N})$-almost everywhere.

We square both sides of (5.14), and then integrate $[d\mathrm{P}_{\mathcal{L}^d}\times d\mathrm{Q}_{\mathcal{L}^N}]$. By way of (4.83), we arrive at the following:

$$\begin{aligned}\sum_{\pi\subseteq\{1,\dots,N\}} \mathrm{E}_{\mathrm{P}\times\mathrm{Q}}\left[\sup_{\substack{\boldsymbol{s}\in[0,R]^N\cap\mathbf{Q}^N\\ \boldsymbol{t}\in[0,R]^M\cap\mathbf{Q}^M}} \left|\mathrm{E}_{\mathrm{P}\times\mathrm{Q}}\left[J_\epsilon(\mu_{\Gamma})\mid \mathscr{F}^\pi(\boldsymbol{s}\otimes\boldsymbol{t})\right]\right|^2\right]& \\ \geq A''\left[\mathcal{I}_{\Gamma}(\epsilon)\right]^2\cdot(\mathrm{P}_{\mathcal{L}^d}\times\mathrm{Q}_{\mathcal{L}^N})\left\{\overline{\mathfrak{X}^{-1}(\{0\})}\cap\Gamma\cap\overline{\mathfrak{S}([0\,,R]^M)}\neq\varnothing\right\}&,\end{aligned} \tag{5.15}$$

where A'' does not depend on $\epsilon > 0$. Thanks to (4.39) and Proposition 4.2, the left-hand side is at most

$$4^{N+M}\sum_{\pi\subseteq\{1,\dots,N\}} \mathrm{E}_{\mathrm{P}\times\mathrm{Q}}\left[\left|J_\epsilon(\mu_{\Gamma})\right|^2\right] \leq A'''\mathcal{I}_{\Gamma}(\epsilon), \tag{5.16}$$

where A''' does not depend on $\epsilon > 0$. This proves then that

$$(\mathrm{P}_{\mathcal{L}^d}\times\mathrm{Q}_{\mathcal{L}^N})\left\{\overline{\mathfrak{X}^{-1}(\{0\})}\cap\Gamma\cap\overline{\mathfrak{S}([0\,,R]^M)}\neq\varnothing\right\} \leq \frac{A_*}{\mathcal{I}_{\Gamma}(\epsilon)}, \tag{5.17}$$

where A_* does not depend on $\epsilon > 0$. According to Fatou's lemma,

$$\liminf_{\epsilon\to 0^+}\mathcal{I}_{\Gamma}(\epsilon) \geq \frac{1}{\left(\mathcal{L}^N(\Gamma)\right)^2} I_{\Phi}^{(N-\alpha M)}(\mu_{\Gamma}), \tag{5.18}$$

which is manifestly infinite if $N-\alpha M > \gamma$; see (3.2). Thus, we have proved that if $N-\alpha M > \gamma$, then

$$\int_{\mathbf{R}^d}\int_{\mathbf{R}^N_+} (\mathrm{P}_{-x}\times\mathrm{Q}_y)\left\{\overline{\mathfrak{X}^{-1}(\{0\})}\cap\Gamma\cap\overline{\mathfrak{S}([0\,,R]^M)}\neq\varnothing\right\}\,dx\,dy \tag{5.19}$$

is zero. Now we argue precisely as we did in the proof of Theorem 3.2, and find that if Γ is a cube in $[1/R\,,R]^N$, then for all $y \in (0\,,1/R)^N$,

$$(\mathrm{P}_0 \times \mathrm{Q}_{\boldsymbol{y}})\left\{\overline{\mathfrak{X}^{-1}(\{0\})} \cap \Gamma \cap \mathfrak{S}(\mathbf{R}^M_+) \neq \varnothing\right\} = 0, \tag{5.20}$$

as long as $N - \alpha M > \gamma$. See the derivation of (4.101) from (4.92). Hence we have $\mathscr{C}_{N-\alpha M}\big(\overline{\mathfrak{X}^{-1}(\{0\})} \cap \Gamma\big) = 0$ almost surely [P]. Since $N - \alpha M$ can be arbitrary close to γ, this proves that a.s. [P],

$$\dim_{_{\rm H}} \left(\overline{\mathfrak{X}^{-1}(\{0\})} \cap \Gamma\right) \leq \gamma. \tag{5.21}$$

Because the preceding is valid a.s. for all $R > 1$ and all cubes $\Gamma \subseteq [1/R\,,R]^N$, we find that

$$\dim_{_{\rm H}} \overline{\mathfrak{X}^{-1}(\{0\})} \leq \gamma \qquad \text{a.s.} \tag{5.22}$$

On the other hand, according to Theorem 3.2, if $R > 1$ and Γ is any cube in $[1/R\,,R]^N$, then a.s. on $\{\mathfrak{X}^{-1}(\{0\}) \cap \Gamma \neq \varnothing\}$,

$$\dim_{_{\rm H}} \left(\mathfrak{X}^{-1}(\{0\}) \cap \Gamma\right) \geq \sup\left\{0 < q < N :\ I_\Phi^{(q)}(\mu_{_\Gamma}) < \infty\right\}, \tag{5.23}$$

and we have seen already that the right-hand side coincides with γ. Let Γ increase and exhaust $\mathbf{R}^N_+$ to complete the proof. ☐

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