

Packing-Dimension Profiles and Fractional Brownian Motion †

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(Received 13 November 2006; revised 23 October 2007)

Abstract

In order to compute the packing dimension of orthogonal projections Falconer and Howroyd (1997) have introduced a family of packing dimension profiles Dim_s that are parametrized by real numbers $s > 0$. Subsequently, Howroyd (2001) introduced alternate s -dimensional packing dimension profiles P-dim_s by using Caratheodory-type packing measures, and proved, among many other things, that $\text{P-dim}_s E = \text{Dim}_s E$ for all integers $s > 0$ and all analytic sets $E \subseteq \mathbf{R}^N$.

The goal of this article is to prove that $\text{P-dim}_s E = \text{Dim}_s E$ for all real numbers $s > 0$ and analytic sets $E \subseteq \mathbf{R}^N$. This answers a question of Howroyd (2001, p. 159). Our proof hinges on a establishing new property of fractional Brownian motion.

1. Introduction

Packing dimension and packing measure were introduced in the early 1980s by Tricot (1982) and Taylor and Tricot (1985) as dual concepts to Hausdorff dimension and Hausdorff measure. Falconer (1990) and Mattila (1995) contain systematic accounts.

It has been known for some time now that some Hausdorff dimension formulas — such as those for orthogonal projections and those for image sets of fractional Brownian motion — do not have packing dimension analogues; see Järvenpää (1994) and Talagrand and Xiao (1996) for precise statements. The problem of computing the packing dimension of orthogonal projections was further investigated by Falconer and Mattila (1996), and Falconer and Howroyd (1996). Their combined results suggested that information given by packing and Hausdorff dimensions is not enough and a new concept of dimension is needed to compute the packing dimension of orthogonal projections or some random sets.

† Research partially supported by NSF grant DMS-0404729.

Falconer and Howroyd (1997) solved the problem of computing the packing dimension of orthogonal projections by introducing a family of *packing dimension profiles* $\{\text{Dim}_s\}_{s>0}$ that we recall in Section 2 below. Falconer and Howroyd (1997) proved that for every analytic set $E \subseteq \mathbf{R}^N$ and every integer $1 \leq m \leq N$,

$$\dim_p(P_V E) = \text{Dim}_m E \quad \text{for } \gamma_{n,m}\text{-almost all } V \in \mathcal{G}_{n,m}, \quad (1.1)$$

where $\gamma_{n,m}$ is the natural orthogonally-invariant measure on the Grassman manifold $\mathcal{G}_{n,m}$ of all m -dimensional subspaces of \mathbf{R}^N , and $P_V E$ denotes the projection of E onto V .

Subsequently, Howroyd (2001) introduced a family $\{\text{B-dim}_s\}_{s>0}$ of box-dimension profiles, together with their regularizations $\{\text{P-dim}_s\}_{s>0}$. The latter are also called packing dimension profiles; see Section 2. Howroyd (2001) showed that $\{\text{P-dim}_s\}_{s>0}$ can be characterized as critical values of a Caratheodory-type packing measure and then used these dimension profiles to compute the [traditional] box and packing dimensions of orthogonal projections. In addition, Howroyd (2001, Corollary 32) proved that for all analytic sets $E \subseteq \mathbf{R}^N$: (i) $\text{P-dim}_s E \geq \text{Dim}_s E$ if $s > 0$; and (ii) if $s \in (0, N)$ is an *integer* then

$$\text{P-dim}_s E = \text{Dim}_s E. \quad (1.2)$$

Finally, $\text{P-dim}_s E$ and $\text{Dim}_s E$ agree for arbitrary $s \geq N$, and their common value is the packing dimension $\dim_p E$.

The principle aim of this note is to prove that (1.2) holds for all real numbers $s \in (0, N)$. Equivalently, we offer the following.

THEOREM 1.1. *Equation (1.2) is valid for all $s > 0$.*

This solves a question of Howroyd (2001, p. 159).

Our derivation is probabilistic, and relies on properties of fractional Brownian motion (*fBM*). In order to explain the connection to *fBM* let $X := \{X(t)\}_{t \in \mathbf{R}^N}$ be a d -dimensional *fBM* with Hurst parameter $H \in (0, 1)$. That is, $X(t) = (X_1(t), \dots, X_d(t))$ for all $t \in \mathbf{R}^N$, where X_1, \dots, X_d are independent copies of a real-valued *fBM* with common Hurst parameter H (Kahane, 1985, Chapter 18). Xiao (1997) proved that for every analytic set $E \subseteq \mathbf{R}^N$,

$$\dim_p X(E) = \frac{1}{H} \text{Dim}_{Hd} E \quad \text{a.s.} \quad (1.3)$$

Here we will derive an alternative expression.

THEOREM 1.2. *For all analytic sets $E \subseteq \mathbf{R}^N$,*

$$\dim_p X(E) = \frac{1}{H} \text{P-dim}_{Hd} E \quad \text{a.s.} \quad (1.4)$$

Thanks to (1.3) and Theorem 1.2, $\text{Dim}_{Hd} E = \text{P-dim}_{Hd} E$ for all integers $d \geq 1$ and all $H \in (0, 1)$. Whence follows Theorem 1.1.

We establish Theorem 1.2 in Section 3, following the introductory Section 2 wherein we introduce some of stated notions of fractal geometry in greater detail. Also we add a Section 4 where we derive yet another equivalent formulation for the s -dimensional packing dimension profile $\text{Dim}_s E$ of an analytic set $E \subseteq \mathbf{R}^N$. We hope to use this formulation of $\text{Dim}_s E$ elsewhere in order to compute the packing dimension of many interesting random sets.

Throughout we will use the letter K to denote an unspecified positive and finite constant whose value may differ from line to line and sometimes even within the same line.

2. Dimension Profiles

In this section we recall briefly aspects of the theories of dimension profiles of Falconer and Howroyd (1997) and Howroyd (2001).

2.1. Packing Dimension via Entropy Numbers

For all $r > 0$ and all bounded sets $E \subseteq \mathbf{R}^N$ let $N_r(E)$ denote the maximum number of disjoint closed balls of radius r whose respective centers are all in E . The [upper] *box dimension* of E is defined as

$$\text{B-}\overline{\dim} E = \limsup_{r \downarrow 0} \frac{\log N_r(E)}{\log(1/r)}. \quad (2.1)$$

We follow Tricot (1982) and define the *packing dimension* of E as the “regularization” of $\text{B-}\overline{\dim} E$. That is,

$$\dim_{\text{p}} E = \inf \left\{ \sup_{k \geq 1} \text{B-}\overline{\dim} F_k : E \subseteq \bigcup_{k=1}^{\infty} F_k \right\}. \quad (2.2)$$

There is also a corresponding notion of the packing dimension of a Borel measure. Indeed, the [lower] *packing dimension* of a Borel measure μ on \mathbf{R}^N is

$$\dim_{\text{p}} \mu = \inf \left\{ \dim_{\text{p}} E : \mu(E) > 0 \text{ and } E \subseteq \mathbf{R}^N \text{ is a Borel set} \right\}. \quad (2.3)$$

One can compute $\dim_{\text{p}} E$ from $\dim_{\text{p}} \mu$ as well: Given an analytic set $E \subseteq \mathbf{R}^N$ let $\mathcal{M}_c^+(E)$ denote the collection of all finite compactly-supported Borel measures on E . Then, according to Hu and Taylor (1994),

$$\dim_{\text{p}} E = \sup \left\{ \dim_{\text{p}} \mu : \mu \in \mathcal{M}_c^+(E) \right\}. \quad (2.4)$$

2.2. The Packing Dimension Profiles of Falconer and Howroyd

Given a finite Borel measure μ on \mathbf{R}^N and an $s \in (0, \infty]$ define

$$F_s^\mu(x, r) := \int_{\mathbf{R}^N} \psi_s \left(\frac{x-y}{r} \right) \mu(dy), \quad (2.5)$$

where for finite $s \in (0, \infty)$,

$$\psi_s(x) := \min(1, |x|^{-s}) \quad \forall x \in \mathbf{R}^N, \quad (2.6)$$

and

$$\psi_\infty(x) := \mathbf{1}_{\{y \in \mathbf{R}^N : |y| \leq 1\}}(x) \quad \forall x \in \mathbf{R}^N. \quad (2.7)$$

The *s-dimensional packing dimension profile* of μ is defined as

$$\text{Dim}_s \mu = \sup \left\{ t \geq 0 : \liminf_{r \downarrow 0} \frac{F_s^\mu(x, r)}{r^t} = 0 \text{ for } \mu\text{-a.a. } x \in \mathbf{R}^N \right\}. \quad (2.8)$$

Packing dimension profiles generalize the packing dimension because $\dim_{\text{p}} \mu = \text{Dim}_s \mu$ for all finite Borel measures μ on \mathbf{R}^N and for all $s \geq N$. See Falconer and Howroyd (1997, p. 272) for a proof.

Falconer and Howroyd (1997) also defined the *s-dimensional packing dimension profile* of a Borel set $E \subseteq \mathbf{R}^N$ by

$$\text{Dim}_s E = \sup \left\{ \text{Dim}_s \mu : \mu \in \mathcal{M}_c^+(E) \right\}. \quad (2.9)$$

2.3. The Packing Dimension Profiles of Howroyd

If $E \subseteq \mathbf{R}^N$ and $s > 0$, then a sequence of triples $(w_i, x_i, r_i)_{i=1}^\infty$ is called a (ψ_s, δ) -packing of E whenever $w_i \geq 0$, $x_i \in E$, $0 < r_i \leq \delta$, and

$$\sup_{i \geq 1} \sum_{j=1}^{\infty} w_j \psi_s \left(\frac{x_i - x_j}{r_j} \right) \leq 1. \quad (2.10)$$

For all $E \subseteq \mathbf{R}^N$, define

$$\mathcal{P}_0^{\alpha, s}(E) := \lim_{\delta \downarrow 0} \left(\sup \sum_{i=1}^{\infty} w_i (2r_i)^\alpha \right), \quad (2.11)$$

where the supremum is taken over all (ψ_s, δ) -packings $(w_i, x_i, r_i)_{i=1}^\infty$ of E . Then the α -dimensional ψ_s -packing measure $\mathcal{P}^{\alpha, s}(E)$ is defined as

$$\mathcal{P}^{\alpha, s}(E) = \inf \left\{ \sum_{k=1}^{\infty} \mathcal{P}_0^{\alpha, s}(E_k) : E \subseteq \bigcup_{k=1}^{\infty} E_k \right\}. \quad (2.12)$$

The s -dimensional packing dimension profile of E can then be defined as

$$\text{P-dim}_s E := \inf \{ \alpha > 0 : \mathcal{P}^{\alpha, s}(E) = 0 \}. \quad (2.13)$$

We will make use of the following two lemmas. They are ready consequences of Lemma 20 and Theorem 22 of Howroyd (2001), respectively.

LEMMA 2.1. *If $E \subseteq \mathbf{R}^N$ and $\mathcal{P}^{\gamma, s}(E) > 0$, then E has non-sigma-finite $\mathcal{P}^{\alpha, s}$ -measure for every $\alpha \in (0, \gamma)$.*

LEMMA 2.2. *Let $A \subseteq \mathbf{R}^N$ be an analytic set of non-sigma-finite $\mathcal{P}^{\alpha, s}$ -measure. Then there exists a compact set $K \subseteq A$ such that $\mathcal{P}_0^{\alpha, s}(K \cap G) = \infty$ for all open sets $G \subseteq \mathbf{R}^N$ with $K \cap G \neq \emptyset$. Moreover, K is also of non-sigma-finite $\mathcal{P}^{\alpha, s}$ -measure.*

2.4. Upper Box Dimension Profiles

Given $r > 0$ and $E \subseteq \mathbf{R}^N$, a sequence of pairs $(w_i, x_i)_{i=1}^k$ is a *size- r weighted ψ_s -packing* of E if: (i) $x_i \in E$; (ii) $w_i \geq 0$; and (iii)

$$\max_{1 \leq i \leq k} \sum_{j=1}^k w_j \psi_s \left(\frac{x_i - x_j}{r} \right) \leq 1. \quad (2.14)$$

Define

$$N_r(E; \psi_s) := \sup \sum_{i=1}^k w_i, \quad (2.15)$$

where the supremum is taken over all size- r weighted ψ_s -packings $(w_i, x_i)_{i=1}^k$ of E . The defined quantity $N_r(E; \psi_s)$ is related to the entropy number $N_r(E)$. In fact, Howroyd (2001, Lemma 5) has shown that $N_r(E; \psi_\infty) = N_{r/2}(E)$ for all $r > 0$ and all $E \subseteq \mathbf{R}^N$. We will use this fact in the proof of Lemma 3.1 below.

The s -dimensional upper box dimension of E is defined as

$$\text{B-dim}_s E := \limsup_{r \downarrow 0} \frac{\log N_r(E; \psi_s)}{\log(1/r)}, \quad (2.16)$$

where $\log 0 := -\infty$. Note in particular that $\text{B-dim}_s \emptyset = -\infty$. It is possible to deduce that $s \mapsto \text{B-dim}_s E$ is non-decreasing.

Define $\mathcal{P}_A(E)$ to be the collection of all probability measures that are supported on a finite number of points in E . For all $\mu \in \mathcal{P}_A(E)$ define

$$J_s(r, \mu) := \max_{x \in \text{supp } \mu} F_s^\mu(x, r) \quad \text{and} \quad I_s(r, \mu) := \int F_s^\mu(x, r) \mu(dx). \quad (2.17)$$

For $E \subseteq \mathbf{R}^N$, define

$$Z_s(r; E) := \inf_{\mu \in \mathcal{P}_A(E)} J_s(r, \mu). \quad (2.18)$$

Howroyd (2001) has demonstrated that for all $s, r > 0$,

$$Z_s(r; E) = \inf_{\mu \in \mathcal{P}_A(E)} I_s(r, \mu) \quad \text{and} \quad N_r(E; \psi_s) = \frac{1}{Z_s(r; E)}. \quad (2.19)$$

Consequently,

$$\text{B-}\overline{\text{dim}}_s E = \limsup_{r \downarrow 0} \frac{\log Z_s(r; E)}{\log r}. \quad (2.20)$$

According to Howroyd (2001, Proposition 8),

$$\text{B-}\overline{\text{dim}}_s E = \text{B-}\overline{\text{dim}} E \quad \forall s \geq N, E \subseteq \mathbf{R}^N. \quad (2.21)$$

Howroyd (2001) also proved that P-dim_s is the regularization of $\text{B-}\overline{\text{dim}}_s$; i.e.,

$$\text{P-dim}_s E = \inf \left\{ \sup_{k \geq 1} \text{B-}\overline{\text{dim}}_s E_k : E \subseteq \bigcup_{k=1}^{\infty} E_k \right\}, \quad (2.22)$$

This is the dimension-profile analogue of (2.2).

3. Proof of Theorem 1.2

Recall that X is a centered, d -dimensional, N -parameter Gaussian random field such that for all $s, t \in \mathbf{R}^N$ and $j, k \in \{1, \dots, d\}$,

$$\text{Cov}(X_j(s), X_k(t)) = \frac{1}{2} (|s|^{2H} + |t|^{2H} - |s - t|^{2H}) \delta_{ij}. \quad (3.1)$$

Throughout, we assume that the process X is constructed in a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$, and that the sample function $t \mapsto X_\omega(t)$ is continuous for almost every $\omega \in \Omega$. According to the general theory of Gaussian processes this can always be arranged.

Our proof of Theorem 1.2 hinges on several lemmas. The first is a technical lemma which verifies the folklore statement that, for every $r > 0$ and $E \subseteq \mathbf{R}^N$, the entropy number $N_r(X(E))$ is a random variable. We recall that $(\Omega, \mathcal{F}, \mathbf{P})$ is assumed to be complete.

LEMMA 3.1. *Let $E \subseteq \mathbf{R}^N$ be a fixed set, and choose and fix some $r > 0$. Then $N_r(X(E))$ and $Z_\infty(r; X(E))$ are non-negative random variables.*

Proof. It follows from (2.19) that $Z_\infty(r; X(E)) = 1/N_{r/2}(X(E))$. Hence it suffices to prove $N_r(X(E))$ is a random variable.

Let $C(\mathbf{R}^N)$ be the space of continuous functions $f : \mathbf{R}^N \rightarrow \mathbf{R}^d$ equipped with the norm

$$\|f\| := \sum_{k=1}^{\infty} 2^{-k} \frac{\max_{|t| \leq k} |f(t)|}{1 + \max_{|t| \leq k} |f(t)|}. \quad (3.2)$$

According to general theory we can assume without loss of generality that $\Omega = C(\mathbf{R}^N)$. It suffices to prove that for all $a > 0$ fixed, $\Theta_a := \{f \in C(\mathbf{R}^N) : N_r(f(E)) > a\}$ is open and hence Borel measurable. For then $\{\omega \in \Omega : N_r(X(E)) > a\} = X^{-1}(\Theta_a)$ is also measurable.

To this end we assume that $N_r(f(E)) > a$, and define $n := [a]$. There necessarily exist $t_1, \dots, t_{n+1} \in E$ such that $|f(t_i) - f(t_j)| > 2r$ for all $1 \leq i \neq j \leq n+1$. Choose and fix $\eta \in (0, 1)$ such that $\eta < \min\{|f(t_i) - f(t_j)| - 2r : \forall 1 \leq i \neq j \leq n+1\}$. We can then find an integer $k_0 > 0$ such that $|t_i| \leq k_0$ for all $i = 1, \dots, n+1$. It follows from our definition of the norm $\|\cdot\|$ that for all $\delta \in (0, \eta 2^{-(k_0+2)})$ and all functions $g \in C(\mathbf{R}^N)$ with $\|g - f\| < \delta$,

$$\max_{1 \leq i \leq n+1} |g(t_i) - f(t_i)| < \frac{\eta}{2}. \quad (3.3)$$

This and the triangle inequality imply $|g(t_i) - g(t_j)| \geq |f(t_i) - f(t_j)| - \eta > 2r$ for all $1 \leq i \neq j \leq n+1$, and hence $N_r(g(E)) > n$. This verifies that $\{f \in C(\mathbf{R}^N) : N_r(f(E)) > a\}$ is an open set. \square

The following lemma is inspired by Lemma 12 of Howroyd (2001). We emphasize that $\mathbb{E}[Z_\infty(r; X(E))]$ is well defined (Lemma 3.1).

LEMMA 3.2. *If $E \subseteq \mathbf{R}^N$ then*

$$\mathbb{E}[Z_\infty(r; X(E))] \leq K Z_{Hd}(r^{1/H}; E) \quad \forall r > 0. \quad (3.4)$$

The constant $K \in (0, \infty)$ depends only on d and H .

Proof. Note that $(\mu \circ X^{-1}) \in \mathcal{P}_A(X(E))$ whenever $\mu \in \mathcal{P}_A(E)$. Hence, $Z_\infty(r; X(E)) \leq I_\infty(r, \mu \circ X^{-1})$. Because

$$I_\infty(r, \mu \circ X^{-1}) = \int \int \mathbf{1}_{\{|X(s) - X(t)| \leq r\}} \mu(ds) \mu(dt) \quad (3.5)$$

for all $r > 0$, we have

$$\begin{aligned} \mathbb{E}[Z_\infty(r; X(E))] &\leq \int \int \mathbb{P}\{|X(s) - X(t)| \leq r\} \mu(ds) \mu(dt) \\ &\leq K \int \int \left(\frac{r^d}{|s - t|^{Hd}} \wedge 1 \right) \mu(ds) \mu(dt) \\ &= K I_{Hd}(r^{1/H}, \mu), \end{aligned} \quad (3.6)$$

where the last inequality follows from the self-similarity and stationarity of the increments of X , and where $K > 0$ is a constant that depends only on d and H . We obtain the desired result by optimizing over all $\mu \in \mathcal{P}_A(E)$. \square

LEMMA 3.3. *For all nonrandom sets $E \subseteq \mathbf{R}^N$,*

$$\text{B-dim} X(E) \geq \frac{1}{H} \text{B-dim}_{Hd} E \quad a.s. \quad (3.7)$$

Proof. Without loss of generality we assume $\text{B-dim}_{Hd} E > 0$, for otherwise there is nothing left to prove. Then for any constant $\gamma \in (0, \text{B-dim}_{Hd} E)$ there exists a sequence $\{r_n\}_{n=1}^\infty$ of positive numbers such that $r_n \downarrow 0$ and $Z_{Hd}(r_n; E) = o(r_n^\gamma)$ as $n \rightarrow \infty$. It follows from Lemma 3.2 and Fatou's lemma that

$$\mathbb{E} \left[\liminf_{r \downarrow 0} \frac{Z_\infty(r; X(E))}{r^{\gamma/H}} \right] \leq \liminf_{n \rightarrow \infty} \frac{\mathbb{E}[Z_\infty(r_n^H; X(E))]}{r_n^\gamma}$$

$$\begin{aligned}
 &\leq K \lim_{n \rightarrow \infty} \frac{Z_{Hd}(r_n; E)}{r_n^\gamma} \\
 &= 0.
 \end{aligned} \tag{3.8}$$

Consequently, (2.20) and (2.21) together imply that $\text{B-dim} X(E) \geq \gamma/H$ a.s. The lemma follows because $\gamma \in (0, \text{B-dim}_{Hd} E)$ is arbitrary. \square

The following Lemma is borrowed from Falconer and Howroyd (1996, Lemma 5).

LEMMA 3.4. *If a set $E \subseteq \mathbf{R}^N$ has the property that $\text{B-dim}(E \cap G) \geq \delta$ for all open sets $G \subseteq \mathbf{R}^N$ such that $E \cap G \neq \emptyset$. Then $\dim_p E \geq \delta$.*

We are ready to prove Theorem 1.2.

Proof of Theorem 1.2. Since $\text{P-dim}_{Hd} E \geq \text{Dim}_{Hd} E$, (1.3) implies that $\dim_p X(E)$ is almost surely bounded above by $\frac{1}{H} \text{P-dim}_{Hd} E$. Consequently, it remains to prove the reverse inequality.

To this end we may assume without loss of generality that $\text{P-dim}_{Hd} E > 0$, lest the inequality becomes trivial. Choose and fix an arbitrary $\alpha \in (0, \text{P-dim}_{Hd} E)$. Lemma 2.1 implies that E has non- σ -infinite $\mathcal{P}^{\alpha, Hd}$ -measure. By Lemma 2.2, there exists a compact set $K \subseteq E$ such that $\mathcal{P}^{\alpha, Hd}(G \cap K) = \infty$ for all open sets $G \subseteq \mathbf{R}^N$ with $G \cap K \neq \emptyset$.

By separability there exists a countable basis of the usual Euclidean topology on \mathbf{R}^N . Let $\{G_k\}_{k=1}^\infty$ be an enumeration of those sets in the basis that intersect K . It follows from Lemma 3.3 that for every $k = 1, 2, \dots$ there exists an event Ω_k of P-measure one such that for all $\omega \in \Omega_k$,

$$\begin{aligned}
 \text{B-dim} X_\omega(G_k \cap K) &\geq \frac{1}{H} \text{B-dim}_{Hd}(G_k \cap K) \\
 &\geq \frac{\alpha}{H}.
 \end{aligned} \tag{3.9}$$

Therefore, $\Omega_0 := \bigcap_{k=1}^\infty \Omega_k$ has full P-measure, and for every $\omega \in \Omega_0$,

$$\begin{aligned}
 \text{B-dim}(X_\omega(K) \cap U) &\geq \text{B-dim}(X_\omega(K \cap X^{-1}(U))) \\
 &\geq \frac{\alpha}{H}.
 \end{aligned} \tag{3.10}$$

The preceding is valid for all open sets U with $X(K) \cap U \neq \emptyset$ because $X^{-1}(U)$ is open and $K \cap X^{-1}(U) \neq \emptyset$. According to Lemma 3.4 this proves that $\dim_p X_\omega(K) \geq \alpha/H$ almost surely. Because $\alpha \in (0, \text{P-dim}_{Hd} E)$ is arbitrary this finishes the proof of Theorem 1.2. \square

4. An Equivalent Definition

Given a Borel set $E \subseteq \mathbf{R}^N$, we define $\mathcal{P}(E)$ as the collection all probability measures μ on \mathbf{R}^N such that $\mu(E) = 1$ [μ is called a probability measure on E]. Define for all Borel sets $E \subseteq \mathbf{R}^N$ and all $s \in (0, \infty]$,

$$\mathcal{I}_s(r; E) := \inf_{\mu \in \mathcal{P}(E)} I_s(r, \mu). \tag{4.1}$$

Thus, the sole difference between \mathcal{I}_s and Z_s is that in the latter we use all finitely-supported [discrete] probability measures on E , whereas in the former we use all probability measures on E . We may also define \mathcal{I}_s using $\mathcal{M}_c^+(E)$ in place of $\mathcal{P}(E)$ in (4.1). Our

next theorem shows that all these notions lead to the same s -dimensional box dimension profile.

THEOREM 4.1. *For all analytic sets $E \subseteq \mathbf{R}^N$ and all $s \in (0, \infty]$,*

$$\overline{\text{B-dim}}_s(E) = \limsup_{r \downarrow 0} \frac{\log \mathcal{Z}_s(r; E)}{\log r}. \quad (4.2)$$

Proof. Because $\mathcal{P}_A(E) \subseteq \mathcal{P}(E)$ it follows immediately that $\mathcal{Z}_s(r; E) \leq Z_s(r; E)$. Consequently,

$$\limsup_{r \downarrow 0} \frac{\log \mathcal{Z}_s(r; E)}{\log r} \geq \overline{\text{B-dim}}_s(E). \quad (4.3)$$

We explain the rest only when $N = 1$; the general case is handled similarly.

Without loss of much generality suppose $E \subseteq [0, 1)$ and μ is a probability measure on E .

For all integers $n \geq 1$ and $i \in \{0, 1, \dots, n-1\}$ define $C_i = C_{i,n} := [i/n, (i+1)/n)$. Then, we can write $I_s(1/n, \mu) = T_1 + T_2$, where

$$\begin{aligned} T_1 &:= \sum_{0 \leq i < n} \sum_{j \in \{i-1, i, i+1\}} \int_{C_i} \int_{C_j} \left(1 \wedge \frac{1}{n|x-y|}\right)^s \mu(dx) \mu(dy), \\ T_2 &:= \sum_{0 \leq i < n} \sum_{j \notin \{i-1, i, i+1\}} \int_{C_i} \int_{C_j} \left(1 \wedge \frac{1}{n|x-y|}\right)^s \mu(dx) \mu(dy). \end{aligned} \quad (4.4)$$

Any interval C_j with $\mu(C_j) = 0$ does not contribute to $I_s(1/n, \mu)$. For every j with $\mu(C_j) > 0$, we choose an arbitrary point $\tau_j \in E \cap C_j$ and denote $w_j := \mu(C_j)$. Then the discrete probability measure ν that puts mass w_j at $\tau_j \in E$ belongs to $\mathcal{P}_A(E)$. For simplicity of notation, in the following we assume $\mu(C_j) > 0$ for all $j = 0, 1, \dots, n-1$.

If $j \notin \{i-1, i, i+1\}$, then $\sup_{x \in C_i} \sup_{y \in C_j} |x-y| \leq 3|\tau_j - \tau_i|$, whence we have

$$T_2 \geq \frac{1}{3^s} \sum_{0 \leq i < n} \sum_{j \notin \{i-1, i, i+1\}} \left(1 \wedge \frac{1}{n|\tau_j - \tau_i|}\right)^s w_i w_j. \quad (4.5)$$

If $j \in \{i-1, i, i+1\}$, then a similar case-by-case analysis can be used. This leads us to the bound,

$$\begin{aligned} I_s\left(\frac{1}{n}, \mu\right) &\geq \frac{1}{3^s} \sum_{0 \leq i, j < n} \sum \left(1 \wedge \frac{1}{n|\tau_j - \tau_i|}\right)^s w_i w_j \\ &= \frac{1}{3^s} \int \int \left(1 \wedge \frac{1/n}{|a-b|}\right)^s \nu(da) \nu(db). \end{aligned} \quad (4.6)$$

Consequently, the right-hand side of (4.6) is at most $3^{-s} Z_s(1/n; E)$. It follows that

$$3^{-s} Z_s\left(\frac{1}{n}; E\right) \leq \mathcal{Z}_s\left(\frac{1}{n}; E\right) \leq Z_s\left(\frac{1}{n}; E\right). \quad (4.7)$$

If r is between $1/n$ and $1/(n+1)$, then $Z_s(r; E)$ is between $Z_s(1/n; E)$ and $Z_s(1/(n+1); E)$. A similar remark applies to \mathcal{Z}_s . Because $\log n \sim \log(n+1)$ as $n \rightarrow \infty$, this proves the theorem. \square

Acknowledgements. We express our thanks to an anonymous referee who carefully read the manuscript and made several constructive remarks.

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