LINEAR FRACTIONAL STABLE SHEETS: WAVELET EXPANSION AND SAMPLE PATH PROPERTIES

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Abstract. In this paper we give a detailed description of the random wavelet series representation of real-valued linear fractional stable sheet introduced in [3]. By using this representation, in the case where the sample paths are continuous, an anisotropic uniform and quasi-optimal modulus of continuity of these paths is obtained as well as an upper bound for their behavior at infinity and around the coordinate axes. The Hausdorff dimensions of the range and graph of these stable random fields are then derived.

1. Introduction and main results

Let $0 < \alpha < 2$ and $H = (H_1, \ldots, H_N) \in (0,1)^N$ be given. We define an α-stable random field $X_0 = \{X_0(t), t \in \mathbb{R}^N\}$ with values in $\mathbb{R}$ by

$$X_0(t) = \int_{\mathbb{R}^N} h_H(t,s)\, Z_\alpha(ds), \tag{1.1}$$

where Z_α is a strictly α-stable random measure on $\mathbb{R}^N$ with Lebesgue measure as its control measure and $\beta(s)$ as its skewness intensity. That is, for every Lebesgue measurable set $A \subseteq \mathbb{R}^N$ with Lebesgue measure $\lambda_N(A) < \infty$, $Z_\alpha(A)$ is a strictly α-stable random variable with scale parameter $\lambda_N(A)^{1/\alpha}$ and skewness parameter $(1/\lambda_N(A)) \int_A \beta(s) ds$. If $\beta(s) \equiv 0$, then Z_α is a symmetric α-stable random measure on $\mathbb{R}^N$. We refer to [20, Chapter 3] for more information on stable random measures and their integrals. Also in (1.1),

$$h_H(t,s) = \kappa \prod_{\ell=1}^{N} \left\{ (t_\ell - s_\ell)_+^{H_\ell - \frac{1}{\alpha}} - (-s_\ell)_+^{H_\ell - \frac{1}{\alpha}} \right\}, \tag{1.2}$$

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where $\kappa > 0$ is a normalizing constant such that the scale parameter of $X_0(1)$, denoted by $\|X_0(1)\|_\alpha$, equals 1, $t_+ = \max\{t, 0\}$ and $0^0 = 1$. Observe that, if $H_1 = \cdots = H_N = \frac{1}{\alpha}$, X_0 is the ordinary stable sheet studied in [9]. In general, the random field X_0 is called a linear fractional α-stable sheet defined on $\mathbb{R}^N$ (or $(N, 1)$-LFSS for brevity) in $\mathbb{R}$ with index H. LFSS is an extension of both linear fractional stable motion (LFSM), which corresponds to the case where $N = 1$, and of ordinary fractional Brownian sheet (FBS) which corresponds to $\alpha = 2$, that is, to replacing the stable measure in (1.1) by a Gaussian random measure.

We will also consider (N, d)-LFSS, with $d > 1$, that is a linear fractional α-stable sheet defined on $\mathbb{R}^N$ and taking its values in $\mathbb{R}^d$. The (N, d)-LFSS that we consider is the stable field $X = \{X(t), t \in \mathbb{R}^N\}$ defined by

$$X(t) = \big(X_1(t), \ldots, X_d(t)\big), \qquad \forall t \in \mathbb{R}^N , \tag{1.3}$$

where $X_1, \ldots, X_d$ are d independent copies of X_0. It is easy to verify by using the representation (1.1) that X satisfies the following scaling property: For any $N \times N$ diagonal matrix $A = (a_{ij})$ with $a_{ii} = a_i > 0$ for all $1 \le i \le N$ and $a_{ij} = 0$ if $i \ne j$, we have

$$\big\{X(At),\, t \in \mathbb{R}^N\big\} \stackrel{d}{=} \bigg\{\prod_{j=1}^{N} a_j^{H_j} X(t),\ t \in \mathbb{R}^N\bigg\}, \tag{1.4}$$

where $\stackrel{d}{=}$ denotes the equality in the sense of finite dimensional distributions, provided that the skewness intensity satisfies $\beta(As) = \beta(s)$ for almost every $s \in \mathbb{R}^N$. Relation (1.4) means that the (N, d)-LFSS X is an operator-self-similar [or operator-scaling] random field in the *time variable* (see [6, 24]). When the indices $H_1, \ldots, H_N$ are not the same, X has different scaling behavior along different directions. This anisotropic nature of X makes it a potential model for various spatial objects, as is already the case for anisotropic Gaussian fields ([7] and [5]). We also mention that one can construct (N, d)-stable random fields which are self-similar in the *space variables* in the sense of [12, 14]. This will not be discussed in this paper.

Similarly to LFSM and FBS, see for instance [13, 21, 11, 23, 2, 1], there are close connections between sample path properties of LFSS and its parameters H and α. In this article we study some of these connections. In all the remainder of this paper we assume that the sample paths of X are continuous, *i.e.* $\min(H_1, \ldots, H_N) > 1/\alpha$. For convenience we even assume that

$$1/\alpha < H_1 \le \cdots \le H_N . \tag{1.5}$$

Of course, there is no loss of generality in the arbitrary ordering of $H_1, \ldots, H_N$. Let us now state our main results.

The following theorem is an improved version of Theorems 1.2 and 1.3 in [3]. Relation (1.6) provides a sharp upper bound for the uniform modulus of continuity of LFSS, while Relation (1.7) gives an upper bound for its asymptotic behavior at infinity and around the coordinate axes.

Theorem 1. *Let Ω_0^* be the event of probability 1 that will be introduced in Corollary 5. Then for every compact set $\mathcal{K} \subseteq \mathbb{R}^N$, all $\omega \in \Omega_0^*$ and any arbitrarily small $\eta > 0$, one has*

$$\sup_{s,t\in\mathcal{K}} \frac{|X_0(s,\omega) - X_0(t,\omega)|}{\sum_{j=1}^N |s_j - t_j|^{H_j - 1/\alpha}\big(1 + \big|\log|s_j - t_j|\big|\big)^{2/\alpha+\eta}} < \infty \tag{1.6}$$

and

$$\sup_{t\in\mathbb{R}^N} \frac{|X_0(t,\omega)|}{\prod_{j=1}^N |t_j|^{H_j}(1 + \big|\log|t_j|\big|)^{1/\alpha+\eta}} < \infty. \tag{1.7}$$

The following result can be viewed as an inverse of (1.6) in Theorem 1.

Theorem 2. *Let Ω_3^* be the event of probability 1 that will be introduced in Lemma 12. Then for all $\omega \in \Omega_3^*$, all vectors $\widehat{u}_n \in \mathbb{R}^{N-1}$ with non-vanishing coordinates, any $n = 1, \dots, N$ and any real numbers $y_1 < y_2$ and $\epsilon > 0$, one has*

$$\sup_{s_n,t_n\in[y_1,y_2]} \frac{|X_0(s_n, \widehat{u}_n, \omega) - X_0(t_n, \widehat{u}_n, \omega)|}{|s_n - t_n|^{H_n - 1/\alpha}\big(1 + \big|\log|s_n - t_n|\big|\big)^{-1/\alpha-\epsilon}} = \infty, \tag{1.8}$$

where, for every real x_n, we have set $(x_n, \widehat{u}_n) = (u_1, \dots, u_{n-1}, x_n, u_{n+1}, \dots, u_N)$.

Observe that Theorems 1 and 2 have already been obtained by Takashima [21] in the particular case of LFSM (*i.e.*, $N = 1$). However, the proofs given by this author can hardly be adapted to LFSS. To establish the above theorems we introduce a wavelet series representation of X_0 and use wavelet methods which are, more or less, inspired from [2]. It is also worth noticing that the event Ω_3^* in Theorem 2 does not depend on $\widehat{u}_n$. This is why the latter theorem cannot be obtained by simply using the fact that LFSS is an LFSM of Hurst parameter H_n along the direction of the n-th axis.

The next theorem gives the Hausdorff dimensions of the range

$$X\big([0,1]^N\big) = \big\{X(t) : t \in [0,1]^N\big\}$$

and the graph

$$\mathrm{Gr}X\big([0,1]^N\big) = \big\{(t, X(t)) : t \in [0,1]^N\big\}$$

of an (N,d)-LFSS X. We refer to [10] for the definition and basic properties of Hausdorff dimension.

The following result extends Theorem 4 in [2] to the linear fractional stable sheets. Unlike the fractional Brownian sheet case, we remark that the Hausdorff dimensions of $X\big([0,1]^N\big)$ and $\mathrm{Gr}X\big([0,1]^N\big)$ are not determined by the uniform Hölder exponent of X on $[0,1]^N$.

Theorem 3. *Let the assumption (1.5) hold. Then, with probability 1,*

$$\dim_{\mathrm{H}} X\big([0,1]^N\big) = \min\left\{d;\quad \sum_{\ell=1}^N \frac{1}{H_\ell}\right\} \tag{1.9}$$

and

$$\begin{aligned}\dim_{\rm H} {\rm Gr} X\big([0,1]^N\big) &= \min\left\{\sum_{\ell=1}^{k}\frac{H_k}{H_\ell}+N-k+(1-H_k)d,\ 1\le k\le N;\ \sum_{\ell=1}^{N}\frac{1}{H_\ell}\right\}\\ &= \begin{cases}\sum_{\ell=1}^N \frac{1}{H_\ell} & \text{if } \sum_{\ell=1}^N \frac{1}{H_\ell}\le d,\\ \sum_{\ell=1}^k \frac{H_k}{H_\ell}+N-k+(1-H_k)d & \text{if } \sum_{\ell=1}^{k-1}\frac{1}{H_\ell}\le d<\sum_{\ell=1}^{k}\frac{1}{H_\ell},\end{cases}\end{aligned} \tag{1.10}$$

where $\sum_{\ell=1}^0 \frac{1}{H_\ell} := 0$.

Remark 4. The second equality in (1.10) can be verified by using (1.5) and some elementary computation; see [2].

In light of Theorem 3 it is a natural question to consider the Hausdorff dimensions of the image $X(E)$ and graph ${\rm Gr}X(E)$, where E is an arbitrary Borel set in $\mathbb{R}^N$. As shown by Wu and Xiao [22] for fractional Brownian sheets, due to the anisotropic nature of X, the Hausdorff dimension of E and the index H alone are not enough to determine $\dim_{\rm H} X(E)$. By combining the methods in Wu and Xiao [22] and Xiao [24] with the moment argument in this paper we determine $\dim_{\rm H} X(E)$ for every nonrandom Borel set $E\subseteq(0,\infty)^N$; see Theorem 20.

We end the Introduction with some notation. Throughout this paper, the underlying parameter spaces are $\mathbb{R}^N$, $\mathbb{R}_+^N=[0,\infty)^N$ or $\mathbb{Z}^N$. A typical parameter, $t\in\mathbb{R}^N$ is written as $t=(t_1,\ldots,t_N)$ or $t=\langle t_j\rangle$ whichever is more convenient. For any $s,t\in\mathbb{R}^N$ such that $s_j<t_j$ $(j=1,\ldots,N)$, the set $[s,t]=\prod_{j=1}^N[s_j,t_j]$ is called a closed interval (or a rectangle). Open or half-open intervals can be defined analogously. We will use capital letters $C, C_1, C_2,\ldots$ to denote positive and finite random variables and use $c, c_1, c_2,\ldots$ to denote unspecified positive and finite constants. Moreover, C and c may not be the same in each occurrence.

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2. Wavelet expansion of LFSS

The goal of this section is to give a detailed description of the wavelet representations of LFSS X_0. First we need to introduce some notation that will be extensively used in all the sequel.

(i) The real-valued function ψ denotes a well chosen compactly supported Daubechies wavelet (see [8, 16]). Contrary to the Gaussian case the fact that ψ is compactly supported will play a crucial role in the proof of Theorem 2 (see the proof of Part (b) of Proposition 14).

(ii) For any $\ell=1,\ldots,N$, the real-valued functions ψ^{H_ℓ} and ψ^{-H_ℓ} respectively denote the left-sided fractional primitive of order $H_\ell+1-1/\alpha$ and the right-sided fractional derivative of order $H_\ell + 1 - 1/\alpha$ of ψ, which are respectively defined for all $x \in \mathbb{R}$ by

$$\psi^{H_\ell}(x) = \int_{\mathbb{R}} (x-y)_+^{H_\ell - 1/\alpha} \psi(y)\, dy \text{ and } \psi^{-H_\ell}(x) = \frac{d^2}{dx^2} \int_{\mathbb{R}} (y-x)_+^{1/\alpha - H_\ell} \psi(y)\, dy. \tag{2.1}$$

Observe that the functions ψ^{H_ℓ} and ψ^{-H_ℓ} are well-defined, continuously differentiable and well-localized provided that ψ has sufficiently many vanishing moments (and thus is smooth enough). By saying that a function $\phi : \mathbb{R} \to \mathbb{R}$ is well-localized we mean that

$$\sup_{x\in\mathbb{R}} (1+|x|)^2 \left\{ |\phi(x)| + \left|\phi'(x)\right| \right\} < \infty\,. \tag{2.2}$$

(iii) $\{\epsilon_{j,k},\ (j,k) \in \mathbb{Z}^N \times \mathbb{Z}^N\}$ will denote the sequence of random variables defined as

$$\epsilon_{j,k} = \int_{\mathbb{R}^N} \prod_{\ell=1}^{N} \left\{ 2^{j_\ell/\alpha} \psi(2^{j_\ell} s_\ell - k_\ell) \right\} Z_\alpha(ds)\,. \tag{2.3}$$

They are centered α-stable random variables all with the same scale parameter

$$\|\epsilon_{j,k}\|_\alpha = \left\{ \int_{\mathbb{R}} |\psi(t)|^\alpha \, dt \right\}^{N/\alpha}\,.$$

and skewness parameter

$$\beta_{j,k} = \|\epsilon_{j,k}\|_\alpha^{-\alpha} \int_{\mathbb{R}^N} \prod_{\ell=1}^{N} \left\{ 2^{j_\ell/\alpha} \psi^{<\alpha>}(2^{j_\ell} s_\ell - k_\ell) \right\} \beta(s)\, ds\,,$$

where $x^{<\alpha>} = |x|^\alpha \mathrm{sgn}(x)$ which is the number having the same sign as x and absolute value $|x|^\alpha$. Moreover, if $L > 0$ is a constant such that the support of ψ is included in $[-L, L]$, then for any integers $p > 2L$, any $r \in \{0, \ldots, p-1\}^N$ and $j \in \mathbb{Z}^N$, $\{\epsilon_{j,r+kp};\ k \in \mathbb{Z}^N\}$ is a sequence of independent random variables.

A consequence of the above properties of the sequence $\{\epsilon_{j,k},\ (j,k) \in \mathbb{Z}^N \times \mathbb{Z}^N\}$ is the following.

Corollary 5. *There exists an event Ω_0^* of probability 1 such that, for any $\eta > 0$, for all $\omega \in \Omega_0^*$ and all $j, k \in \mathbb{Z}^N \times \mathbb{Z}^N$,*

$$|\epsilon_{j,k}(\omega)| \le C(\omega) \prod_{l=1}^{N} \left\{ (1+|j_l|)^{1/\alpha+\eta} (1+|k_l|)^{1/\alpha} \log^{1/\alpha+\eta}(2+|k_l|) \right\}\,,$$

where C is a finite positive random variable.

Proof We apply Lemma 22.□

It is worth noticing that, for every $\ell = 1, \ldots, N$, the functions ψ^{H_ℓ} and ψ^{-H_ℓ} can be defined equivalently to (2.1), up to a multiplicative constant, but in the Fourier domain by (see *e.g.* [19])

$$\widehat{\psi^{H_\ell}}(\xi) = \mathrm{e}^{i\,\mathrm{sgn}(\xi)(H_\ell - 1/\alpha + 1)\frac{\pi}{2}} \frac{\widehat{\psi}(\xi)}{|\xi|^{H_\ell - 1/\alpha + 1}} \tag{2.4}$$

and

$$\widehat{\psi^{-H_\ell}}(\xi) = \mathrm{e}^{i\,\mathrm{sgn}(\xi)(H_\ell-1/\alpha+1)\frac{\pi}{2}} |\xi|^{H_\ell-1/\alpha+1} \widehat{\psi}(\xi). \tag{2.5}$$

It follows from Parseval's Formula, (2.4), (2.5) and the orthonormality (in $L^2(\mathbb{R})$) of the sequence $\{2^{j/2}\psi(2^j\cdot -k),\ j,k\in\mathbb{Z}\}$ that ψ^{H_ℓ} and ψ^{-H_ℓ} satisfy, for all $(J,K)\in\mathbb{Z}^2$ and $(J',K')\in\mathbb{Z}^2$, up to a multiplicative constant,

$$\int_{\mathbb{R}} \psi^{H_\ell}(2^J x - K)\overline{\psi^{-H_\ell}(2^{J'}x - K')}\,dx = 2^{-J}\delta(J,K;J',K'), \tag{2.6}$$

where $\delta(J,K;J',K') = 1$ when $(J,K) = (J',K')$ and 0 otherwise. By putting together (2.4), (2.5) and the fact that $\widehat{\psi}(\xi) = O(\xi^2)$ as $|\xi|\to 0$, another useful property is obtained: for every $\ell = 1,\dots,N$, the first moment of the functions ψ^{H_ℓ} and ψ^{-H_ℓ} vanish, namely one has

$$\int_{\mathbb{R}} \psi^{H_\ell}(u)\,du = \int_{\mathbb{R}} \psi^{-H_\ell}(u)\,du = 0. \tag{2.7}$$

We are now in position to state the main results of this section.

Proposition 6. *Let Ω_1^* be the event of probability 1 that will be introduced in Lemma 21. For every $n\in\mathbb{N}$, $M>0$ and $t\in\mathbb{R}^N$ we set*

$$U_{n,M}(t) = \sum_{(j,k)\in D^N_{n,M}} 2^{-\langle j,H\rangle}\epsilon_{j,k}\prod_{l=1}^N \left\{\psi^{H_l}(2^{j_l}t_l - k_l) - \psi^{H_l}(-k_l)\right\}\ , \tag{2.8}$$

where the random variables $\{\epsilon_{j,k},\ (j,k)\in\mathbb{Z}^N\times\mathbb{Z}^N\}$ are defined by (2.3) and

$$D^N_{n,M} = \{(j,k)\in\mathbb{Z}^N\times\mathbb{Z}^N\ :\ \text{for all } l=1,\dots,N\ |j_l|\le n \text{ and } |k_l|\le M2^{n+1}\}\,. \tag{2.9}$$

Then for every $\omega\in\Omega_1^*$ the functional sequence $(U_{n,M}(\cdot,\omega))_{n\in\mathbb{N}}$ is a Cauchy sequence in the Hölder space $C^\gamma(\mathcal{K})$ for every $\gamma\in[0,H_1-1/\alpha)$ and compact set $\mathcal{K}\subseteq[-M,M]^N$. We denote its limit by

$$\sum_{(j,k)\in\mathbb{Z}^N\times\mathbb{Z}^N} 2^{-\langle j,H\rangle}\epsilon_{j,k}\prod_{l=1}^N \left\{\psi^{H_l}(2^{j_l}t_l - k_l) - \psi^{H_l}(-k_l)\right\}.$$

Proposition 7. *With probability 1, the following holds for all $t\in\mathbb{R}^N$*

$$X_0(t) = \sum_{(j,k)\in\mathbb{Z}^N\times\mathbb{Z}^N} 2^{-\langle j,H\rangle}\epsilon_{j,k}\prod_{l=1}^N \left\{\psi^{H_l}(2^{j_l}t_l - k_l) - \psi^{H_l}(-k_l)\right\}. \tag{2.10}$$

Remark 8. By the definition of X_0 and by Proposition 6, the both sides of (2.10) are continuous in t with probability 1. Hence, to prove Proposition 7, it is sufficient to show that

$$\left\{\sum_{(j,k)\in\mathbb{Z}^N\times\mathbb{Z}^N} 2^{-\langle j,H\rangle}\epsilon_{j,k}\prod_{l=1}^N \left\{\psi^{H_l}(2^{j_l}t_l - k_l) - \psi^{H_l}(-k_l)\right\},\ t\in\mathbb{R}^N\right\}$$

is a modification of X_0. This is a natural extension of the wavelet series representations both of LFSM and FBS (see [4, 1, 2]) and will be called the *random wavelet series representation* of LFSS.

Assume for a while that Proposition 6 holds and let us prove Proposition 7.

Proof of Proposition 7 Let us fix $l \in \{1, \ldots, N\}$. For any $(j_l, k_l) \in \mathbb{Z} \times \mathbb{Z}$ and $s_l \in \mathbb{R}$ we set

$$\psi_{j_l,k_l}(s_l) = 2^{j_l/\alpha}\psi(2^{j_l}s_l - k_l). \tag{2.11}$$

Since $\{\psi_{j,k}, j \in \mathbb{Z}, k \in \mathbb{Z}\}$ is an unconditional basis of $L^\alpha(\mathbb{R})$ (see [15]) and, for every fixed $t_l \in \mathbb{R}$, the function $s_l \mapsto (t_l - s_l)_+^{H_l - 1/\alpha} - (-s_l)_+^{H_n - 1/\alpha} \in L^\alpha(\mathbb{R}) \cap L^2(\mathbb{R})$, one has

$$(t_l - s_l)_+^{H_l-1/\alpha} - (-s_l)_+^{H_l-1/\alpha} = \sum_{j_l \in \mathbb{Z}} \sum_{k_l \in \mathbb{Z}} \kappa_{l,j,k}(t_l)\psi_{j_l,k_l}(s_l) \ , \tag{2.12}$$

where the convergence of the series in the RHS of (2.12), as a function of s_l, holds in $L^\alpha(\mathbb{R})$. Next by using the Hölder inequality and the $L^2(\mathbb{R})$ orthonormality of the sequence $\{2^{j_l(1/2-1/\alpha)}\psi_{j_l,k_l}, j_l \in \mathbb{Z}, k_l \in \mathbb{Z}\}$, one can prove that

$$\begin{aligned} \kappa_{l,j_l,k_l}(t_l) &= 2^{j_l(1-1/\alpha)} \int_{\mathbb{R}} \{(t_l - s_l)_+^{H_l-1/\alpha} - (-s_l)_+^{H_l-1/\alpha}\} \psi(2^{j_l}s_l - k_l)\, ds_l \\ &= 2^{-j_l, H_l} \left\{\psi^{H_l}(2^{j_l}t_l - k_l) - \psi^{H_l}(-k_l)\right\}. \end{aligned} \tag{2.13}$$

By inserting (2.12) into (1.1) for every $l = 1, \ldots, N$, we get that for any fixed $t \in \mathbb{R}^N$, the series (2.10) converges in probability to $X_0(t)$ and Proposition 7 follows from Remark 8. □

From now on our goal will be to prove Proposition 6. We need some preliminary results.

Proof of Proposition 6. For the sake of simplicity we suppose that $N = 2$. The proof for the general case is similar. The space $C^\gamma(\mathcal{K})$ is endowed with the norm

$$\|f\|_\gamma = \sup_{x \in \mathcal{K}} |f(x)| + |f|_\gamma \quad \text{with} \quad |f|_\gamma = \sup_{x \neq y \in \mathcal{K}} \frac{|f(x) - f(y)|}{\|x - y\|^\gamma} \ ,$$

where $\| \cdot \|$ denotes the Euclidean norm in $\mathbb{R}^2$. For every $n \in \mathbb{N}$ we set $D_n^c = (\mathbb{Z}^2 \times \mathbb{Z}^2) \setminus D_{n,M}^2$. Let us define $F_n(x, y) = F_n(x, y; \psi^{H_1}; M, \phi, \delta, \beta, \eta)$ and $E(x, y) = E(x, y; \phi, \delta, \beta, \eta)$ by

$$F_n(x, y) = A_n(x, y) + B_n(x, y) \ ,$$

where $A_n(x, y)$ and $B_n(x, y)$ are defined in Lemma 23 in the Appendix, and

$$E(x, y) = \sum_{(J,K) \in \mathbb{Z}^2} 2^{-J\delta} \frac{|\phi(2^J x - K) - \phi(2^J y - K)|}{|x - y|^\beta} (3 + |J|)^{1/\alpha+\eta}(3 + |K|)^{1/\alpha+\eta}.$$

Using (2.8), the triangle inequality, Lemma 21, one has for any $n, p \in \mathbb{N}$ and $s_1, s_2, t_1, t_2 \in [-M, M]$,

$$
\begin{aligned}
&\frac{|U_{n+p,M}(s_1,s_2) - U_{n+p,M}(t_1,t_2) - U_{n,M}(s_1,s_2) + U_{n,M}(t_1,t_2)|}{(|s_1-t_1|^2+|s_2-t_2|^2)^{\beta/2}} \\
&\le \sum_{(j,k)\in D_n^c} 2^{-\langle j,H\rangle}\,|\epsilon_{j,k}| \frac{\left|\prod_{l=1}^2 \left(\psi^{H_l}(2^{j_l}s_l-k_l)-\psi^{H_l}(-k_l)\right) - \prod_{l=1}^2 \left(\psi^{H_l}(2^{j_l}t_l-k_l)-\psi^{H_l}(-k_l)\right)\right|}{(|s_1-t_1|^2+|s_2-t_2|^2)^{\beta/2}} \\
&\le F_n(s_1,t_1;\psi^{H_1};H_1,\beta,\eta)E(s_2,0;\psi^{H_2};H_2,0,\eta) + E(s_1,t_1;\psi^{H_1};H_1,\beta,\eta)F_n(s_2,0;\psi^{H_2};H_2,0,\eta) \\
&\quad + F_n(s_2,t_2;\psi^{H_2};H_2,\beta,\eta)E(t_1,0;\psi^{H_1};H_1,0,\eta) + E(s_2,t_2;\psi^{H_2};H_2,\beta,\eta)F_n(t_1,0;\psi^{H_1};H_1,0,\eta).
\end{aligned}
$$

By Lemma 23, we have that $\sup_{x,y\in[-M,M]} F_n(x,y) \to 0$ as $n\to\infty$ and $\sup_{x,y\in[-M,M]} E(x,y) < \infty$; hence the last display yields that $\sup_{p\ge 0} |U_{n+p,M} - U_{n,M}|_\gamma \to 0$ as $n \to \infty$. Observing that $U_{n,M}$ vanishes on the axes, the same result holds with $|\cdot|_\gamma$ replaced by $\|\cdot\|_\gamma$ and Proposition 6 is proved. □

Remark 9. Proposition 6 is much easier to prove in the Gaussian case. Indeed, in this case, using the fact that the $\epsilon_{j,k}$'s are *independent* $\mathcal{N}(0,1)$ Gaussian random variables one can easily show that the sequence $(U_{n,M})_{n\in\mathbb{N}}$ is weakly relatively compact in the space $C(\mathcal{K})$. We refer to the proof of Proposition 3 in [2] for more details.

From now on we will always identify the LFSS X_0 with its random wavelet series representation (2.10).

3. Uniform modulus of continuity and behavior as $|t_\ell| \to 0$ or ∞

The goal of this section is to prove Theorem 1. An immediate consequence of Proposition 6 is that X_0 is locally C^γ for any $\gamma \in (0, H_1 - 1/\alpha)$, almost surely. Theorem 1 completes this result by providing a sharper estimate on the uniform modulus of continuity, see (1.6), and the behavior at infinity and around the axes, see (1.7). As in our note [3], these results are obtained by using the representation (2.10). However, we improved the modulus of continuity estimate by relying on the independence present in the coefficients $\{\epsilon_{j,k},\ j,k \in \mathbb{Z}^N \times \mathbb{Z}^N\}$, see Lemma 22. If this independence is not taken into account, an alternative result (i.e., Lemma 21) may be used, resulting in a less precise estimate. The latter result holds in a quite general framework since they can be extended to a more general class of random wavelet series, see Remark 10 below.

Proof of Theorem 1. It follows from (2.10), Corollary 5 and Lemma 24 that for every $\omega \in \Omega_0^*$ and every $s, t \in K$, the triangle inequality implies

$$\begin{aligned}
\big|X_0(s,\omega) - X_0(t,\omega)\big| &\le \sum_{n=1}^{N} \big|X_0(t_1,\dots,t_{n-1},s_n,,\dots,s_N;\omega) - X_0(t_1,\dots,t_n,s_{n+1},\dots,s_N;\omega)\big| \\
&\le C_1(\omega)\sum_{n=1}^{N}\bigg(\prod_{\ell=1}^{n-1} T_{H_\ell,1/\alpha,\eta}(t_\ell;\psi^{H_\ell})\bigg)\times\bigg(\prod_{\ell=n+1}^{N} T_{H_\ell,1/\alpha,\eta}(s_\ell;\psi^{H_\ell})\bigg) \\
&\qquad \times S_{H_n,1/\alpha,\eta}(t_n,s_n;\psi^{H_n}) \qquad (3.1)\\
&\le C_2(\omega)\sum_{n=1}^{N} |t_n - s_n|^{H_n-1/\alpha}\big(1+\big|\log|t_n-s_n|\big|\big)^{2/\alpha+2\eta}.
\end{aligned}$$

This shows (1.6).

Similarly, using (2.10), Corollary 5 and Lemma 24, we obtain, for every $\omega \in \Omega_0^*$ and every $t \in \mathbb{R}$,

$$\big|X_0(t,\omega)\big| \le C_3(\omega)\prod_{\ell=1}^{N} T_{H_\ell,1/\alpha,\eta}(t_\ell;\psi^{H_\ell}) \le C_4(\omega)\prod_{\ell=1}^{N}\big(1+\big|\log|t_\ell|\big|\big)^{1/\alpha+\eta}|t_\ell|^{H_\ell}. \qquad (3.2)$$

The proof of Theorem 1 is finished. □

Remark 10. Clearly Proposition 6 holds more generally for any process $Y = \{Y(t),\, t \in \mathbb{R}^N\}$ having a wavelet series representation of the form

$$Y(t) = \sum_{(j,k)\in\mathbb{Z}^N\times\mathbb{Z}^N} c_{j,k}\lambda_{j,k}\prod_{l=1}^{N}\left\{\phi_l(2^{j_l}t_l - k_l) - \phi_l(-k_l)\right\},$$

where the ϕ_l's are well-localized functions, $\{c_{j,k},\, j,k \in \mathbb{Z}^N\}$ is a sequence of complex-valued coefficients satisfying $|c_{j,k}| \le c2^{-\langle j,H\rangle}$ ($c > 0$ being a constant) and $\{\lambda_{j,k},\, j,k \in \mathbb{Z}^N\}$ is a sequence of complex-valued random variables satisfying $\sup_{j,k}\mathbb{E}[|\lambda_{j,k}|^\nu] < \infty$ for all $0 < \nu < \alpha$. We can also show that (1.7) holds with probability 1 for such a process Y. In contrast, for this more general class of process, we cannot show (1.6) but a less precise estimate for the uniform modulus of continuity. Namely, as announced in our note [3], with probability 1,

$$\sup_{s,t\in\mathcal{K}} \frac{|X_0(s,\omega) - X_0(t,\omega)|}{\sum_{j=1}^{N}|s_j - t_j|^{H_j-1/\alpha-\eta}} < \infty$$

for all compact sets $\mathcal{K} \subseteq \mathbb{R}^N$.

4. Optimality of the modulus of continuity estimate

The goal of this section is to prove Theorem 2. For every $n \in \{1,\dots,N\}$ and $(j_n,k_n) \in \mathbb{N}\times\mathbb{Z}$, let $G_{j_n,k_n} = \{G_{j_n,k_n}(\widehat{u}_n),\, \widehat{u}_n \in \mathbb{R}^{N-1}\}$ be the α-stable field defined as the following wavelet transformation:

$$G_{j_n,k_n}(\widehat{u}_n) = 2^{j_n(1+H_n)} \int_{\mathbb{R}} X_0(s_n, \widehat{u}_n) \psi^{-H_n}(2^{j_n} s_n - k_n)\, ds_n, \tag{4.1}$$

where the notation $(s_n, \widehat{u}_n)$ is introduced in Theorem 2. By using (1.7) and the fact that the wavelet ψ^{-H_n} is well-localized, the process $\{G_{j_n,k_n}(u),\, u \in \mathbb{R}^{N-1}\}$ is well-defined and its trajectories are continuous, almost surely. The proof of Theorem 2 mainly relies on the following Lemmas 11 and 12.

Lemma 11. *Let Ω_0^* be the event of probability 1 in Corollary 5 and let $n \in \{1, \dots, N\}$. Suppose that there exist $(u_n, \widehat{u}_n) \in \mathbb{R}^N$, $\rho > 0$, $\epsilon > 0$ and $\omega \in \Omega_0^*$ such that*

$$\sup_{s_n,t_n \in [u_n-\rho, u_n+\rho]} \frac{|X_0(s_n, \widehat{u}_n, \omega) - X_0(t_n, \widehat{u}_n, \omega)|}{|s_n - t_n|^{H_n - 1/\alpha}\big(1 + \big|\log|s_n - t_n|\big|\big)^{-1/\alpha-\epsilon}} < \infty. \tag{4.2}$$

Then one has

$$\limsup_{j_n \to \infty} (j_n 2^{-j_n})^{1/\alpha} \max\left\{|G_{j_n,k_n}(\widehat{u}_n, \omega)| \,:\, k_n \in \mathbb{Z},\, |u_n - 2^{-j_n} k_n| \le \rho/8\right\} = 0. \tag{4.3}$$

Lemma 12. *Let Ω_3^* be the event of probability 1 defined as $\Omega_3^* = \Omega_0^* \cap \Omega_2^*$, where Ω_0^* and Ω_2^* are respectively the events defined in Corollary 5 and Lemma 13. For all $\omega \in \Omega_3^*$, $n \in \{1, \dots, N\}$, all integers $j_n \in \mathbb{N}$, real numbers $z_1 < z_2$ and all $0 < \tau_1 < \tau_2$, one has*

$$\liminf_{j_n \to \infty} (j_n 2^{-j_n})^{1/\alpha} \inf_{\widehat{u}_n \in [\tau_1,\tau_2]^{N-1}} \max\left\{|G_{j_n,k_n}(\widehat{u}_n, \omega)|;\ k_n \in [2^{j_n} z_1, 2^{j_n} z_2] \cap \mathbb{Z}\right\} > 0. \tag{4.4}$$

Before proving these lemmas, we show how they yield Theorem 2.

Proof of Theorem 2. For the sake of simplicity we only consider the case where $\widehat{u}_n$ have positive and non-vanishing coordinates. The general case is similar. Suppose *ad absurdum* that there exists $\omega \in \Omega_3^*$ such that (1.8) is not satisfied. Then, for some $n \in \{1, \dots, N\}$, there exists $\widehat{u}_n \in \mathbb{R}^{N-1}$ with positive and non vanishing coordinates, some real number u_n, $\rho > 0$ and $\epsilon > 0$ arbitrary small such that (4.2) holds. By Lemma 11, this implies (4.3). Then the conclusion of Lemma 12 leads to a contradiction. This proves Theorem 2. □

Proof of Lemma 11. Let $j_n \in \mathbb{N}$ and $k_n \in \mathbb{Z}$ be such that

$$|u_n - 2^{-j_n} k_n| \le \rho/8. \tag{4.5}$$

It follows from (4.1) and (2.7) that $G_{j_n,k_n}(\widehat{u}_n, \omega)$ can be written as

$$2^{j_n(1+H_n)} \int_{\mathbb{R}} \Big(X_0(s_n, \widehat{u}_n, \omega) - X_0(2^{-j_n} k_n, \widehat{u}_n, \omega)\Big) \psi^{-H_n}(2^{j_n} s_n - k_n)\, ds_n.$$

Hence, we have

$$\begin{aligned} \big|G_{j_n,k_n}(\widehat{u}_n, \omega)\big| &\le 2^{j_n(1+H_n)} \int_{\mathbb{R}} \big|X_0(s_n, \widehat{u}_n, \omega) - X_0(2^{-j_n} k_n, \widehat{u}_n, \omega)\big|\, \big|\psi^{-H_n}(2^{j_n} s_n - k_n)\big|\, ds_n \\ &= 2^{j_n(1+H_n)} \left\{A_{j_n,k_n}(\widehat{u}_n, \omega) + B_{j_n,k_n}(\widehat{u}_n, \omega)\right\}\ , \end{aligned} \tag{4.6}$$

where

$$A_{j_n,k_n}(\widehat{u}_n,\omega) = \int_{|s_n-u_n|\le\rho/2} \big|X_0(s_n,\widehat{u}_n,\omega) - X_0(2^{-j_n}k_n,\widehat{u}_n,\omega)\big|\,\big|\psi^{-H_n}(2^{j_n}s_n-k_n)\big|\,ds_n \tag{4.7}$$

and

$$B_{j_n,k_n}(\widehat{u}_n,\omega) = \int_{|s_n-u_n|>\rho/2} \big|X_0(s_n,\widehat{u}_n,\omega) - X_0(2^{-j_n}k_n,\widehat{u}_n,\omega)\big|\,\big|\psi^{-H_n}(2^{j_n}s_n-k_n)\big|\,ds_n. \tag{4.8}$$

Let us now give a suitable upper bound for $A_{j_n,k_n}(\widehat{u}_n,\omega)$. It follows from (4.7) and (4.2) that $A_{j_n,k_n}(\widehat{u}_n,\omega)$ is at most

$$C_5(\omega)\int_{\mathbb{R}} \big|s_n-2^{-j_n}k_n\big|^{H_n-1/\alpha}\big(1+\big|\log|s_n-2^{-j_n}k_n|\big|\big)^{-1/\alpha-\epsilon}\big|\psi^{-H_n}(2^{j_n}s_n-k_n)\big|\,ds_n. \tag{4.9}$$

We claim that

$$\sup_{j_n\ge1}\int_{\mathbb{R}} |x|^{H_n-1/\alpha}\big(1/j_n+\big|\log 2-(\log|x|)/j_n\big|\big)^{-1/\alpha-\epsilon}\,\big|\psi^{-H_n}(x)\big|\,dx<\infty \tag{4.10}$$

and differ its proof after we have shown (4.3).

By setting $x=2^{j_n}s_n-k_n$ in the integral in (4.9) and using (4.10), one obtains, for all $j_n\ge1$ and $k_n\in\mathbb{Z}$ satisfying (4.5),

$$A_{j_n,k_n}(\widehat{u}_n,\omega)\le C_6(\omega)2^{j_n(-1-H_n+1/\alpha)}j_n^{-1/\alpha-\epsilon}\,. \tag{4.11}$$

In order to derive an upper bound for $B_{j_n,k_n}(\widehat{u}_n,\omega)$, we use the fact that ψ^{-H_n} is a well-localized function and (4.5) to get

$$\begin{aligned}
B_{j_n,k_n}(\widehat{u}_n,\omega) &\le c\int_{|s_n-u_n|>\rho/2}\big|X_0(s_n,\widehat{u}_n,\omega)-X_0(2^{-j_n}k_n,\widehat{u}_n,\omega)\big|\,\Big(1+|2^{j_n}s_n-k_n|\Big)^{-2}ds_n\\
&\le c\int_{|s_n-u_n|>\rho/2}\big|X_0(s_n,\widehat{u}_n,\omega)-X_0(2^{-j_n}k_n,\widehat{u}_n,\omega)\big|\\
&\qquad\qquad\times\Big(1+2^{j_n}\Big(|s_n-u_n|-|u_n-2^{-j_n}k_n|\Big)\Big)^{-2}\,ds_n\\
&\le c\,2^{-2j_n}\int_{|s_n-u_n|>\rho/2}\big|X_0(s_n,\widehat{u}_n,\omega)-X_0(2^{-j_n}k_n,\widehat{u}_n,\omega)\big|\,\big|s_n-u_n\big|^{-2}\,ds_n.
\end{aligned}$$

This last inequality, together with (1.7), implies that, since $\omega\in\Omega_0^*$,

$$B_{j_n,k_n}(\widehat{u}_n,\omega)\le C_7(\omega)\,2^{-2j_n},$$

where C_7 is a random variable that does not depend on the integers j_n and k_n satisfying (4.5). Hence, putting together the last inequality, (4.11) and (4.6) one obtains (4.3).

Finally, to conclude the proof of the lemma, it remains to show (4.10). We separate the integral in (4.10) into two domains, $|x|>2^{j_n/2}$ and $|x|\le2^{j_n/2}$. We bound $\big(1/j_n+\big|\log 2-$

$(\log|x|)/j_n|\big)^{-1/\alpha-\epsilon}$ from above by $j_n^{1/\alpha+\epsilon}$ on the first domain, and by $\big((\log 2)/2\big)^{-1/\alpha-\epsilon}$ on the second domain, yielding that the integral in (4.10) is at most

$$j_n^{1/\alpha+\epsilon}\int_{|x|>2^{j_n/2}}\big|x\big|^{H_n-1/\alpha}\big|\psi^{-H_n}(x)\big|\,dx+\big((\log 2)/2\big)^{-1/\alpha-\epsilon}\int_{\mathbb{R}}\big|x\big|^{H_n-1/\alpha}\big|\psi^{-H_n}(x)\big|\,dx\;.$$

Using that $H_n-1/\alpha\in(0,1)$ and that ψ^{-H_n} is well localized, we thus get (4.10). □

In order to prove Lemma 12, we first prove a weaker result, namely the following lemma.

Lemma 13. *There exists Ω_2^*, an event of probability 1, such that for all $\omega\in\Omega_2^*$, $n\in\{1,\dots,N\}$ and real numbers $M>1$, $z_1<z_2$, $0<\tau_1<\tau_2$, one has*

$$\liminf_{j_n\to\infty}(j_n2^{-j_n})^{1/\alpha}\nu(n,j_n;M;z_1,z_2;\tau_1,\tau_2;\omega)>0\;, \tag{4.12}$$

where

$$\begin{aligned}&\nu(n,j_n;M;z_1,z_2;\tau_1,\tau_2;\omega)\\&=\min_{\widehat{k}_n\in[M^{j_n}\tau_1,M^{j_n}\tau_2]^{N-1}\cap\mathbb{Z}^{N-1}}\max\left\{\big|G_{j_n,k_n}(M^{-j_n}\widehat{k}_n,\omega)\big|;\;k_n\in\Big[2^{j_n}z_1,\,2^{j_n}z_2\Big]\cap\mathbb{Z}\right\}.\end{aligned}\tag{4.13}$$

In order to prove Lemma 13 we need to show that the the random variables $G_{j_n,k_n}(\widehat{u}_n)$ satisfy some nice properties, namely the following proposition.

Proposition 14. *Let $\widehat{u}_n\in\mathbb{R}^{N-1}$ be an arbitrary fixed vector with non-vanishing coordinates, then the following results hold:*

(a) $\{G_{j_n,k_n}(\widehat{u}_n),\;(j_n,k_n)\in\mathbb{N}\times\mathbb{Z}\}$ is a sequence of strictly α-stable random variables with identical scale parameters given by

$$\|G_{j_n,k_n}(\widehat{u}_n)\|_\alpha=\|\psi\|_{L^\alpha(\mathbb{R})}\prod_{l\neq n}\Big\|(u_l-\cdot)_+^{H_l-1/\alpha}-(-\cdot)_+^{H_l-1/\alpha}\Big\|_{L^\alpha(\mathbb{R})} \tag{4.14}$$

(b) Let $L>0$ be a constant such that the support of ψ is included in $[-L,L]$. Then for all integers $p>2L$ and $j_n\geq 0$, $\{G_{j_n,q_np}(\widehat{u}_n);\;q_n\in\mathbb{Z}\}$ is a sequence of independent random variables.

Proposition 14 is in fact a straightforward consequence of the following proposition and the fact that any two functions $s_n\mapsto\psi(2^{j_n}s_n-q_np)$ with different values of q_n have disjoint supports.

Proposition 15. *For every vector $\widehat{u}_n$ with non-vanishing coordinates and for every $(j_n,k_n)\in\mathbb{N}\times\mathbb{Z}$ one has almost surely*

$$G_{j_n,k_n}(\widehat{u}_n)=\int_{\mathbb{R}^N}\left[2^{j_n/\alpha}\psi(2^{j_n}s_n-k_n)\prod_{l\neq n}\Big\{(u_l-s_l)_+^{H_l-1/\alpha}-(-s_l)_+^{H_l-1/\alpha}\Big\}\right]dZ_\alpha(s)\;. \tag{4.15}$$

Proof of Proposition 15. As in (3.2), we have

$$\sup_{t\in\mathbb{R}^N} \frac{\sum_{(j,k)\in\mathbb{Z}^N\times\mathbb{Z}^N} 2^{-\langle j,H\rangle}\,|\epsilon_{j,k}|\Big|\prod_{l=1}^N \{\psi^{H_l}(2^{j_l}t_l-k_l)-\psi^{H_l}(-k_l)\}\Big|}{\prod_{j=1}^N |t_j|^{H_j}\big(1+\big|\log|t_j|\big|\big)^{1/\alpha+\eta}} < \infty. \tag{4.16}$$

It follows from Propositions 7 and 6, (4.16), the Dominated Convergence Theorem, (2.6), (2.7) and (2.13) that for any $\widehat{u}_n\in\mathbb{R}^{N-1}$ one has almost surely, for any increasing sequence $(D_m)_{m\in\mathbb{N}}$ of finite sets in $\mathbb{Z}\times\mathbb{Z}$ such that $\cup_m D_m=\mathbb{Z}\times\mathbb{Z}$,

$$\begin{aligned} G_{j_n,k_n}(\widehat{u}_n) &= \lim_{m\to\infty}\sum_{(j,k)\in D_m^N} 2^{j_n-\langle \widehat{j}_n,\widehat{H}_n\rangle}\epsilon_{j,k} \\ &\qquad\times\int_{\mathbb{R}}\prod_{l=1}^N\Big[\psi^{H_l}(2^{j_l}s_l-k_l)-\psi^{H_l}(-k_l)\Big]\psi^{-H_n}(2^{j_n}s_n-k_n)\,ds_n \\ &= \lim_{m\to\infty}\sum_{(\widehat{j}_n,\widehat{k}_n)\in D_m^{N-1}}\prod_{l\neq n}\kappa_{l,j_l,k_l}(u_l)\,\epsilon_{(j_n,\widehat{j}_n);(k_n,\widehat{k}_n)}, \end{aligned} \tag{4.17}$$

where $\kappa_{l,j_l,k_l}(u_l)$ is defined in (2.13). On the other hand, it follows from (2.12) that

$$\psi_{j_n,k_n}(s_n)\prod_{l\neq n}\Big\{(u_l-s_l)_+^{H_l-1/\alpha}-(-s_l)_+^{H_l-1/\alpha}\Big\} = \sum_{(\widehat{j}_n,\widehat{k}_n)\in\mathbb{Z}^{2(N-1)}}\prod_{l\neq n}\kappa_{l,j_l,k_l}(u_l)\prod_{l=1}^N\psi_{j_l,k_l}(s_l), \tag{4.18}$$

where for all fixed $\widehat{u}_n\in\mathbb{R}^{N-1}$ the convergence of the series in the RHS (4.18), as a function of $s\in\mathbb{R}^N$, holds in $L^\alpha(\mathbb{R}^N)$. Next using (4.18) and (2.3) one has, for every fixed $\widehat{u}_n\in\mathbb{R}^{N-1}$,

$$\begin{aligned} &\int_{\mathbb{R}^N}\left[\psi_{j_n,k_n}(s_n)\Bigg(\prod_{l\neq n}\Big\{(u_l-s_l)_+^{H_l-1/\alpha}-(-s_l)_+^{H_l-1/\alpha}\Big\}\Bigg)\right]dZ_\alpha(s) \\ &\qquad = \sum_{(\widehat{j}_n,\widehat{k}_n)\in\mathbb{Z}^{2(N-1)}}\prod_{l\neq n}\kappa_{l,j_l,k_l}(u_l)\epsilon_{(j_n,\widehat{j}_n);(k_n,\widehat{k}_n)}, \end{aligned} \tag{4.19}$$

where the convergence of the series holds in probability. Finally, putting together (4.17), (4.19) and (2.11), one obtains the proposition. □

We are now in position to prove Lemma 13.

Proof of Lemma 13. For any constants $M,c_1>0$, $n\in\{1,\dots,N\}$, integer $j_n\geq 0$ and rational numbers $r_1<r_2$, $0<\theta_1<\theta_2$ and $\zeta>0$, let $\Gamma(n,j_n)=\Gamma(n,j_n;M,c_1;r_1,r_2;\theta_1,\theta_2;\zeta)$ be the event defined as

$$\Gamma(n,j_n;M,c_1;r_1,r_2;\theta_1,\theta_2)=\Big\{\omega\,:\,\nu(n,j_n;M;r_1,r_2;\theta_1,\theta_2;\omega)\leq(c_1\,j_n2^{-j_n})^{-1/\alpha}\Big\}, \tag{4.20}$$

First we will show that, there exists c_1 large enough such that

$$\sum_{j_n\in\mathbb{N}}\mathbb{P}\Big(\Gamma(n,j_n;M,c_1;r_1,r_2;\theta_1,\theta_2)\Big)<\infty. \tag{4.21}$$

Using (A.2), (4.14) and that $\big\|(u_l-\cdot)_+^{H_l-1/\alpha}-(-\cdot)_+^{H_l-1/\alpha}\big\|_{L^\alpha(\mathbb{R})}$ is increasing with $|u_l|$ and non-zero for $u_l \neq 0$, we have

$$c_2 := \min_{n=1,\dots,N} \inf_{t\geq 1} \inf_{(j_n,k_n)\in\mathbb{N}\times\mathbb{Z}} \inf_{\widehat{u}_n\in[\theta_1,\theta_2]^{N-1}} t^\alpha P(|G_{j_n,k_n}(\widehat{u}_n)|>t)>0\ . \tag{4.22}$$

Observe finally that

$$\nu(n,j_n;M;r_1,r_2;\theta_1,\theta_2;\omega)$$
$$\geq \min_{\widehat{k}_n\in[M^{j_n}\theta_1,M^{j_n}\theta_2]^{N-1}\cap\mathbb{Z}^{N-1}} \max\Big\{\big|G_{j_n,q_np}(M^{-j_n}\widehat{k}_n,\omega)\big|;\ q_n\in\Big[\frac{2^{j_n}r_1}{p},\ \frac{2^{j_n}r_2}{p}\Big]\cap\mathbb{Z}\Big\}.$$

It follows from Proposition 14 and (4.22) and this inequality that

$$\begin{aligned}\mathbb{P}\Big(\Gamma(n,j_n)\Big) &\leq \sum_{\widehat{k}_n} \prod_{q_n\in[\frac{2^{j_n}r_1}{p},\frac{2^{j_n}r_2}{p}]\cap\mathbb{Z}} \mathbb{P}\Big(|G_{j_n,q_np}(M^{-j_n}\widehat{k}_n)|\leq (c_1\,j_n2^{-j_n})^{-1/\alpha}\Big)\\ &\leq c_3\,M^{(N-1)j_n}\Big(1-c_2\,j_n\,2^{-j_n}/c_1\Big)^{c_4 2^{j_n}},\end{aligned} \tag{4.23}$$

where the summation is taken over all $\widehat{k}_n\in[M^{j_n}\theta_1,M^{j_n}\theta_2]^{N-1}\cap\mathbb{Z}^{N-1}$ and the constants c_2, c_3 and c_4 do not depend on j_n. Using the last inequality one can prove that (4.21) holds by choosing $c_1>0$ large enough. Hence the Borel-Cantelli Lemma implies that, for such a constant c_1,

$$\mathbb{P}\Big(\bigcup_{m\in\mathbb{N}}\bigcap_{j_n\geq m}\Gamma^c(n,j_n;M,c_1;r_1,r_2;\theta_1,\theta_2)\Big)=1,$$

where $\Gamma^c(n,j_n;M,c_1,r_1,r_2;\theta_1,\theta_2)$ denotes the complement event of $\Gamma(n,j_n;M,c_1;r_1,r_2;\theta_1,\theta_2)$. But this implies that the event

$$\Big\{\omega:\liminf_{j_n\to\infty}(j_n2^{-j_n})^{1/\alpha}\nu(n,j_n;M;r_1,r_2;\theta_1,\theta_2;\omega)>0\Big\}$$

has probability 1. Finally setting Ω_2^* as the intersection of such sets over $\Big\{(M;r_1,r_2;\theta_1,\theta_2)\in\mathbb{Q}^5\ :\ M>0,\ r_1<r_2 \text{ and } 0<\theta_1<\theta_2\Big\}$, one obtains the lemma. $\square$

The following proposition will allow us to derive Lemma 12 starting from Lemma 13. Roughly speaking it means that the increments of the random field $\{G_{j_n,k_n}(\widehat{u}_n),\widehat{u}_n\in[\tau_1,\tau_2]^{N-1}\}$ can be bound uniformly in the indices j_n and k_n,

Proposition 16. *Let Ω_0^* be the event of probability 1 that was introduced in Corollary 5. Then for any reals $z_1<z_2$, $0<\tau_1<\tau_2$ and $\eta>0$ arbitrarily small, there exists an almost surely finite random variable $C_8>0$ such that for every $n\in\{1,\dots,N\}$, $j_n\in\mathbb{N}$, $k_n\in[2^{j_n}z_1,\,2^{j_n}z_2]$, $\widehat{u}_n\in[\tau_1,\,\tau_2]^{N-1}$, $\widehat{v}_n\in[\tau_1,\,\tau_2]^{N-1}$ and $\omega\in\Omega_0^*$, one has*

$$\big|G_{j_n,k_n}(\widehat{u}_n,\omega)-G_{j_n,k_n}(\widehat{v}_n,\omega)\big|\leq C_8(\omega)\,2^{j_nH_n}\sum_{l\neq n}|u_l-v_l|^{H_l-1/\alpha-\eta}. \tag{4.24}$$

Proof. Lemma 24 applied to (3.1) shows that, for all $\omega \in \Omega_0^*$ and any $\eta > 0$, there exists $C(\omega) > 0$ such that, for every $n \in \{1, \dots, N\}$, $s_n \in \mathbb{R}$, $\widehat{u}_n \in [\tau_1, \tau_2]^{N-1}$, $\widehat{v}_n \in [\tau_1, \tau_2]^{N-1}$,

$$\big|X_0(s_n, \widehat{u}_n, \omega) - X_0(s_n, \widehat{v}_n, \omega)\big| \leq C(\omega)\Big(\sum_{l \neq n} |u_l - v_l|^{H_l - 1/\alpha - \eta}\Big)|s_n|^{H_n}(1 + |\log(|s_n|)|)^{1/\alpha+\eta}. \quad (4.25)$$

Let $\zeta > 0$ be arbitrary small and consider the integral

$$I(j_n, k_n) = 2^{j_n} \int_{\mathbb{R}} (1 + |s_n|)^{H_n + \zeta} \left|\psi^{-H_n}(2^{j_n} s_n - k_n)\right| ds_n \,.$$

By setting $x = 2^{j_n} s_n - k_n$ we derive that

$$\begin{aligned}
\sup_{j_n \in \mathbb{N}} \max_{k_n \in [2^{j_n} z_1,\, 2^{j_n} z_2]} I(j_n, k_n) &= \sup_{j_n \in \mathbb{N}} \max_{k_n \in [2^{j_n} z_1,\, 2^{j_n} z_2]} \int_{\mathbb{R}} \Big(1 + 2^{-j_n}|x + k_n|\Big)^{H_n + \zeta} \left|\psi^{-H_n}(x)\right| dx \\
&\leq \int_{\mathbb{R}} \Big(1 + |x| + \max\{|z_1|, |z_2|\}\Big)^{H_n + \zeta} \left|\psi^{-H_n}(x)\right| dx < \infty. \quad (4.26)
\end{aligned}$$

The inequality (4.24) then follows from (4.1), (4.25) and (4.26). □

We are now in position to prove Lemma 12.

Proof of Lemma 12. We set

$$\widetilde{\nu}(n, j_n; z_1, z_2; \tau_1, \tau_2; \omega) = \inf_{\widehat{u}_n \in [\tau_1, \tau_2]^{N-1}} \max\Big\{|G_{j_n,k_n}(\widehat{u}_n, \omega)|;\ k_n \in [2^{j_n} z_1, 2^{j_n} z_2] \cap \mathbb{Z}\Big\}. \quad (4.27)$$

In view of Lemma 13 it is sufficient to show that there exists $\gamma > 0$ small enough and $M > 0$ such that, for all $n \in \{1, \dots, N\}$, $\omega \in \Omega_3$ and reals $z_1 < z_2$, $0 < \tau_1 < \tau_2$, one has

$$\lim_{j_n \to \infty} 2^{-j_n(1/\alpha - \gamma)} \Big|\widetilde{\nu}(n, j_n; M; z_1, z_2; \tau_1, \tau_2; \omega) - \nu(n, j_n; z_1, z_2; \tau_1, \tau_2; \omega)\Big| = 0. \quad (4.28)$$

As the function $f_{j_n}(\cdot) = \max\Big\{\big|G_{j_n,k_n}(\cdot, \omega)\big|;\ k_n \in [2^{j_n} z_1,\, 2^{j_n} z_2]\Big\}$ is continuous, there exists $\widehat{u}_n^0(j_n) \in [\tau_1,\, \tau_2]^{N-1}$ such that

$$f_{j_n}(\widehat{u}_n^0(j_n)) = \inf\Big\{f_{j_n}(\widehat{u}_n);\ \widehat{u}_n \in [\tau_1, \tau_2]^{N-1}\Big\}. \quad (4.29)$$

Moreover, when j_n is big enough, one has for some $\widehat{k}_n^0(j_n) \in [M^{j_n}\tau_1,\, M^{j_n}\tau_2]^{N-1} \cap \mathbb{Z}^{N-1}$,

$$\|M^{-j_n}\widehat{k}_n^0(j_n) - \widehat{u}_n^0(j_n)\|_\infty \leq M^{-j_n} \,. \quad (4.30)$$

Then it follows from Proposition 16 that there exists a constant $c_5 > 0$ (independent of (j_n, k_n)) such that the following inequality holds

$$\big|G_{j_n,k_n}(M^{-j_n}\widehat{k}_n^0(j_n), \omega) - G_{j_n,k_n}(\widehat{u}_n^0(j_n), \omega)\big| \leq c_5\, 2^{j_n H_n} M^{-j_n(H_1 - 1/\alpha - \eta)}.$$

The last inequality implies that

$$f_{j_n}(M^{-j_n}\widehat{k}_n^0(j_n)) \leq f_{j_n}(\widehat{u}_n^0(j_n)) + c_5\, 2^{j_n H_n} M^{-j_n(H_1 - 1/\alpha - \eta)}. \quad (4.31)$$

By using (4.29) and (4.31) one obtains that

$$\begin{aligned} f_{j_n}(\widehat{u}_n^0(j_n)) &\le \min\Big\{f_{j_n}(M^{-j_n}\widehat{k}_n);\ \widehat{k}_n \in [M^{j_n}\tau_1, M^{j_n}\tau_2]^{n-1}\Big\} \\ &\le f_{j_n}(\widehat{u}_n^0(j_n)) + c_5\, 2^{j_n H_n} M^{-j_n(H_1-1/\alpha-\eta)}. \end{aligned} \tag{4.32}$$

Let us choose M large enough so that

$$\frac{H_N - 1/\alpha}{H_1 - 1/\alpha} < \frac{\log M}{\log 2}.$$

and then, using (1.5), we choose $\eta > 0$ and $\gamma > 0$ small enough so that $2^{j_n H_n} M^{-j_n(H_1-1/\alpha-\eta)} = o(2^{-j_n(1/\alpha-\gamma)})$ as $j_n \to \infty$. Finally combining this with (4.31), we obtain (4.28). This proves Lemma 12. □

5. Proof of Theorem 3

As usual, the proof of Theorem 3 is divided into proving the upper and lower bounds separately. The proofs of the lower bounds rely on the standard capacity argument and the following Lemma 17. However, the proofs of the upper bounds are significantly different from that in [2], due to the fact that both $\dim_{\mathrm{H}} X\big([0,1]^N\big)$ and $\dim_{\mathrm{H}} \mathrm{Gr} X\big([0,1]^N\big)$ are not determined by the exponent for the uniform modulus of continuity of X. Our argument is based on the moment method in [12]. Combining this argument with the methods in [24], we are able to determine the Hausdorff dimension of the image $X(E)$ for all nonrandom Borel sets $E \subseteq (0,\infty)^N$.

We start by proving some results on the scale parameter of the stable random variable $X_0(s) - X_0(t)$ and the moments of the supremum of stable random fields. Since $\big\|X_0(s) - X_0(t)\big\|_\alpha$ can be used as a pseudometric for characterizing the regularity properties of X_0 via metric entropy methods (cf. [20, Chapter 12]), these results will be useful for studying other properties of LFSS X as well.

Lemma 17 is an extension of Lemma 3.4 in [2] for fractional Brownian sheets.

Lemma 17. *For any constant $\varepsilon > 0$, there exist positive and finite constants c_6 and c_7 such that for all $s, t \in [\varepsilon, 1]^N$,*

$$c_6 \sum_{\ell=1}^N \big|s_\ell - t_\ell\big|^{H_\ell} \le \big\|X_0(s) - X_0(t)\big\|_\alpha \le c_7 \sum_{\ell=1}^N \big|s_\ell - t_\ell\big|^{H_\ell}. \tag{5.1}$$

Proof. To prove the upper bound in (5.1), we use induction on N. When $N = 1$, X_0 is an (H,α)-linear fractional stable motion and one can verify directly that (5.1) holds as an equality. Suppose the upper bound in (5.1) holds for any linear fractional stable sheet with n parameters. We now show that it holds for a linear fractional stable sheet X_0 with $n+1$ parameters.

It follows from the representation (1.1) that, for any $s, t \in [\varepsilon, 1]^{n+1}$, $\big\|X_0(s) - X_0(t)\big\|_\alpha^\alpha$ is a constant multiple of the following integral:

$$
\begin{aligned}
&\int_{\mathbb{R}^{n+1}} \bigg| \prod_{\ell=1}^{n+1} \Big\{(t_\ell - r_\ell)_+^{H_\ell - \frac{1}{\alpha}} - (-r_\ell)_+^{H_\ell - \frac{1}{\alpha}}\Big\} - \prod_{\ell=1}^{n+1} \Big\{(s_\ell - r_\ell)_+^{H_\ell - \frac{1}{\alpha}} - (-r_\ell)_+^{H_\ell - \frac{1}{\alpha}}\Big\} \bigg|^\alpha dr \\
&\le c \int_{\mathbb{R}^n} \bigg| \prod_{\ell=1}^{n} \Big\{(t_\ell - r_\ell)_+^{H_\ell - \frac{1}{\alpha}} - (-r_\ell)_+^{H_\ell - \frac{1}{\alpha}}\Big\} - \prod_{\ell=1}^{n} \Big\{(s_\ell - r_\ell)_+^{H_\ell - \frac{1}{\alpha}} - (-r_\ell)_+^{H_\ell - \frac{1}{\alpha}}\Big\} \bigg|^\alpha dr \\
&\qquad \times \int_{\mathbb{R}} \Big\{(t_{n+1} - r_{n+1})_+^{H_{n+1} - \frac{1}{\alpha}} - (-r_{n+1})_+^{H_{n+1} - \frac{1}{\alpha}}\Big\}^\alpha dr_{n+1} \\
&+ c \int_{\mathbb{R}^n} \bigg[\prod_{\ell=1}^{n} \Big\{(s_\ell - r_\ell)_+^{H_\ell - \frac{1}{\alpha}} - (-r_\ell)_+^{H_\ell - \frac{1}{\alpha}}\Big\} \bigg]^\alpha \\
&\qquad \times \int_{\mathbb{R}} \Big|(t_{n+1} - r_{n+1})_+^{H_{n+1} - \frac{1}{\alpha}} - (s_{n+1} - r_{n+1})_+^{H_{n+1} - \frac{1}{\alpha}}\Big|^\alpha dr_{n+1} \\
&\le c \left\{ \bigg(\sum_{\ell=1}^{n} |s_\ell - t_\ell|^{H_\ell} \bigg)^\alpha + |t_{n+1} - s_{n+1}|^{H_{n+1}\alpha} \right\},
\end{aligned}
$$

where, in deriving the last inequality, we have used the induction hypothesis, the fact that the function $t \mapsto \int_{\mathbb{R}} \{(t-r)_+^{H-1/\alpha} - (-r)_+^{H-1/\alpha}\}^\alpha dr$ is locally uniformly bounded for any $H > 1/\alpha$ and that, by a change of variable $r_{n+1} = t_{n+1} + |t_{n+1} - s_{n+1}|u$, the last integral in the previous display is less than $|t_{n+1} - s_{n+1}|^{\alpha H_{n+1}}$ up to a multiplicative constant. Hence we have proved the upper bound in (5.1).

For proving the lower bound in (5.1), we define the stable field $Y = \{Y(t), t \in \mathbb{R}_+^N\}$ by

$$
Y(t) = \int_{[0,\, t]} h_{_H}(t, r)\, Z_\alpha(dr), \tag{5.2}
$$

where the function $h_{_H}(t, r)$ is defined in (1.2). Then by using (1.1) again we can write

$$
\big\|X_0(s) - X_0(t)\big\|_\alpha \ge \big\|Y(t) - Y(s)\big\|_\alpha. \tag{5.3}
$$

To proceed, we use the same argument as in [2, pp. 428–429] to decompose Y as a sum of independent stable random fields. For every $t \in [\varepsilon, 1]^N$, we decompose the rectangle $[0, t]$ into the following disjoint union:

$$
[0,\, t] = [0, \varepsilon]^N \cup \bigcup_{j=1}^N R(t_j) \cup \Delta(\varepsilon, t), \tag{5.4}
$$

where $R(t_j) = \{r \in [0,\, 1]^N : 0 \le r_i \le \varepsilon \text{ if } i \neq j,\, \varepsilon < r_j \le t_j\}$ and $\Delta(\varepsilon, t)$ can be written as a union of $2^N - N - 1$ sub-rectangles of $[0, t]$. It follows from (5.2) and (5.4) that for every $t \in [\varepsilon, 1]^N$,

$$\begin{aligned} Y(t) &= \int_{[0,\varepsilon]^N} h_H(t,r)\, Z_\alpha(dr) + \sum_{j=1}^N \int_{R(t_j)} h_H(t,r)\, Z_\alpha(dr) + \int_{\Delta(\varepsilon,t)} h_H(t,r)\, Z_\alpha(dr) \\ &:= Y(\varepsilon, t) + \sum_{j=1}^N Y_j(t) + Z(\varepsilon, t). \end{aligned} \tag{5.5}$$

Since the processes $\{Y(\varepsilon,t),\, t \in \mathbb{R}^N\}$, $\{Y_j(t),\, t \in \mathbb{R}^N\}$ $(1 \le j \le N)$ and $\{Z(\varepsilon,t),\, t \in \mathbb{R}^N\}$ are defined by the stochastic integrals with respect to Z_α over disjoint sets, they are independent. Only the $Y_j(t)$'s will be useful for proving the lower bound in (5.1).

Now let $s, t \in [\varepsilon,\, 1]^N$ and $j \in \{1, \ldots, N\}$ be fixed. Without loss of generality, we assume $s_j \le t_j$. Then

$$\big\|Y_j(t) - Y_j(s)\big\|_\alpha^\alpha = \int_{R(s_j)} \big(h_H(t,r) - h_H(s,r)\big)^\alpha\, dr + \int_{R(s_j,t_j)} h_H^\alpha(t,r)\, dr, \tag{5.6}$$

where $R(s_j, t_j) = \{r \in [0,1]^N : 0 \le r_i \le \varepsilon \text{ if } i \neq j,\, s_j < r_j \le t_j\}$. By (5.6) and some elementary calculations we derive

$$\begin{aligned} \big\|Y_j(t) - Y_j(s)\big\|_\alpha^\alpha &\ge \int_{R(s_j,t_j)} h_H^\alpha(t,r)\, dr \\ &= \int_{[0,\varepsilon]^{N-1}} \prod_{k\neq j} (t_k - r_k)^{\alpha H_k - 1} \int_{s_j}^{t_j} (t_j - r_j)^{\alpha H_j - 1}\, dr \\ &\ge c\, |t_j - s_j|^{\alpha H_j}, \end{aligned} \tag{5.7}$$

where $c > 0$ is a constant depending on ε, α and $H_k\, (1 \le k \le N)$ only. The lower bound in (5.1) follows from (5.5), (5.6) and (5.7). □

In order to estimate $\mathbb{E}\big[\sup_{t\in T} |X_0(t) - X_0(a)|\big]$ for all intervals $T = [a, b] \subseteq [\varepsilon, 1]^N$, we will make use of a general moment inequality of Móricz [17] for the maximum partial sums of multi-indexed random variables. This approach has the advantage that it is applicable to non-stable random fields as well. Another way for proving Lemma 19 below is to establish sharp upper bounds for the tail probability $\mathbb{P}\big[\sup_{t\in T} |X_0(t) - X_0(a)| > u\big]$ by modifying the arguments in [18].

First we adapt some notation from [17] to our setting. Let $\{\xi_k,\, k \in \mathbb{N}^N\}$ be a sequence of random variables. For any $m \in \mathbb{Z}_+^N$ ($\mathbb{Z}_+$ is the set of nonnegative integers) and $k \in \mathbb{N}^N$, let $R = R(m,k) = (m, m+k] \cap \mathbb{Z}_+^N$, which will also be called a rectangle in $\mathbb{Z}_+^N$, and we denote

$$S(R) = S(m,k) = \sum_{p\in R} \xi_p \quad \text{and} \quad M(R) = \max_{1\le q\le k} \big|S(m,q)\big|. \tag{5.8}$$

It can be verified that $M(R) \le \max_{Q\subseteq R} \big|S(Q)\big| \le 2^N M(R)$, where the maximum is taken over all rectangles $Q \subseteq R$. Let $f(R)$ be a nonnegative function of the rectangle R with left-lower vertex in $\mathbb{Z}_+^N$. We call f superadditive if for every rectangle $R = R(m, k)$ the inequality

$$f(R_{j1}) + f(R_{j2}) \le f(R) \tag{5.9}$$

holds for every $1 \le j \le N$ and $1 \le q_j < k_j$, where

$$R_{j1} = R\big((m_1, \dots, m_N),\ (k_1, \dots, k_{j-1}, q_j, k_{j+1}, \dots, k_N)\big)$$

and

$$R_{j2} = R\big((m_1, \dots, m_{j-1}, m_j + q_j, m_{j+1}, \dots, m_N),\ (k_1, \dots, k_{j-1}, k_j - q_j, k_{j+1}, \dots, k_N)\big).$$

In other words, $R_{j1} \cup R_{j2} = R$ is a disjoint decomposition of R by a hyperplane which is perpendicular to the jth axis. Together with the nonnegativity of f, (5.9) implies that, for every fixed $m \in \mathbb{Z}_+^N$, $f(R(m, k))$ is nondecreasing in each variable k_j $(1 \le j \le N)$.

The following moment inequality for the maximum $M(R)$ follows from Corollary 1 in [17].

Lemma 18. *Let $\beta > 1$ and $\gamma \ge 1$ be given constants. If there exists a nonnegative and superadditive function $f(R)$ of the rectangle R in $\mathbb{Z}_+^N$ such that $\mathbb{E}\big[|S(R)|^\gamma\big] \le f^\beta(R)$ for every R, then*

$$\mathbb{E}\big[M(R)^\gamma\big] \le \Big(\frac{5}{2}\Big)^N \big(1 - 2^{(1-\beta)/\gamma}\big)^{N\gamma} f^\beta(R) \tag{5.10}$$

for every rectangle R in $\mathbb{Z}_+^N$.

It is useful to notice that the constant in (5.10) is independent of R. Applying Lemma 18 to the linear fractional stable sheets, we obtain

Lemma 19. *Let the assumption (1.5) hold. Then there exists a positive and finite constant c_8 such that for all rectangles $T = [a, b] \subseteq [\varepsilon, 1]^N$,*

$$\mathbb{E}\bigg(\sup_{t \in T} \big|X_0(t) - X_0(a)\big|\bigg) \le c_8 \sum_{j=1}^N (b_j - a_j)^{H_j}. \tag{5.11}$$

Proof. For all $n \in \mathbb{N}$ we define a grid in $[a, b]$ with mesh 2^{-n} by the collection of points

$$\tau_n(p) = \langle a_j + (p_j - 1)(b_j - a_j)2^{-n} \rangle, \quad p \in R(0, \langle 2^n \rangle) = \{1, \dots, 2^n\}^N .$$

We rank these points using the lexicographical order, that is, $p \in R(0, \langle 2^n \rangle)$ has rank $k(p) = p_1 + 2^n(p_2 - 1) + \cdots + 2^{(N-1)n}(p_N - 1) \in \{1, \dots, 2^{Nn}\}$. Observing that, for all $p \in R(0, \langle 2^n \rangle)$ and all $p' \in R(0, \langle p \rangle) \setminus \{p\}$, we have $k(p') \le k(p)$. Now we define a sequence $\{\xi_p,\ p \in R(0, \langle 2^n \rangle))\}$ of random variables by induction on the rank of $p \in R(0, \langle 2^n \rangle)$ as follows: $\xi_{\langle 1 \rangle} = 0$ (rank 1), and for any $p \in R(0, \langle 2^n \rangle)$ having rank $k(p) \in \{2, \dots, 2^{Nn}\}$, assuming that $\xi_{p'}$ is already defined for all p' having rank at most $k(p) - 1$, we define ξ_p by the relationship

$$\sum_{p' \in R(0,p)} \xi_{p'} = X_0(\tau_n(p)) - X_0(a) . \tag{5.12}$$

This equation means that $\{\xi_p\}$ defines a signed measure with finite support in $R(0,\langle 2^n\rangle)$ and whose repartition function is given by $X_0(\tau_n(p)) - X_0(a)$ for $p \in R(0,\langle 2^n\rangle)$. Hence (5.12) can be extended to any rectangle $R(m,k) \subseteq R(0,\langle 2^n\rangle)$ for $m \neq 0$ as follows

$$\sum_{p\in R(m,k)} \xi_p = \sum_{i=1}^{2^{\ell-1}} \left\{X_0(\tau_n(q(i))) - X_0(\tau_n(r(i)))\right\} \ , \tag{5.13}$$

where $\ell \in \{1,\dots,N\}$ is the number of non-zero coordinates of m and, for all $i \in \{1,\dots,2^{\ell-1}\}$, $q(i)$ and $r(i)$ are the points of $\mathbb{Z}_+^N$ satisfying, for all $j \in \{1,\dots,N\}$, $|q_j(i) - r_j(i)| = k_j$ if $m_j \neq 0$ and $q_j(i) = r_j(i)$ otherwise.

We are now ready to prove (5.11). By the continuity of the sample function $X_0(t)$ and the monotone convergence theorem, since the set $\cup_{n\ge 1}\{\tau_n(p) \ : \ p \in R(0,\langle 2^n\rangle)\}$ is dense in $[a,b]$, it is sufficient to show that for all integers $n \ge 1$,

$$\mathbb{E}\Big(\max_{p\in R(0,\langle 2^n\rangle)} \big|X_0(\tau_n(p)) - X_0(a)\big|\Big) \le c_9 \sum_{j=1}^N (b_j - a_j)^{H_j}, \tag{5.14}$$

where $c_9 > 0$ is a finite constant independent of $[a,b] \subseteq [\varepsilon,1]^N$ and n. Because of (5.12), we see that this can be done by applying Lemmas 17 and 18.

It follows from Lemma 22 in the Appendix that for any strictly α-stable random variable Z with scale parameter 1 and every $\gamma < \alpha$, we have $\mathbb{E}(|Z|^\gamma) \le c_{10}$, where c_{10} depends on α and γ only. This fact, (5.12), (5.13) and Lemma 17 imply that for every $1 < \gamma < \alpha$ and every rectangle $R = R(m,k) \subseteq R(0,\langle 2^n\rangle)$,

$$\begin{aligned}\mathbb{E}\Big(\Big|\sum_{p\in R} \xi_p\Big|^\gamma\Big) &\le c_{11} \left[\sum_{j=1}^N \left(\frac{k_j(b_j-a_j)}{2^n}\right)^{H_j}\right]^\gamma \\ &\le \left[c_{12} \sum_{j=1}^N \left(\frac{k_j(b_j-a_j)}{2^n}\right)^{H_j/H_1}\right]^{H_1\gamma},\end{aligned} \tag{5.15}$$

where the last inequality follows from Hölder's inequality. For every rectangle $R = R(m,k)$ included in $R(0,\langle 2^n\rangle)$, let

$$f(R) = c_{12} \sum_{j=1}^N \left(\frac{k_j(b_j-a_j)}{2^n}\right)^{H_j/H_1}.$$

Note that, under assumption (1.5), we have $\alpha \in (1,2)$, $H_1\alpha > 1$ and $H_j \ge H_1$ for every $j = 1,\dots,N$. Hence the inequality $x^{H_j/H_1} + y^{H_j/H_1} \le (x+y)^{H_j/H_1}$ for all $x, y > 0$ implies that f is superadditive.

We take $\gamma \in (1, \alpha)$ such that $\beta = \gamma H_1 > 1$ and apply Lemma 18 to derive

$$\begin{aligned}\mathbb{E}\Big(\sup_{k\in R(0,\langle 2^n\rangle)}\Big|\sum_{p\in R(0,k)}\xi_p\Big|^{\gamma}\Big) &\le c_{13}\Big[\sum_{j=1}^{N}(b_j-a_j)^{H_j/H_1}\Big]^{H_1\gamma}\\ &\le c_{13}\Big[\sum_{j=1}^{N}(b_j-a_j)^{H_j}\Big]^{\gamma},\end{aligned}\tag{5.16}$$

where $c_{13} > 0$ is a finite constant independent of $[a, b]$ and n. It can be seen that (5.14) follows from (5.12), (5.16) and Hölder's inequality. This finishes the proof of Lemma 19. □

We now proceed to prove Theorem 3.

Proof of Theorem 3. We only prove (5.28), which is done by modifying the proof of Theorem 4 in [2] and by making use of Lemmas 17 and 19. The formula (1.10) can be proven using similar arguments and we leave it to the interested reader.

First we prove the lower bound in (5.28). Let $\varepsilon \in (0,\, 1)$ be given and let $I = [\varepsilon,\, 1]^N$. We will prove that for every $0 < \gamma < \min\{d,\ \sum_{\ell=1}^N \frac{1}{H_\ell}\}$, $\dim_{\rm H} X(I) \ge \gamma$ almost surely. By Frostman's theorem, it is sufficient to show that we have

$$\mathbb{E}\int_I\int_I \frac{1}{\|X(s)-X(t)\|^{\gamma}}\, dsdt < \infty,\tag{5.17}$$

where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^d$.

It is known that for any d-dimensional distribution function F in $\mathbb{R}^d$ with characteristic function φ and any $\gamma > 0$, we have

$$2^{\gamma/2-1}\Gamma\Big(\frac{\gamma}{2}\Big)\int_{\mathbb{R}^d}\|x\|^{-\gamma}\, F(dx) = (2\pi)^{-d/2}\int_0^{+\infty}u^{\gamma-1}du\int_{\mathbb{R}^d}\exp\Big(-\frac{\|x\|^2}{2}\Big)\varphi(ux)\, dx.\tag{5.18}$$

This equality can be verified by replacing φ in the right side of (5.18) by its expression as a Fourier integral and then performing a routine calculation. Applying (5.18) to the distribution of the stable random variable $\xi = \big(X(s)-X(t)\big)/\|X(s)-X(t)\|_\alpha$ and using Fubini's theorem, we obtain

$$\begin{aligned}\mathbb{E}\big(\|\xi\|^{-\gamma}\big) &\le c_{14}\int_{\mathbb{R}^d}\exp\Big(-\frac{\|x\|^2}{2}\Big)dx\int_0^{\infty}u^{\gamma-1}\exp\Big(-c_{15}|u|^{\alpha}\|x\|^{\alpha}\Big)du\\ &= c_{16}\int_{\mathbb{R}^d}\exp\Big(-\frac{\|x\|^2}{2}\Big)\|x\|^{-\gamma}dx < \infty,\end{aligned}\tag{5.19}$$

where the last integral is convergent because $\gamma < d$. Combining (5.19) with Lemma 17 yields

$$\mathbb{E}\int_I\int_I\frac{1}{\|X(s)-X(t)\|^{\gamma}}\, dsdt \le \int_I\int_I\frac{1}{\big(\sum_{\ell=1}^N|s_\ell-t_\ell|^{H_\ell}\big)^{\gamma}}\, dsdt < \infty,\tag{5.20}$$

where the finiteness of the last integral is proved in [2, p. 432]. This proves (5.17) and hence the lower bound in (5.28).

To prove the upper bound in (5.28), we use the covering argument in [12] and [2]. Since clearly $\dim_{\mathrm{H}} X\big([0,1]^N\big) \le d$ a.s. and Hausdorff dimension is σ-stable [10], it is sufficient to show that for every $\varepsilon \in (0,1)$,

$$\dim_{\mathrm{H}} X\big([\varepsilon,1]^N\big) \le \sum_{j=1}^{N} \frac{1}{H_j} \qquad \text{a.s.} \tag{5.21}$$

This will be done by using a covering argument.

For any integer $n \ge 2$, we divide $[\varepsilon,1]^N$ into m_n sub-rectangles $\{R_{n,i}\}$ with sides parallel to the axes and side-lengths n^{-1/H_j} $(j=1,\dots,N)$, respectively. Then

$$m_n \le c\, n^{\sum_{j=1}^{N} \frac{1}{H_j}} \tag{5.22}$$

and $X\big([\varepsilon,1]^N\big)$ can be covered by $X(R_{n,i})$ $(1 \le i \le m_n)$. Denote the lower-left vertex of $R_{n,i}$ by $a_{n,i}$. Note that the image $X(R_{n,i})$ is contained in a rectangle in $\mathbb{R}^d$ with sides parallel to the axes and side lengths at most $2\sup_{s\in R_{n,i}} \big|X_k(s) - X_k(a_{n,i})\big|$ $(k=1,\dots,d)$, respectively. Hence each $X(R_{n,i})$ can be covered by at most

$$\prod_{k=1}^{d} \left[\frac{2\sup_{s\in R_{n,i}} \big|X_k(s) - X_k(a_{n,i})\big|}{n^{-1}} + 1\right]$$

cubes of side-lengths n^{-1}. In this way, we have obtained a $(\sqrt{d}\, n^{-1})$-covering for $X\big([\varepsilon,1]^N\big)$.

By Lemma 19, we derive that for every $1 \le i \le m_n$ and $1 \le k \le d$,

$$\mathbb{E}\bigg(\sup_{s\in R_{n,i}} \big|X_k(s) - X_k(a_{n,i})\big|\bigg) \le c\, n^{-1}. \tag{5.23}$$

It follows from (5.22), (5.23) and the independence of $X_1,\dots,X_d$ that for any $\gamma > \sum_{j=1}^{N} \frac{1}{H_j}$, we have

$$\begin{aligned}
&\mathbb{E}\bigg\{\sum_{i=1}^{m_n} \prod_{k=1}^{d} \left[\frac{2\,\sup_{s\in R_{n,i}} \big|X_k(s) - X_k(a_{n,i})\big|}{n^{-1}} + 1\right] \Big(\sqrt{d}\, n^{-1}\Big)^{\gamma}\bigg\} \\
&\quad \le c\, n^{\sum_{j=1}^{N} \frac{1}{H_j}}\, n^{-\gamma} \to 0 \qquad \text{as } \ n \to \infty.
\end{aligned} \tag{5.24}$$

This and Fatou's lemma imply that $\dim_{\mathrm{H}} X\big([\varepsilon,1]^N\big) \le \gamma$ almost surely. By letting $\gamma \downarrow \sum_{j=1}^{N} \frac{1}{H_j}$ along rational numbers, we derive (5.21). This completes the proof of Theorem 3. □

The above method can be extended to determine the Hausdorff dimension of the image $X(E)$ for every nonrandom Borel set $E \subseteq (0,\infty)^N$, thus extending the results in Wu and Xiao [22] and Xiao [24] for anisotropic Gaussian random fields to (N,d)-LFSS.

For this purpose, let us first recall from [24] the definition of a Hausdorff-type dimension which is more convenient to capture the anisotropic nature of X.

For a fixed $(H_1, \ldots, H_N) \in (0,1)^N$, let ρ be the metric on $\mathbb{R}^N$ defined by

$$\rho(s,t) = \sum_{j=1}^{N} |s_j - t_j|^{H_j}, \qquad \forall\, s,t \in \mathbb{R}^N. \tag{5.25}$$

For any $\beta > 0$ and $E \subseteq \mathbb{R}^N$, define the β-dimensional Hausdorff measure [in the metric ρ] of E by

$$\mathcal{H}^{\beta}_{\rho}(E) = \lim_{\delta \to 0} \inf \left\{ \sum_{n=1}^{\infty} (2r_n)^{\beta} : \, E \subseteq \bigcup_{n=1}^{\infty} B_{\rho}(r_n),\, r_n \le \delta \right\}, \tag{5.26}$$

where $B_\rho(r)$ denotes a closed (or open) ball of radius r in the metric space $(\mathbb{R}^N, \rho)$. Then $\mathcal{H}^{\beta}_{\rho}$ is a metric outer measure and all Borel sets are $\mathcal{H}^{\beta}_{\rho}$-measurable. The corresponding Hausdorff dimension of E is defined by

$$\dim^{\rho}_{\mathrm{H}} E = \inf \big\{ \beta > 0 : \, \mathcal{H}^{\beta}_{\rho}(E) = 0 \big\}. \tag{5.27}$$

We refer to [24] for more information on the history and basic properties of $\mathcal{H}^{\beta}_{\rho}$ and $\dim^{\rho}_{\mathrm{H}}$.

Theorem 20. *Let the assumption (1.5) hold. Then, for every nonrandom Borel set $E \subseteq (0,\infty)^N$,*

$$\dim_{\mathrm{H}} X(E) = \min \big\{ d;\ \dim^{\rho}_{\mathrm{H}} E \big\} \qquad a.s. \tag{5.28}$$

Proof. The proof is a modification of those of Theorem 3 above and Theorem 6.11 in [24]. For any $\gamma > \dim^{\rho}_{\mathrm{H}} E$, there is a covering $\{B_\rho(r_n), n \ge 1\}$ of E such that $\sum_{n=1}^{\infty} (2r_n)^{\gamma} \le 1$. Note that $X(E) \subseteq \cup_{n=1}^{\infty} X\big(B_\rho(r_n)\big)$ and we can cover each $X\big(B_\rho(r_n)\big)$ as in the proof of Theorem 3. The same argument shows that $\dim_{\mathrm{H}} X(E) \le \gamma$ almost surely, which yields the desired upper bound for $\dim_{\mathrm{H}} X(E)$.

By using the Frostman lemma for $\mathcal{H}^{\beta}_{\rho}$ (Lemma 6.10 in [24]) and the capacity argument in the proof of Theorem 3, one can show $\dim_{\mathrm{H}} X(E) \ge \min \big\{ d;\ \dim^{\rho}_{\mathrm{H}} E \big\}$ almost surely. We omit the details. □

Appendix A. Technical lemmas

The following lemma allows to control the growth of an arbitrary sequence of strictly α-stable random variables having the same scale parameter.

Lemma 21. *Let $\{\epsilon_\lambda,\, \lambda \in \mathbb{Z}^d\}$ be an arbitrary sequence of strictly α-stable random variables having the same scale parameter. Then, there exists an event Ω_1^* of probability 1, such that for any $\eta > 0$ and any $\omega \in \Omega_1^*$,*

$$|\epsilon_\lambda(\omega)| \le C(\omega) \prod_{l=1}^{d} (3 + |\lambda_l|)^{1/\alpha + \eta}, \tag{A.1}$$

where $C > 0$ is an almost surely finite random variable, only depending on η.

Proof This lemma simply follows from the fact that for any $\nu \in ((1/\alpha+\eta)^{-1}, \alpha)$ one has

$$\mathbb{E}\left(\sup_{\lambda\in\mathbb{Z}^d} \frac{|\epsilon_\lambda|^\nu}{\prod_{j=1}^d (3+|\lambda_j|)^{\nu(1/\alpha+\eta)}}\right) \le c \sum_{\lambda\in\mathbb{Z}^d} \prod_{j=1}^d (3+|\lambda_j|)^{-\nu(1/\alpha+\eta)} < \infty.$$

□

Lemma 22. *Let $\alpha \in (0,2)$. There exists a constant c_{17} depending only on α such that for any strictly α-stable random variable Z with scale parameter $\|Z\|_\alpha > 0$ and skewness parameter $\beta \in [-1,1]$ and all $t \ge \|Z\|_\alpha$,*

$$c_{17}^{-1} \|Z\|_\alpha^\alpha t^{-\alpha} \le \mathbb{P}(|Z|>t) \le c_{17} \|Z\|_\alpha^\alpha t^{-\alpha}. \tag{A.2}$$

Let $N \ge 1$. Suppose now that $\{Z_{j,k},\ j_\ell \ge 1,\ k_\ell \ge 2 \ \text{ for } \ \ell = 1,\dots,N\}$ is a sequence of strictly α-stable random variables such that

(i) For all $j \in (\mathbb{N}\setminus\{0\})^N$, $\{Z_{j,k},\ k_\ell \ge 2 \ \text{ for } \ \ell=1,\dots,N\}$ are independent;

(ii) For all $j \in (\mathbb{N}\setminus\{0\})^N$ and $k \in (\mathbb{N}\setminus\{0,1\})^N$, $\|Z_{j,k}\|_\alpha \le 1$.

Then, with probability 1, one has, for any $\gamma > 0$,

$$\sup\left\{|Z_{j,k}| \prod_{\ell=1}^N j_\ell^{-1/\alpha-\gamma} k_\ell^{-1/\alpha} \log^{-1/\alpha-\gamma}(k_\ell) \ :\ j_\ell \ge 1,\ k_\ell \ge 2 \ \text{ for } \ \ell=1,\dots,N\right\} < \infty\,. \tag{A.3}$$

Proof Relation (A.2) follows from Property 1.2.15 in [20]. Let us now show (A.3) for $N=1$, the proof for $N>1$ is similar. By using (A.2), we obtain, for all $j \ge 1$ and $n \ge 1$,

$$\mathbb{P}\big(\max\{|Z_{j,2}|,\dots,|Z_{j,n}|\} > u_{j,n}\big) \le 1-(1-c_{17}\, u_{j,n}^{-\alpha})^n,$$

where $u_{j,n} = j^{1/\alpha+\gamma} n^{1/\alpha} \log^{1/\alpha+\gamma} n$. Defining $n_m = [\exp(m)]$, we obtain

$$\mathbb{E}\left[\sum_{j\ge1}\sum_{m\ge1} \mathbb{1}_{\max\{|Z_{j,2}|,\dots,|Z_{j,n_m}|\}>u_{j,n_m}}\right] = \sum_{j\ge1}\sum_{m\ge1} \mathbb{P}\Big(\max\{|Z_{j,2}|,\dots,|Z_{j,n_m}|\} > u_{j,n_m}\Big) < \infty\,.$$

Thus the random variable $\sum_{j\ge1}\sum_{m\ge1} \mathbb{1}_{\max\{|Z_{j,2}|,\dots,|Z_{j,n_m}|\}>u_{j,n_m}}$ is a.s. finite. As a consequence there exists an a.s. finite positive random variable C such that

$$\max\{|Z_{j,2}|,\dots,|Z_{j,n_m}|\} \le C\, u_{j,n_m} \quad \text{for all} \quad j\ge1,\ m\ge1\,.$$

Let $m(k)$ is the unique integer satisfying $n_{m(k)} \le k < n_{m(k)+1}$. Thus for all $j \ge 1, k\ge 2$, we have

$$|Z_{j,k}| \le C\, u_{j,n_{m(k)+1}} = C\, j^{1/\alpha+\gamma}\, n_{m(k)+1}^{1/\alpha}\, \log^{1/\alpha+\gamma}(n_{m(k)+1})\,,$$

Observe now that we have, for all $k \ge 2$,

$$n_{m(k)+1} \le \exp(m(k)+1) \le \mathrm{e}\,(n_{m(k)}+1) \le \mathrm{e}\,(k+1)\,.$$

Relation (A.3) follows from the last two displays. □

Lemma 23. *For any $M > 0$, $\eta > 0$ small enough, $\delta \in (1/\alpha + \eta, 1)$, $\beta \in [0, \delta - 1/\alpha - \eta)$, any well-localized function ϕ and $x, y \in \mathbb{R}$, let $A_n(x,y) := A_n(x,y; M, \phi, \delta, \beta, \eta)$ be the quantity defined as*

$$A_n(x,y) = \sum_{|J|\le n} \sum_{|K|>M2^{n+1}} 2^{-J\delta} \frac{|\phi(2^J x - K) - \phi(2^J y - K)|}{|x-y|^\beta} (3+|J|)^{1/\alpha+\eta}(3+|K|)^{1/\alpha+\eta} \quad \text{(A.4)}$$

and let $B_n(x,y) := B_n(x,y;\phi;\delta,\beta,\eta)$ be the quantity defined as

$$B_n(x,y) = \sum_{|J|\ge n+1} \sum_{K\in\mathbb{Z}} 2^{-J\delta} \frac{|\phi(2^J x - K) - \phi(2^J y - K)|}{|x-y|^\beta} (3+|J|)^{1/\alpha+\eta}(3+|K|)^{1/\alpha+\eta}, \quad \text{(A.5)}$$

with the convention that $A_n(x,x) = B_n(x,x) = 0$ for any $x \in \mathbb{R}$. These quantities converge to 0, uniformly in $x, y \in [-M, M]$, as n goes to infinity.

Proof. Let $x, y \in [-M, M]$ and $J_0 \ge -\log_2(2M)$ be the unique integer such that

$$2^{-J_0-1} < |x-y| \le 2^{-J_0}. \quad \text{(A.6)}$$

Let us first prove that $A_n(x,y)$ converges to 0, uniformly in x, y as n goes to infinity. From now on we suppose that J is an arbitrary integer satisfying $|J| \le n$. We need to derive suitable upper bounds for the quantity

$$A_n^{(J)}(x,y) = \sum_{|K|>M2^{n+1}} \frac{|\phi(2^J x - K) - \phi(2^J y - K)|}{|x-y|^\beta}(3+|K|)^{1/\alpha+\eta}. \quad \text{(A.7)}$$

For this purpose, we consider two cases $J \le J_0$ and $J \ge J_0 + 1$ separately. First we suppose that

$$J \le J_0. \quad \text{(A.8)}$$

Using the Mean Value Theorem, (2.2), (A.6) and (A.8) one obtains that

$$\begin{aligned} |\phi(2^J x - K) - \phi(2^J y - K)| &\le c\, 2^J |x-y| \sup_{u\in I}(3+|u|)^{-2} \\ &\le c\, 2^J |x-y| (2 + |2^J x - K|)^{-2}, \end{aligned}$$

where I denotes the compact interval with end-points $2^J x - K$ and $2^J y - K$, whose length is at most 1 by (A.6) and (A.8). Next the last inequality and (A.7) entail that

$$A_n^{(J)}(x,y) \le c\, 2^J |x-y|^{1-\beta} \sum_{|K|>M2^{n+1}} \frac{(3+|K|)^{1/\alpha+\eta}}{(2+|2^J x - K|)^2}. \quad \text{(A.9)}$$

On the other hand, using that $|x| \le M$ and $|J| \le n$, for all $|K| > M2^{n+1}$, one gets

$$\frac{(3+|K|)^{1/\alpha+\eta}}{(2+|2^J x - K|)^2} \le \frac{(3+|K|)^{1/\alpha+\eta}}{(2+|K| - M2^n)^2} \le c\left(1+|K|\right)^{-(2-1/\alpha-\eta)}. \quad \text{(A.10)}$$

Putting together (A.9), (A.10) and (A.6), one obtains that

$$A_n^{(J)}(x,y) \le c\, 2^{J_0(\beta-1)} 2^{J - n(1-1/\alpha-\eta)}. \quad \text{(A.11)}$$

Let us now study the second case where

$$J \geq J_0 + 1. \tag{A.12}$$

It follows from (A.6), (A.12) and (A.7) that

$$A_n^{(J)}(x,y) \leq 2^{J\beta} \sum_{|K|>M2^{n+1}} \Big\{ |\phi(2^J x - K)| + |\phi(2^J x - K)| \Big\} (3+|K|)^{1/\alpha+\eta}. \tag{A.13}$$

On the other hand, using (2.2) and the fact that $|J| \leq n$ one has, for any real $u \in [-M, M]$ and any $K \in \mathbb{Z}$ satisfying $|K| > M2^{n+1}$,

$$\big|\phi(2^J u - K)\big| \leq c\,(3 + |2^J u - K|)^{-2} \leq c\,(3+|K| - M2^n)^{-2} \leq c_{18}\,(3+|K|)^{-2}. \tag{A.14}$$

Combining (A.13) with (A.14) one gets that

$$A_n^{(J)}(x,y) \leq c_{19}\, 2^{J\beta - n(1-1/\alpha-\eta)}\,. \tag{A.15}$$

It follows from (A.4), (A.7), (A.11) and (A.15) that

$$\begin{aligned} A_n(x,y) &\leq c\, 2^{-n(1-1/\alpha-\eta)} \left[2^{J_0(\beta-1)} \sum_{J=-\infty}^{J_0} 2^{J(1-\delta)} (3+|J|)^{1/\alpha+\eta} + \sum_{J=J_0+1}^{\infty} 2^{J(\beta-\delta)} (3+|J|)^{1/\alpha+\eta} \right] \\ &\leq c\, 2^{-n(1-1/\alpha-\eta)} 2^{J_0(\beta-\delta)} (3+|J_0|)^{1/\alpha+\eta} \\ &\leq c_{20}\, 2^{-n(1-1/\alpha-\eta)}\,, \end{aligned}$$

where we used Lemma 26 to bound the series and then that $2^{-J_0} \leq 2M$, by (A.6). Since c_{20} does not depend on (x,y), the last inequality proves that $A_n(x,y)$ converges to 0, uniformly in $x, y \in [-M, M]$ as n goes to infinity.

Let us now prove that $B_n(x,y)$ converges to 0, uniformly in x, y as n goes to infinity. In all the sequel J denotes an arbitrary integer satisfying $|J| \geq n+1$. First, we derive a suitable upper bound for the quantity

$$B_n^{(J)}(x,y) = \sum_{K\in\mathbb{Z}} \frac{|\phi(2^J x - K) - \phi(2^J y - K)|}{|x-y|^\beta} (3+|K|)^{1/\alpha+\eta}. \tag{A.16}$$

As above, we distinguish two cases: $J \leq J_0$ and $J \geq J_0 + 1$. First we suppose that (A.8) is verified. As in (A.9), we have $B_n^{(J)}(x,y) \leq c\, 2^J |x-y|^{1-\beta} \sum_{K\in\mathbb{Z}} (3+|K|)^{1/\alpha+\eta} (2 + |2^J x - K|)^{-2}$. Next, using (A.6) and Lemma 25 and the fact that $|x| \leq M$, one obtains that

$$B_n^{(J)}(x,y) \leq c\, 2^{J + J_0(\beta-1)} (1 + 2^J)^{1/\alpha+\eta}. \tag{A.17}$$

Now let us suppose that (A.12) is verified. By using this relation, (A.6), the triangle inequality, (2.2), Lemma 25 and the fact that $x, y \in [-M, M]$, one gets

$$\begin{aligned} B_n^{(J)}(x,y) &\leq 2^{J\beta} \sum_{K\in\mathbb{Z}} \Big\{ |\phi(2^J x - K)| + |\phi(2^J y - K)| \Big\} (3+|K|)^{1/\alpha+\eta} \\ &\leq c\, 2^{J\beta} \sum_{K\in\mathbb{Z}} \Big\{ (3+|2^J x - K|)^{-2} + (3+|2^J y - K|)^{-2} \Big\} (3+|K|)^{1/\alpha+\eta} \\ &\leq c\, 2^{J\beta} \Big\{ (1+2^J|x|)^{1/\alpha+\eta} + (1+2^J|y|)^{1/\alpha+\eta} \Big\} \\ &\leq c\, 2^{J(\beta+1/\alpha+\eta)} \end{aligned} \tag{A.18}$$

Since $2^{-J_0} \leq M$, for all $n \geq \log_2(2M)$, we have $-n \leq J_0$, and thus, by (A.17),

$$\begin{aligned} \sum_{J\leq -n} 2^{-J\delta}(3+|J|)^{1/\alpha+\eta} B_n^{(J)}(x,y) &\leq c\, 2^{J_0(\beta-1)} \sum_{J\leq -n} 2^{J(1-\delta)}(3+|J|)^{1/\alpha+\eta} \\ &\leq c\, 2^{n(\delta-1)}(1+n)^{1/\alpha+\eta}\,, \end{aligned} \tag{A.19}$$

where we used Lemma 26 and $2^{-J_0} \leq M$. Applying Lemma (26) with (A.17) and (A.18) yields

$$\sum_{J\in\mathbb{Z}} 2^{-J\delta}(3+|J|)^{1/\alpha+\eta} B_n^{(J)}(x,y) \leq c\, 2^{J_0(\beta+1/a+\eta-\delta)}(3+|J_0|)^{1/\alpha+\eta}\,, \tag{A.20}$$

and for any $n \geq J_0$,

$$\sum_{J\geq n} 2^{-J\delta}(3+|J|)^{1/\alpha+\eta} B_n^{(J)}(x,y) \leq c\, 2^{n(\beta+1/a+\eta-\delta)}(3+n)^{1/\alpha+\eta}\,. \tag{A.21}$$

Since $\beta + 1/a + \eta - \delta < 0$, the function $t \mapsto 2^{t(\beta+1/a+\eta-\delta)}(3+t)^{1/\alpha+\eta}$ is decreasing for t large enough, and hence for n large enough, either $n \geq J_0$ and we may apply (A.21), or $n \leq J_0$ and we may apply (A.20) whose right-hand side is smaller than the right-hand side of (A.21). Hence (A.21) holds for all n large enough independently of J_0. This, with (A.19), shows that $B_n(x,y)$ converges uniformly in x, y, as n goes to infinity. $\square$

Lemma 24. *Let ϕ be a well-localized function i.e. a function satisfying the condition (2.2). For any $\delta \in (0,1)$, $\gamma \in (0,\delta)$ and $\eta \geq 0$, define*

$$S_{\delta,\gamma,\eta}(x,y;\phi) = \sum_{(J,K)\in\mathbb{Z}^2} 2^{-J\delta}|\phi(2^J x - K) - \phi(2^J y - K)|(3+|J|)^{\gamma+\eta}(3+|K|)^{\gamma}\log^{\gamma+\eta}(2+|K|) \tag{A.22}$$

and

$$T_{\delta,\gamma,\eta}(x;\phi) = \sum_{(J,K)\in\mathbb{Z}^2} 2^{-J\delta}\left|\phi(2^J x - K) - \phi(-K)\right|(3+|J|)^{\gamma+\eta}(3+|K|)^{\gamma}\log^{\gamma+\eta}(2+|K|). \tag{A.23}$$

Then, there exists a constant $c>0$, only depending on δ, γ and ϕ, such that the inequalities

$$S_{\delta,\gamma,\eta}(x,y;\phi) \le c\,|y-x|^{\delta-\gamma}\big[|y-x|^{\gamma}+|x|^{\gamma}+|y|^{\gamma}\big] \\ \times \big(1+\big|\log|y-x|\big|\big)^{2\gamma+2\eta}\left\{\log^{\gamma+\eta}(2+|x|)+\log^{\gamma+\eta}(2+|y|)\right\} \quad \text{(A.24)}$$

and

$$T_{\delta,\gamma,\eta}(x;\phi) \le c\big(1+\big|\log|x|\big|\big)^{\gamma+\eta}\,|x|^{\delta} \quad \text{(A.25)}$$

hold for all $x,y\in\mathbb{R}$ (with the convention that $0^a\times\log^b 0=0$ for all $a,b>0$).

Proof. We only prove (A.24), the proof of (A.25) is similar. By (2.2), there is a constant $c>0$ such that, for all $J,K\in\mathbb{Z}$ and $x,y\in\mathbb{R}$,

$$\big|\phi(2^Jx-K)-\phi(2^Jy-K)\big| \le c\,\left\{(2+|2^Jx-K|)^{-2}+(2+|2^Jy-K|)^{-2}\right\}. \quad \text{(A.26)}$$

The quantity $|\phi(2^Jx-K)-\phi(2^Jy-K)|$ can be bounded more sharply when the condition $2^J|x-y|\le 1$ holds, namely by using (2.2) and the Mean Value Theorem one obtains that

$$\begin{aligned} |\phi(2^Jx-K)-\phi(2^Jy-K)| &\le c\,2^J|x-y|\sup_{u\in I}(3+|2^Ju-K|)^{-2} \\ &\le c\,2^J|x-y|(2+|2^Jx-K|)^{-2}\,, \end{aligned} \quad \text{(A.27)}$$

where I denotes the compact interval whose end-points are x and y. From now on we will assume that $x\ne y$ (Relation (A.24) is trivial otherwise) and let $J_0\in\mathbb{Z}$ be the unique integer satisfying

$$1/2<2^{J_0}|y-x|\le 1. \quad \text{(A.28)}$$

The inequalities (A.26) and (A.27) entail that

$$S_{\delta,\gamma,\eta}(x,y;\phi)\le c\left(A_{J_0}|x-y|+B_{J_0}\right), \quad \text{(A.29)}$$

where

$$A_{J_0}=\sum_{J\le J_0}\sum_{K\in\mathbb{Z}}2^{J(1-\delta)}(2+|2^Jx-K|)^{-2}(3+|J|)^{\gamma+\eta}(3+|K|)^{\gamma}\log^{\gamma+\eta}(2+|K|)$$

and

$$B_{J_0}=\sum_{J>J_0}\sum_{K\in\mathbb{Z}}2^{-J\delta}\Big\{(2+|2^Jx-K|)^{-2}+(2+|2^Jy-K|)^{-2}\Big\}(3+|J|)^{\gamma+\eta}(3+|K|)^{\gamma}\log^{\gamma+\eta}(2+|K|).$$

Lemma 25 and Lemma 26 yield

$$A_{J_0}\le c\,2^{J_0(1-\delta)}\,(1+|x|^{\gamma}2^{J_0\gamma})\,(1+|J_0|)^{2\gamma+2\eta}\,\log^{\gamma+\eta}(2+|x|)$$

and, since $\gamma-\delta<0$,

$$B_{J_0}\le c\,2^{-J_0\delta}\,(1+(|x|^{\gamma}+|y|^{\gamma})2^{J_0\gamma})\,(1+|J_0|)^{2\gamma+2\eta}\left\{\log^{\gamma+\eta}(2+|x|)+\log^{\gamma+\eta}(2+|y|)\right\}.$$

Inserting these two bounds into (A.29) and using (A.28), we get (A.24) and the proof is finished. □

Lemma 25. *For any $\gamma \in [0,1)$ and $\eta \geq 0$, there exists a constant $c > 0$ such that, for all $u \in \mathbb{R}$,*

$$\sum_{k\in\mathbb{Z}} (2+|u-k|)^{-2}(1+|k|)^{\gamma} \log^{\eta}(2+|k|) \leq c\,(1+|u|)^{\gamma}\ \log^{\eta}(2+|u|)\ .$$

Proof Put $k' = [u] - k$, where $[u]$ is the integer part of u. Hence

$$\begin{aligned}\sum_{k\in\mathbb{Z}} \frac{(1+|k|)^{\gamma}\log^{\eta}(2+|k|)}{(2+|u-k|)^2} &= \sum_{k'\in\mathbb{Z}} (2+|u-[u]+k'|)^{-2}(1+|[u]-k'|)^{\gamma}\log^{\eta}(2+|[u]-k'|) \\ &\leq \sum_{k'\in\mathbb{Z}} (1+|k'|)^{-2}(2+|u|+|k'|)^{\gamma}\log^{\eta}(2+|u|+|k'|)\ .\end{aligned}$$

The result then follows by observing that $(2+|u|+|k'|)^{\gamma} \leq (1+|u|)^{\gamma}(2+|k'|)^{\gamma}$, $\log^{\eta}(2+|u|+|k'|) \leq c\ \log^{\eta}(2+|u|)\log^{\eta}(2+|k'|)\}$ and $\gamma - 2 < -1$. □

Lemma 26. *Let $\theta \neq 0$ and $\gamma \in \mathbb{R}$. Set $c := \sum_{n\geq 0} 2^{-|\theta|n}(1+n)^{|\gamma|} < \infty$. Then for any $n_0 < n_1$ in $\{0, \pm1, \pm2, \ldots, \pm\infty\}$,*

$$\sum_{n=n_0}^{n_1} 2^{n\theta}(1+|n|)^{\gamma} \leq c \begin{cases} 2^{n_0\theta}(1+|n_0|)^{\gamma} & \text{if } \theta < 0 \\ 2^{n_1\theta}(1+|n_1|)^{\gamma} & \text{if } \theta > 0. \end{cases} \tag{A.30}$$

Proof. Take *e.g.* $\theta < 0$ and write

$$\sum_{n=n_0}^{n_1} 2^{n\theta}(1+|n|)^{\gamma} \leq 2^{n_0\theta}(1+|n_0|)^{\gamma} \sum_{m\geq 0} 2^{m\theta}\left(\frac{1+|m+n_0|}{1+|n_0|}\right)^{\gamma}\ .$$

Now observe that

$$\frac{1}{1+|m|} \leq \frac{1+|m+n_0|}{1+|n_0|} \leq 1+|m|$$

so that $\sup_{n_0}\sum_{m\geq 0} 2^{m\theta}\left(\frac{1+|m+n_0|}{1+|n_0|}\right)^{\gamma} < \infty$ for any $\gamma \in \mathbb{R}$.□

References

[1] A. Ayache. Hausdorff dimension of the graph of the fractional Brownian sheet. *Rev. Mat. Iberoamericana*, 20(2):395–412, 2004. ISSN 0213-2230.

[2] A. Ayache and Y. Xiao. Asymptotic properties and hausdorff dimensions of fractional brownian sheets. *J. Fourier Anal. Appl.*, 11:407–439, 2005.

[3] A. Ayache, F. Roueff, and Y. Xiao. Local and asymptotic properties of linear fractional stable sheets. *C. R. Acad. Sci. Paris, Ser. I.*, 344(6):389–394, 2007.

[4] A. Benassi and D. Roux. Elliptic self-similar stochastic processes. *Rev. Mat. Iberoamericana*, 19(3):767–796, 2003. ISSN 0213-2230.

[5] D.A. Benson, M.M. Meerschaert, and B. Baeumer. Aquifer operator–scaling and the effect on solute mixing and dispersion. *Water Resour. Res.*, 42, 2006.

[6] H. Biermé, M.M. Meerschaert and H.-P. Scheffler. Operator scaling stable random fields. *Stoch. Process. Appl.*, 117:312–332, 2007.

[7] A. Bonami and A. Estrade. Anisotropic analysis of some Gaussian models. *J. Fourier Anal. Appl.*, 9(3):215–236, 2003. ISSN 1069-5869.

[8] I. Daubechies. *Ten Lectures on Wavelets.* SIAM Philadelphia, 1992. CBMS-NSF series, Volume 61.

[9] W. Ehm. Sample function properties of multi-parameter stable processes. *Z. Wahrsch. verw Gebiete*, 56(2):195–228, 1981.

[10] K. J. Falconer. *Fractal Geometry: Mathematical Foundations and Applications.* John Wiley and Sons, New York, 1990.

[11] N. Kôno and M. Maejima. Hölder continuity of sample paths of some self-similar stable processes. *Tokyo J. Math.*, 14:93–100, 1991.

[12] H. Lin and Y. Xiao. Dimension properties of the sample paths of self-similar processes. *Acta Math. Sinica*, 10:289–300, 1994.

[13] M. Maejima. A self-similar process with nowhere bounded sample paths. *Z. Wahrsch. Verw. Gebiete*, 65:115–119, 1983.

[14] J. D. Mason and Y. Xiao. Sample path properties of operator-self-similar Gaussian random fields. *Teor. Veroyatnost. i Primenen.*, 46(1):94–116, 2001. ISSN 0040-361X.

[15] Y. Meyer. *Ondelettes et opérateurs*, volume 1. Hermann, Paris, 1990.

[16] Y. Meyer. *Wavelets and Operators*, volume 1. Cambridge University Press, 1992.

[17] F. Moricz. A general moment inequality for the maximum of the rectangular partial sums of multiple series. *Acta Math. Hung.*, 41(3–4):337–346, 1983.

[18] J. Rosinski and G. Samorodnitsky. Distributions of subadditive functionals of sample paths of infinitely divisible processes. *Ann. Probab.*, 21:996–1104, 1993.

[19] S.G. Samko, A.A. Kilbas, and O.I. Marichev. *Fractional integrals and derivatives.* Gordon and Breach Science Publishers, Yverdon, 1993. ISBN 2-88124-864-0.

[20] G. Samorodnitsky and M. S. Taqqu. *Stable non-Gaussian processes: stochastic models with infinite variance.* Chapman and Hall, 1994.

[21] K. Takashima. Sample path properties of ergodic self-similar processes. *Osaka J. Math.*, 26 (1):159–189, 1989. ISSN 0030-6126.

[22] D. Wu and Y. Xiao. Geometric properties of fractional Brownian sheets. *J. Fourier Anal. Appl.*, 13(1):1–37, 2007. ISSN 1069-5869.

[23] Y. Xiao. Properties of local nondeterminism of gaussian and stable random fields and their applications. *Ann. Fac. Sci. Toulouse Math.*, XV:157–19, 2006.

[24] Y. Xiao. Sample path properties of anisotropic Gaussian random fields. *Submitted*, 2007.

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