Uniform Modulus of Continuity of Random Fields

Yimin Xiao *
Michigan State University

June 2, 2009

Abstract

A sufficient condition for the uniform modulus of continuity of a random field $X = \{X(t), t \in \mathbb{R}^N\}$ is provided. The result is applicable to random fields with heavy-tailed distribution such as stable random fields.

Running head: Uniform Modulus of Continuity of Random Fields

2000 AMS Classification numbers: 60G60, 60G52, 60G17, 60G18.

Key words: Random fields, uniform modulus of continuity, chaining method, maximal moment index, stable random fields, harmonizable fractional stable fields.

1 Introduction

Sample path continuity and Hölder regularity of stochastic processes have been studied by many authors. The celebrated theorems of Kolmorogov [cf. Khoshnevisan (2002)] and Garsia (1972) provide uniform modulus of continuity for rather general stochastic processes and random fields. For Gaussian processes, a powerful chaining argument leads to much deeper results; see the recent books of Talagrand (2006), Marcus and Rosen (2006) and Adler and Taylor (2007). Some of the results on uniform modulus of continuity have been extended to non-Gaussian processes provided the tail probabilities of the increments of the process decay fast enough [see Csáki and Csörgő (1992), Kwapień and Rosiński (2004)].

In recent years, there has been increasing interest in sample path continuity of stochastic processes with heavy-tailed distributions such as certain infinitely divisible processes including particularly stable processes. See Samorodnitsky and Taqqu (1994), Marcus and Rosiński (2005) and the references therein for more information. Since many of such processes do not have finite second moment, the aforementioned results on the uniform modulus of continuity do not apply. There have been only a few results on uniform modulus of continuity for stable processes; see Bernard (1970), Takashima (1989), Kôno and Maejima (1991a, 1991b), Biermé and Lacaux (2009), Ayache, Roueff and Xiao (2007, 2009).

*Research supported in part by the NSF grant DMS-0706728.

The objective of this paper is to study uniform modulus of continuity of a real-valued random field $X = \{X(t), t \in \mathbb{R}^N\}$ with heavy-tailed distributions. In Section 2, we modify the chaining argument to prove a general result on uniform modulus of continuity for X. Compared with Kolmogorov's continuity theorem, Dudley's entropy theorem or the theorems in Csáki and Csörgő (1992), our result does not assume exponential-type tail probabilities nor higher moments, hence it can be applied to wider classes of random fields. In Section 3, we introduce the notion of maximal moment index for a sequence of random variables and study its basic properties. We show that the maximal moment index is closely related to the uniform Hölder exponent of the random field X. In Section 4, we apply the results in Sections 2 and 3 to stable random fields.

Throughout this paper, we will use K to denote an unspecified positive constant which may differ in each occurrence. Some specific constants will be denoted by $K_1, K_2, \ldots$.

Acknowledgment The author thanks Professors Michael B. Marcus and Jan Rosiński for stimulating discussions. He is grateful to the anonymous referee for his/her comments and suggestions which have led to significant improvement of the manuscript.

This paper was finished during the authors visit to the Statistical & Applied Mathematical Sciences Institute (SAMSI). He thanks the staff of SAMSI for their support and the good working conditions.

2 A general result on uniform modulus of continuity

Let $X = \{X(t), t \in T\}$ be a real-valued random field on a compact metric space (T, d). In this section we establish a general result on uniform modulus of continuity of X. Similar to the proofs of Kolmogorov's continuity theorem and Dudley's entropy theorem [see Khoshnevisan (2002), Talagrand (2006), Marcus and Rosen (2006) or Adler and Taylor (2007)], our method is also based on the powerful chaining argument. However, in order to be able to apply the method to random fields with heavy-tailed distributions such as stable random fields, we will make two modifications. One is that we will rely more on the geometry of the parameter set T and choose approximating chains more carefully so that, at the n-th step of the chain, there are $\kappa_0 N_n$ pairs of random variables, where N_n is the 2^{-n}-entropy number of T and κ_0 is a constant, instead of $N_n N_{n-1}$ pairs as in Talagrand (2006) or Adler and Taylor (2007). This is important for deriving correct rate function for the modulus of continuity for processes with heavy-tailed distributions. The second modification is, instead of trying to control the tail probabilities of the increment of $X(t)$ over every step of the chain as in the aforementioned references, we control the γ-th moments of the maximum increments over the steps. Here $\gamma \in (0, 1]$ is a constant. This latter argument extends an idea of Kôno and Maejima (1991b) who studied the modulus of continuity of an H-self-similar α-stable process with stationary increments and satisfying $\alpha \in (1, 2)$ and $\frac{1}{\alpha} < H < 1$.

We assume that there is a sequence $\{D_n, n \geq 1\}$ of finite subsets of T satisfying the following conditions:

(i). There exists a positive integer κ_0 depending only on (T, d) such that for every $\tau_{n+1} \in D_{n+1}$, the number of points $\tau_n \in D_n$ satisfying $d(\tau_{n+1}, \tau_n) \leq 2^{-n}$ is at most κ_0. Such a point τ_n is called a D_n-neighbor of τ_{n+1}. We denote by $O_n(\tau_{n+1})$ the set of all D_n-neighbors of τ_{n+1}.

(ii). [Chaining property] For every $s, t \in T$ with $d(s,t) \le 2^{-n}$, there exist two sequences $\{\tau_p(s), p \ge n\}$ and $\{\tau_p(t), p \ge n\}$ such that $\tau_n(s) = \tau_n(t)$ (i.e., the two chains have the same starting point) and, for every $p \ge n$, both $\tau_p(s)$ and $\tau_p(t)$ are in D_p,

$$d(\tau_p(t),\, t) \le 2^{-p}, \qquad d(\tau_p(s),\, s) \le 2^{-p}$$

and

$$\tau_p(t) \in O_p\big(\tau_{p+1}(t)\big), \qquad \tau_p(s) \in O_p\big(\tau_{p+1}(s)\big).$$

If, in addition, $s \in D := \bigcup_{k=1}^{\infty} D_k$ (or $t \in D$), then there exists an integer $q \ge n$ such that $\tau_p(s) = s$ (or $\tau_p(t) = t$) for all $p \ge q$.

Condition (ii) implies that, for every n, T is covered by d-balls with radius 2^{-n} and centers in D_n; and the collection of all the points in $\{D_n, n \ge 1\}$ is dense in T, i.e., $\overline{\bigcup_{n=1}^{\infty} D_n} = T$.

When T is a closed interval in $\mathbb{R}^N$ and d is a metric which is equivalent to the ℓ^∞ metric $|s-t|_\infty = \max_{1\le j\le N} |s_j - t_j|$, it is convenient to choose a sequence $\{D_n, n \ge 1\}$ satisfying the above conditions. For example, if $T = [0,1]^N$ and d is the ℓ^∞ metric, then, for any integer $n \ge 1$, one can take D_n to be the collection of all vertices of dyadic cubes of order n which are contained in T. Note that in this case, $\{D_n, n \ge 1\}$ is increasing (i.e., $D_n \subset D_{n+1}$ for every $n \ge 1$) and for every $\tau_n \in D_n$, the number of D_{n-1}-neighbors of τ_n is at most $\kappa_0 := 3^N - 1$.

The following is the main result of this section.

Theorem 2.1 *Let $X = \{X(t), t \in T\}$ be a real-valued random field on a compact metric space (T, d) and let $\{D_n, n \ge 1\}$ be a sequence of finite subsets of T satisfying Conditions (i) and (ii). Suppose $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing continuous function such that $\sigma(2h) \le K_1 \sigma(h)$ for some constant $K_1 > 0$. If there exist constants $K_2 > 0$, $\gamma \in (0,1]$ and $\varepsilon_0 > 0$ such that $\lim_{h\to 0} \sigma(h)\big(\log(1/h)\big)^{(1+\varepsilon_0)/\gamma} = 0$ and*

$$\sum_{p=n}^{\infty} \mathbb{E}\bigg(\max_{\tau_p \in D_p} \max_{\tau'_{p-1} \in O_{p-1}(\tau_p)} \big|X(\tau_p) - X(\tau'_{p-1})\big|^\gamma \bigg) \le K_2\, \sigma(2^{-n})^\gamma. \tag{2.1}$$

Then X has a continuous version, still denoted by X, such that for all $\varepsilon > 0$

$$\lim_{h\to 0+} \frac{\sup_{t\in T} \sup_{d(s,t)\le h} \big|X(t) - X(s)\big|}{\sigma(h)\, \big(\log(1/h)\big)^{(1+\varepsilon)/\gamma}} = 0, \qquad a.s. \tag{2.2}$$

Proof Given any $s, t \in D = \bigcup_{k=1}^{\infty} D_k$ with $d(s,t) \le 2^{-n}$, let $\{\tau_p(s), n \le p \le q\}$ and $\{\tau_p(t), n \le p \le q\}$ be the two approximating chains to s and t given by Condition (ii). The triangle inequality and the fact $\tau_n(s) = \tau_n(t)$ imply

$$\begin{aligned} \big|X(t) - X(s)\big| &\le \sum_{p=n+1}^{q} \big|X(\tau_p(t)) - X(\tau_{p-1}(t))\big| + \sum_{p=n+1}^{q} \big|X(\tau_p(s)) - X(\tau_{p-1}(s))\big| \\ &\le 2 \sum_{p=n+1}^{\infty} \max_{\tau_p \in D_p} \max_{\tau'_{p-1} \in O_{p-1}(\tau_n)} \big|X(\tau_p) - X(\tau'_{p-1})\big|. \end{aligned} \tag{2.3}$$

It is helpful to note that, for each $p \ge n+1$, the maximum in (2.3) is taken over at most $\kappa_0\, N_p$ increments, where N_p denotes the cardinality of D_p.

For any integer $n \geq 1$, let

$$\Delta_n = \sup_{s,t \in D} \sup_{d(s,t) \leq 2^{-n}} \big|X(t) - X(s)\big|.$$

It follows from (2.3), the elementary inequality $|a+b|^\gamma \leq |a|^\gamma + |b|^\gamma$ ($\gamma \in (0,1]$ and $a, b \in \mathbb{R}$) and (2.1) that

$$\mathbb{E}\big(\Delta_n^\gamma\big) \leq 2^\gamma \sum_{p=n+1}^{\infty} \mathbb{E}\bigg(\max_{\tau_p \in D_p} \max_{\tau'_{p-1} \in O_{p-1}(\tau_p)} \big|X(\tau_p) - X(\tau'_{p-1})\big|^\gamma\bigg) \leq 2^\gamma K_2 \sigma(2^{-n})^\gamma.$$

Hence Markov's inequality gives that for all integers $n \geq 1$ and real numbers $u > 0$

$$\mathbb{P}\Big(\Delta_n \geq \sigma(2^{-n})\, u\Big) \leq 2^\gamma K_2\, u^{-\gamma}. \tag{2.4}$$

For any $\varepsilon \in (0, \varepsilon_0)$ and $\eta > 0$, by taking $u = \eta (\log 2^n)^{(1+\varepsilon)/\gamma}$, we derive from (2.4) that

$$\mathbb{P}\Big(\Delta_n \geq \eta\, \sigma(2^{-n})\, (\log 2^n)^{(1+\varepsilon)/\gamma}\Big) \leq \frac{K_3}{n^{1+\varepsilon}},$$

which is summable. The Borel-Cantelli lemma implies almost surely

$$\Delta_n \leq \eta\, \sigma(2^{-n})\, (\log 2^n)^{(1+\varepsilon)/\gamma} \tag{2.5}$$

for all n large enough. Therefore, $X(t)$ is a.s. uniformly continuous in $D = \bigcup_{k=1}^\infty D_k$. Since D is dense in T, it is standard to derive from (2.5) that X has a continuous version (still denoted by X) such that for all $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{\sup_{t\in T} \sup_{d(s,t) \leq 2^{-n}} \big|X(t) - X(s)\big|}{\sigma(2^{-n})\, (\log 2^n)^{(1+\varepsilon)/\gamma}} = 0, \qquad \text{a.s.} \tag{2.6}$$

By (2.6), the properties of σ and a monotonicity argument we derive (2.2). This finishes the proof of Theorem 2.1. □

From now on, we will not distinguish X from its continuous version. The following result follows directly from Theorem 2.1 and is often more convenient to use.

Corollary 2.2 *Let $X = \{X(t), t \in T\}$ be a real-valued random field on a compact metric space (T, d) and let $\{D_n, n \geq 1\}$ be as in Theorem 2.1. Assume that there exist constants $\gamma \in (0,1]$, $\delta > 0$ and $K > 0$ such that*

$$\mathbb{E}\bigg(\max_{\tau_n \in D_n} \max_{\tau'_{n-1} \in O_{n-1}(\tau_n)} \big|X(\tau_n) - X(\tau'_{n-1})\big|^\gamma\bigg) \leq K\, 2^{-\delta\gamma n} \tag{2.7}$$

for all integers $n \geq 1$. Then for all $\varepsilon > 0$,

$$\lim_{h\to 0+} \frac{\sup_{t\in T} \sup_{d(s,t)\leq h} \big|X(t) - X(s)\big|}{h^\delta \big(\log(1/h)\big)^{(1+\varepsilon)/\gamma}} = 0, \qquad a.s. \tag{2.8}$$

Next we apply Theorem 2.1 to a random field $X = \{X(t), t \in [0,1]^N\}$ which may be anisotropic. Corollary 2.3 below is applicable to anisotropic stable random fields including linear fractional stable sheets considered in Ayache, Roueff and Xiao (2007, 2009), harmonizable fractional stable sheets [cf, Xiao (2006, 2008)], operator-scaling stable fields with stationary increments in Biermé and Lacaux (2009) and others infinitely divisible fields.

Given a constant vector $(H_1, \ldots, H_N) \in (0,1]^N$, we consider the metric ρ on $\mathbb{R}^N$ defined by:

$$\rho(s,t) = \max_{1\le j\le N} |s_j - t_j|^{H_j}, \qquad \forall\, s,\, t \in \mathbb{R}^N. \tag{2.9}$$

For every $n \ge 1$, define

$$\widetilde{D}_n = \Big\{ \Big(\frac{k_1}{2^{n/H_1}}, \ldots, \frac{k_N}{2^{n/H_N}} \Big),\ 1 \le k_j \le \lfloor 2^{n/H_j} \rfloor,\ \forall 1 \le j \le N \Big\}. \tag{2.10}$$

Then

$$\#\widetilde{D}_n \le 2^{Qn} \quad \text{and} \quad \overline{\bigcup_{n=1}^{\infty} \widetilde{D}_n} = [0,1]^N.$$

In the above and in the sequel, $\#D$ denotes the cardinality of D and $Q = \sum_{j=1}^N \frac{1}{H_j}$. It is easy to see that the sequence $\{\widetilde{D}_n, n \ge 1\}$ satisfies Conditions (i) and (ii) on the compact metric space $([0,1]^N, \rho)$.

Corollary 2.3 *Let $X = \{X(t), t \in [0,1]^N\}$ be a real-valued random field and assume $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies the conditions of Theorem 2.1. Let $\{\widetilde{D}_n, n \ge 1\}$ be the sequence defined above. If there exist constants $\gamma \in (0,1]$ and $K > 0$ such that*

$$\sum_{p=n}^{\infty} \mathbb{E}\Big(\max_{\tau_p \in \widetilde{D}_p} \max_{\tau'_{p-1} \in O_{p-1}(\tau_p)} \big|X(\tau_p) - X(\tau'_{p-1})\big|^{\gamma} \Big) \le K\, \sigma\big(2^{-n}\big)^{\gamma}$$

for all integers $n \ge 1$, then for any $\varepsilon > 0$,

$$\lim_{h\to 0+} \sup_{t\in[0,1]^N} \sup_{|s-t|\le h} \frac{\big|X(t) - X(s)\big|}{\sigma\big(\sum_{j=1}^N |t_j - s_j|^{H_j}\big) \,\big|\log\big(\sum_{j=1}^N |t_j - s_j|^{H_j}\big)\big|^{(1+\varepsilon)/\gamma}} = 0, \quad a.s. \tag{2.11}$$

Proof Note that the conditions of Theorem 2.1 are satisfied. By (2.2) we have that for all $\varepsilon > 0$

$$\lim_{h\to 0+} \frac{\sup_{t\in[0,1]^N,\, \rho(s,t)\le h} \big|X(t) - X(s)\big|}{\sigma(h)\,\big(\log(1/h)\big)^{\frac{1+\varepsilon}{\gamma}}} = 0, \qquad \text{a.s.} \tag{2.12}$$

For any $s,t \in [0,1]^N$, there is an integer n such that $2^{-n-1} \le \rho(s,t) < 2^{-n}$. Then (2.12), together with the properties of σ, implies that

$$\big|X(t) - X(s)\big| \le \sigma\big(2^{-n}\big)\, n^{\frac{1+\varepsilon}{\gamma}} \le K_1\, \sigma\big(\rho(s,t)\big)\, \big(\log(1/\rho(s,t))\big)^{\frac{1+\varepsilon}{\gamma}}, \quad \text{a.s.}$$

for all $s,t \in [0,1]^N$ such that $\rho(s,t)$ is small. Since $\varepsilon > 0$ is arbitrary, this proves (2.11). □

3 Maximal moment index

It is clear that condition (2.1) [or (2.7)] is essential for Theorem 2.1. In many cases such as when $X = \{X(t), t \in T\}$ is a Gaussian or stable random field, or has stationary increments, one can normalize the random variables $X(\tau_n) - X(\tau'_{n-1})$ so that (2.7) is reduced to conditions on the maximal γ-moments of a sequence of "homogeneous" random variables.

Motivated by this, we introduce the notion of maximal γ-moment index for a (not necessarily stationary) sequence of random variables which, in turn, provides some sufficient conditions for (2.7) to hold.

Definition 3.1 *Let $\{\xi_k, k \ge 1\}$ be a sequence of random variables such that, for some positive constants γ and K_γ, $\mathbb{E}\big(|\xi_k|^\gamma\big) = K_\gamma$ for all $k \ge 1$. Let $M_n(\gamma) = \mathbb{E}\big(\max_{1\le k\le n} |\xi_k|^\gamma\big)$. Then the maximal γ-moment (upper) index of $\{\xi_k, k \ge 1\}$ is defined by*

$$\theta_\gamma = \limsup_{n\to\infty} \frac{\log M_n(\gamma)}{\log n}. \tag{3.1}$$

If $\gamma = 1$, we simply call θ_1 the maximal moment index of $\{\xi_k, k \ge 1\}$.

Some remarks are in order.

- If $0 < \gamma < \beta$, then Jensen's inequality implies that $\theta_\gamma \le \frac{\gamma}{\beta}\,\theta_\beta$.
- Since $\mathbb{E}(|\xi_1|^\gamma) \le M_n(\gamma) \le \mathbb{E}(\sum_{k=1}^n |\xi_k|^\gamma)$, we clearly have $\theta_\gamma \in [0, 1]$.
- The maximal γ-moment index carries some information about the dependence structure and distributional properties of $\{\xi_k, k \ge 1\}$. Usually the choice of γ is determined by the heaviness of the tail probabilities of $\{\xi_k, k \ge 1\}$. For a fixed γ, smaller value of θ_γ indicates more dependence among $\{\xi_k, k \ge 1\}$. This can be seen through the two extreme examples of stationary processes $\{\xi_k, k \ge 1\}$: If $\xi_k \equiv \xi$, then $\theta_\gamma = 0$; while if ξ_k $(k \ge 1)$ are i.i.d. and satisfy the conditions (3.9) and (3.11) below, then $\theta_\gamma = \gamma/\alpha$ for $\gamma \in (0, \alpha)$. Further evidence can be found in Samorodnitsky (2004) where it is shown that the partial maxima of long-range dependent stable processes grow slower than those of short-range dependent processes.

The following consequence of Theorem 2.1 shows the usefulness of maximal γ-moment index in determining the uniform modulus of continuity.

Corollary 3.2 *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a continuous random field with values in $\mathbb{R}$ and let $\{D_n, n \ge 1\} \subseteq [0, 1]^N$ be defined by*

$$D_n = \left\{ \Big(\frac{k_1}{2^n}, \cdots, \frac{k_N}{2^n}\Big) :\ 1 \le k_j \le 2^n, 1 \le j \le N \right\}. \tag{3.2}$$

We assume there exist constants $\gamma \in (0, 1]$ and $H > 0$ such that for all $s, t \in [0, 1]^N$

$$\mathbb{E}\big(|X(s) - X(t)|^\gamma\big) \le K\, |s - t|^{H\gamma}. \tag{3.3}$$

Consider the normalized random variables

$$\frac{X(\tau_p) - X(\tau'_{p-1})}{\big[\mathbb{E}(|X(\tau_p) - X(\tau'_{p-1})|^\gamma)\big]^{1/\gamma}}, \qquad \forall\ \tau_p \in D_p,\ \tau'_{p-1} \in O_{p-1}(\tau_p) \ \ and \ \ \forall\ p \geq 1, \tag{3.4}$$

and number them according to the order D_1, $D_2 \backslash D_1, \ldots$ and denote the sequence by $\{\xi_k, k \geq 1\}$. If $\{\xi_k, k \geq 1\}$ has a maximal γ-moment index $\theta := \theta_\gamma$ and $H\gamma > N\theta$, then for every $\varepsilon > 0$,

$$\limsup_{h \to 0+} \frac{\sup_{t \in [0,1]^N} \sup_{|s-t| \leq h} \big|X(t) - X(s)\big|}{h^{H - \frac{N\theta}{\gamma} - \varepsilon}} = 0, \qquad a.s. \tag{3.5}$$

Namely, $X(t)$ is uniformly Hölder continuous on $[0,1]^N$ of all orders $< H - \frac{N\theta}{\gamma}$.

Proof For any $\varepsilon > 0$, it follows from (3.1) that

$$M_{2^{Nn}}(\gamma) = \mathbb{E}\Big(\max_{1 \leq k \leq \# D_n} |\xi_k|^\gamma\Big) \leq 2^{n(N\theta + \gamma\varepsilon)}$$

for all n large enough. Combining this with (3.3) we derive

$$\mathbb{E}\left(\max_{\tau_n \in D_n} \max_{\tau'_{n-1} \in O_{n-1}(\tau_n)} \big|X(\tau_n) - X(\tau'_{n-1})\big|^\gamma\right) \leq K\, 2^{-(H - \frac{N\theta}{\gamma} - \varepsilon)\gamma n}$$

for all integers n large enough. Hence (3.5) follows from Corollary 2.2. □

As an example, we show that the following multiparameter version of Kolmogorov's continuity theorem follows from Corollary 3.2.

Corollary 3.3 [**Kolmogorov's continuity theorem**] *Let $X = \{X(t), t \in [0,1]^N\}$ be a real-valued stochastic process. Suppose there exist positive constants β, K and δ such that*

$$\mathbb{E}\big(|X(s) - X(t)|^\beta\big) \leq K\, |s-t|^{N+\delta}, \qquad \forall\ s, t \in [0,1]^N. \tag{3.6}$$

Then X has a version which is uniformly Hölder conditions on $[0,1]^N$ of all orders $< \frac{\delta}{\beta}$.

Proof For any integer $n \geq 1$, let D_n be defined by (3.2) and let $\{\xi_k, k \geq 1\}$ be the sequence of the normalized random variables in (3.4). By (3.6), we see that (3.3) is satisfied with $\gamma = \beta$ and $H = \frac{N+\delta}{\beta}$. In order to verify the second condition in Corollary 3.2, we note that $\mathbb{E}\big(|\xi_k|^\gamma\big) = 1$ for every $k \geq 1$ and use the trivial bound $\mathbb{E}\big(\max_{1 \leq k \leq m} |\xi_k|^\gamma\big) \leq m$ for all integers $m \geq 1$ to derive $\theta_\gamma \leq 1$. Then it is clear that the conclusion of Corollary 3.3 follows from Corollary 3.2. □

In the rest of this section, we study the maximal moment indices for several classes of random variables. The following lemma on Gaussian sequences will be useful in the next section.

Lemma 3.4 *Let $\{\xi_k, k \geq 1\}$ be a sequence of jointly Gaussian random variables with mean 0 and variance 1. Then the following statements hold:*

(i) There is a universal constant $K_4 > 0$ such that

$$\mathbb{E}\Big(\max_{1\le k\le n} |\xi_k|\Big) \le K_4 \sqrt{\log n}. \tag{3.7}$$

In particular, for every $\gamma \in (0,1]$, the maximal γ-moment index of $\{\xi_k, k \ge 1\}$ is 0.

(ii) If $|\mathbb{E}(\xi_j \xi_k)| \le \delta$ for a constant $\delta \in (0,1)$ and for all $1 \le j < k \le n$, then there exists a constant K_5 such that $\mathbb{E}\big(\max_{1\le k\le n} |\xi_k|\big) \ge K_5 \sqrt{\log n}$.

Proof This lemma is well known, and we include a proof for completeness. Part (i) can be proved by using the metric entropy method. For any fixed integer $n \ge 1$, let $T = \{1, \ldots, n\}$ equipped with the canonical metric $d(i,j) = \big[\mathbb{E}(\xi_i - \xi_j)^2\big]^{1/2}$. Then the d-diameter of T is at most 2. For any $\varepsilon \in (0,1)$, the ε-covering number $N_d(T, \varepsilon) \le n$. Hence Dudley's entropy theorem [cf. e.g., Marcus and Rosen (2006, Theorem 6.1.2) or Adler and Taylor (2007, Theorem 1.3.3)] gives

$$\mathbb{E}\Big(\max_{1\le k\le n} |\xi_k|\Big) \le K \int_0^1 \sqrt{\log n}\, d\varepsilon, \tag{3.8}$$

which yields (3.7).

Under the condition of (ii), we have $d(i,j) = 2(1 - \mathbb{E}(\xi_i \xi_j)) \ge 2(1-\delta)$ for all $i \ne j$. Hence the conclusion of (ii) follows from the Sudakov minoration [see Talagrand (2006, Lemma 2.1.2)]. □

Dudley's entropy theorem has been extended to stochastic processes in Orlicz spaces, see Talagrand (2006, p.30) for related references. Let $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a convex function such that $\Psi(0) = 0$ and $\Psi(r) > 0$ if $r \ne 0$. For a random variable ξ, the Orlicz norm of ξ is defined as

$$\|\xi\|_\Psi = \inf\left\{ c > 0 : \mathbb{E}\Psi\big(\frac{\xi}{c}\big) \le 1 \right\}.$$

If $\{\xi_k, k \ge 1\}$ is a sequence of random variables such that $\|\xi_k\|_\Psi \equiv K < \infty$, then (1.53) in Talagrand (2006) implies that $\mathbb{E}\big(\max_{1\le k\le n} |\xi_k|\big) \le K\, \Psi^{-1}(n)$. This result extends Part (i) of Lemma 3.4. By taking $\Psi(r) = r^p$ $(p \ge 1)$, one gets an upper bound for $\mathbb{E}\big(\max_{1\le k\le n} |\xi_k|\big)$ when $\{\xi_k, k \ge 1\}$ is a sequence of random variables with the same finite p-th absolute moments.

The following lemma is applicable to α-stable random variables.

Lemma 3.5 *Let $\{\xi_k, k \ge 1\}$ be a sequence of random variables. The following statements hold:*

(i) If there exist positive constants α and K_6 such that

$$\mathbb{P}\big(|\xi_k| \ge u\big) \le K_6\, u^{-\alpha}, \qquad \forall\, k \ge 1 \ \text{ and } \ u > 0, \tag{3.9}$$

then for any $\gamma \in (0, \alpha)$ and $\varepsilon > 0$ there is a finite constant K_7 such that

$$\mathbb{E}\Big(\max_{1\le k\le n} |\xi_k|^\gamma\Big) \le K_7\, n^{\gamma/\alpha}\, (\log n)^{(1+\varepsilon)\gamma/\alpha} \tag{3.10}$$

for all integers $n \ge 2$. Consequently, for any $\gamma < \alpha$, we have $\theta_\gamma \le \gamma/\alpha$.

(ii) If there exists a positive constant K_8 such that

$$\mathbb{P}\big(|\xi_k| \geq u\big) \geq K_8\, u^{-\alpha}, \qquad \forall\ k \geq 1 \ \ \text{and} \ \ u > 0 \tag{3.11}$$

and for all integers $n \geq 1$ and $u > 0$

$$\mathbb{P}\Big(\max_{1\leq k\leq n} |\xi_k| \leq u\Big) \leq \prod_{k=1}^{n} \mathbb{P}\big(|\xi_k| \leq u\big), \tag{3.12}$$

then, for any $\gamma \in (0, \alpha)$, there is a constant $K_9 > 0$ such that

$$\mathbb{E}\Big(\max_{1\leq k\leq n} |\xi_k|^{\gamma}\Big) \geq K_9\, n^{\gamma/\alpha} \tag{3.13}$$

for all integers $n \geq 1$.

Remark 3.6 Clearly (3.12) is satisfied if the random variables ξ_k $(k \geq 1)$ are independent or, more generally, if the random variables $|\xi_k|$ $(k \geq 1)$ are negatively orthand dependent, that is, for all $n \geq 1$ and $x_1, \ldots, x_n > 0$,

$$\mathbb{P}\Big(|\xi_1| \leq x_1, \ldots, |\xi_n| \leq x_n\Big) \leq \prod_{k=1}^{n} \mathbb{P}\big(|\xi_k| \leq x_k\big).$$

Moreover, by modifying the proof of Part (ii) of Lemma 3.5, one can show that, if we assume in (i) that the random variables $|\xi_k|$ $(k \geq 1)$ satisfies

$$\mathbb{P}\Big(\max_{1\leq k\leq n} |\xi_k| \leq u\Big) \geq \prod_{k=1}^{n} \mathbb{P}\big(|\xi_k| \leq u\big),$$

then (3.10) can be improved to $\mathbb{E}\big(\max_{1\leq k\leq n} |\xi_k|^{\gamma}\big) \leq K\, n^{\gamma/\alpha}$.

Proof of Lemma 3.5 Let $\gamma \in (0, \alpha)$ and $\varepsilon > 0$ be given constants. To prove Part (i), let $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a nondecreasing and convex function satisfying $\Phi(0) = 0$ and

$$\Phi(x) \sim \frac{x^{\alpha/\gamma}}{(\log x)^{1+\varepsilon}} \qquad \text{as} \ \ x \to \infty. \tag{3.14}$$

Here and in the sequel, $f(x) \sim g(x)$ means $f(x)/g(x) \to 1$ as $x \to \infty$ or $x \to 0$. Then the inverse function of $\Phi(x)$, denoted by $\Phi^{-1}(x)$, is nonnegative, nondecreasing and concave on $[0, \infty)$. Moreover,

$$\Phi^{-1}(x) \sim \frac{\gamma}{\alpha}\, x^{\gamma/\alpha}\, (\log x)^{(1+\varepsilon)\gamma/\alpha} \qquad \text{as} \ \ x \to \infty. \tag{3.15}$$

By (3.9) and (3.14) we derive $\mathbb{E}\big(\Phi(|\xi_k|^{\gamma})\big) \leq K_{10}$ for all $k \geq 1$, where K_{10} is a positive constant depending on K_6, α, ε and γ only. This and Jensen's inequality together imply

$$\begin{aligned}
\mathbb{E}\Big(\max_{1\leq k\leq n} |\xi_k|^{\gamma}\Big) &\leq \Phi^{-1}\Big[\mathbb{E}\Phi\Big(\max_{1\leq k\leq n} |\xi_k|^{\gamma}\Big)\Big] \\
&\leq \Phi^{-1}\Big[\sum_{1\leq k\leq n} \mathbb{E}\big(\Phi(|\xi_k|^{\gamma})\big)\Big] = \Phi^{-1}(K_{10}\, n).
\end{aligned}$$

Combining this and (3.15) yields (3.10).

To prove Part (ii), we write

$$\mathbb{E}\Big(\max_{1\le k\le n}|\xi_k|^\gamma\Big) = \gamma\int_0^\infty u^{\gamma-1}\mathbb{P}\Big(\max_{1\le k\le n}|\xi_k|>u\Big)\,du. \tag{3.16}$$

It follows from (3.12) and (3.11) that

$$\begin{aligned}\mathbb{P}\Big(\max_{1\le k\le n}|\xi_k|>u\Big) &\ge 1-\prod_{k=1}^n\Big(1-\mathbb{P}\big(|\xi_k|>u\big)\Big)\\ &\ge 1-\Big(1-K_8\,u^{-\alpha}\Big)^n\end{aligned} \tag{3.17}$$

for all $u > K_8^{1/\alpha}$. By using the elementary inequality

$$1-(1-x)^n \ge \frac{1}{2}nx \qquad \forall\ 0\le x\le 1-(\frac{1}{2})^{1/(n-1)},$$

we derive from (3.17) that

$$\mathbb{P}\Big(\max_{1\le k\le n}|\xi_k|>u\Big) \ge \frac{1}{2}\,nu^{-\alpha} \tag{3.18}$$

for all $u \ge K\,\big(1-(\frac{1}{2})^{1/(n-1)}\big)^{-1/\alpha} \asymp n^{1/\alpha}$. Combining (3.16), (3.17) and (3.18) we obtain

$$\mathbb{E}\Bigg(\max_{1\le k\le n}|\xi_k|^\gamma\Bigg) \ge K\,n\int_{n^{1/\alpha}}^\infty u^{\gamma-\alpha-1}\,du = K\,n^{\gamma/\alpha}. \tag{3.19}$$

This finishes the proof of Lemma 3.5. □

Part (ii) of Lemma 3.5 indicates that, in general, it is difficult to determine an optimal lower bound for $\mathbb{E}\big(\max_{1\le k\le n}|\xi_k|^\gamma\big)$ without additional information about the dependence structure of $\{\xi_k, k\ge 1\}$. We point out that, even for stationary stable sequences, it is an open problem in general to determine sharp (i.e., up to constant factors) upper and lower bounds for $\mathbb{E}\big(\max_{1\le k\le n}|\xi_k|^\gamma\big)$.

Here are some partial results. For symmetric stable or infinitely divisible sequences, some lower bounds can be derived from Theorem 2.3.1 and Theorem 5.3.2 in Talagrand (2006). For example, if $\{\xi_k\}$ is a stationary sequence of symmetric α-stable (SαS) random variables such that $\alpha\in(1,2]$ and there exists a constant $\delta>0$ such that $\|\xi_k-\xi_j\|_\alpha\ge\delta$ for all $k\ne j$ (where $\|\xi\|_\alpha$ denotes the scale parameter of ξ), then $\mathbb{E}\big(\max_{1\le k\le n}|\xi_k|\big)\ge K(\log n)^{(\alpha-1)/\alpha}$. On the other hand, Samorodnitsky (2004) studied the rate of growth of the partial maxima $M_n=\max\{|\xi_k|:1\le k\le n\}$ of a stationary α-stable sequence $\{\xi_k,k\ge 1\}$ based on the ergodic theoretical properties of the underlying flow. He discovered that (i) if the stationary SαS process $\{\xi_k,k\ge1\}$ is generated by a dissipative flow then M_n grows always at the rate of $n^{1/\alpha}$ and (ii) if the stationary SαS process $\{\xi_k,k\ge1\}$ is generated by a conservative flow then M_n grows at the rate slower than $n^{1/\alpha}$. Samorodnitsky (2004, p.1440) conjectured that many other important properties of $\{\xi_k,k\ge1\}$ will also change as the underlying flow changes from being dissipative to being conservative. In particular, we believe that, if $X=\{X(t),t\in\mathbb{R}\}$ is a (self-similar) SαS process with stationary increments, then the uniform modulus of continuity of X depends on the nature of the flow generating the stationary sequence $\xi_k = X(k+1)-X(k)$ $(k \ge 0)$. This idea will be pursued further elsewhere.

Combining Lemma 3.5 with the proof of Theorem 2.1, we have the following result on modulus of continuity for general self-similar processes with stationary increments, which is an improvement of Corollary 3.2. In the special case of $N = 1$ and $X = \{X(t), t \in \mathbb{R}\}$ being an α-stable process, it recovers Theorem 2 in Kôno and Maejima (1991b).

Corollary 3.7 *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a real-valued H-self-similar random field with stationary increments. Let $V = \{v_\ell, 1 \le \ell \le 2^N - 1\}$ be the set of vertices of $[0,1]^N$, excluding 0. If there exists a constant $\alpha > \frac{N}{H}$ such that*

$$\mathbb{P}\big(|X(v_\ell)| \ge u\big) \le K\, u^{-\alpha}, \qquad \forall\, v_\ell \in V \ \ \text{and} \ \ u > 0, \tag{3.20}$$

then for any $\varepsilon > 0$,

$$\limsup_{h\to 0+} \frac{\sup_{t\in[0,1]^N} \sup_{|s-t|\le h} \big|X(t) - X(s)\big|}{h^{H-\frac{N}{\alpha}} \big(\log 1/h\big)^{\frac{2+\varepsilon}{\alpha}}} = 0, \qquad a.s. \tag{3.21}$$

Proof Again we use the ℓ^∞ metric in $\mathbb{R}^N$. For every $n \ge 1$, let D_n be defined as in (3.2). Then the sequence $\{D_n, n \ge 1\}$ satisfies the conditions in Section 2. The self-similarity and stationarity of the increments of X imply that, for every $\tau_n \in D_n$ and $\tau'_{n-1} \in O_{n-1}(\tau_n)$, there is a $v_\ell \in V$ such that

$$X(\tau_n) - X(\tau'_{n-1}) \overset{d}{=} |\tau_n - \tau'_{n-1}|^H X(v_\ell).$$

Hence we apply Lemma 3.5 to the normalized sequence $\{\frac{X(\tau_n)-X(\tau'_{n-1})}{|\tau_n-\tau'_{n-1}|^H}\}$ to derive that for any $0 < \gamma < \min\{1, \alpha\}$

$$\mathbb{E}\bigg(\max_{\tau_n\in D_n} \max_{\tau'_{n-1}\in O_{n-1}(\tau_n)} \big|X(\tau_n) - X(\tau'_{n-1})\big|^\gamma\bigg) \le K\, 2^{-(H-\frac{N}{\alpha})\gamma n} \big(\log 2^{Nn}\big)^{(1+\varepsilon)\gamma/\alpha}. \tag{3.22}$$

Hence (3.21) follows from (3.22) and Theorem 2.1. □

4 Applications to stable random fields

In this section we apply the results in Sections 2 and 3 to harmonizable-type α-stable random fields with $0 < \alpha < 2$. Similar methods can be applied to other types of stable random fields, or more generally, infinitely divisible processes.

4.1 Harmonizable-type stable random fields with stationary increments

Let $X = \{X(t),\, t \in \mathbb{R}^N\}$ be a real-valued stable random field defined by

$$X(t) = \mathrm{Re} \int_{\mathbb{R}^N} \big(e^{i\langle t,x\rangle} - 1\big)\, \widetilde{M}_\alpha(dx), \tag{4.1}$$

where $\widetilde{M}_\alpha$ is a rotationally invariant α-stable random measure on $\mathbb{R}^N$ with control measure Δ, which satisfies

$$\int_{\mathbb{R}^N} (1 \wedge |x|^\alpha)\, \Delta(dx) < \infty. \tag{4.2}$$

This condition assures that stochastic integral in (4.1) is well-defined, see Samorodnitsky and Taqqu (1994, Chapter 6) for further information. The measure Δ is called the spectral measure of X and its density function, when it exists, is called the spectral density of X.

It can be verified that the stable random field X defined by (4.1) has stationary increments and $X(0) = 0$. Denote the scale parameter of $X(t)$ by $\|X(t)\|_\alpha$. Then for all $t \in \mathbb{R}^N$,

$$\|X(t)\|_\alpha^\alpha = 2^{\alpha/2} \int_{\mathbb{R}^N} \big(1 - \cos\langle t, x\rangle\big)^{\alpha/2}\, \Delta(dx). \tag{4.3}$$

Similar to Gaussian processes, this function plays an important role in studying sample path properties of stable random field X defined by (4.1).

For simplicity, we will assume that the spectral measure Δ is absolutely continuous and its density function $f(x)$ satisfies the following condition

$$f(x) \le K_{11}\, |x|^{-(\alpha H + N)}, \qquad \forall\, x \in \mathbb{R}^N \ \text{ with } \ |x| \ge K_{12}, \tag{4.4}$$

where $K_{11}, K_{12} > 0$ and $H \in (0,1)$ are constants. As shown by Theorem 4.5 below, the parameter H determines the smoothness of the sample function $X(t)$.

Now we provide some examples of stable random fields satisfying the condition (4.4).

Example 4.1 [Harmonizable fractional stable motion] Let $H \in (0,1)$ and $\alpha \in (0,2]$ be given constants. The harmonizable fractional stable field $\widetilde{Z}^H = \{\widetilde{Z}^H(t), t \in \mathbb{R}^N\}$ with values in $\mathbb{R}$ is defined by (4.1) with spectral density

$$f_{H,\alpha}(x) = c(\alpha, H, N)\, \frac{1}{|x|^{\alpha H + N}}, \tag{4.5}$$

where $c(\alpha, H, N) > 0$ is a normalizing constant such that the scale parameter of $\widetilde{Z}^H(e_1)$ equals 1, where $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^N$. Hence we have $\|\widetilde{Z}^H(t)\|_\alpha = |t|^H$ for all $t \in \mathbb{R}^N$.

It is easy to verify that the α-stable random field $\widetilde{Z}^H$ is H-self-similar with stationary and isotropic increments [or *stationary increments in the strong sense* in terms of Samorodnitsky and Taqqu (1994, p.392)]. The random field $\widetilde{Z}^H$ is a stable analogue of fractional Brownian motion B^H of index H and serves as an important representative for understanding harmonizable-type stable random fields. Even though the results on local times of $\widetilde{Z}^H$ [Nolan (1989) and Xiao (2008)] and uniform modulus of continuity [see Theorem 4.5 below] show that $\widetilde{Z}^H$ shares many sample path properties with B^H, little has been known about the sharpness of these results and the existing tools do not seem to be capable for attacking these problems. It is an interesting (and challenging) task to develop new methods for studying the fine structures of $\widetilde{Z}^H$ and other stable random fields.

Example 4.2 [Fractional Riesz-Bessel α-stable motion] Consider the stable random field $X = \{X(t), t \in \mathbb{R}^N\}$ in $\mathbb{R}$ defined by (4.1) with spectral density

$$f_{\gamma,\eta}(x) = \frac{c(\alpha, \gamma, \eta, N)}{|x|^{2\gamma}(1 + |x|^2)^\eta}, \tag{4.6}$$

where η and γ are positive constants satisfying

$$\eta + \gamma > \frac{N}{2}, \quad 0 < 2\gamma < \alpha + N$$

and $c(\alpha, \gamma, \eta, N) > 0$ is a normalizing constant. When $\alpha = 2$ [i.e., the Gaussian case] such density functions were consider by Anh et al. (1999). Since $f_{\gamma,\eta}$ involves both the Fourier transforms of the Riesz kernel and the Bessel kernel, Anh et al. (1999) called the corresponding Gaussian random field X the fractional Riesz-Bessel motion with indices η and γ. They showed that these Gaussian random fields can be used for modeling simultaneously long range dependence and intermittency.

In analogy to the terminology in Anh et al. (1999), we call X the fractional Riesz-Bessel α-stable motion with indices η and γ. Clearly, the spectral density $f_{\gamma,\eta}(x)$ in (4.6) satisfies (4.4) with $H = (2(\eta+\gamma) - N)/\alpha$. Moreover, since the spectral density $f_{\gamma,\eta}(x)$ is regularly varying at infinity of order $2(\eta + \gamma) > N$, by modifying the proof of Theorem 1 in Pitman [17] we can show that, if $2(\gamma + \eta) - N < \alpha$, then $\|X(t)\|_\alpha$ is regularly varying at 0 of order $(2(\eta + \gamma) - N)/\alpha$ and

$$\|X(t)\|_\alpha \sim |t|^{(2(\eta+\gamma)-N)/\alpha}, \qquad \text{as } |t| \to 0. \tag{4.7}$$

We will see that the modulus of continuity of X is determined by (4.7).

The following result provides information on the maximal moment of harmonizable-type stable random fields. It is more precise than Part (i) of Lemma 3.5.

Proposition 4.3 *Let $X = \{X(t), t \in [0,1]^N\}$ be an α-stable random field defined by (4.1) with spectral density satisfying (4.4). Then for every $0 < \gamma < \min\{1, \alpha\}$, $\eta > 0$ and $n \ge 1$,*

$$\mathbb{E}\Big(\max_{\tau_n \in D_n} \max_{\tau'_{n-1} \in O_{n-1}(\tau_n)} \big| X(\tau_n) - X(\tau'_{n-1}) \big|^\gamma \Big) \le K\, 2^{-H\gamma n}\, n^{\frac{(1+\eta)\gamma}{\alpha}}, \tag{4.8}$$

where $\{D_n, n \ge 1\}$ is the sequence defined in (3.2).

In order to prove Proposition 4.3, we will make use of a LePage-type representation for X, which allows us to view X as a mixture of Gaussian process. This powerful idea was due to Marcus and Pisier (1984) and has become a standard tool for studying sample path regularity of stable processes. For more information on series representations of infinitely divisible processes and their applications, see Rosiński (1989, 1990), Kôno and Maejima (1991a), Samorodnitsky and Taqqu (1994), Marcus and Rosinski (2005), Biermé and Lacaux (2009), just to mention a few.

We need some notation. Let μ be an arbitrary probability on $\mathbb{R}^N$ which is equivalent to the Lebesgue measure λ_N. Denote by $\varphi = d\mu/d\lambda_N$ the Radon-Nikodym derivative of μ, so $\mu(dx) = \varphi(x)\, dx$. Assume that

- $\{\Gamma_j, j \ge 1\}$ is a sequence of Poisson arrival times with intensity 1;
- $\{g_j, j \ge 1\}$ is a sequence of i.i.d. complex valued Gaussian random variables such that $g_j \overset{d}{=} e^{i\theta} g_j$ for all $\theta \in \mathbb{R}$ and $\mathbb{E}(|\mathrm{Re}\, g_1|^\alpha) = 1$;
- $\{\xi_j, j \ge 1\}$ is a sequence of i.i.d. random variables with values in $\mathbb{R}^N$ and density function φ;

- the sequences $\{\Gamma_j, j \geq 1\}$, $\{g_j, j \geq 1\}$ and $\{\xi_j, j \geq 1\}$ are independent. We denote the expectations with respect to $\{\Gamma_j, j \geq 1\}$, $\{g_j, j \geq 1\}$ and $\{\xi_j, j \geq 1\}$ by $\mathbb{E}_\Gamma$, $\mathbb{E}_g$ and $\mathbb{E}_\xi$, respectively.

The following lemma is from Biermé and Lacaux (2009); see also Kôno and Maejima (1991a) and Marcus and Pisier (1984).

Lemma 4.4 *For any family of complex-valued functions $h(t, \cdot) \in L^\alpha(\mathbb{R}^N, dx)$ $(0 < \alpha < 2)$, let $Z = \{Z(t), t \in \mathbb{R}^N\}$ be the α-stable random field defined by*

$$Z(t) = \mathrm{Re} \int_{\mathbb{R}^N} h(t,x)\, \widetilde{Z}_\alpha(dx), \qquad \forall t \in \mathbb{R}^N,$$

where $\widetilde{Z}_\alpha$ is a complex-valued, rotationally invariant α-stable random measure on $(\mathbb{R}^N, \mathcal{B}(\mathbb{R}^N))$ with Lebesgue control measure. Then

$$\Big\{Z(t), t \in \mathbb{R}^N\Big\} \stackrel{d}{=} \Big\{Y(t), t \in \mathbb{R}^N\Big\},$$

where $\stackrel{d}{=}$ means equality in finite dimensional distributions and

$$Y(t) = C_\alpha\, \mathrm{Re}\bigg(\sum_{j=1}^\infty \Gamma_j^{-1/\alpha} \varphi(\xi_j)^{-1/\alpha}\, h(t, \xi_j)\, g_j\bigg). \tag{4.9}$$

In the above, for every $t \in \mathbb{R}^N$, the random series (4.9) converges almost surely and C_α is the constant given by

$$C_\alpha = \bigg(\frac{1}{2\pi}\int_0^\pi \big|\cos\theta\big|^\alpha\, d\theta\bigg)^{1/\alpha} \bigg(\int_0^\infty \frac{\sin\theta}{\theta^\alpha}\, d\theta\bigg)^{-1/\alpha}.$$

Proof of Proposition 4.3 Choose two positive constants η and β such that $N - \frac{2\alpha(1-H)}{2-\alpha} < \beta < N$. Let $\varphi : \mathbb{R}^N \backslash\{0\} \to [0, \infty)$ be the function defined by

$$\varphi(x) = \begin{cases} K_{13}\, |x|^{-\beta} & \text{if } |x| \leq 3, \\ K_{14}\, \big(|x|^N (\log |x|)^{1+\eta}\big)^{-1} & \text{if } |x| > 3, \end{cases} \tag{4.10}$$

where the constants K_{13} and K_{14} are chosen such that $\int_{\mathbb{R}^N} \varphi(x)\, dx = 1$.

By Lemma 4.4, the stable random field X defined by (4.1) with spectral density $f(x)$ has the same finite dimensional distributions as

$$Y(t) = C_\alpha\, \mathrm{Re}\bigg(\sum_{j=1}^\infty \Gamma_j^{-1/\alpha} \varphi(\xi_j)^{-1/\alpha}\, h(t, \xi_j)\, g_j\bigg), \tag{4.11}$$

where the function $h(t,x)$ is defined by $h(t,x) = \big(e^{i\langle t,x\rangle} - 1\big) f(x)^{1/\alpha}$. Hence it is sufficient to prove (4.8) for Y.

Conditional on $\{(\xi_j, \Gamma_j),\ j \ge 1\}$, Y is a Gaussian random field with incremental variance given by

$$
\begin{aligned}
\mathbb{E}_g\Big[\big(Y(t)-Y(s)\big)^2\Big] &= 2\,C_\alpha^2 \sum_{j=1}^{\infty} \Gamma_j^{-2/\alpha} \varphi(\xi_j)^{-2/\alpha} \big(1-\cos\langle t-s, \xi_j\rangle\big)\, f(\xi_j)^{2/\alpha} \\
&\le K \sum_{j=1}^{\infty} \Gamma_j^{-2/\alpha} \varphi(\xi_j)^{-2/\alpha} \big\{1 \wedge |t-s|^2 |\xi_j|^2\big\}\, |\xi_j|^{-2(H+\frac{N}{\alpha})},
\end{aligned}
\tag{4.12}
$$

where the inequality follows from (4.4) and $K > 0$ is a constant.

Hence by using (4.12) and Lemma 3.4, we have

$$
\begin{aligned}
&\mathbb{E}_g\bigg(\max_{\tau_n \in D_n} \max_{\tau'_{n-1} \in O_{n-1}(\tau_n)} \big|Y(\tau_n) - Y(\tau'_{n-1})\big|^{\gamma}\bigg) \\
&\le K \max_{\tau_n \in D_n} \max_{\tau'_{n-1} \in O_{n-1}(\tau_n)} \Big[\mathbb{E}_g\big(Y(\tau_n) - Y(\tau'_{n-1})\big)^2\Big]^{\gamma/2} n^{\gamma/2} \\
&\le K \bigg[\sum_{j=1}^{\infty} \Gamma_j^{-2/\alpha} \varphi(\xi_j)^{-2/\alpha} \big\{1 \wedge 2^{-2n} |\xi_j|^2\big\}\, |\xi_j|^{-2(H+\frac{N}{\alpha})}\bigg]^{\gamma/2} n^{\gamma/2}.
\end{aligned}
\tag{4.13}
$$

It remains to show that (4.8) follows from taking expectations on both sides of (4.13) with respect to $\{\xi_j, j \ge 1\}$ and $\{\Gamma_j, j \ge 1\}$.

Note that, for every $j \ge 1$, Γ_j is a Gamma random variable with density function

$$
p(x) = \frac{x^{j-1}\, e^{-x}}{(j-1)!}, \qquad \forall\ x \ge 0.
$$

It is elementary to verify that there is a constant $K > 0$ such that

$$
\mathbb{E}_\Gamma\big(\Gamma_j^{-2/\alpha}\big) \le K\, j^{-2/\alpha}, \qquad \forall\ j > 2/\alpha \tag{4.14}
$$

and

$$
\mathbb{E}_\Gamma\big(\Gamma_j^{-\gamma/\alpha}\big) \le K \qquad \forall\ 1 \le j \le 2/\alpha. \tag{4.15}
$$

Since $\gamma \in (0,1)$, we use the elementary inequality $(x+y)^{\gamma/2} \le x^{\gamma/2} + y^{\gamma/2}$ and Jensen's inequality to derive

$$
\begin{aligned}
&\mathbb{E}_{\Gamma,\xi}\bigg\{\bigg(\sum_{j=1}^{\infty} \Gamma_j^{-2/\alpha} \varphi(\xi_j)^{-2/\alpha} \big\{1 \wedge 2^{-2n} |\xi_j|^2\big\}\, |\xi_j|^{-2(H+\frac{N}{\alpha})}\bigg)^{\gamma/2}\bigg\} \\
&\le \mathbb{E}_{\Gamma,\xi}\bigg(\sum_{j=1}^{\lfloor 2/\alpha \rfloor} \Gamma_j^{-\gamma/\alpha} \varphi(\xi_j)^{-\gamma/\alpha} \big\{1 \wedge 2^{-\gamma n} |\xi_j|^{\gamma}\big\}\, |\xi_j|^{-\gamma(H+\frac{N}{\alpha})}\bigg) \\
&\qquad + \bigg[\mathbb{E}_{\Gamma,\xi}\bigg(\sum_{j=\lfloor 2/\alpha \rfloor + 1}^{\infty} \Gamma_j^{-2/\alpha} \varphi(\xi_j)^{-2/\alpha} \big\{1 \wedge 2^{-2n} |\xi_j|^2\big\}\, |\xi_j|^{-2(H+\frac{N}{\alpha})}\bigg)\bigg]^{\gamma/2} \\
&:= I_1 + I_2.
\end{aligned}
\tag{4.16}
$$

Since I_1 and I_2 can be estimated by using the same method, we only consider I_2 below. Taking the expectation $\mathbb{E}_\xi$ first, we have

$$\begin{aligned}
&\mathbb{E}_\xi\bigg(\varphi(\xi_j)^{-2/\alpha}\left\{1\wedge 2^{-2n}|\xi_j|^2\right\}|\xi_j|^{-2(H+\frac{N}{\alpha})}\bigg)\\
&=\int_{\mathbb{R}^N}\varphi(x)^{1-\frac{2}{\alpha}}\left(1\wedge 2^{-2n}|x|^2\right)\frac{dx}{|x|^{2(H+\frac{N}{\alpha})}}\\
&=2^{-2n}\int_{|x|\le 2^n}\varphi(x)^{1-\frac{2}{\alpha}}\frac{dx}{|x|^{2(H+\frac{N}{\alpha}-1)}}+\int_{|x|>2^n}\varphi(x)^{1-\frac{2}{\alpha}}\frac{dx}{|x|^{2(H+\frac{N}{\alpha})}}\\
&:=J_1+J_2.
\end{aligned}\tag{4.17}$$

By (4.10) and a change of variables, we derive

$$\begin{aligned}
J_1&=K\,2^{-2n}\bigg(\int_0^3\frac{dr}{r^{2H+(N-\beta)(\frac{2}{\alpha}-1)-1}}\\
&\qquad\qquad+\int_3^{2^n}\frac{dr}{r^{2H-1}|\log r|^{(1+\eta)(1-\frac{2}{\alpha})}}\bigg)\\
&\le K\,2^{-2Hn}\,n^{-(1+\eta)(1-\frac{2}{\alpha})}.
\end{aligned}\tag{4.18}$$

Note that, because of the choice of β, the first integral is constant. Similarly, we also have

$$J_2\le K\,2^{-2Hn}\,n^{-(1+\eta)(1-\frac{2}{\alpha})}.\tag{4.19}$$

By (4.14), (4.17), (4.18) and (4.19), we obtain

$$\begin{aligned}
I_2&\le\Bigg[\mathbb{E}_\Gamma\bigg(K\sum_{j=\lfloor 2/\alpha\rfloor+1}^{\infty}\Gamma_j^{-2/\alpha}2^{-2Hn}\,n^{-(1+\eta)(1-\frac{2}{\alpha})}\bigg)\Bigg]^{\gamma/2}\\
&\le K\,2^{-\gamma Hn}\,n^{-\gamma(1+\eta)(\frac{1}{2}-\frac{1}{\alpha})}\Bigg[\sum_{j=\lfloor 2/\alpha\rfloor+1}^{\infty}\mathbb{E}_\Gamma\left(\Gamma_j^{-2/\alpha}\right)\Bigg]^{\gamma/2}\\
&\le K\,2^{-\gamma Hn}\,n^{-\gamma(1+\eta)(\frac{1}{2}-\frac{1}{\alpha})}.
\end{aligned}\tag{4.20}$$

Similarly, we can derive

$$I_1\le K\,2^{-\gamma Hn}\,n^{-\gamma(1+\eta)(\frac{1}{2}-\frac{\gamma}{\alpha})}.\tag{4.21}$$

Combining (4.16), (4.20) and (4.21) we obtain

$$\begin{aligned}
&\mathbb{E}_{\Gamma,\xi}\Bigg\{\bigg(\sum_{j=1}^{\infty}\Gamma_j^{-2/\alpha}\varphi(\xi_j)^{-2/\alpha}\left\{1\wedge 2^{-2n}|\xi_j|^2\right\}|\xi_j|^{-2(H+\frac{N}{\alpha})}\bigg)^{\gamma/2}\Bigg\}\\
&\le K\,2^{-\gamma Hn}\,n^{-\gamma(1+\eta)(\frac{1}{2}-\frac{\gamma}{\alpha})}.
\end{aligned}\tag{4.22}$$

Finally (4.8) follows from (4.13) and (4.22). This proves Proposition 4.3. □

The following is a consequence of Proposition 4.3 and Theorem 2.1.

Theorem 4.5 *Let* $X = \{X(t), t \in \mathbb{R}^N\}$ *be an* α*-stable random field defined by (4.1) with spectral density satisfying (4.4). Then for any* $\varepsilon > 0$,

$$\limsup_{h\to 0+} \frac{\sup_{t\in[0,1]^N}\sup_{|s-t|\le h}\big|X(t)-X(s)\big|}{h^H\big(\log 1/h\big)^{(2+\varepsilon)/\alpha}} = 0, \qquad a.s. \tag{4.23}$$

Proof It follows from Proposition 4.3 that for all $\eta > 0$ and integers $n \ge 1$,

$$\sum_{p=n}^{\infty} \mathbb{E}\bigg(\max_{\tau_p\in D_p}\max_{\tau'_{p-1}\in O_{p-1}(\tau_p)} \big|X(\tau_p)-X(\tau'_{p-1})\big|^{\gamma}\bigg) \le K\, 2^{-H\gamma n}\, n^{\frac{\gamma(1+\eta)}{\alpha}}.$$

Thus, X satisfies (2.1) with $\sigma(h) = h^H(\log 1/h)^{\frac{1+\eta}{\alpha}}$. Since $\eta > 0$ and $0 < \gamma < \alpha$ are arbitrary, (4.23) follows from Theorem 2.1. □

4.2 Harmonizable fractional stable sheets

For any given $0 < \alpha < 2$ and $\vec{H} = (H_1, \dots, H_N) \in (0,1)^N$, we define the harmonizable fractional stable sheet $\widetilde{Z}^{\vec{H}} = \{\widetilde{Z}^{\vec{H}}(t), t \in \mathbb{R}^N_+\}$ with values in $\mathbb{R}$ by

$$\widetilde{Z}^{\vec{H}}(t) = \mathrm{Re}\int_{\mathbb{R}^N}\prod_{j=1}^{N}\frac{e^{it_jx_j}-1}{|x_j|^{H_j+\frac{1}{\alpha}}}\,\widetilde{Z}_\alpha(d\lambda), \tag{4.24}$$

where $\widetilde{Z}_\alpha$ is a complex-valued random measure as in Lemma 4.4.

From (4.24) it follows that $\widetilde{Z}^{\vec{H}}$ has the following operator-scaling property: For any $N \times N$ diagonal matrix $E = (b_{ij})$ with $b_{ii} = b_i > 0$ for all $1 \le i \le N$ and $b_{ij} = 0$ if $i \ne j$, we have

$$\big\{\widetilde{Z}^{\vec{H}}(Et),\, t\in\mathbb{R}^N\big\} \stackrel{d}{=} \bigg\{\Big(\prod_{j=1}^{N} b_j^{H_j}\Big)\widetilde{Z}^{\vec{H}}(t),\ t\in\mathbb{R}^N\bigg\}. \tag{4.25}$$

Along each direction of $\mathbb{R}^N_+$, $\widetilde{Z}^{\vec{H}}$ becomes a real-valued harmonizable fractional stable motion [cf. Samorodnitsky and Taqqu (1994, Chapter 7)]. When the indices $H_1, \dots, H_N$ are not the same, $\widetilde{Z}^{\vec{H}}$ has different scaling behavior along different directions and this anisotropic nature induces some interesting geometric and analytic properties for $\widetilde{Z}^{\vec{H}}$. Note that $\widetilde{Z}^{\vec{H}}$ does not have stationary increments in the ordinary sense and, thus, is different from the operator-scaling stable fields considered in Biermé and Lacaux (2009). See Xiao (2006, 2008) for further information on sample path properties of stable random fields.

The following result gives the uniform modulus of continuity for $\widetilde{Z}^{\vec{H}}$, which is significantly different from the result for linear fractional stable sheets obtained in Ayache, Roueff and Xiao (2007, 2009).

Theorem 4.6 *For any arbitrarily small* $\varepsilon > 0$, *one has*

$$\lim_{h\to 0}\sup_{t\in[0,1]^N, |s-t|\le h}\frac{|\widetilde{Z}^{\vec{H}}(s)-\widetilde{Z}^{\vec{H}}(t)|}{\sum_{j=1}^N |s_j-t_j|^{H_j}\big|\log\big(\sum_{j=1}^N|s_j-t_j|^{H_j}\big)\big|^{(2+\varepsilon)/\alpha}} = 0, \qquad a.s. \tag{4.26}$$

Proof Let ρ be the metric on $\mathbb{R}^N$ defined in (2.9) and let $\{\widetilde{D}_n, n \geq 1\}$ be the sequence defined in (2.10). We claim that for all integers $n \geq 1$, $0 < \gamma < \min\{1, \alpha\}$ and $\eta > 0$,

$$\mathbb{E}\bigg(\max_{\tau_n \in \widetilde{D}_n} \max_{\tau'_{n-1} \in O_{n-1}(\tau_n)} \big| \widetilde{Z}^{\vec{H}}(\tau_n) - \widetilde{Z}^{\vec{H}}(\tau'_{n-1}) \big|^{\gamma} \bigg) \leq K\, 2^{-\gamma n}\, n^{\frac{(1+\eta)\gamma}{\alpha}}. \tag{4.27}$$

Before proving (4.27), we note that it implies

$$\sum_{p=n}^{\infty} \mathbb{E}\bigg(\max_{\tau_p \in \widetilde{D}_p} \max_{\tau'_{p-1} \in O_{p-1}(\tau_p)} \big| \widetilde{Z}^{\vec{H}}(\tau_p) - \widetilde{Z}^{\vec{H}}(\tau'_{p-1}) \big|^{\gamma} \bigg) \leq K\, 2^{-\gamma n}\, n^{\frac{(1+\eta)\gamma}{\alpha}}.$$

Hence $\widetilde{Z}^{\vec{H}}$ satisfies the conditions of Corollary 2.3 with $\sigma(h) = h\big(\log 1/h\big)^{(1+\eta)/\alpha}$. Thus, (4.26) follows from (2.11).

It remains to prove (4.27). Since this is similar to the proof of Proposition 4.3, we only provide a sketch of it. Let η and β be two positive constants with β satisfying

$$\max_{1 \leq j \leq N} \left\{ 1 - \frac{2\alpha(1 - H_j)}{2 - \alpha} \right\} < \beta < 1.$$

Define the density function $\varphi : \mathbb{R}^N \to [0, \infty)$ by $\varphi(x) = \prod_{j=1}^N \varphi_j(x_j)$, where $x = (x_1, \ldots, x_N)$ and

$$\varphi_j(x_j) = \begin{cases} K_{15}\, |x_j|^{-\beta} & \text{if } |x_j| \leq 3, \\ K_{16}\, \big(|x_j| (\log |x|)^{1+\eta}\big)^{-1} & \text{if } |x_j| > 3. \end{cases}$$

In the above, the constants K_{15} and K_{16} are chosen such that $\int_{\mathbb{R}^N} \varphi(x)\, dx = 1$. By Lemma 4.4, $\widetilde{Z}^{\vec{H}}$ has the same finite dimensional distributions as

$$Y(t) = C_\alpha \, \mathrm{Re} \bigg(\sum_{j=1}^{\infty} \Gamma_j^{-1/\alpha} \varphi(\xi_j)^{-1/\alpha}\, h(t, \xi_j)\, g_j \bigg),$$

where the function $h(t, x)$ is now defined by

$$h(t, x) = \prod_{j=1}^N \frac{e^{i t_j x_j} - 1}{|x_j|^{H_j + \frac{1}{\alpha}}}.$$

Again, by conditional on $\{(\xi_j, \Gamma_j),\, j \geq 1\}$, we have

$$\mathbb{E}_g \Big[\big(Y(t) - Y(s)\big)^2 \Big] = C_\alpha^2 \sum_{j=1}^{\infty} \Gamma_j^{-2/\alpha} \varphi(\xi_j)^{-2/\alpha} \big| h(t, \xi_j) - h(s, \xi_j) \big|^2. \tag{4.28}$$

If follows from (4.28) and Lemma 3.4 that

$$\begin{aligned} &\mathbb{E}\bigg(\max_{\tau_n \in \widetilde{D}_n} \max_{\tau'_{n-1} \in O_{n-1}(\tau_n)} \big| Y(\tau_n) - Y(\tau'_{n-1}) \big|^{\gamma} \bigg) \\ &\leq K\, n^{\gamma/2} \mathbb{E}_{\Gamma, \xi} \bigg\{ \bigg(\sum_{j=1}^{\infty} \Gamma_j^{-2/\alpha} \varphi(\xi_j)^{-2/\alpha} \max_{\tau_n \in \widetilde{D}_n} \max_{\tau'_{n-1} \in O_{n-1}(\tau_n)} \big| h(\tau_n, \xi_j) - h(\tau'_{n-1}, \xi_j) \big|^2 \bigg)^{\gamma/2} \bigg\}. \end{aligned} \tag{4.29}$$

It can be verified that for every s, t and $x \in \mathbb{R}^N$

$$\begin{aligned} \big|h(t,x) - h(s,x)\big|^2 &= \bigg|\prod_{k=1}^{N}\big(e^{it_k x_k} - 1\big) - \prod_{k=1}^{N}\big(e^{is_k x_k} - 1\big)\bigg|^2 \prod_{k=1}^{N} \frac{1}{|x_k|^{2H_k + \frac{2}{\alpha}}} \\ &\le K \sum_{\ell=1}^{N} \bigg[\big\{1 \wedge (t_\ell - s_\ell)^2 x_\ell^2\big\} \bigg(\prod_{k<\ell}\big\{1 \wedge t_k^2 x_k^2\big\} \prod_{k>\ell}\big\{1 \wedge s_k^2 x_k^2\big\}\bigg)\bigg] \prod_{k=1}^{N} \frac{1}{|x_k|^{2H_k + \frac{2}{\alpha}}}. \end{aligned} \tag{4.30}$$

By using (4.30) and an argument similar to the proof of Proposition 4.3, one can derive

$$\begin{aligned} &\mathbb{E}_{\Gamma,\xi}\bigg\{\bigg(\sum_{j=1}^{\infty} \Gamma_j^{-2/\alpha} \varphi(\xi_j)^{-2/\alpha} \max_{\tau_n \in \widetilde{D}_n} \max_{\tau'_{n-1} \in O_{n-1}(\tau_n)} \big|h(\tau_n, \xi_j) - h(\tau'_{n-1}, \xi_j)\big|^2\bigg)^{\gamma/2}\bigg\} \\ &\le K\, 2^{-\gamma n}\, n^{-\gamma(1+\eta)(\frac{1}{2} - \frac{\gamma}{\alpha})}. \end{aligned}$$

This, together with (4.29), proves (4.27). □

References

[1] Adler, R. J. and Taylor, J. E. (2007), *Random Fields and Geometry.* Springer, New York.

[2] V. V. Anh, J. M. Angulo and M. D. Ruiz-Medina (1999), Possible long-range dependence in fractional random fields. *J. Statist. Plann. Inference* **80**, 95–110.

[3] Ayache, A., Roueff, F. and Xiao, Y. (2007), Local and asymptotic properties of linear fractional stable sheets. *C. R. Acad. Sci. Paris, Ser. A.* **344**, 389–394.

[4] Ayache, A., Roueff, F. and Xiao, Y. (2009), Linear fractional stable sheets: wavelet expansion and sample path properties. *Stoch. Process. Appl.* **119**, 1168-1197.

[5] Bernard, P. (1970), Quelques propriétés des trajectoires des fonctions aléatoires stables sur $\mathbb{R}^k$. *Ann. Inst. H. Poincaré Sect. B (N.S.)* **6**, 131–151.

[6] Biermé, H. and Lacaux, C. (2009), Hölder regularity for operator scaling stable random fields. *Stoch. Process. Appl.* to appear.

[7] Csáki, E. and Csörgő, M. (1992), Inequalities for increments of stochastic processes and moduli of continuity. *Ann. Probab.* **20**, 1031–1052.

[8] Garsia, A. M. (1972), Continuity properties of Gaussian processes with multidimensional time parameter. *Proc. 6th Berkeley Symp. Math. Statist. Probability* **II**, pp. 369–374, University of California Press, Berkeley.

[9] Khoshnevisan, D. (2002), *Multiparameter Processes: An Introduction to Random Fields.* Springer, New York.

[10] Kôno, N. and Maejima, M. (1991a), Self-similar stable processes with stationary increments. In *Stable processes and related topics (Ithaca, NY, 1990)*, 275–295, Progr. Probab., 25, Birkhäuser Boston, Boston, MA.

[11] Kôno, N. and Maejima, M. (1991b), Hölder continuity of sample paths of some self-similar stable processes. *Tokyo J. Math.* **14**, 93–100.

[12] Kwapień, S. and Rosiński, J. (2004), Sample Hölder continuity of stochastic processes and majorizing measures. In: *Seminar on Stochastic Analysis, Random Fields and Applications IV*, pp. 155–163, Progr. Probab., **58**, Birkhäuser, Basel.

[13] Marcus, M. B. and Pisier, G. (1984), Characterizations of almost surely continuous p-stable random Fourier series and strongly stationary processes. *Acta Math.* **152**, 245–301.

[14] Marcus, M. B. and Rosen, J. (2006), *Markov Processes, Gaussian Processes, and Local Times.* Cambridge University Press, Cambridge.

[15] Marcus, M. B. and Rosinski, J. (2005), Continuity and boundedness of infinitely divisible processes: a Poisson point process approach. *J. Theoret. Probab.* **18**, 109–160.

[16] Nolan, J. (1989), Local nondeterminism and local times for stable processes. *Probab. Th. Rel. Fields* **82**, 387–410.

[17] Pitman, E. J. G. (1968), On the behavior of the characteristic function of a probability sidtribution in the neighbourhood of the origin. *J. Australian Math. Soc. Series A* **8**, 422–443.

[18] Rosiński, J. (1989), On path properties of certain infinitely divisible processes. *Stochastic Proc. Appl.* **33**, 73–87.

[19] Rosiński, J. (1990), On series representations of infinitely divisible random vectors. *Ann. Probab.* **18**, 405–430.

[20] Samorodnitsky, G. (2004), Extreme value theory, ergodic theory and the boundary between short memory and long memory for stationary stable processes. *Ann. Probab.* **32**, 1438–1468.

[21] Samorodnitsky, G. and Taqqu, M. S. (1994), *Stable non-Gaussian Random Processes: Stochastic models with infinite variance.* Chapman & Hall, New York.

[22] Takashima, K. (1989), Sample path properties of ergodic self-similar processes. *Osaka Math. J.* **26**, 159–189.

[23] Talagrand, M. (2006), *Generic Chaining.* Springer-Verlag, New York.

[24] Xiao, Y. (2006), Properties of local nondeterminism of Gaussian and stable random fields and their applications. *Ann. Fac. Sci. Toulouse Math.* **XV**, 157–193.

[25] Xiao, Y. (2008), Properties of strong local nondeterminism and local times of stable random fields. *Seminar on Stochastic Analysis, Random Fields and Applications VI*, to appear.

Yimin Xiao. Department of Statistics and Probability, A-413 Wells Hall, Michigan State University, East Lansing, MI 48824, U.S.A.
E-mail: `xiao@stt.msu.edu`
URL: `http://www.stt.msu.edu/~xiaoyimi`