Multivariate Operator-Self-Similar Random Fields

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Abstract

Multivariate random fields whose distributions are invariant under operator-scalings in both time-domain and state space are studied. Such random fields are called operator-self-similar random fields and their scaling operators are characterized. Two classes of operator-self-similar stable random fields $X = \{X(t), t \in \mathbb{R}^d\}$ with values in $\mathbb{R}^m$ are constructed by utilizing homogeneous functions and stochastic integral representations.

Keywords: Random fields, operator-self-similarity, anisotropy, Gaussian random fields, stable random fields, stochastic integral representation.

1. Introduction

A self-similar process $X = \{X(t), t \in \mathbb{R}\}$ is a stochastic process whose finite-dimensional distributions are invariant under suitable scaling of the time-variable t and the corresponding $X(t)$ in the state space. It was first studied rigorously by Lamperti [16] under the name "semi-stable" process. Recall that an $\mathbb{R}^m$-valued process X is called self-similar if it is stochastically continuous (i.e. continuous in probability at each $t \in \mathbb{R}$) and for every constant $r > 0$, there exist a positive number $b(r)$ and a vector $a(r) \in \mathbb{R}^m$ such that

$$\{X(rt), t \in \mathbb{R}\} \stackrel{d}{=} \{b(r)X(t) + a(r), t \in \mathbb{R}\}, \tag{1.1}$$

where $\stackrel{d}{=}$ means equality of all finite-dimensional distributions. Lamperti [16] showed that if X is *proper* (see below for the definition) then $b(r) = r^H$ for some $H \geq 0$, which is called the self-similarity index or the Hurst index in the literature.

Self-similar processes have been under extensive investigations during the past four decades due to their theoretical importance (e.g. they often arise in functional

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limit theorems) and their applications as stochastic models in a wide range of scientific areas including physics, engineering, biology, insurance risk theory, economics, mathematical finance, just to mention a few.

The notion of self-similarity has been extended in two ways. The first extension is to allow scaling in the state space $\mathbb{R}^m$ by linear operators (namely, $b(r)$ in (1.1) is allowed to be a linear operator on $\mathbb{R}^m$) and the corresponding processes are called operator-self-similar processes in the literature. More specifically, Laha and Rohatgi [15] first extended Lamperti's notion of self-similarity by allowing $b(r)$ in (1.1) to be in the set of nonsingular positive-definite self-adjoint linear operators on $\mathbb{R}^m$. Hudson and Mason [12] subsequently allowed $b(r)$ to be an arbitrary linear operator on $\mathbb{R}^m$. The operator-self-similarity defined by Sato [28] has an additional assumption that $a(r) \equiv 0$ in (1.1). Thus the operator-self-similarity in the sense of Sato [28] is stronger than that in Hudson and Mason [12]. Various examples of operator-self-similar Gaussian and non-Gaussian processes have been constructed and studied by Hudson and Mason [12], Sato [28], Maejima and Mason [17], Mason and Xiao [19], Didier and Pipiras [9]. The aforementioned extensions to operator-self-similarity is useful for establishing functional limit theorems for multivariate time series and their statistical inference [21].

The second extension is for random fields (i.e., multi-parameter stochastic processes) which is to allow scaling by linear operators on the multiparameter "time"-variable $t \in \mathbb{R}^d$. This was done by Biermé, Meerschaert and Scheffler [4]. In their terminology, a real-valued random field $X = \{X(t),\, t \in \mathbb{R}^d\}$ is called operator-scaling if there exist a linear operator E on $\mathbb{R}^d$ with positive real parts of the eigenvalues and some constant $\beta > 0$ such that for all constant $r > 0$,

$$\big\{X(r^E t),\, t \in \mathbb{R}^d\big\} \stackrel{d}{=} \big\{r^\beta X(t),\, t \in \mathbb{R}^d\big\}. \tag{1.2}$$

In the above and in the sequel, r^E is the linear operator on $\mathbb{R}^d$ defined by $r^E = \sum_{n=0}^{\infty} \frac{(\ln r)^n E^n}{n!}$. A typical example of Gaussian random fields satisfying (1.2) is fractional Brownian sheets introduced by Kamont [13] and other examples have been constructed in [4, 31]. We mention that (1.2) leads to *anisotropy in the "time"-variable* t, which is a distinct property from those of one-parameter processes. Several authors have proposed to apply such random fields for modeling phenomena in spatial statistics, stochastic hydrology and imaging processing (see [5, 2, 8]).

In this paper, we further extend the notions of operator-self-similarity and operator-scaling to multivariate random fields by combining the aforementioned two approaches. That is, we will allow scaling of the random field in both "time"-domain and state space by linear operators. This is mainly motivated by the increasing interest in *multivariate random field models* in spatial statistics as well as in applied areas such as environmental, agricultural, and ecological sciences, where multivariate measurements are performed routinely. See Wackernagel [30], Chilés and Delfiner [6] and their combined references for further information. We also believe that the random field models constructed by Zhang [34], Gneiting, Kleiber and Schlather [11], Apanasovich and Genton [1] are locally operator-self-similar and their tangent fields are operator-self-similar in the sense of Definition 1.1 below. This problem will be investigated in a subsequent paper.

Throughout this paper, let $X = \{X(t), t \in \mathbb{R}^d\}$ be a random field with values in $\mathbb{R}^m$, where $d \geq 2$ and $m \geq 2$ are fixed integers. In the probability literature, $\mathbb{R}^d$ is often referred to as the "time"-domain (or parameter space), $\mathbb{R}^m$ as the state space and X a (d, m)-random field. We will be careful not to confuse the terminology with the space-time random fields in geostatistics.

The following definition is a natural extension of the wide-sense operator-self-similarity and operator-self-similarity in Sato [28] for one-parameter processes to (d, m)-random fields.

Definition 1.1 *Let E be a $d \times d$ matrix whose eigenvalues have positive real parts. A (d, m)-random field $X = \{X(t), t \in \mathbb{R}^d\}$ is called wide-sense operator-self-similar (w.o.s.s.) with time-variable scaling exponent E, if for any constant $r > 0$ there exist an $m \times m$ matrix $B(r)$ (which is called a state space scaling operator) and a function $a_r(\cdot) : \mathbb{R}^d \to \mathbb{R}^m$ (both $B(r)$ and $a_r(\cdot)$ are non-random) such that*

$$\big\{X(r^E t),\, t \in \mathbb{R}^d\big\} \overset{d}{=} \big\{B(r)X(t) + a_r(t), t \in \mathbb{R}^d\big\}. \tag{1.3}$$

If, in addition, $a_r(t) \equiv 0$, then X is called operator-self-similar (o.s.s.) with scaling exponent E.

Remark 1.1 Here are some remarks about Definition 1.1.

(i) If a random field X is w.o.s.s., then the consistency in (1.3) implies

$$B(r_1 r_2) = B(r_1)B(r_2) = B(r_2)B(r_1), \qquad \forall\, r_1, r_2 > 0 \tag{1.4}$$

and for all $r_1, r_2 > 0$ and $t \in \mathbb{R}^d$,

$$a_{r_1 r_2}(t) = B(r_1)a_{r_2}(t) + a_{r_1}(r_2^E t) = B(r_2)a_{r_1}(t) + a_{r_2}(r_1^E t). \tag{1.5}$$

(ii) One can also define operator-self-similarity for random fields by extending the analogous notion in Hudson and Mason [12]. Namely, we say that a (d, m)-random field $X = \{X(t), t \in \mathbb{R}^d\}$ is operator-self-similar (o.s.s.) in the sense of Hudson and Mason with time-variable scaling exponent E, if for any constant $r > 0$ there exist an $m \times m$ matrix $B(r)$ and a vector $a(r) \in \mathbb{R}^m$ such that

$$\big\{X(r^E t),\, t \in \mathbb{R}^d\big\} \overset{d}{=} \big\{B(r)X(t) + a(r), t \in \mathbb{R}^d\big\}. \tag{1.6}$$

Since the function $a(r)$ does not depend on $t \in \mathbb{R}^d$, (1.6) is stronger than w.o.s.s. in Definition 1.1, but is weaker than the operator-self-similarity.

Recall that a probability measure μ on $\mathbb{R}^m$ is *full* if its support is not contained in any proper hyperplane in $\mathbb{R}^m$. We say that a (d, m)-random field $X = \{X(t),\, t \in \mathbb{R}^d\}$ is *proper* if for each $t \neq 0$, the distribution of $X(t)$ is full. Then one can verify (see e.g. [12, p.282]) that for a proper w.o.s.s. random field, its space-scaling operator $B(r)$ must be nonsingular for all $r > 0$.

We remark that proper w.o.s.s. random fields are special cases of *group self-similar processes* introduced by Kolodyński and Rosiński [14] and can be studied by using their general framework. To recall their definition, let G be a group of transformations of a set T and, for each $(g, t) \in G \times T$, let $C(g, t) : \mathbb{R}^m \to \mathbb{R}^m$ be a bijection such that

$$C(g_1 g_2, t) = C(g_1, g_2(t)) \circ C(g_2, t), \qquad \forall g_1, g_2 \in G \text{ and } t \in T,$$

and $C(e, t) = I$. Here e is the unit element of G and I is the identity operator on $\mathbb{R}^m$. In other words, C is a cocycle for the group action $(g, t) \mapsto g(t)$ of G on T. According to Kolodyński and Rosiński [14], a stochastic process $\{X(t), t \in T\}$ taking values in $\mathbb{R}^m$ is called *G-self-similar with cocycle C* if

$$\{X\big(g(t)\big),\, t \in T\} \stackrel{d}{=} \{C(g, t)X(t),\, t \in T\}. \tag{1.7}$$

Now we take $T = \mathbb{R}^d$ and $G = \{r^E : r > 0\}$ which is a subgroup of invertible linear operators on $\mathbb{R}^d$. It is clear that if a proper (d, m)-random field $X = \{X(t), t \in \mathbb{R}^d\}$ is w.o.s.s. in the sense of Definition 1.1, then it is G-self-similar with cocycle C, where for each $g = r^E \in G$ and $t \in \mathbb{R}^d$, $C(g, t) : \mathbb{R}^m \to \mathbb{R}^m$ is defined by $C(g, t)(w) = B(r)w + a_r(t)$. Note that $C(g, t)$ is a bijection since X is proper; and it is a cocycle because of (1.4) and (1.5).

In [14], Kolodyński and Rosiński consider a strictly stable process $X = \{X(t), t \in T\}$ with values in $\mathbb{R}^m$ which is G-self-similar with cocycle C and characterize the minimal spectral representation of X (which is a kind of stochastic integral representation and always exists for strictly stable processes) in terms of a nonsingular action L of G on a measure space $(S, \mathcal{B}(S), \mu)$, where S is a Borel subset of a Polish space equipped with its Borel σ-algebra $\mathcal{B}(S)$ and μ is a σ-finite measure, and a cocycle $c : G \times S \to \{-1, 1\}$ relative to L (see Section 3 of [14] for details). They also construct strictly stable processes which are G-self-similar with cocycle C by using nonsingular actions L of G on S and $\{-1, 1\}$-valued cocycle c relative to L (see Section 4 of [14]). Their general framework provides a unified treatment for stochastic processes with various invariance properties (such as stationarity, isotropy, and self-similarity) and is particularly powerful when combined with methods from ergodic theory to study probabilistic and statistical properties of G-self-similar strictly stable processes. See, Rosiński [22, 23], Roy and Samorodnitsky [24] and Samorodnitsky [26] for recent results on stationary stable processes and random fields. It would be very interesting to pursue further this line of research for o.s.s. or more general G-self-similar stable random fields.

The main objective of the present paper is to characterize the permissible forms for the state space scaling operator (or simply the space-scaling operator) $B(r)$, which provides corresponding information on the cocycle $C(g, t)$. We will also construct two types of proper o.s.s. symmetric α-stable (d, m)-random fields by using stochastic integrals of matrix-valued deterministic functions with respect to vector-valued symmetric α-stable (SαS) random measures. Our construction method is somewhat different and less general than that of Kolodyński and Rosiński [14] who use stochastic integrals of real-valued deterministic functions with respect to a real-valued strictly stable random measure and who only require their deterministic integrands to satisfy certain recurrence equation involving a non-singular action L of G on S and a cocycle $c : G \times S \to \{-1, 1\}$ relative to L. See Proposition 4.1 in [14] for details. The deterministic integrands in our constructions are given in terms of Θ-homogeneous functions (see Definition 2.6 in [4] or Section 2 below). Hence the resulting o.s.s. stable (d, m)-random fields in this paper are natural multivariate extensions of the familiar linear and harmonizable fractional stable fields. To explore the connections between these o.s.s. stable random fields and the G-self-similar stable random fields in Proposition 4.1 of Kolodyński and Rosiński [14], we determine the non-singular action L of $G = \{r^E, r > 0\}$ on the measure space $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \lambda_d)$ and the cocycle $c : G \times \mathbb{R}^d \to \{-1, 1\}$ relative to L for the o.s.s. SαS random fields constructed in Theorems 2.5 and 2.6. These preliminary results may be helpful for applying the powerful tools developed in Rosiński [22, 23] to study operator-self-similar SαS random fields.

The rest of this paper is divided into three sections. In Section 2, we provide some preliminaries and state the main results of this paper. Theorem 2.1 proves that, under some standard conditions, the space-scaling operator $B(r)$ in (1.3) must be of the form $B(r) = r^D$ for some $D \in M(\mathbb{R}^m)$, which will be called the state space scaling exponent (or the space-scaling exponent) of X. Theorem 2.2 is an analogous result for o.s.s. random fields. Theorems 2.5 and 2.6 provide general ways for constructing proper moving-average-type and harmonizable-type o.s.s. stable (d, m)-random fields with prescribed operator-self-similarity exponents. We also describe the connection between these random fields and the G-self-similar stable random fields in [14]. In Section 3 we characterize the forms of the space-scaling operators and prove Theorems 2.1 and 2.2. The proofs of Theorem 2.5 and 2.6 are given in Section 4. It will be clear that the arguments in Hudson and Mason [12], Maejima and Mason [17] and Biermé, Meerschaert and Scheffler [4] play important roles throughout this paper.

We end this section with some notation. For any integer $n \geq 1$, we use λ_n to denote the Lebesgue measure on $\mathbb{R}^n$ and $\mathcal{B}(\mathbb{R}^n)$ the Borel algebra. The Euclidean norm and inner product in $\mathbb{R}^n$ are denoted by $|\cdot|$ and $\langle \cdot, \cdot \rangle$, respectively. Let $\mathrm{End}(\mathbb{R}^n)$ be the set of all linear operators on $\mathbb{R}^n$ or, equivalently, $n \times n$ matrices. The set of invertible linear operators in $\mathrm{End}(\mathbb{R}^n)$ is denoted by $\mathrm{Aut}(\mathbb{R}^n)$. Let $Q(\mathbb{R}^n)$ be the set of $A \in \mathrm{Aut}(\mathbb{R}^n)$ such that all eigenvalues of A have positive real parts. Let $M(\mathbb{R}^n)$ be the set of $A \in \mathrm{End}(\mathbb{R}^n)$ such that all eigenvalues of A have nonnegative real parts and every eigenvalue of A with real part equal to zero (if it exists) is a simple root of the minimal polynomial of A.

We will use $C_0, C_1, C_2, \cdots$ to denote unspecified positive finite constants which may not necessarily be the same in each occurrence.

2. Main results

Throughout this paper, $E \in Q(\mathbb{R}^d)$ is a fixed $d \times d$ matrix. E^* is the adjoint of E; and $\alpha \in (0, 2]$ is a constant.

Our first result characterizes the form of the space-scaling operator $B(r)$ for a w.o.s.s. random field.

Theorem 2.1 *Let $X = \{X(t), t \in \mathbb{R}^d\}$ be a stochastically continuous and proper w.o.s.s. random field with values in $\mathbb{R}^m$ and time-variable scaling exponent $E \in Q(\mathbb{R}^d)$. There exist a matrix $D \in M(\mathbb{R}^m)$ and a function $b_r(t) : (0, \infty) \times \mathbb{R}^d \to \mathbb{R}^m$ which is continuous at every $(r, t) \in (0, \infty) \times \mathbb{R}^d$ such that for all constants $r > 0$*

$$\left\{X(r^E t),\, t \in \mathbb{R}^d\right\} \stackrel{d}{=} \left\{r^D X(t) + b_r(t),\, t \in \mathbb{R}^d\right\}. \tag{2.1}$$

Furthermore, $X(0) = a$ a.s. for some constant vector $a \in \mathbb{R}^m$) if and only if $D \in Q(\mathbb{R}^m)$. In this latter case, we define $b_0(t) \equiv a$ for all $t \in \mathbb{R}^d$, then the function $(r, t) \mapsto b_r(t)$ is continuous on $[0, \infty) \times \mathbb{R}^d$.

The operator D will be called the state space scaling exponent (or space-scaling exponent). For a given time-variable scaling exponent $E \in Q(\mathbb{R}^d)$, the corresponding exponent D may not be unique. In order to emphasize the roles of the linear operators E and D, we call X w.o.s.s. with exponents (E, D), or simply (E, D)-w.o.s.s. By combining Theorem 2.1 with Lemma 2.4 in [28], we derive readily the following corollary. Of course, (2.2) also follows from (1.5).

Corollary 2.1 *Under the conditions of Theorem 2.1, the function $b_r(t)$ is uniquely determined by E and D. Furthermore,*

$$b_{r_1 r_2}(t) = b_{r_1}(r_2^E t) + r_1^D b_{r_2}(t) = b_{r_2}(r_1^E t) + r_2^D b_{r_1}(t) \tag{2.2}$$

for all $r_1, r_2 > 0$ and $t \in \mathbb{R}^d$.

The following corollary expresses the function $b_r(t)$ in terms of a function of t and the scaling exponents E and D.

Corollary 2.2 *Under the conditions of Theorem 2.1, there exists a continuous function $b(\cdot) : \mathbb{R}^d \backslash \{0\} \to \mathbb{R}^m$ such that the function $b_r(t)$ satisfies*

$$b_r(t) = b(r^E t) - r^D b(t), \qquad \forall r > 0 \ \text{ and } t \in \mathbb{R}^d \backslash \{0\}. \tag{2.3}$$

If, in addition, $D \in Q(\mathbb{R}^m)$ and $X(0) = a$ a.s., where $a \in \mathbb{R}^m$ is a constant vector, then we can extend the definition of $b(\cdot)$ to $\mathbb{R}^d$ by defining $b(0) = a$ such that (2.3) holds for all $r > 0$ and $t \in \mathbb{R}^d$.

The proof of this corollary is based on the polar coordinate representation of $t \in \mathbb{R}^d \backslash \{0\}$ under operator E given in [4, p.317] (the definition is recalled below) and will be given in Section 3.

The next theorem is an analogue of Theorem 2.1 for o.s.s. random fields.

Theorem 2.2 *Let $X = \{X(t), t \in \mathbb{R}^d\}$ be a stochastically continuous and proper random field with values in $\mathbb{R}^m$.*

(i) If X is o.s.s. with time-variable scaling exponent $E \in Q(\mathbb{R}^d)$, then there exists a matrix $D \in M(\mathbb{R}^m)$ such that for all $r > 0$

$$\{X(r^E t),\, t \in \mathbb{R}^d\} \stackrel{d}{=} \{r^D X(t),\, t \in \mathbb{R}^d\}. \tag{2.4}$$

Moreover, $D \in Q(\mathbb{R}^m)$ if and only if $X(0) = 0$ a.s.

(ii) If X is o.s.s. with time-variable scaling exponent $E \in Q(\mathbb{R}^d)$ in the sense of Hudson and Mason, then there exist a matrix $D \in M(\mathbb{R}^m)$ and a continuous function $b(r) : (0, \infty) \to \mathbb{R}^m$ such that for all constants $r > 0$

$$\{X(r^E t),\, t \in \mathbb{R}^d\} \stackrel{d}{=} \{r^D X(t) + b(r),\, t \in \mathbb{R}^d\}. \tag{2.5}$$

A (d, m)-random field $X = \{X(t), t \in \mathbb{R}^d\}$ is called operator-self-similar with exponents (E, D) (or (E, D)-o.s.s.) if (2.4) holds. By Corollary 2.2, we see that if X is a w.o.s.s. (d, m)-random field as in Theorem 2.1, then the (d, m)-random field $Y = \{X(t) - b(t), t \in \mathbb{R}^d\backslash\{0\}\}$ is operator-self-similar with exponents (E, D). Using the terminology of Sato [28], we also call the function $b(t)$ in Corollary 2.2 the drift function of (d, m)-random field X.

Recall that a (d, m)-random field X is said to have stationary increments if for all $h \in \mathbb{R}^d$,

$$\{X(t + h) - X(h),\, t \in \mathbb{R}^d\} \stackrel{d}{=} \{X(t),\, t \in \mathbb{R}^d\}. \tag{2.6}$$

Now we turn to construction of interesting examples of stable o.s.s. (d, m)-random fields with stationary increments, by using stochastic integrals with respect to a stable random measure. We refer to Samorodnitsky and Taqqu [27] for a systematic account on the latter. For simplicity we will only consider symmetric α-stable (SαS) random fields and the main idea comes from [4], [19] and [17]. By using stochastic integral with respect to a strictly stable random measure one can extend the construction to obtain strictly stable o.s.s. (d, m)-random fields. Kolodyński and Rosiński [14] use this more general approach.

For any given operators $E \in Q(\mathbb{R}^d)$ and $D \in Q(\mathbb{R}^m)$ we construct (E, D)-o.s.s. α-stable random fields by using stochastic integrals with respect to a symmetric α-stable random vector measure (when $\alpha = 2$ the resulting o.s.s. random fields are Gaussian). For this purpose, we recall briefly the definitions of stochastic integrals with respect to vector-valued α-stable random measures.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the underlying probability space and let $L^0(\Omega)$ be the set of all $\mathbb{R}^m$-valued random vectors defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Let S_{m-1} be the unit sphere in $\mathbb{R}^m$ with the Borel algebra $\mathcal{B}(S_{m-1})$.

Let K be a σ-finite measure on $\mathbb{R}^d \times S_{m-1}$ such that for any $A \in \mathcal{B}(\mathbb{R}^d)$, $K(A \times \cdot)$ is a symmetric finite measure on $(S_{m-1}, \mathcal{B}(S_{m-1}))$. Denote

$$\mathcal{M} := \{A \in \mathcal{B}(\mathbb{R}^d) : K(A, S_{m-1}) < \infty\}.$$

We first give the definition of a vector-valued symmetric α-stable (SαS) random measure.

Definition 2.1 *An $\mathbb{R}^m$-valued $S\alpha S$ random measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ with control measure K is an independently scattered σ-additive $\mathbb{R}^m$-valued set function $M : \mathcal{M} \to L^0(\Omega)$ such that, for every $A \in \mathcal{M}$, the random vector $(M^1(A), \cdots, M^m(A))$ is jointly $S\alpha S$ with spectral measure $K(A, \cdot)$. Here, the meaning of "independently scattered" and "σ-additive" is the same as in Section 3.3 of [27].*

One can apply Kolmogorov's extension theorem to show that $\mathbb{R}^m$-valued SαS random measure M in Definition 2.1 exists, with finite-dimensional distributions characterized by

$$\begin{aligned} &\mathbb{E}\exp\left\{ i\sum_{j=1}^{k}\langle\theta_j, M(A_j)\rangle \right\} \\ &\qquad = \exp\left\{ -\int_{\mathbb{R}^d}\int_{S_{m-1}} \left| \sum_{j=1}^{k}\sum_{l=1}^{m} s_l\theta_{j,l}\mathbf{1}_{A_j}(x)\right|^{\alpha} K(\mathrm{d}x, \mathrm{d}s)\right\}, \end{aligned} \tag{2.7}$$

where $A_j \in \mathcal{M}$ and $\theta_j = (\theta_{j,1}, \cdots, \theta_{j,m}) \in \mathbb{R}^m$ for all $k \geq 1$ and $j = 1, \cdots, k$.

In this paper, unless stated otherwise, the control measure K will always be assumed to have the form $K(A, B) = \lambda_d(A)\Gamma(B)$ for all $A \in \mathcal{B}(\mathbb{R}^d)$ and $B \in \mathcal{B}(S_{m-1})$, where λ_d is the Lebesgue measure on $\mathbb{R}^d$ and $\Gamma(\cdot)$ is the normalized uniform measure on S_{m-1} such that for all $\theta = (\theta_1, \cdots, \theta_m) \in \mathbb{R}^m$,

$$\int_{S_{m-1}} \left|\sum_{l=1}^{m} s_l\theta_l\right|^{\alpha} \Gamma(\mathrm{d}s) = |\theta|^{\alpha}.$$

Therefore, for disjoint sets $A_j \in \mathcal{M}$, $j = 1, 2, \cdots, k$, Eq. (2.7) can be written as

$$\mathbb{E}\exp\left\{ i\sum_{j=1}^{k}\langle\theta_j,\, M(A_j)\rangle \right\} = \exp\left\{ -\sum_{j=1}^{k}\lambda_d(A_j)|\theta_j|^{\alpha}\right\}. \tag{2.8}$$

For any real $m \times m$ matrix Q, let $\|Q\| := \max_{|x|=1}|Qx|$ be the operator norm of Q. It is easy to see that for $Q_1,\ Q_2 \in \mathrm{End}(\mathbb{R}^m)$, $\|Q_1Q_2\| \leq \|Q_1\| \cdot \|Q_2\|$. The following theorem is an extension of Theorem 4.1 in [17] and defines stochastic integrals of matrix-valued functions with respect to a vector-valued SαS random measure.

Theorem 2.3 *Let $\{Q(u), u \in \mathbb{R}^d\}$ be a family of real $m \times m$-matrices. If $Q(u)$ is $\mathcal{B}(\mathbb{R}^d)$-measurable and $\int_{\mathbb{R}^d}\|Q(u)\|^{\alpha}\mathrm{d}u < \infty$, then the stochastic integral*

$$I(Q) := \int_{\mathbb{R}^d} Q(u)\, M(\mathrm{d}u)$$

is well defined and it is a symmetric α-stable vector in $\mathbb{R}^m$ with characteristic function

$$\mathbb{E}\Big[\mathrm{e}^{i\langle\theta, I(Q)\rangle}\Big] = \exp\left\{ -\int_{\mathbb{R}^d} \left|Q(u)^*\theta\right|^{\alpha}\mathrm{d}u\right\}, \qquad \forall\, \theta \in \mathbb{R}^m. \tag{2.9}$$

It follows from (2.9) and Lemma 3.2 below that if the matrix $Q(u)$ is invertible for u in a set of positive λ_d-measure, then the distribution of $I(Q)$ is full. This fact is useful for constructing proper SαS random fields.

One can also define stochastic integrals of complex matrix-valued functions with respect to a complex vector-valued SαS random measure $\widetilde{M}$ defined as follows. Let $\overline{M}$ be an $\mathbb{R}^{2m}$-valued SαS-random measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ with control measure $K = \lambda_d \times \Gamma$, where Γ is the normalized uniform measure on S_{2m-1}. Define the complex-valued SαS-random measures $\widetilde{M}_k = \overline{M}_k + i\overline{M}_{m+k}$ for all $k = 1, \cdots, m$. Then $\widetilde{M} = (\widetilde{M}_1, \cdots, \widetilde{M}_m)$ is a $\mathbb{C}^m$-valued SαS-random measure with control measure K. Its real and imaginary parts are $\widetilde{M}_R = (\overline{M}_1, \cdots, \overline{M}_m)$ and $\widetilde{M}_I = (\overline{M}_{m+1}, \cdots, \overline{M}_{2m})$, respectively. The following theorem defines stochastic integrals of complex matrix-valued functions with respect to $\widetilde{M}$.

Theorem 2.4 *Let $\{\widetilde{Q}_1(u),\, u \in \mathbb{R}^d\}$ and $\{\widetilde{Q}_2(u),\, u \in \mathbb{R}^d\}$ be two families of real $m \times m$-matrices. Let $\widetilde{Q}(u) = \widetilde{Q}_1(u) + i\widetilde{Q}_2(u)$ for all $u \in \mathbb{R}^d$. If $\widetilde{Q}_1(u)$ and $\widetilde{Q}_2(u)$ are $\mathcal{B}(\mathbb{R}^d)$-measurable and $\int_{\mathbb{R}^d}(\|\widetilde{Q}_1(u)\|^\alpha + \|\widetilde{Q}_2(u)\|^\alpha)\,\mathrm{d}u < \infty$, then*

$$\widetilde{I}(\widetilde{Q}) := \mathrm{Re} \int_{\mathbb{R}^d} \widetilde{Q}(u)\, \widetilde{M}(\mathrm{d}u)$$

is well defined and it is a symmetric α-stable vector in $\mathbb{R}^m$ with its characteristic function given by

$$\begin{aligned} &\mathbb{E}\Big[\mathrm{e}^{i\langle \theta, \widetilde{I}(\widetilde{Q})\rangle}\Big] \\ &= \exp\bigg\{ - \int_{\mathbb{R}^d} \bigg(\sqrt{\big|\widetilde{Q}_1(u)^*\theta\big|^2 + \big|\widetilde{Q}_2(u)^*\theta\big|^2} \bigg)^\alpha \mathrm{d}u \bigg\}, \qquad \forall\, \theta \in \mathbb{R}^m. \end{aligned} \tag{2.10}$$

It follows from (2.10) and Lemma 3.2 below that if the matrix $\widetilde{Q}_1(u)$ or $\widetilde{Q}_2(u)$ is invertible for u in a set of positive λ_d-measure, then the distribution of $\widetilde{I}(\widetilde{Q})$ is full.

Based on the above stochastic integrals, we can construct moving-average type or harmonizable-type α-stable random fields by choosing suitable functions Q and $\widetilde{Q}$. In order to obtain o.s.s. random fields, we will make use of the Θ-homogeneous functions and the (β, Θ)-admissible functions as in [4].

Suppose $\Theta \in Q(\mathbb{R}^d)$ with real parts of the eigenvalues $0 < a_1 < a_2 < \cdots < a_p$ for $p \le d$. Let q denote the trace of Θ. It follows from [4, p.314] that every $x \in \mathbb{R}^d \setminus \{0\}$ can be written uniquely as $x = \tau(x)^\Theta l(x)$ for some radial part $\tau(x) > 0$ and some direction $l(x) \in \Sigma_0$ such that the functions $x \mapsto \tau(x)$ and $x \mapsto l(x)$ are continuous, where $\Sigma_0 = \{x \in \mathbb{R}^d,\, \tau(x) = 1\}$. It is well-known that $\tau(x) = \tau(-x)$ and $\tau(r^\Theta x) = r\tau(x)$ for all $r > 0$ and $x \in \mathbb{R}^d$. Moreover, Σ_0 is compact; $\tau(x) \to \infty$ as $|x| \to \infty$ and $\tau(x) \to 0$ as $|x| \to 0$. In addition, Lemma 2.2 in [4] shows that there exists a constant $C_0 \ge 1$ such that for all $x,\, y \in \mathbb{R}^d$

$$\tau(x+y) \le C_0(\tau(x) + \tau(y)). \tag{2.11}$$

For convenience, we call $(\tau(x), l(x))$ the polar coordinates of x under operator Θ. According to Definition 2.6 in [4], a function $\phi : \mathbb{R}^d \to \mathbb{C}$ is said to be Θ-homogeneous if $\phi(r^\Theta x) = r\phi(x)$ for all $r > 0$ and $x \in \mathbb{R}^d \setminus \{0\}$. Obviously, if ϕ is Θ-homogeneous, continuous on $\mathbb{R}^d$ and takes positive values on $\mathbb{R}^d \setminus \{0\}$, then $\phi(0) = 0$,

$$M_\phi = \max_{\theta \in \Sigma_0} \phi(\theta) > 0 \quad \text{and} \quad m_\phi = \min_{\theta \in \Sigma_0} \phi(\theta) > 0. \tag{2.12}$$

Let $\beta > 0$. Recall from Definition 2.7 in [4] that a function $\psi : \mathbb{R}^d \to [0, \infty)$ is called (β, Θ)-admissible, if $\psi(x) > 0$ for all $x \neq 0$ and for any $0 < A < B$ there exists a positive constant $C_1 > 0$ such that, for $A \leq |y| \leq B$,

$$\tau(x) \leq 1 \;\Rightarrow\; |\psi(x + y) - \psi(y)| \leq C_1 \tau(x)^\beta.$$

For any given matrices $E \in Q(\mathbb{R}^d)$ and $D \in Q(\mathbb{R}^m)$, Theorem 2.5 provides a class of moving-average-type o.s.s. α-stable random fields with prescribed self-similarity exponents (E, D).

Theorem 2.5 *Suppose $\phi : \mathbb{R}^d \mapsto [0, \infty)$ is an E-homogeneous, (β, E)-admissible function for some constant $\beta > 0$. Let q be the trace of E, H be the maximum of the real parts of the eigenvalues of $D \in Q(\mathbb{R}^m)$ and let I be the identity operator in $\mathbb{R}^m$. If $H < \beta$, then the random field*

$$X_\phi(x) = \int_{\mathbb{R}^d} \left[\phi(x - y)^{D - qI/\alpha} - \phi(-y)^{D - qI/\alpha} \right] M(\mathrm{d}y), \quad x \in \mathbb{R}^d \tag{2.13}$$

is well defined, where the stochastic integral in (2.13) is defined as in Theorem 2.3. Furthermore, $X_\phi = \{X_\phi(x), x \in \mathbb{R}^d\}$ is a stochastically continuous (E, D)-o.s.s. $S\alpha S$-random field with stationary increments.

Remark 2.1 We can choose E and D to ensure that the SαS-random field X is proper. A sufficient condition is that q/α is not an eigenvalue of D. This implies that, for every $x \in \mathbb{R}^d$, the operator $\phi(x - y)^{D - qI/\alpha} - \phi(-y)^{D - qI/\alpha}$ is invertible for y in a subset of $\mathbb{R}^d$ with positive Lebesgue measure, which ensures that the distribution of $X_\phi(x)$ is full.

When $m = 1$ and $D = HI$, Theorem 2.5 reduces to Theorem 3.1 in Biermé, Meerschaert and Scheffler [4]. For a general $D \in Q(\mathbb{R}^m)$, the following example of X_ϕ is instructive. Let $E = (e_{ij})$ be the diagonal matrix in $Q(\mathbb{R}^d)$ with $e_{jj} = \gamma_j^{-1}$, where $\gamma_j \in (0, 1)$ $(1 \leq j \leq d)$ are constants. It can be verified that there exists a constant $C_2 \geq 1$ such that the corresponding radial part $\tau(x)$ satisfies

$$C_2^{-1} \sum_{j=1}^d |x_j|^{\gamma_j} \leq \tau(x) \leq C_2 \sum_{j=1}^d |x_j|^{\gamma_j} \tag{2.14}$$

for all $x \in \mathbb{R}^d$. Note that the function $\phi(x) = \sum_{j=1}^d |x_j|^{\gamma_j}$ is E-homogeneous and (β, E)-admissible with $\beta = 1$. This latter assertion follows from (2.14) and the elementary inequality $|x + y|^\gamma \leq |x|^\gamma + |y|^\gamma$ if $\gamma \in (0, 1)$. Let $D \in Q(\mathbb{R}^m)$ be as in Theorem 2.5, then $X_\phi = \{X_\phi(x),\, x \in \mathbb{R}^d\}$ defined by

$$X_\phi(x) = \int_{\mathbb{R}^d} \left[\left(\sum_{j=1}^d |x_j - y_j|^{\gamma_j} \right)^{D - qI/\alpha} - \left(\sum_{j=1}^d |y_j|^{\gamma_j} \right)^{D - qI/\alpha} \right] M(\mathrm{d}y)$$

is an (E, D)-o.s.s. $S\alpha S$ random field with stationary increments. Moreover, since $H < 1$ and $q/\alpha > 1$ (we have assumed $d \geq 2$ in this paper), we see that X_ϕ is proper.

Similarly to Theorem 2.5, we can construct harmonizable-type o.s.s. $S\alpha S$ stable random fields as follows.

Theorem 2.6 *Suppose $\psi : \mathbb{R}^d \mapsto [0, \infty)$ is a continuous, E^*-homogeneous function such that $\psi(x) \neq 0$ for $x \neq 0$. Let q be the trace of E and let I be the identity operator in $\mathbb{R}^m$. If $D \in Q(\mathbb{R}^m)$ and its maximal real part of the eigenvalues $H < a_1$, where a_1 is the minimal real part of the eigenvalues of E, then the random field*

$$\widetilde{X}_\psi(x) = \mathrm{Re} \int_{\mathbb{R}^d} \left(\mathrm{e}^{i\langle x,y\rangle} - 1\right) \psi(y)^{-D-qI/\alpha}\, \widetilde{M}(\mathrm{d}y), \quad \forall x \in \mathbb{R}^d, \tag{2.15}$$

is well defined, where the stochastic integral in (2.15) is defined as in Theorem 2.4. Furthermore, $\widetilde{X}_\psi = \{\widetilde{X}_\psi(x), x \in \mathbb{R}^d\}$ is a stochastically continuous, proper (E, D)-o.s.s. $S\alpha S$-random field with stationary increments.

Remark 2.2 Unlike in Theorem 2.5, $\widetilde{X}_\psi$ in Theorem 2.6 is always proper.

Theorem 2.6 is a multivariate extension of Theorem 4.1 and Corollary 4.2 of Biermé, Meerschaert and Scheffler [4]. To give a representative of the harmonizable-type o.s.s. in Theorem 2.6, again we take $E = (e_{ij}) \in Q(\mathbb{R}^d)$ to be the diagonal matrix as above. Let $\psi(x) = \sum_{j=1}^d |x_j|^{\gamma_j}$, which is E^*-homogeneous. Then, for any $D \in Q(\mathbb{R}^m)$ with its maximal real parts of the eigenvalues $H < \min\{\gamma_j^{-1}\}$, the SαS-random field $\widetilde{X}_\psi = \{\widetilde{X}_\psi(x),\, x \in \mathbb{R}^d\}$ defined by

$$\widetilde{X}_\psi(x) = \mathrm{Re} \int_{\mathbb{R}^d} \frac{\mathrm{e}^{i\langle x,y\rangle} - 1}{\left(\sum_{j=1}^d |y_j|^{\gamma_j}\right)^{D+qI/\alpha}} \widetilde{M}(\mathrm{d}y) \tag{2.16}$$

is proper and (E, D)-o.s.s. with stationary increments. In the special case of $D = I$, the stable random field $\widetilde{X}_\psi$ has been studied in Xiao [32]. We believe that the argument in proving Theorem 3.4 in [32] can be applied to show that $\widetilde{X}_\psi$ has the property of strong local nondeterminism, which is useful for establishing the joint continuity of the local times of $\widetilde{X}_\psi$.

The o.s.s. SαS (d, m)-random fields in Theorems 2.5 and 2.6 provide concrete examples for the G-self-similar stable random fields in Proposition 4.1 of Kolodyński and Rosiński [14]. Recall that the o.s.s. SαS random fields in Theorems 2.5 and 2.6 are G-self-similar with cocycle C, where $G = \{r^E, r > 0\}$ and $C(r, t) = r^D$ for every $r > 0$ and $t \in \mathbb{R}^d$. In the following we provide non-singular actions of $G = \{r^E, r > 0\}$ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \lambda_d)$ and cocycles $c : G \times \mathbb{R}^d \to \{-1, 1\}$ (or $\{z \in \mathbb{C} : |z| = 1\}$ in the complex case) such that the integrands in (2.13) and (2.15) satisfy the recurrence equation (4.1) in Kolodyński and Rosiński [14].

For the o.s.s. SαS random field X_ϕ in Theorem 2.5, the non-singular action of G on $\mathbb{R}^d$ is $L_r(s) = r^E s$, and the cocycle $c(r, x) \equiv 1$. A change of variable shows that

$$\frac{\mathrm{d}(\lambda_d \circ L_{r^{-1}})}{\mathrm{d}\lambda_d} = r^q, \tag{2.17}$$

where q is the trace of E. By using (2.17) and the E-homogeneity of ϕ one can verify that the family of integrands $\{f_x, x \in \mathbb{R}^d\}$ in Theorem 2.5, where

$$f_x(y) = \phi(x-y)^{D-qI/\alpha} - \phi(-y)^{D-qI/\alpha}$$

is a matrix-valued function, satisfies

$$f_{r^E x}(y) = c(r, L_{r^{-1}}y)\left\{\frac{\mathrm{d}(\lambda_d \circ L_{r^{-1}})}{\mathrm{d}\lambda_d}\right\}^{1/\alpha} C(r,x) f_x \circ L_{r^{-1}}(y), \qquad \forall y \in \mathbb{R}^d, \quad (2.18)$$

which is an analogue of the recurrence equation (4.1) in Kolodyński and Rosiński [14].

For the o.s.s. SαS random field $\widetilde{X}_\psi$ in Theorem 2.6, the non-singular action of G on $\mathbb{R}^d$ is $\widetilde{L}_r(s) = r^{E^*}s$ and the cocycle $c(r,x) \equiv 1$. Then, by using (2.17) and the E^*-homogeneity of ψ one can verify that the family of integrands $\{\widetilde{f}_x, x \in \mathbb{R}^d\}$, where

$$\widetilde{f}_x(y) = \left(\mathrm{e}^{i\langle x,y\rangle} - 1\right)\psi(y)^{-D-qI/\alpha},$$

satisfies the recurrence equation (2.18) with L being replaced by $\widetilde{L}$.

3. Characterization of space-scaling exponents: Proofs of Theorems 2.1 and 2.2

In this section, we prove Theorem 2.1. The main idea of our proof is originated from [12] and [28]. We will make use of the following lemmas which are taken from [28] and [29], respectively.

Lemma 3.1 ([28, Lemma 2.6])
For any integer $n \geq 1$, $H \in Q(\mathbb{R}^n)$ if and only if $\lim_{r\downarrow 0} r^H x = 0$ for every $x \in \mathbb{R}^n$. $H \in M(\mathbb{R}^n)$ if and only if $\limsup_{r\downarrow 0} |r^H x| < \infty$ for every $x \in \mathbb{R}^n$.

Lemma 3.2 ([29, Proposition 1])
A probability measure μ on $\mathbb{R}^n$ is not full if and only if there exists a vector $y \in \mathbb{R}^n\backslash\{0\}$ such that $|\widehat{\mu}(cy)| = 1$ for all $c \in \mathbb{R}$, where $\widehat{\mu}$ is the characteristic function of μ.

For $r > 0$ and $E \in Q(\mathbb{R}^d)$ fixed, define G_r to be the set of $A \in \mathrm{Aut}(\mathbb{R}^m)$ such that $\{X(r^E t),\, t \in \mathbb{R}^d\} \stackrel{d}{=} \{AX(t) + b(t),\, t \in \mathbb{R}^d\}$, for some function $b : \mathbb{R}^d \to \mathbb{R}^m$. Let $G = \bigcup_{r>0} G_r$.

Lemma 3.3 *The set G is a subgroup of $Aut(\mathbb{R}^m)$. In particular, the identity matrix $I \in G_1$; $A \in G_r$ implies $A^{-1} \in G_{1/r}$; $A \in G_r$ and $B \in G_s$ imply $AB \in G_{sr}$.*

Proof. This can be verified by using the above definition and the proof is elementary. We omit the details here. □

Lemma 3.4 *The following statements are equivalent:*
(1) *There exist a sequence $\{r_n, n \geq 1\}$ with $r_n \downarrow 0$ and $A_n \in G_{r_n}$ such that A_n tends*

to $A \in Aut(\mathbb{R}^m)$.
(2) $\{X(t),\, t \in \mathbb{R}^d\} \overset{d}{=} \{X(0) + \phi(t),\, t \in \mathbb{R}^d\}$, *where* ϕ *is unique and continuous on* $\mathbb{R}^d$.
(3) $G = G_s$ *for all* $s > 0$.
(4) $G_s \cap G_r \neq \emptyset$ *for some distinct* $s, r > 0$.

Proof. (1)⇒(2). Assume (1) holds then we have that $\{X(r_n^E t),\, t \in \mathbb{R}^d\} \overset{d}{=} \{A_n X(t) + b_{r_n}(t),\ t \in \mathbb{R}^d\}$. By Lemma 3.1 and the stochastic continuity of X, we derive that there is a function $b(t)$ such that $\{X(t),\, t \in \mathbb{R}^d\} \overset{d}{=} \{A^{-1}X(0) - A^{-1}b(t),\, t \in \mathbb{R}^d\}$ and, in particular, $X(0) \overset{d}{=} A^{-1}X(0) - A^{-1}b(0)$. This yields (2) with $\phi(t) = A^{-1}b(0) - A^{-1}b(t)$. The continuity of ϕ follows from the stochastic continuity of X and the uniqueness of ϕ follows from Lemma 2.4 in [28].

(2)⇒(3) Suppose (2) holds and $A \in G_r$. Then

$$\{X(0) + \phi(r^E t),\, t \in \mathbb{R}^d\} \overset{d}{=} \{X(r^E t),\, t \in \mathbb{R}^d\} \overset{d}{=} \{AX(t) + b_r(t),\, t \in \mathbb{R}^d\}.$$

Hence for all positive numbers $s \neq r$,

$$\begin{aligned}\{X(s^E t),\, t \in \mathbb{R}^d\} &\overset{d}{=} \{X(0) + \phi(s^E t),\, t \in \mathbb{R}^d\} \\ &\overset{d}{=} \{AX(t) + b_r(t) - \phi(r^E t) + \phi(s^E t),\, t \in \mathbb{R}^d\}.\end{aligned}$$

Thus $A \in G_s$, which shows $G_r \subset G_s$. By symmetry, we also have $G_s \subset G_r$. Therefore $G_r = G_s$ for all $s \neq r$, and hence $G_r = G$.

(3)⇒(4) This is obvious.

(4)⇒(1) Now we assume (4) holds for some $s < r$. Let $A \in G_s \cap G_r$. Since $\{X(s^E t),\, t \in \mathbb{R}^d\} \overset{d}{=} \{AX(t) + b_s(t),\, t \in \mathbb{R}^d\}$ and $\{X(r^E t),\, t \in \mathbb{R}^d\} \overset{d}{=} \{AX(t) + b_r(t),\, t \in \mathbb{R}^d\}$, we obtain that $\{X(s^E t),\, t \in \mathbb{R}^d\} \overset{d}{=} \{X(r^E t) + \psi(t),\, t \in \mathbb{R}^d\}$ for some function $\psi : \mathbb{R}^d \to \mathbb{R}^m$. Then

$$\{X((s/r)^E t),\, t \in \mathbb{R}^d\} \overset{d}{=} \{X(t) + \psi(r^{-E} t),\, t \in \mathbb{R}^d\}. \tag{3.1}$$

This shows that $I \in G_{s/r}$. Let $c_n = (s/r)^n$. By iterating (3.1) we derive that

$$\{X(c_n^E t),\, t \in \mathbb{R}^d\} \overset{d}{=} \{X(t) + \psi_n(t),\, t \in \mathbb{R}^d\},$$

where $\psi_n(t) = \sum_{i=0}^{n-1} \psi(c_i^E r^{-E} t)$. Hence $I \in G_{c_n}$ for all $n \geq 0$. Since $c_n \to 0$ and $I \in \mathrm{Aut}(\mathbb{R}^m)$, we arrive at (1). □

Lemma 3.5 *Assume* $G \neq G_s$ *for some* $s > 0$. *If* $A_n \in G_{r_n}$, $A \in Aut(R^m)$ *and* $A_n \to A$ *as* $n \to \infty$, *then the sequence* $\{r_n\}$ *converges to some* $r > 0$ *as* $n \to \infty$ *and* $A \in G_r$.

Proof. Suppose that $\{r_{n_k}\}$ is a subsequence of $\{r_n\}$ and that $\{r_{n_k}\}$ converges to some $r \in [0, \infty]$. Then $0 < r < \infty$. In fact if $r = 0$, then $A_{n_k} \to A$ and Lemma

3.4 imply $G = G_s$ for all $s > 0$, which is a contradiction to the assumption. On the other hand, if $r \to \infty$, then $A_{n_k}^{-1} \in G_{r_{n_k}^{-1}} \to A^{-1}$ and $r_{n_k}^{-1} \to 0$. By Lemma 3.4, we also get a contradiction. It follows from

$$\{X(r_{n_k}^E t),\, t \in \mathbb{R}^d\} \stackrel{d}{=} \{A_{n_k} X(t) + b_{r_{n_k}}(t),\, t \in \mathbb{R}^d\}$$

and the stochastic continuity of X that

$$\{X(r^E t),\, t \in \mathbb{R}^d\} \stackrel{d}{=} \{A X(t) + b_r(t),\, t \in \mathbb{R}^d\}$$

for some function b_r. Therefore $A \in G_r$ and hence from Lemma 3.4 we infer that all convergent subsequences of $\{r_n\}$ have the same limit r. Consequently, $\{r_n\}$ converges to $r > 0$. □

From Lemma 3.4 and Lemma 3.5, we derive the following result.

Corollary 3.1 *If $G \neq G_s$ for some $s > 0$, then G_1 is not a neighborhood of I in G.*

Proof. From Lemma 3.4, the assumption that $G \neq G_s$ for some $s > 0$ implies $G_r \cap G_1 = \emptyset$ for all $r \neq 1$. Therefore, to prove the corollary, it is enough to show that there exists a sequence $A_n \in G_{r_n}$ such that $r_n \neq 1$ and $A_n \to I$ as $n \to \infty$. This can be proved as follows.

Let $\{r_n\}$ be a sequence with $r_n \neq 1$ and $r_n \to 1$ as $n \to \infty$. Take $B_n \in G_{r_n}$. Then by the convergence of types theorem (see, e.g., [29, p.55]), $\{B_n\}$ is pre-compact in $\mathrm{Aut}(\mathbb{R}^m)$. Hence we can find a subsequence $\{B_{n_k}\}$ such that $B_{n_k} \to B \in \mathrm{Aut}(\mathbb{R}^m)$. By Lemma 3.5, we have $B \in G_1$ and thus $B^{-1} \in G_1$. Furthermore, by Lemma 3.3, $G_{r_{n_k}} \ni B^{-1} B_{n_k} \to I \in G_1$. Let $A_k = B^{-1} B_{n_k}$, then the sequence $\{A_k\}_k$ is what we need. □

Using the above results, we give the proof of Theorem 2.1 as follows.

Proof of Theorem 2.1. From Lemma 3.4, we only need to consider two cases.

Case 1: $G = G_s$ for all $s > 0$. By Part (2) of Lemma 3.4, we derive that for all constant $c > 0$,

$$\begin{aligned}\big\{X(r^E t),\, t \in \mathbb{R}^d\big\} &\stackrel{d}{=} \{X(0) + \phi(r^E t),\, t \in \mathbb{R}^d\} \\ &\stackrel{d}{=} \{X(t) + \phi(r^E t) - \phi(t),\, t \in \mathbb{R}^d\big\}.\end{aligned}$$

Hence (2.1) holds with $D = \mathbf{0}$, which is the matrix with all entries equal 0, and $b_r(t) = \phi(r^E t) - \phi(t)$.

Case 2: $\{G_s, s > 0\}$ is a disjoint family. In this case, G is a closed subgroup of $\mathrm{Aut}(\mathbb{R}^m)$. Define η: $G \to \mathbb{R}$ by $\eta(A) = \ln s$ if $A \in G_s$. It is well-defined and, from Lemma 3.3 and Lemma 3.5, is a continuous homomorphism between the group G and the group $(\mathbb{R}, +)$. Let $T(G)$ be the tangent space to G at the identity I. It is well-known that the image of $T(G)$ under the exponential map is a neighborhood of the identity of G; see [7, p.110.]. Therefore, by Corollary 3.1, there exists $A \in T(G)$ such that $\mathrm{e}^A \notin G_1$. Furthermore, by the same arguments used in the proof of

Theorem 2.1 of [12, p.288], we know there is a $D \in \text{End}(\mathbb{R}^m)$ such that $s^D \in G_s$ for every $s > 0$. This implies that

$$\{X(r^E t),\, t \in \mathbb{R}^d\} \stackrel{d}{=} \{r^D X(t) + b_r(t),\, t \in \mathbb{R}^d\}. \tag{3.2}$$

for some function $b_r(t)$. Note that the linear operators r^E and r^D are continuous on $r \in (0, \infty)$. By the convergence of types theorem, it is not hard to see that $b_r(t)$ is continuous in $(r,t) \in (0, \infty) \times \mathbb{R}^d$. In order to verify the fact $D \in M(\mathbb{R}^m)$, we let $\{X_0(t), t \in \mathbb{R}^d\}$ be the symmetrization of $\{X(t), t \in \mathbb{R}^d\}$ and let $\mu(t)$ be the distribution of $X_0(t)$. Then by (3.2)

$$\mu(r^E t) = r^D \mu(t),$$

for all $r > 0$ and $t \in \mathbb{R}^d$. Therefore, the characteristic function of $\mu(t)$, denoted by $\widehat{\mu}_t(z)\,(z \in \mathbb{R}^m)$, satisfies

$$\widehat{\mu}_{r^E t}(z) = \widehat{\mu}_t(r^{D^*} z) \tag{3.3}$$

for every $r > 0$ and $t \in \mathbb{R}^d$, where D^* is the adjoint of D. Suppose $D \notin M(\mathbb{R}^m)$, then $D^* \notin M(R^m)$ either. By Lemma 3.1, we can find $r_n \to 0$ and $z_0 \in \mathbb{R}^m$ such that $|r_n^{D^*} z_0| \to \infty$. Let $\alpha_n = |r_n^{D^*} z_0|^{-1}$. Then by choosing a subsequence if necessary, we have that $\alpha_n r_n^{D^*} z_0$ converges to some $z_1 \in \mathbb{R}^m$ with $|z_1| = 1$. From (3.3), it follows that for all $c \in \mathbb{R}$

$$\widehat{\mu}_{r_n^E t}(c\,\alpha_n z_0) = \widehat{\mu}_t(c\,\alpha_n r_n^{D^*} z_0). \tag{3.4}$$

Letting $n \to \infty$, since Lemma 3.1 implies $r_n^E t \to 0$, by the continuity of $\widehat{\mu}_t(\cdot)$, we have that $\widehat{\mu}_t(cz_1) = \widehat{\mu}_0(0) = 1$ for all $c \in \mathbb{R}$. It follows from Lemma 3.2 that $X(t)$ is not full in $\mathbb{R}^m$. This contradicts the hypothesis that X is proper. Consequently, the matrix D in (3.2) belongs to $M(\mathbb{R}^m)$ and the function $b_r(t)$ is continuous in $(0, \infty) \times \mathbb{R}^d$.

Now we prove that $X(0) = a$ a.s. for some constant vector $a \in \mathbb{R}^m$) if and only if $D \in Q(\mathbb{R}^m)$. From Lemma 3.1, it can be shown that, if X is a stochastically continuous w.o.s.s. random field and $D \in Q(\mathbb{R}^m)$, then $X(0) = const$, a.s. Considering the converse assertion, we note that, in this case, the symmetrization of $\{X(t), t \in \mathbb{R}^d\}$, i.e. $\{X_0(t), t \in \mathbb{R}^d\}$, satisfies $X_0(0) = 0$ a.s. If $D \notin Q(\mathbb{R}^m)$, then by Lemma 3.1, we can find $r_n \to 0$ and z_0 such that $|r_n^{D^*} z_0|$ does not converge to 0. Let $\alpha_n = |r_n^{D^*} z_0|^{-1}$. Then choosing a subsequence if necessary, by the fact $D \in M(\mathbb{R}^m)$, we have that α_n converges to a finite $\alpha > 0$ and that $\alpha_n r_n^{D^*} z_0$ converges to some $z_1 \in \mathbb{R}^m$ with $|z_1| = 1$. By using (2.1) and the same argument as that leads to (3.3) and (3.4) we derive

$$\widehat{\mu}_{r_n^E t}(c\,\alpha_n z_0) = \widehat{\mu}_t(c\,\alpha_n r_n^{D^*} z_0) \tag{3.5}$$

for all $c \in \mathbb{R}$. Letting $n \to \infty$, we have that $\widehat{\mu}_t(cz_1) = \widehat{\mu}_0(c\alpha z_0) = 1$. Then by Lemma 3.2, $X(t)$ is not full in $\mathbb{R}^m$. This contradiction implies $D \in Q(\mathbb{R}^m)$.

The last assertion follows from the stochastic continuity of X and (2.1). This finishes the proof of Theorem 2.1. □

Proof of Corollary 2.2. For every $t \in \mathbb{R}^d \backslash \{0\}$ we use polar coordinate decomposition under the operator E to write it as $t = \tau_E(t)^E l(t)$. We define $b(t) = b_{\tau_E(t)}\big(l(t)\big)$ for $t \in \mathbb{R}^d \backslash \{0\}$. Then from (2.2) we derive that for all $r > 0$ and $t \in \mathbb{R}^d \backslash \{0\}$,

$$b_{r\tau_E(t)}\big(l(t)\big) = b_r\big(\tau_E(t)^E l(t)\big) + r^D b_{\tau_E(t)}\big(l(t)\big),$$

which can be rewritten as

$$b\big((r\tau_E(t))^E l(t)\big) = b_r\big(t\big) + r^D b\big(t\big).$$

This implies $b_r(t) = b\big(r^E t\big) - r^D b(t)$ for all $r > 0$ and $t \in \mathbb{R}^d \backslash \{0\}$. In the case when $X(0) = a$ a.s., (2.1) implies $b_r(0) = a - r^D a$, which shows that (2.3) still holds for $t = 0$. □

Proof of Theorem 2.2. The proof is similar to the proof of Theorem 2.1, with some minor modifications. For proving Part (i), we define G_r to be the set of $A \in \mathrm{Aut}(\mathbb{R}^m)$ such that $\{X(r^E t),\, t \in \mathbb{R}^d\} \stackrel{d}{=} \{AX(t),\, t \in \mathbb{R}^d\}$; and for proving Part (ii), we define G_r to be the set of $A \in \mathrm{Aut}(\mathbb{R}^m)$ such that $\{X(r^E t),\, t \in \mathbb{R}^d\} \stackrel{d}{=} \{AX(t) + b(r),\, t \in \mathbb{R}^d\}$, for some function $b : (0, \infty) \to \mathbb{R}^m$. The rest of the proof follow similar lines as in the proof of Theorem 2.1 and is omitted. □

We end this section with two more propositions. Proposition 3.1 shows that, if a (d, m)-random field X is w.o.s.s. with time-variable scaling exponent E, then along each direction of the eigenvectors of E, X is an ordinary one-parameter operator-self-similar process as defined by Sato [28]. Proposition 3.2 discusses the relationship between w.o.s.s. random fields and o.s.s. random fields in the sense of Hudson and Mason (see (ii) in Remark 1.1).

Proposition 3.1 *Let $X = \{X(t), t \in \mathbb{R}^d\}$ be a stochastically continuous and proper (E, D)-w.o.s.s. random field with values in $\mathbb{R}^m$. Let λ be a positive eigenvalue of E and $\xi \in \mathbb{R}^d$ satisfy $E\xi = \lambda\xi$. Denote $\widetilde{b}_r(u) = b_r(u\xi)$ for all $u \in \mathbb{R}$. Then the following statements hold.*

- *(i) There exists a continuous function $f(u)$ from $\mathbb{R} \backslash \{0\}$ to $\mathbb{R}^m$, such that $\widetilde{b}_r(u) = f(ur^\lambda) - r^D f(u)$ for all $u \neq 0$ and $r > 0$.*
- *(ii) If $D \in Q(\mathbb{R}^m)$, then $f(u)$ can be defined at $u = 0$ such that $f(u)$ is continuous in $\mathbb{R}$. Moreover, the stochastic process $Y = \{Y(u),\, u \in \mathbb{R}\}$ defined by $Y(u) = X(u\xi) - f(u)$ satisfies that for any $r > 0$*

$$\big\{Y(ru),\, u \in \mathbb{R}\big\} \stackrel{d}{=} \big\{r^{D/\lambda} Y(u),\, u \in \mathbb{R}\big\}.$$

Proof. By Corollary 2.1, we have that

$$b_{r_1 r_2}(u\xi) = b_{r_1}(r_2^E u\xi) + r_1^D b_{r_2}(u\xi)$$

for all $r_1, r_2 > 0$. Since $E\xi = \lambda\xi$ and $r_2^E u\xi = ur_2^\lambda \xi$, we have

$$b_{r_1 r_2}(u\xi) = b_{r_1}(r_2^\lambda u\xi) + r_1^D b_{r_2}(u\xi). \tag{3.6}$$

Define $f(u) = b_{u^{1/\lambda}}(\xi)$ for $u > 0$ and $f(u) = b_{|u|^{1/\lambda}}(-\xi)$ for $u < 0$. Then the continuity of $f(u)$ on $\mathbb{R}\backslash\{0\}$ follows from the continuity of $b_r(t)$. Moreover, from (3.6) it follows that

$$\widetilde{b}_{r_1}(r_2^\lambda) = -r_1^D f(r_2^\lambda) + f(r_1^\lambda r_2^\lambda), \tag{3.7}$$
$$\widetilde{b}_{r_1}(-r_2^\lambda) = -r_1^D f(-r_2^\lambda) + f(-r_1^\lambda r_2^\lambda). \tag{3.8}$$

Writing $u = r_2^\lambda$ or $-r_2^\lambda$ and $r = r_1$, we see that (3.7) and (3.8) yield that

$$\widetilde{b}_r(u) = f(ur^\lambda) - r^D f(u) \tag{3.9}$$

for all $r > 0$, $u \neq 0$. This proves (i).

Suppose $D \in Q(\mathbb{R}^m)$. Lemma 3.1 implies that $r^D X(\xi) \to 0$ and $r^D X(-\xi) \to 0$ in probability as $r \to 0$. Theorem 2.1 and the convergence of types theorem indicate that, as $r \to 0+$, the limits of $b_r(\xi)$ and $b_r(-\xi)$ exist and coincide. Hence, we can define $f(0) := \lim_{r\to 0} b_r(\xi)$. Then $f(u)$ is continuous in $\mathbb{R}$. Combining (2.1) and (3.9) yields that for all $r > 0$, $u \in \mathbb{R}$,

$$\begin{aligned}\big\{X(r^\lambda u\xi),\, u \in \mathbb{R}\big\} &= \{X(r^E u\xi),\, u \in \mathbb{R}\} \\ &\stackrel{d}{=} \big\{r^D X(u\xi) + f(ur^\lambda) - r^D f(u),\, u \in \mathbb{R}\big\}.\end{aligned}$$

Hence for the process $Y = \{Y(u), u \in \mathbb{R}\}$ defined by $Y(u) = X(u\xi) - f(u)$, we have $\{Y(r^\lambda u), u \in \mathbb{R}\} \stackrel{d}{=} \{r^D Y(u), u \in \mathbb{R}\}$. Equivalently, Y is D/λ-o.s.s. This finishes the proof. $\square$

Proposition 3.2 *Let $X = \{X(t), t \in \mathbb{R}^d\}$ be a stochastically continuous and proper (E, D)-w.o.s.s. random field with values in $\mathbb{R}^m$. Suppose E has two different positive eigenvalues λ_1 and λ_2. Then X is o.s.s. in the sense of Hudson and Mason if and only if $b_r(t)$ in (2.1) only depends on r and $|t|$ for all $r > 0$ and $t \in \mathbb{R}^d$.*

Proof. The “necessity ” part is obvious, because, for every (E, D)-o.s.s. random field in the sense of Hudson and Mason, the function $b(r)$ does not depend on t.

In the following, we prove the sufficiency. Suppose $b_r(t)$ only depends on r and $|t|$ for all $r > 0$ and $t \in \mathbb{R}^d$. Then we can find a function g on $\mathbb{R}^2$ such that $b_r(t) = g(r, |t|)$. By Corollary 2.1, we have that for all $r_1, r_2 > 0$ and $t \in \mathbb{R}^d$

$$g(r_1 r_2, |t|) = g(r_1, |r_2^E t|) + r_1^D g(r_2, |t|). \tag{3.10}$$

Let ξ_1, ξ_2 be the eigenvectors of E corresponding to λ_1 and λ_2, respectively. Without loss of generality, we assume $|\xi_1| = |\xi_2| = 1$ and $\lambda_2 < \lambda_1$. Then from (3.10), we have that

$$g(r_1 r_2, 1) = g(r_1, r_2^{\lambda_1}) + r_1^D g(r_2, 1),$$
$$g(r_1 r_2, 1) = g(r_1, r_2^{\lambda_2}) + r_1^D g(r_2, 1),$$

where we have used the facts $r^E\xi_1 = r^{\lambda_1}\xi_1$ and $r^E\xi_2 = r^{\lambda_2}\xi_2$. Therefore, we derive that $g(r, u^{\lambda_1}) = g(r, u^{\lambda_2})$for any $r > 0$ and $u \geq 0$ and hence, for all $n \geq 1$,

$$g(r, u) = g(r, u^{\lambda_2/\lambda_1}) = g(r, u^{\lambda_2^n/\lambda_1^n}). \tag{3.11}$$

Note that by Theorem 2.1, $g(r, u)$ is continuous on $(0, \infty) \times [0, \infty)$. Therefore,

$$g(r, 0) = \lim_{u \to 0} g(r, u) \tag{3.12}$$

and for any $u > 0$, letting $n \to \infty$, from (3.11) we get that

$$g(r, u) = g(r, 1). \tag{3.13}$$

Combining (3.12) with (3.13), we obtain that $g(r, 0) = g(r, 1)$ and hence for all $r > 0$ and $u \geq 0$, $g(r, u) = g(r, 1)$. This means $b_r(t) = g(r, 1)$ is independent of t. Hence the random field X is o.s.s. in the sense of Hudson and Mason. $\square$

4. Construction of o.s.s. stable random fields: Proofs of Theorems 2.3–2.6

This section is concerned with constructing (E, D)-o.s.s. random fields by using stochastic integrals with respect to SαS random measures. In particular, we prove the remaining theorems in Section 2.

Note that Theorem 2.3 is a multiparameter extension of Theorem 4.1 in [17] and can be proved by using essentially the same argument with some modifications. Hence the proof of Theorem 2.3 is omitted here. In the following, we first prove Theorem 2.4.

Proof of Theorem 2.4. We divide the proof into two steps.

(1) When $\widetilde{Q}(u)$ is a simple function of the form

$$\widetilde{Q}(u) = \widetilde{Q}_1(u) + i\widetilde{Q}_2(u) = \sum_{j=1}^{k} R_j \mathbf{1}_{A_j}(u) + i \sum_{j=1}^{k} I_j \mathbf{1}_{A_j}(u), \tag{4.1}$$

where $R_j, I_j \in \mathrm{End}(\mathbb{R}^m)$ and $A_j, j = 1, 2, \cdots, k$ are pairwise disjoint sets in $\mathcal{M}$, we define

$$\widetilde{I}(\widetilde{Q}) = \sum_{j=1}^{k} \Big(R_j \widetilde{M}_R(A_j) - I_j \widetilde{M}_I(A_j) \Big).$$

Then for any $\theta \in \mathbb{R}^m$, from (2.8), we obtain that

$$\begin{aligned}\mathbb{E}\Big[\mathrm{e}^{i\langle\theta, \widetilde{I}(\widetilde{Q})\rangle}\Big] &= \exp\bigg\{ - \sum_{j=1}^{k} \Big(\big|R_j^*\theta\big|^2 + \big|I_j^*\theta\big|^2 \Big)^{\alpha/2} \lambda(A_j) \bigg\} \\ &= \exp\bigg\{ - \int_{\mathbb{R}^d} \Big(\big|\widetilde{Q}_1(u)^*\theta\big|^2 + \big|\widetilde{Q}_2(u)^*\theta\big|^2 \Big)^{\alpha/2} \mathrm{d}u \bigg\}. \end{aligned} \tag{4.2}$$

(2). When $\{\widetilde{Q}(u)\}$ fulfills $\int_{\mathbb{R}^d}\big(\|\widetilde{Q}_1(u)\|^\alpha + \|\widetilde{Q}_2(u)\|^\alpha\big)\mathrm{d}u < \infty$, we can choose a sequence of simple functions $\{\widetilde{Q}^{(n)}(u) = \widetilde{Q}_1^{(n)}(u) + i\widetilde{Q}_2^{(n)}(u)\}$ of the form (4.1) such that as $n \to \infty$,

$$\int_{\mathbb{R}^d} \big\|\widetilde{Q}_1(u)^* - \widetilde{Q}_1^{(n)}(u)^*\big\|^\alpha \mathrm{d}u \to 0 \tag{4.3}$$

and

$$\int_{\mathbb{R}^d} \big\|\widetilde{Q}_2(u)^* - \widetilde{Q}_2^{(n)}(u)^*\big\|^\alpha \mathrm{d}u \to 0. \tag{4.4}$$

By the linearity of $\widetilde{I}(\cdot)$ we have

$$\widetilde{I}(\widetilde{Q}^{(n)}) - \widetilde{I}(\widetilde{Q}^{(\ell)}) = \widetilde{I}(\widetilde{Q}^{(n)} - \widetilde{Q}^{(\ell)}),$$

and $\mathbb{E}(\mathrm{e}^{i\langle\theta, \widetilde{I}(\widetilde{Q}^{(n)} - \widetilde{Q}^{(\ell)})\rangle})$ equals

$$\begin{aligned}
&\exp\left\{ -\int_{\mathbb{R}^d} \Big(\big|\big(\widetilde{Q}_1^{(n)}(u)^* - \widetilde{Q}_1^{(\ell)}(u)^*\big)\theta\big|^2 + \big|\big(\widetilde{Q}_2^{(n)}(u)^* - \widetilde{Q}_2^{(\ell)}(u)^*\big)\theta\big|^2\Big)^{\alpha/2} \mathrm{d}u \right\} \\
&\ge \exp\left\{ -\int_{\mathbb{R}^d} \big|\big(\widetilde{Q}_1^{(n)}(u)^* - \widetilde{Q}_1^{(\ell)}(u)^*\big)\theta\big|^\alpha \mathrm{d}u - \int_{\mathbb{R}^d} \big|\big(\widetilde{Q}_2^{(n)}(u)^* - \widetilde{Q}_2^{(\ell)}(u)^*\big)\theta\big|^\alpha \mathrm{d}u \right\}
\end{aligned}$$

which converges to 1 as $\ell, n \to \infty$ by (4.3) and (4.4). Thus $\widetilde{I}(\widetilde{Q}^{(n)}) - \widetilde{I}(\widetilde{Q}^{(\ell)}) \to 0$ in probability as $\ell, n \to \infty$, and $\widetilde{I}(\widetilde{Q}^{(n)})$ converges to an $\mathbb{R}^m$-valued random vector in probability. It is easy to see that the limit does not depend on the choice of $\{\widetilde{Q}^{(n)}\}$. Therefore, we can define $\widetilde{I}(\widetilde{Q})$ as the limit of $\widetilde{I}(\widetilde{Q}^{(n)})$, and hence

$$\begin{aligned}
\mathbb{E}\Big(\mathrm{e}^{i\langle\theta, \widetilde{I}(\widetilde{Q})\rangle}\Big) &= \lim_{n\to\infty} \mathbb{E}\Big(\mathrm{e}^{i\langle\theta, \widetilde{I}(\widetilde{Q}^{(n)})\rangle}\Big) \\
&= \exp\left\{ -\int_{\mathbb{R}^d} \Big(\sqrt{\big|\widetilde{Q}_1(u)^*\theta\big|^2 + \big|\widetilde{Q}_2(u)^*\theta\big|^2}\Big)^\alpha \mathrm{d}u \right\}.
\end{aligned}$$

The proof of Theorem 2.4 is completed. □

In order to prove Theorem 2.5 and Theorem 2.6, we will use the following change of variable formula from [4].

Lemma 4.1 ([4, Proposition 2.3]) Let $E \in Q(\mathbb{R}^d)$ be fixed and let $(\tau(x), l(x))$ be the polar coordinates of x under the operator E. Denote $\Sigma_0 := \{\tau(x) = 1\}$. Then there exists a unique finite Radon measure σ on Σ_0 such that for all $f \in L^1(\mathbb{R}^d, \mathrm{d}x)$,

$$\int_{\mathbb{R}^d} f(x)\,\mathrm{d}x = \int_0^\infty \int_{\Sigma_0} f(r^E\theta)\sigma(\mathrm{d}\theta) r^{q-1}\mathrm{d}r.$$

We also need the following lemma which is due to Maejima and Mason [17]. For more precise estimates on $\|r^D\|$ see Mason and Xiao [19].

Lemma 4.2 *Let $D \in Q(\mathbb{R}^m)$ and let $h > 0$ and $H > 0$ be the minimal and maximal real parts of the eigenvalues of D, respectively. Then for any $\delta > 0$, there exist positive constants C_3 and C_4 such that*

$$\|r^D\| \leq \begin{cases} C_3\, r^{h-\delta}, & \textit{if } 0 < r \leq 1, \\ C_4\, r^{H+\delta}, & \textit{if } r > 1. \end{cases}$$

Now we are in position to prove Theorem 2.5.
Proof of Theorem 2.5. We divide the proof into four parts.

(i). First we show that the stochastic integral in (2.13) is well defined. By Theorem 2.3, it suffices to show that for all $x \in \mathbb{R}^d$

$$\Upsilon_\phi^\alpha(x) = \int_{\mathbb{R}^d} \left\| \phi(x-y)^{D-qI/\alpha} - \phi(-y)^{D-qI/\alpha} \right\|^\alpha \mathrm{d}y < \infty. \tag{4.5}$$

Let $(\tau(x), l(x))$ be the polar coordinates of x under operator E. By the fact that ϕ is E-homogeneous, we see that

$$\phi(y) = \tau(y)\phi(l(y)) \qquad \forall\ y \in \mathbb{R}^d.$$

Then by (2.12), we have that

$$m_\phi \tau(y) \leq \phi(y) \leq M_\phi \tau(y). \tag{4.6}$$

Therefore, there exists a constant $C_5 > 0$ such that

$$\left\|\phi(y)^{D-qI/\alpha}\right\|^\alpha \leq C_5 \left\|\tau(y)^{D-qI/\alpha}\right\|^\alpha.$$

Note that

$$M_1 = \sup_{m_\phi \leq r \leq M_\phi} \|r^E\| > 0 \quad \text{and} \quad M_2 = \sup_{1/M_\phi \leq r \leq 1/m_\phi} \|r^E\| > 0$$

are finite because r^E is continuous in r and $\|r^E\| \neq 0$ for all $r > 0$, and that

$$0 < m = \inf_{y \in \Sigma_0} |y| \leq M = \sup_{y \in \Sigma_0} |y| < \infty,$$

since Σ_0 is compact and $0 \notin \Sigma_0$. Therefore, from

$$\phi^{-E}(y)y = \phi^{-E}(y)\tau(y)^E l(y) = (\phi^{-1}(y)\tau(y))^E l(y)$$

and (4.6), it follows that

$$0 < \frac{m}{M_1} \leq \left|\phi^{-E}(y)y\right| \leq MM_2 < \infty. \tag{4.7}$$

Since ϕ is (β, E)-admissible, for any z with $\frac{m}{M_1} \leq |z| \leq MM_2$ there exists a positive constant $C_1 > 0$ such that

$$|\phi(x+z) - \phi(z)| \leq C_1 \tau(x)^\beta \tag{4.8}$$

for all $x \in \mathbb{R}^d$ with $\tau(x) \le 1$. For any $\gamma > 0$, on the set $\{y \in \mathbb{R}^d : \tau(y) \le \gamma\}$, we have

$$\Big\|\phi(x-y)^{D-qI/\alpha} - \phi(-y)^{D-qI/\alpha}\Big\|^{\alpha} \le 2\Big\|\phi(x-y)^{D-qI/\alpha}\Big\|^{\alpha} + 2\Big\|\phi(-y)^{D-qI/\alpha}\Big\|^{\alpha}.$$

Consequently, by Lemma 4.1, Lemma 4.2 and the fact $\tau(-y) = \tau(y)$, there exist constants $C_6 > 0$ and $0 < \delta < \alpha h$ such that,

$$\begin{aligned}\int_{\tau(y)\le\gamma} \Big\|\phi(-y)^{D-qI/\alpha}\Big\|^{\alpha} \mathrm{d}y \le& \int_{\tau(y)\le\gamma} C_5 \Big\|\tau(y)^{D-qI/\alpha}\Big\|^{\alpha} \mathrm{d}y \\ \le& \, C_6 \int_{\tau(y)\le\gamma} \tau(y)^{\alpha h - q - \delta} \mathrm{d}y < \infty.\end{aligned}$$

At the same time, (2.11) implies

$$\begin{aligned}\big\{y \in \mathbb{R}^d : \tau(x+y) \le \gamma\big\} \subset& \big\{y \in \mathbb{R}^d : \tau(y) \le C_0(\gamma + \tau(-x))\big\} \\ =& \big\{y \in \mathbb{R}^d : \tau(y) \le C_0\,(\gamma + \tau(x))\big\}.\end{aligned}$$

Consequently we derive that

$$\begin{aligned}&\int_{\tau(y)\le\gamma} \Big\|\phi(x-y)^{D-qI/\alpha}\Big\|^{\alpha} \mathrm{d}y = \int_{\tau(x+y)\le\gamma} \Big\|\phi(-y)^{D-qI/\alpha}\Big\|^{\alpha} \mathrm{d}y \\ &\le C_6 \int_{\tau(y)\le C_0(\gamma+\tau(x))} \tau(y)^{\alpha h - q - \delta}\, \mathrm{d}y < \infty.\end{aligned}$$

Combining the above shows that for any $\gamma > 0$

$$\int_{\tau(y)\le\gamma} \Big\|\phi(x-y)^{D-qI/\alpha} - \phi(-y)^{D-qI/\alpha}\Big\|^{\alpha} \mathrm{d}y < \infty. \tag{4.9}$$

Next we consider the integral on the set $\{y \in \mathbb{R}^d : \tau(y) > \gamma\}$ for sufficiently large γ such that $\phi(-y)^{-1}\tau(x) < 1$, $C_1\phi(-y)^{-\beta}\tau(x)^{\beta} < 1/2$ and $\phi(-y) > 1$. This is possible because of (4.6). Note that for any $3/2 > u > 1/2$, from the fact

$$\frac{\mathrm{d}s^{D-qI/\alpha}}{\mathrm{d}s} = \frac{\mathrm{d}}{\mathrm{d}s}\mathrm{e}^{\ln s(D-qI/\alpha)} = (D - qI/\alpha)s^{\alpha D-(1+q/\alpha)I}$$

and Lemma 4.2, there exists $C_7 > 0$ such that

$$\begin{aligned}\big\|u^{D-qI/\alpha} - I\big\| \le& \Big\|D - \frac{qI}{\alpha}\Big\| \int_{1\wedge u}^{1\vee u} \big\|s^{D-(1+q/\alpha)I}\big\|\mathrm{d}s \\ \le& \, C_7 \Big\|D - \frac{qI}{\alpha}\Big\| \cdot |u-1|.\end{aligned} \tag{4.10}$$

Since ϕ is E-homogenous and $\phi(-y) > 0$, we have

$$\begin{aligned}&\Big\|\phi(x-y)^{D-qI/\alpha} - \phi(-y)^{D-qI/\alpha}\Big\| \\ &\quad\le \Big\|\phi(-y)^{D-qI/\alpha}\Big\| \cdot \Big\|\phi(\phi^{-E}(-y)x - \phi^{-E}(-y)y)^{D-qI/\alpha} - I\Big\|.\end{aligned} \tag{4.11}$$

On the other hand, $\tau(\phi^{-E}(-y)x) = \phi^{-1}(-y)\tau(x) < 1$ and $\phi(-\phi^{-E}(-y)y) = 1$, we can use (4.7) and (4.8) to derive

$$\begin{aligned}\left|\phi(\phi^{-E}(-y)x - \phi^{-E}(-y)y) - 1\right| &\leq C_1\left[\tau(\phi^{-E}(-y)x)\right]^{\beta}\\ &= C_1\phi^{-\beta}(-y)\tau(x)^{\beta}.\end{aligned} \tag{4.12}$$

Since the last term is less than 1/2, we can apply (4.10) with $u = \phi(\phi^{-E}(-y)x - \phi^{-E}(-y)y)$. Hence, we derive from (4.11), (4.10), (4.12) and Lemma 4.2 that for some $0 < \delta_1 < (\beta - H)\alpha$

$$\begin{aligned}&\left\|\phi(x-y)^{D-qI/\alpha} - \phi(-y)^{D-qI/\alpha}\right\|^{\alpha}\\ &\quad\leq C_7^{\alpha}\left\|\phi(-y)^{D-qI/\alpha}\right\|^{\alpha}\cdot\left\|D - \frac{q}{\alpha}I\right\|^{\alpha}\left|\phi(\phi^{-E}(-y)x - \phi^{-E}(-y)y) - 1\right|^{\alpha}\\ &\quad\leq C_8\left\|\phi(-y)^{D-qI/\alpha}\right\|^{\alpha}\cdot\phi(-y)^{-\alpha\beta}\tau(x)^{\alpha\beta}\\ &\quad\leq C_9\,\phi(-y)^{\alpha H+\delta_1-q-\alpha\beta}\,\tau(x)^{\alpha\beta}\\ &\quad\leq C_{10}\,\tau(y)^{\alpha H+\delta_1-q-\alpha\beta}\,\tau(x)^{\alpha\beta}.\end{aligned}$$

This and Lemma 4.1 yield

$$\begin{aligned}&\int_{\tau(y)>\gamma}\left\|\phi(x-y)^{D-qI/\alpha} - \phi(-y)^{D-qI/\alpha}\right\|^{\alpha}\mathrm{d}y\\ &\qquad\leq C_{11}\,\tau(x)^{\alpha\beta}\int_{\gamma}^{\infty} r^{-\alpha(\beta-H)+\delta_1-1}\,dr < \infty.\end{aligned} \tag{4.13}$$

Combining (4.9) and (4.13), we get (4.5) which shows that X_ϕ is well defined.

(ii). To show the stochastic continuity of the α-stable random field X_ϕ, it is sufficient to verify that $\mathbb{E}(\exp\{i\langle\theta, X_\phi(x+x_0) - X_\phi(x_0)\rangle\}) \to 1$ for all $x_0 \in \mathbb{R}^d$ and $\theta \in \mathrm{R}^m$. By Theorem 2.2 it is enough to prove that for every $x_0 \in \mathbb{R}^d$, we have

$$\int_{\mathbb{R}^d}\left\|\phi(x_0+x-y)^{D-qI/\alpha} - \phi(x_0-y)^{D-qI/\alpha}\right\|^{\alpha}\mathrm{d}y \to 0 \qquad \text{as } x \to 0. \tag{4.14}$$

By a change of variables, (4.14) holds if

$$\Upsilon_\phi^{\alpha}(x) \to 0 \quad \text{as } x \to 0.$$

From the continuity of ϕ and the continuity of the function $(r, D) \to r^D$ (see Meerschaert and Scheffler [20, p.30]), we have that

$$\left\|\phi(x-y)^{D-qI/\alpha} - \phi(-y)^{D-qI/\alpha}\right\| \to 0 \quad \text{as } x \to 0$$

for every $y \in \mathbb{R}^d\backslash\{0\}$. Moreover, from the argument in Part (i), it follows that for a sufficiently large $\gamma > 0$, there exists a positive constant C_{12} such that $\left\|\phi(x-y)^{D-qI/\alpha} - \phi(-y)^{D-qI/\alpha}\right\|^{\alpha}$ is bounded by

$$\begin{aligned}\Phi(x,y) = &\,C_{12}\mathbf{1}_{\{\tau(y)\leq\gamma\}}(\tau(y)^{\alpha h-\delta-q} + \tau(x-y)^{\alpha h-\delta-q})\\ &+C_{12}\tau(y)^{\alpha H+\delta_1-qI-\alpha\beta}\tau(x)^{\alpha\beta}\mathbf{1}_{\{\tau(y)>\gamma\}},\end{aligned}$$

where $0 < \delta < \alpha h$ and $0 < \delta_1 < \alpha(\beta - H)$. It is easy to see that $\Phi(x, y) \to \Phi(y)$ a.e. as $x \to 0$, where

$$\Phi(y) = 2C_{12}\mathbf{1}_{\{\tau(y) \le \gamma\}}\tau(y)^{\alpha h - \delta - q},$$

and that

$$\int_{\mathbb{R}^d} \Phi(x, y)\mathrm{d}y \to \int_{\mathbb{R}^d} \Phi(y)\mathrm{d}y.$$

By the generalized dominated convergence theorem (see [10, p.492]), (4.14) holds.

(iii). In order to show that for all $r > 0$

$$\big\{X_\phi(r^E x),\, x \in \mathbb{R}^d\big\} \stackrel{d}{=} \big\{r^D X_\phi(x),\, x \in \mathbb{R}^d\big\},$$

we note that, by Theorem 2.3, it is sufficient to prove that for all $k \ge 1$, $x_j \in \mathbb{R}^d$ and $\theta_j \in \mathbb{R}^m$ $(j = 1, 2, \cdots, k)$

$$\begin{aligned} &\int_{\mathbb{R}^d} \left| \sum_{j=1}^k \Big(\phi(r^E x_j - y)^{D - qI/\alpha} - \phi(-y)^{D-qI/\alpha}\Big)^* \theta_j \right|^\alpha \mathrm{d}y \\ &= \int_{\mathbb{R}^d} \left| \sum_{j=1}^k r^D \Big(\phi(x_j - y)^{D - qI/\alpha} - \phi(-y)^{D-qI/\alpha}\Big)^* \theta_j \right|^\alpha \mathrm{d}y. \end{aligned} \tag{4.15}$$

This can be verified by an appropriate change of variables. By the E-homogeneity of ϕ, we have

$$\begin{aligned} &\int_{\mathbb{R}^d} \left| \sum_{j=1}^k \Big(\phi(r^E x_j - y)^{D - qI/\alpha} - \phi(-y)^{D-qI/\alpha}\Big)^* \theta_j \right|^\alpha \mathrm{d}y \\ &= \int_{\mathbb{R}^d} \left| \sum_{j=1}^k \Big(r^{D-qI/\alpha}\Big(\phi(x_j - r^{-E}y)^{D - qI/\alpha} - \phi(-r^{-E}y)^{D-qI/\alpha}\Big)\Big)^* \theta_j \right|^\alpha \mathrm{d}y \\ &= \int_{\mathbb{R}^d} \left| \sum_{j=1}^k \Big(r^{D-qI/\alpha}\Big(\phi(x_j - y)^{D - qI/\alpha} - \phi(-y)^{D-qI/\alpha}\Big)\Big)^* \theta_j \right|^\alpha r^q\mathrm{d}y \\ &= \int_{\mathbb{R}^d} \left| \sum_{j=1}^k \Big(r^{D}\Big(\phi(x_j - y)^{D - qI/\alpha} - \phi(-y)^{D-qI/\alpha}\Big)\Big)^* \theta_j \right|^\alpha \mathrm{d}y. \end{aligned}$$

This proves (4.15) and thus X is an (E, D)-o.s.s. random field.

(iv). In the same way, we can verify that $X_\phi(x)$ has stationary increments. The details are omitted. □

Finally, we prove Theorem 2.6.

Proof of Theorem 2.6. The proof is essentially an extension of the proofs of Theorem 4.1 and Corollary 4.2 in [4]. We only show that the stable random field $\widetilde{X}_\psi$ is well defined. Then properness of the $\widetilde{X}_\psi$ follows from the fact that for the matrix $\psi(y)^{-D-qI/\alpha}$ is invertible for every $y \in \mathbb{R}^d\backslash\{0\}$. The verification of the rest conclusions on $\widetilde{X}_\psi$ is left to the reader.

By Theorem 2.4, it suffices to show that

$$\Upsilon_\psi(x) := \int_{\mathbb{R}^d} \Big(|1-\cos\langle x,y\rangle|^\alpha + |\sin\langle x,y\rangle|^\alpha\Big) \Big\|\psi(y)^{-D-qI/\alpha}\Big\|^\alpha \mathrm{d}y < \infty.$$

Let $(\tau_1(x), l_1(x))$ be the polar coordinates of x under the operator E^*. By (2.12) and Lemma 4.2, there exist $0<\delta<[\frac{\alpha}{1+\alpha}(a_1-H)]\wedge(\alpha h)$ and $C_{13}>0$, such that

$$\mathbf{1}_{\{\tau_1(y)\geq 1\}} \Big\|\psi(y)^{-D-qI/\alpha}\Big\|^\alpha \leq C_{13}\tau_1(y)^{-\alpha h+\delta-q},$$

and

$$\mathbf{1}_{\{\tau_1(y)< 1\}} \Big\|\psi(y)^{-D-qI/\alpha}\Big\|^\alpha \leq C_{13}\tau_1(y)^{-\alpha H-\delta-q}.$$

Then by Lemma 4.1, $\Upsilon_\psi(x)$ is bounded by

$$\begin{aligned} &C_{13}\int_1^\infty\int_{\Sigma_0}\Big(|1-\cos\langle x, r^{E^*}\theta\rangle|^\alpha + |\sin\langle x, r^{E^*}\theta\rangle|^\alpha\Big) r^{-\alpha h+\delta-1}\sigma(\mathrm{d}\theta)\mathrm{d}r \\ &+C_{13}\int_0^1\int_{\Sigma_0}\Big(|1-\cos\langle x, r^{E^*}\theta\rangle|^\alpha + |\sin\langle x, r^{E^*}\theta\rangle|^\alpha\Big) r^{-\alpha H-\delta-1}\sigma(\mathrm{d}\theta)\mathrm{d}r. \end{aligned}$$

Note that there is a constant $C_{14}>0$ such that

$$\big|1-\cos\langle x, r^{E^*}\theta\rangle\big|^\alpha + \big|\sin\langle x, r^{E^*}\theta\rangle\big|^\alpha \leq C_{14}\big(1+|x|^\alpha\big)\big(r^{\alpha(a_1-\delta)}\wedge 1\big).$$

Therefore

$$\Upsilon_\psi(x) \leq C_{15}\left(1+|x|^\alpha\right)\sigma(\Sigma_0)\left[\int_1^\infty r^{-\alpha h+\delta-1}\mathrm{d}r + \int_0^1 r^{\alpha(a_1-H)-(1+\alpha)\delta-1}\mathrm{d}r\right].$$

Since $0<\delta<\big[\frac{\alpha}{1+\alpha}(a_1-H)\big]\wedge(\alpha h)$ and σ is a finite measure on Σ_0, we have $\Upsilon_\psi(x)<\infty$ for every $x\in\mathbb{R}^d$. This proves that $\widetilde{X}_\psi$ is a well-defined stable random field. □

The moving-average-type and harmonizable-type o.s.s. stable random fields are quite different (e.g., even in the special case of $D=I$, the regularity properties of X_ϕ and $\widetilde{X}_\psi$ are different.) From both theoretical and applied points of view, it is important to investigate the sample path regularity and fractal properties of the (E,D)-o.s.s. $S\alpha S$-random fields X_ϕ and $\widetilde{X}_\psi$. We believe that many sample path properties such as Hölder continuity and fractal dimensions of X_ϕ and $\widetilde{X}_\psi$ are determined mostly by the real parts of the eigenvalues of E and D. It would be interesting to find out the precise connections. We refer to Mason and Xiao [19], Biermé and Lacaux [3] and Xiao [32, 33] for related results in some special cases.

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