PACKING DIMENSION OF THE RANGE OF A LÉVY PROCESS

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Abstract. Let $\{X(t)\}_{t\geq 0}$ denote a Lévy process in $\mathbf{R}^d$ with exponent Ψ. Taylor (1986) proved that the packing dimension of the range $X([0\,,1])$ is given by the index

$$\gamma' = \sup \quad \alpha \geq 0 : \liminf_{r\to 0^+} \int_0^1 \frac{\mathrm{P}\{|X(t)| \leq r\}}{r^\alpha}\, dt = 0 \quad . \tag{0.1}$$

We provide an alternative formulation of γ' in terms of the Lévy exponent Ψ. Our formulation, as well as methods, are Fourier-analytic, and rely on the properties of the Cauchy transform. We show, through examples, some applications of our formula.

1. Introduction

Let $X := \{X(t)\}_{t\geq 0}$ denote a d-dimensional Lévy process (Bertoin, 1998; Sato, 1999) which starts at the origin. Define Ψ to be the Lévy exponent of X, normalized so that $\mathrm{E}[\exp(iz \cdot X(t))] = \exp(-t\Psi(z))$ for all $t \geq 0$ and $z \in \mathbf{R}^d$, and let $\dim_{\mathrm{P}}$ denote the packing dimension (Tricot, 1982; Sullivan, 1984). S. J. Taylor (1986) has proved that with probability one, $\dim_{\mathrm{P}} X([0\,,1]) = \gamma'$, where γ' is the index of Hendricks (1983); see (0.1).

Usually, one defines a Lévy process by constructing its Lévy exponent Ψ. From this perspective, formula (0.1) is difficult to apply in concrete settings. Primarily this is because the small-r behavior of $\int_0^1 \mathrm{P}\{|X(t)| \leq r\}\, dt$ is only well-understood when X is a nice Lévy process. For instance, when X is a subordinator γ' can be shown to be equal to the Blumenthal and Getoor (1961) upper index β (Fristedt and Taylor, 1992; Bertoin, 1999); see also Theorem 3.3 below. When X is a general Lévy process Pruitt and Taylor (1996) find several quantitative relationships between γ' and other known fractal indices of Lévy processes.

The principle goal of this article is to describe $\gamma' = \dim_{\mathrm{P}} X([0\,,1])$ more explicitly than (0.1), and solely in terms of the Lévy exponent Ψ. For all $r > 0$ define

$$W(r) := \int_{\mathbf{R}^d} \frac{\kappa(x/r)}{\prod_{j=1}^d (1 + x_j^2)}\, dx, \tag{1.1}$$

where κ is the following well-known function (Orey, 1967; Kesten, 1969):

$$\kappa(z) := \mathrm{Re}\left(\frac{1}{1 + \Psi(z)}\right) \qquad \text{for all } z \in \mathbf{R}^d. \tag{1.2}$$

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This function is symmetric (i.e., $\kappa(-z) = \kappa(z)$ for all $z \in \mathbf{R}^d$) and satisfies the pointwise bounds $0 \le \kappa \le 1$, which we use tacitly throughout. The following contains our formula for γ'.

Theorem 1.1. *For all d-dimensional Lévy processes X,*

$$\dim_{\mathrm{P}} X([0\,,1]) = \sup\left\{\alpha \ge 0 : \liminf_{r\to 0^+} \frac{W(r)}{r^\alpha} = 0\right\} = \limsup_{r\to 0^+} \frac{\log W(r)}{\log r}, \tag{1.3}$$

almost surely, where $\sup \varnothing := 0$.

Xiao (2004, Question 4.16) has asked if we can write $\dim_{\mathrm{P}} X([0\,,1])$ explicitly in terms of Ψ. Theorem 1.1 answers this question in the affirmative.

The following is one of the many consequences of Theorem 1.1.

Theorem 1.2. *Let X be a d-dimensional Lévy process that has a non-trivial, non-degenerate Gaussian part. That is, $X = G + Y$, where G is a non-degenerate Gaussian Lévy process, and Y is an independent pure-jump Lévy process. Then,* $\dim_{\mathrm{P}} X([0\,,1]) = \dim_{\mathrm{P}} G([0\,,1])$ *and* $\dim_{\mathrm{H}} X([0\,,1]) = \dim_{\mathrm{H}} G([0\,,1])$ *a.s., where* $\dim_{\mathrm{H}}$ *denotes the Hausdorff dimension.*

We do not know of a direct proof of this result, although it is a very natural statement. However, some care is needed as the result can fail when G is degenerate (Example 4.1). Our methods will make clear that in general we can say only that $\dim X([0\,,1]) \ge \dim G([0\,,1])$ a.s., where "dim" stands for either "$\dim_{\mathrm{P}}$" or "$\dim_{\mathrm{H}}$."

We also mention the following ready consequence of Theorem 1.1:

Corollary 1.3. *Let X be a Lévy process in* $\mathbf{R}^d$ *and* $X'(t) := X(t) - X''(t)$*, where* X'' *is an independent copy of X. Then,* $\dim_{\mathrm{P}} X([0\,,1]) \ge \dim_{\mathrm{P}} X'([0\,,1])$ *a.s.*

It has been shown that Corollary 1.3 continues to hold if we replace $\dim_{\mathrm{P}}$ by $\dim_{\mathrm{H}}$ everywhere; see Khoshnevisan, Xiao, and Zhong (2003) and/or (1.4) below. Thus we have further confirmation of the somewhat heuristic observation of Kesten (1969, p. 7) that the range of X is larger than the range of its symmetrization.

Theorem 1.1 is proved in Section 2. Our proof also yields the following almost-sure formula for the Hausdorff dimension of $X([0,1])$:

$$\dim_{\mathrm{H}} X([0\,,1]) = \sup\left\{\alpha \ge 0 : \limsup_{r\to 0^+} \frac{W(r)}{r^\alpha} = 0\right\} = \liminf_{r\to 0^+} \frac{\log W(r)}{\log r}; \tag{1.4}$$

see Remark 2.4. Recently, Khoshnevisan, Xiao, and Zhong (2003) established an equivalent formulation of this formula. Whereas their derivation is long and complicated, ours is direct and fairly elementary. Section 3 contains non-trivial examples wherein we compute $\dim_{\mathrm{P}} X([0\,,1])$ for anisotropic Lévy processes X. Finally, Theorem 1.2 is proved in Section 4.

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2. The Incomplete Renewal Measure

Define U to be the *incomplete renewal measure* of X. That is, for all Borel sets $A \subset \mathbf{R}^d$,

$$U(A) := \int_0^1 \mathrm{P}\{X(t) \in A\}\, dt. \tag{2.1}$$

We may deduce from (0.1) that

$$\dim_{\mathrm{P}} X([0\,,1]) = \limsup_{r\to 0^+} \frac{\log U(B(0\,,r))}{\log r}, \tag{2.2}$$

where $B(a\,,r) := \{z \in \mathbf{R}^d : |z-a| \le r\}$ for all $a \in \mathbf{R}^d$ and $r \ge 0$ so that $\int_0^1 \mathrm{P}\{|X(t)| \le r\}\, dt = U(B(0,r))$, and where $|y| := \max_{1\le j\le d} |y_j|$ is the ℓ^∞-norm of $y \in \mathbf{R}^d$.

Let ζ denote an independent, mean-one exponential random variable. The killed occupation measure of $B(0\,,r)$ can then be defined by

$$T(r) := \int_0^\zeta \mathbf{1}_{B(0,r)}(X(t))\, dt \qquad \forall\, r > 0, \tag{2.3}$$

where $\mathbf{1}_A$ denotes the indicator function of A.

Proposition 2.1. *For all $r > 0$,*

$$\frac{e}{4^d(e-1)} \mathrm{E}[T(r)] \le U(B(0\,,r)) \le e\,\mathrm{E}[T(r)]. \tag{2.4}$$

In order to prove this we first recall the notion of weak unimodality (Khoshnevisan and Xiao, 2003).

Definition 2.2. A Borel measure μ on $\mathbf{R}^d$ is *c-weakly unimodal* if $c > 0$ is a constant that satisfies $\sup_{a\in\mathbf{R}^d} \mu(B(a\,,\,r)) \le c\mu(B(0\,,r))$ for all $r > 0$.

The following is a variant of Lemma 4.1 of Khoshnevisan and Xiao (2003).

Lemma 2.3. *U is 4^d-weakly unimodal.*

Proof. Let us fix $a \in \mathbf{R}^d$ and $r > 0$, and define $\sigma := \inf\{s > 0 : |X(s) - a| \le r\}$, where $\inf \varnothing := \infty$. Clearly, σ is a stopping time, and

$$U(B(a\,,r)) = \mathrm{E}\left[\int_\sigma^1 \mathbf{1}_{B(a,r)}(X(s))\, ds \;;\; \sigma \le 1\right]. \tag{2.5}$$

Thanks to the triangle inequality, the strong Markov property implies that

$$\begin{aligned} U\big(B(a\,,r)\big) &= \mathrm{E}\left[\int_0^{(1-\sigma)^+} \mathbf{1}_{B(0,r)}(X(u+\sigma) - a)\, du\right] \\ &\le \mathrm{E}\left[\int_0^1 \mathbf{1}_{B(0,2r)}(X(u+\sigma) - X(\sigma))\, du\right] = U\big(B(0\,,2r)\big). \end{aligned} \tag{2.6}$$

Euclidean topology in the ℓ^∞-norm dictates that there are points $z_1, \ldots, z_{4^d} \in B(0\,,2r)$ that have the property that $\cup_{i=1}^{4^d} B(z_i\,,r/2) = B(0\,,2r)$. According to (2.6), we have the following "volume doubling" property:

$$U(B(0\,,2r)) \le \sum_{i=1}^{4^d} U(B(z_i\,,r/2)) \le 4^d\, U(B(0\,,r)). \tag{2.7}$$

The desired result follows from this and (2.6). □

Proof of Proposition 2.1. Note that

$$U(B(0\,,r)) \le e \int_0^1 \mathrm{P}\{|X(t)| \le r\}\, e^{-t}\, dt \le e\,\mathrm{E}[T(r)]. \tag{2.8}$$

This proves the upper bound in (2.4). To prove the other half we note that

$$\mathrm{E}[T(r)] = \int_0^\infty \mathrm{P}\{|X(t)| \le r\}\, e^{-t}\, dt \le \sum_{j=0}^\infty e^{-j}\, \mathrm{E}[U(B(X(j)\,, r))], \tag{2.9}$$

thanks to the Markov property. By Lemma 2.3, $\mathrm{E}[U(B(X(j)\,,r))] \le 4^d U(B(0\,,r))$ for every $j \ge 0$. The lower bound in (2.4) follows from this and (2.9). □

Proof of Theorem 1.1. We derive only the second identity of (1.3); the first is manifestly an equivalent statement.

Let $(\mathscr{F} f)(z) := \int_{\mathbf{R}^d} e^{iz\cdot x} f(x)\, dx$ denote the Fourier transform of $f \in L^1(\mathbf{R}^d)$. For all fixed $r > 0$ and $x \in \mathbf{R}^d$ define

$$\phi_r(x) = \prod_{j=1}^d \frac{1-\cos(2rx_j)}{2\pi r x_j^2}. \tag{2.10}$$

Then $\phi_r(x) \ge 0$, and $(\mathscr{F}\phi_r)(z) = \prod_{j=1}^d (1-|z_j|/(2r))^+$ for all $z \in \mathbf{R}^d$ (Durrett, 1996, p. 94). As usual, $a^+ := \max(a\,,0)$ for all $a \in \mathbf{R}$. Evidently, $\phi_r \in L^1(\mathbf{R}^d)$, and $0 \le \mathscr{F}\phi_r \le 1$ pointwise.

Note that $z \in B(0\,,r)$ implies that $1-(2r)^{-1}|z_j| \ge \frac{1}{2}$. This implies that $\mathbf{1}_{B(0,\,r)}(z) \le 2^d\,(\mathscr{F}\phi_r)(z)$ for all $z \in \mathbf{R}^d$. Therefore, by the Fubini–Tonelli theorem,

$$\mathrm{E}[T(r)] \le 2^d \int_{\mathbf{R}^d} \kappa(x)\phi_r(x)\, dx \le 2^d W(r). \tag{2.11}$$

The last inequality follows from the elementary bound

$$\frac{1-\cos(2u)}{2\pi u^2} \le \frac{1}{1+u^2} \qquad \text{for all } u \in \mathbf{R}. \tag{2.12}$$

This can be verified by considering $|u| \le (\pi-1)^{-1/2}$ and $|u| > (\pi-1)^{-1/2}$ separately. Thanks to (2.2) and Proposition 2.1,

$$\dim_{\mathrm{P}} X([0\,,1]) \ge \limsup_{r\to 0^+} \frac{\log W(r)}{\log r} \qquad \text{a.s.} \tag{2.13}$$

In order to establish the converse inequality we introduce the process $\{S(t)\}_{t\ge 0}$ defined by $S(t) := (S_1(t)\,,\ldots,S_d(t))$, where $S_1,\ldots,S_d$ are independent symmetric Cauchy processes in $\mathbf{R}$, all with the same characteristic function $\mathrm{E}[e^{izS_1(t)}] = e^{-t|z|}$. We assume further that S is independent of X. Then for all $\lambda > 0$, $\mathrm{E}\big[\exp\{iX(t)\cdot S(\lambda)\}\big] = \mathrm{E}\big[\exp\{-\lambda\sum_{j=1}^d |X_j(t)|\}\big]$. On the other hand, the scaling property of S implies

$$\mathrm{E}\left[e^{iX(t)\cdot S(\lambda)}\right] = \mathrm{E}\left[e^{-t\Psi(S(\lambda))}\right] = \frac{1}{\pi^d}\int_{\mathbf{R}^d} \frac{e^{-t\Psi(\lambda x)}}{\prod_{j=1}^d (1+x_j^2)}\, dx. \tag{2.14}$$

For all $r, k > 0$ and $x \in \mathbf{R}^d$, $\exp\{-(k/r)\sum_{j=1}^d |x_j|\} \le \mathbf{1}_{B(0,r)}(x) + e^{-k}\mathbf{1}_{B(0,r)^c}(x)$. Therefore, $\mathbf{1}_{B(0,r)}(x) \ge \mathrm{E}[\exp\{ix\cdot S(k/r)\}] - e^{-k}\mathbf{1}_{B(0,r)^c}(x)$, whence

$$\mathrm{E}[T(r)] \ge \int_0^\infty \mathrm{E}\Big[e^{iX(t)\cdot S(k/r)}\Big]\, e^{-t}\, dt - e^{-k}(1-\mathrm{E}[T(r)]). \tag{2.15}$$

This and (2.14) together with the fact that the quantity in (2.14) imply that

$$(1-e^{-k})\mathrm{E}[T(r)] \ge -e^{-k} + \frac{1}{\pi^d} W\big(\frac{r}{k}\big). \tag{2.16}$$

Now we choose $k = r^{-\varepsilon}$, for an arbitrary small $\varepsilon > 0$, to find that the inequality in (2.13) is an equality. This completes our proof. □

Remark 2.4. From the proof of Theorem 1.1 we see that $\mathrm{E}[T(r)]$ and $W(r)$ are roughly comparable; i.e., for all $\varepsilon, r > 0$ sufficiently small,

$$\frac{1}{\pi^d} W(r^{1+\varepsilon}) - \exp(-r^{-\varepsilon}) \le \mathrm{E}[T(r)] \le 2^d W(r). \tag{2.17}$$

Thanks to Proposition 2.1 this yields (1.4).

3. Some Examples

We illustrate the utility of Theorem 1.1 by specializing it to a large class of examples.

3.1. **Anisotropic Examples.** It is possible to construct examples of anisotropic Lévy "stable-like" processes whose $\dim_{\mathrm{P}} X([0\,,1])$ are computable. The following furnishes most of the basic technical background that we shall need.

Theorem 3.1. *Let X be a Lévy process in $\mathbf{R}^d$ with Lévy exponent Ψ. Suppose there exist constants $\beta_1, \ldots, \beta_d$ such that*

$$2 \ge \beta_1 \ge \cdots \ge \beta_d > 0 \quad \textit{and} \quad \lim_{\|z\| \to \infty} \frac{1}{\ln \|z\|} \ln \mathrm{Re} \left(\frac{\sum_{j=1}^d |z_j|^{\beta_j}}{1 + \Psi(z)} \right) = 0. \tag{3.1}$$

In the above, $\|z\|$ denotes the ℓ^2-norm of $z \in \mathbf{R}^d$. If $N := \max\{1 \le j \le d :\ \beta_j = \beta_1\}$, then almost surely,

$$\dim_{\mathrm{P}} X([0\,,1]) = \begin{cases} \beta_1, & \textit{if } \ \beta_1 \le N, \\ 1 + \beta_2 \left(1 - \dfrac{1}{\beta_1}\right), & \textit{otherwise.} \end{cases} \tag{3.2}$$

Proof. Throughout this proof we write c and C for generic constants whose values can change between lines. In order to simplify the exposition somewhat, we note that it is sufficient to prove (3.2) under the following [slightly] stronger form of (3.1):

$$\frac{c}{1 + \sum_{j=1}^d |z_j|^{\beta_j}} \le \mathrm{Re} \left(\frac{1}{1 + \Psi(z)} \right) \le \frac{C}{1 + \sum_{j=1}^d |z_j|^{\beta_j}} \quad \text{for all } z \in \mathbf{R}^d. \tag{3.3}$$

First we consider the case when $\beta_1 \le N$. Condition (3.3) implies that if $r \in (0\,,1)$, then

$$W(r) \ge c\, r^{\beta_1} \int_{\mathbf{R}^d} \frac{dx}{\left(1 + \sum_{j=1}^d |x_j|^{\beta_j}\right) \prod_{j=1}^d (1 + x_j^2)} = C\, r^{\beta_1}. \tag{3.4}$$

Hence we have $\lim_{r \to 0} r^{-\alpha} W(r) = \infty$ for all $\alpha > \beta_1$. It follows from this and Theorem 1.1 that $\dim_{\mathrm{P}} X([0\,,1]) \le \beta_1$ a.s.

Recall that $N := \max\{1 \le j \le d :\ \beta_j = \beta_1\}$. From this it follows that

$$\begin{aligned} W(r) &\le c \int_{\mathbf{R}^N} \frac{dx}{\left(1 + \|x/r\|^{\beta_1}\right) \prod_{j=1}^N \left(1 + x_j^2\right)} \\ &= c\, r^{\beta_1} \left[\int_{\|x\| \le 1} \frac{dx}{r^{\beta_1} + \|x\|^{\beta_1}} + C \right] \le c\, r^{\beta_1} \log(1/r). \end{aligned} \tag{3.5}$$

In the above, $\log(1/r)$ accounts for the case that $\beta_1 = N$. The preceding bound implies that $\lim_{r\to 0} r^{-\alpha} W(r) = 0$ for every $\alpha < \beta_1$. This leads to the lower bound, $\dim_{\rm P} X([0\,,1]) \geq \beta_1$ a.s.

Next we consider the case when $\beta_1 > N$. This implies $N = 1$ and $\beta_1 > \beta_2$. In order to prove that $\dim_{\rm P} X([0\,,1]) \leq 1 + \beta_2(1 - \beta_1^{-1})$ a.s. we first derive a lower bound for $W(r)$. We do this by restricting the integral to the domain $D := \{x \in \mathbf{R}^d : |x_j| \leq 1 \text{ for } 3 \leq j \leq d\}$. More precisely, by (3.3) we have

$$
\begin{aligned}
W(r) &\geq c \int_D \frac{dx}{\left(1 + \sum_{j=1}^d |x_j/r|^{\beta_j}\right) \prod_{j=1}^d \left(1 + x_j^2\right)} \\
&\geq c \int_D \frac{dx}{\left(1 + \sum_{j=1}^d |x_j/r|^{\beta_j}\right) \prod_{j=1}^2 \left(1 + x_j^2\right)}.
\end{aligned}
\tag{3.6}
$$

Let $(x_3\,, \ldots, x_d) \in [-1\,,1]^{d-2}$ be fixed and let $A := 1 + \sum_{j=3}^d |x_j/r|^{\beta_j}$. Consider

$$
\begin{aligned}
\mathscr{I} &:= \int_{\mathbf{R}^2} \frac{dx_1\, dx_2}{\left(A + |x_1/r|^{\beta_1} + |x_2/r|^{\beta_2}\right) \prod_{j=1}^2 (1 + x_j^2)} \\
&\geq r^{\beta_2} \int_{r^{1-(\beta_2/\beta_1)}}^\infty dx_1 \int_0^\infty \frac{dx_2}{\left(r^{\beta_2} A + r^{\beta_2-\beta_1} x_1^{\beta_1} + x_2^{\beta_2}\right) \prod_{j=1}^2 (1 + x_j^2)}.
\end{aligned}
\tag{3.7}
$$

Observe that $r^{\beta_2-\beta_1} x_1^{\beta_1} \geq 1$ for all $x_1 \geq r^{1-(\beta_2/\beta_1)}$. On the other hand, $r^{\beta_2} A \leq d-1$ for all $r \in (0\,,1)$. It follows from these facts, and a change of variables, that

$$
\begin{aligned}
\mathscr{I} &\geq c\, r^{\beta_2} \int_{r^{1-(\beta_2/\beta_1)}}^\infty \frac{dx_1}{1 + x_1^2} \int_1^\infty \frac{1}{\left(r^{\beta_2-\beta_1} x_1^{\beta_1} + x_2^{\beta_2}\right) x_2^2}\, dx_2 \\
&\geq c\, r^{\beta_2} \int_{r^{1-(\beta_2/\beta_1)}}^\infty \frac{dx_1}{1 + x_1^2} \cdot \frac{1}{r^{\beta_2-\beta_1} x_1^{\beta_1}} \geq C\, r^{\beta_1} \int_{r^{1-(\beta_2/\beta_1)}}^1 \frac{dx_1}{x_1^{\beta_1}} \\
&\geq c\, r^{1+\beta_2(1-\beta_1^{-1})}.
\end{aligned}
\tag{3.8}
$$

Combine this with (3.6) and (3.7) to deduce that $W(r) \geq c\, r^{1+\beta_2(1-\beta_1^{-1})}$ for all $r \in (0\,,1)$, whence $\lim_{r\to 0^+} r^{-\alpha} W(r) = \infty$ for all $\alpha > 1 + \beta_2(1 - \beta_1^{-1})$. This implies that $\dim_{\rm P} X([0\,,1]) \leq 1 + \beta_2(1 - \beta_1^{-1})$ a.s.

Now we derive the lower bound for $\dim_{\rm P} X([0\,,1])$. It follows from (3.3) that

$$
\begin{aligned}
W(r) &\leq c \int_{\mathbf{R}^2} \frac{dx_1\, dx_2}{\left(1 + |x_1/r|^{\beta_1} + |x_2/r|^{\beta_2}\right) \prod_{j=1}^2 \left(1 + x_j^2\right)} \\
&= 4c\, r^{\beta_2} \int_0^\infty \frac{dx_1}{1 + x_1^2} \int_0^\infty \frac{dx_2}{\left(B + x_2^{\beta_2}\right)\left(1 + x_2^2\right)},
\end{aligned}
\tag{3.9}
$$

where $B := r^{\beta_2} + r^{\beta_2-\beta_1} x_1^{\beta_1}$. It remains to verify that the last expression in (3.9) is at most $c\, r^{1+\beta_2(1-\beta_1^{-1})} \log(1/r)$, where $\log(1/r)$ appears because of the possibility that $\beta_2 = 1$.

By breaking the dx_2-integral according to whether $|x_2| \leq 1$ or $|x_2| > 1$, and after a change of variables, we can verify the following elementary inequalities:

(i) If $B \leq 1$, then $\int_0^\infty (B + x_2^{\beta_2})^{-1} (1 + x_2^2)^{-1}\, dx_2 \leq c g(x_1)$, where: $g(x_1) = 1$ if $\beta_2 < 1$, $\log(B^{-1})$ if $\beta_2 = 1$, and $B^{(1/\beta_2)-1}$ if $\beta_2 < 1$.

(ii) If $B > 1$, then $\int_0^\infty (B + x_2^{\beta_2})^{-1} (1 + x_2^2)^{-1}\, dx_2 \leq c/B$.

We return to (3.9) and split the dx_1-integral respectively over the intervals $\{x_1 : B \le 1\}$ and $\{x_1 : B > 1\}$. It follows from (3.9), (i) and (ii), and a direct computation, that $W(r) \le cr^{1+\beta_2(1-\beta_1^{-1})}\log(1/r)$. Hence, Theorem 1.1 implies that $\dim_{\mathrm{P}} X([0\,,1]) \ge 1+\beta_2(1-\beta_1^{-1})$ a.s. This finishes the proof Theorem 3.1. □

In the following, we apply Theorem 3.1 to operator-stable Lévy processes in $\mathbf{R}^d$ with $d \ge 2$. Let us first recall from Sharpe (1969) that a non-degenerate distribution μ on $\mathbf{R}^d$ is called *operator-stable* if there exist sequences of independent identically distributed random vectors $\{X_n\}$ in $\mathbf{R}^d$, nonsingular linear operators $\{A_n\}$, and vectors $\{a_n\}$ in $\mathbf{R}^d$ such that $\{A_n \sum_{k=1}^n X_k - a_n\}$ converges in law to μ. A distribution μ on $\mathbf{R}^d$ is called *full* if it is not supported on any $(d-1)$-dimensional hyper-plane. Sharpe (1969) proves that a full distribution μ in $\mathbf{R}^d$ is operator-stable if and only if there exists a non-singular linear operator B on $\mathbf{R}^d$ such that $\mu^t = t^B\mu * \delta(b(t))$ for all $t > 0$ and some $b(t) \in \mathbf{R}^d$. Here, μ^t denotes the t-fold convolution power of μ, and $t^B\mu(dx) := \mu(t^{-B}dx)$ is the image measure of μ under the action of the linear operator $t^B := \sum_{n=0}^\infty (\log t)^n B^n/n!$. In the above, B and $\{b(t), t > 0\}$ are called a *stability exponent* and the family of shifts of μ, respectively. The set of all possible exponents of an operator-stable law is characterized by Holmes et al. (1982); see also Meerschaert and Scheffler (2001, Theorem 7.2.11). By analogy with the one-dimensional case, an operator-stable distribution μ satisfying $\mu^t = t^B\mu$ will be called strictly operator-stable; see Sharpe (1969, p. 64).

A stochastic process $Y := \{Y(t)\}_{t\in\mathbf{R}_+}$ with values in $\mathbf{R}^d$ is said to be *operator self-similar* if there exists a linear operator B on $\mathbf{R}^d$ such that for every $c > 0$, $\{Y(ct)\}_{t\ge0} \overset{d}{=} \{c^B\, Y(t)\}_{t\ge0}$, where "$\overset{d}{=}$" denotes the equality of finite-dimensional distributions. The linear operator B is called a *self-similarity exponent* of Y. Let $X = \{X(t)\}_{t\ge0}$ be a Lévy process in $\mathbf{R}^d$ starting from 0 such that the distribution of $X(t)$ is full for every $t > 0$. Hudson and Mason (1982, Theorem 7) proved that X is operator self-similar if and only if the distribution of $X(1)$, $\nu := \mathrm{P}\circ(X(1))^{-1}$, is strictly operator-stable. In this case, every stability exponent B of ν is also a self-similarity exponent of X. Hence, from now on we will call a Lévy process X in $\mathbf{R}^d$ operator-stable if the distribution of $X(1)$ is full and strictly operator-stable; and refer to B simply as an *exponent* of X.

Operator-stable Lévy processes are scaling limits of d-dimensional random walks that are normalized by linear operators (Meerschaert and Scheffler, 2001, Chapter 11). All d-dimensional strictly stable Lévy processes of index α are operator-stable with exponent $B := \alpha^{-1}I$, where I denotes the $(d\times d)$ identity matrix.

More generally, let $X_1, \ldots, X_d$ be independent stable Lévy processes in $\mathbf{R}$ with respective indices $\alpha_1, \ldots, \alpha_d \in (0\,,2]$. Define $X(t) := (X_1(t)\,, \ldots, X_d(t))$. One can then verify that X is an operator-stable Lévy process whose exponent B is the $(d\times d)$ diagonal matrix $\mathrm{diag}(\alpha_1^{-1}, \alpha_2^{-1}, \ldots, \alpha_d^{-1})$. These processes were first introduced by Pruitt and Taylor (1969) under the title of *Lévy processes with stable components*. These processes have been used to construct various counterexamples (Hendricks, 1983).

Let X be an operator-stable Lévy process in $\mathbf{R}^d$ with exponent B. Factor the minimal polynomial of B into $q_1(x)\,, \cdots, q_p(x)$ where all roots of $q_i(x)$ have real part a_i, and $a_i < a_j$ for $i < j$. Define $\alpha_i := a_i^{-1}$, so that $\alpha_1 > \cdots > \alpha_p$, and note that $0 < \alpha_i \le 2$ (Meerschaert and Scheffler, 2001, Theorem 7.2.1). Define

$V_i := \mathrm{Ker}(q_i(B))$ and $d_i := \dim(V_i)$. Then $d_1 + \cdots + d_p = d$, and $V_1 \oplus \cdots \oplus V_p$ is a direct-sum decomposition of $\mathbf{R}^d$ into B-invariant subspaces. We may write B as $B = B_1 \oplus \cdots \oplus B_p$, where $B_i : V_i \to V_i$ and every eigenvalue λ of B_i has the property that $\mathrm{Re}\,\lambda = a_i$. We can apply Theorem 3.1 for operator-stable Lévy processes to obtain a wholly different proof of the following theorem of Meerschaert and Xiao (2005, Theorem 3.2).

Theorem 3.2 (Meerschaert and Xiao (2005))**.** *Let X be an operator-stable Lévy process in $\mathbf{R}^d$ as described above. Then almost surely,*

$$\dim_{\mathrm{P}} X([0\,,1]) = \begin{cases} \alpha_1 & \text{if } \alpha_1 \le d_1, \\ 1 + \alpha_2\left(1 - \dfrac{1}{\alpha_1}\right) & \text{otherwise.} \end{cases} \tag{3.10}$$

Proof. Define $\beta_j := \alpha_\ell$ where ℓ is determined by $\sum_{i=0}^{\ell-1} d_i < j \le \sum_{i=\ell-1}^{\ell} d_i$, and $d_0 := 0$. Because Meerschaert and Xiao (2005) have established (3.1), Theorem 3.1 implies (3.10) with $N := d_1$. □

3.2. **Subordinators.** Let us consider the special case that X is a [non-negative] subordinator. We conclude this article by showing that our Theorem 1.1 includes the well known formula for $\dim_{\mathrm{P}} X([0\,,1])$; see Fristedt and Taylor (1992) and Bertoin (1999, §5.1.2). Let Φ denote the Laplace exponent of X, normalized so that $\mathrm{E}[\exp(-\lambda X(t))] = \exp(-t\Phi(\lambda))$ for all $\lambda, t \ge 0$. The following is an immediately consequence of Theorem 1.1.

Theorem 3.3 (Fristedt and Taylor (1992); Bertoin (1999))**.** *With probability one,*

$$\dim_{\mathrm{P}} X([0\,,1]) = \limsup_{\lambda\to\infty} \frac{\log \Phi(\lambda)}{\log \lambda}. \tag{3.11}$$

Proof. Let $S = \{S(t)\}_{t\ge 0}$ denote an independent Cauchy process in $\mathbf{R}$ such that $\mathrm{E}[\exp(i\xi S(t))] = \exp(-t|\xi|)$ for all $t \ge 0$ and $\xi \in \mathbf{R}$. Then,

$$e^{-t\Phi(\lambda)} = \mathrm{E}\big[e^{-\lambda X(t)}\big] = \mathrm{E}\big[e^{iX(t)S(\lambda)}\big] = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e^{-t\Psi(\lambda z)}}{1+z^2}\,dz. \tag{3.12}$$

Multiply both sides by e^{-t} and integrate $[dt]$ to find that

$$\frac{1}{1+\Phi(\lambda)} = \frac{1}{\pi} W\big(\frac{1}{\lambda}\big). \tag{3.13}$$

A direct appeal to Theorem 1.1 finishes the proof. □

4. Proof of Theorem 1.2

Throughout, $\|x\| := (x_1^2 + \cdots + x_d^2)^{1/2}$ for all $x \in \mathbf{R}^d$. This is the usual ℓ^2-norm on $\mathbf{R}^d$, and should not be confused with the ℓ^∞-norm $|x| = \max_{1\le j\le d} |x_j|$ that we have used so far.

Because of the Lévy–Khintchine formula (Bertoin, 1998), we can write X as $X = G + Y$, where G is a Gaussian Lévy process and Y is an independent pure-jump Lévy process. Thanks to the centered-ball inequality of Anderson (1955), $a \mapsto \mathrm{P}\{|G(t) - a| \le r\}$ is maximized at the origin. Apply this, conditionally on Y, to find that

$$\mathrm{P}\{|X(t)| \le r\} \le \mathrm{P}\{|G(t)| \le r\} \qquad \text{for all } t, r > 0. \tag{4.1}$$

It follows from (0.1) and (2.2) that $\dim_{\rm P} G([0\,,1]) \le \dim_{\rm P} X([0\,,1])$ a.s. The analogous bound for $\dim_{\rm H}$ follows from this and the formula of Pruitt (1969).

In order to prove the converse bound we appeal to Theorem 1.1. Recall that

$$\Psi_X(z) = O(\|z\|^2) \quad \text{as } \|z\| \to \infty. \tag{4.2}$$

(Bochner, 1955, eq. (3.4.14), p. 67). The subscript X refers to the process X, the subscript G to the process G, etc. Therefore, there exists a constant C such that

$$\kappa_X(z) \ge \frac{C}{1+\|z\|^2} \qquad \text{for all } z \in \mathbf{R}^d. \tag{4.3}$$

The non-degeneracy of G implies that $\|z\|^{-2}\Psi_G(z)$ is bounded below uniformly for all $z \in \mathbf{R}^d$. Because $\operatorname{Re}\Psi_Y(z) \ge 0$, it follows that there exists a constant c such that $\kappa_X(z) \ge c\kappa_Y(z)$ for all $z \in \mathbf{R}^d$. Therefore, $W_X(r) \ge cW_G(r)$, and Theorem 1.1 shows that $\dim_{\rm P} X([0\,,1]) \le \dim_{\rm P} G([0\,,1])$ a.s. The analogous bounds for $\dim_{\rm H}$ follows from (1.4), Remark 2.4 and Proposition 2.1. This completes the proof. □

We conclude this section by mentioning a simple example wherein Theorem 1.2 fails because G is degenerate.

Example 4.1. Let Y be an isotropic stable Lévy process in $\mathbf{R}^2$ with index $\alpha \in (1\,,2]$. Let G_1 be an independent one-dimensional Brownian motion, and define $G(t) := (G_1(t)\,,0)$. Then, $X := G + Y$ has the form of the process in Theorem 1.2, but now G is degenerate. Direct calculations show that $\Psi_Y(z) = c\|z\|^\alpha$ for some $c > 0$, and $\Psi_G(z) = c' z_1^2$ for some $c' > 0$. It follows readily from this discussion that $\Psi_X(z) = c\|z\|^\alpha + c' z_1^2$, whence it follows that

$$\frac{A_1}{|z_1|^2 + |z_2|^\alpha} \le \kappa_X(z) \le \frac{A_2}{|z_1|^2 + |z_2|^\alpha} \qquad \text{for all } z := (z_1\,, z_2) \in \mathbf{R}^2, \tag{4.4}$$

where A_1 and A_2 are universal constants. Theorem 3.1 implies that with probability one, $\dim_{\rm P} X([0\,,1]) = 1 + \alpha(1 - \frac{1}{2}) = 1 + (\alpha/2)$. On the other hand, according to Theorem 1.1, with probability one $\dim_{\rm P} Y([0\,,1]) = \alpha$, whereas $\dim_{\rm P} G([0\,,1]) = 1$. Therefore, if $\alpha \in (1\,,2)$ then $\dim_{\rm P} X([0\,,1])$ is almost surely *strictly* greater than both $\dim_{\rm P} Y([0\,,1])$ and $\dim_{\rm P} G([0\,,1])$.

Despite the preceding, it is not always necessary that G is non-degenerate, viz.,

Example 4.2. Let $Y := \{Y(t)\}_{t\in\mathbf{R}_+}$ be a Lévy process in $\mathbf{R}^d$ with characteristic exponent $\Psi(\xi) = \|\xi\|^2 L(\xi)$, where $L : \mathbf{R}^d \to \mathbf{C}$ is slowly varying at infinity. Such exponents can be constructed via the Lévy–Khintchine formula. For any Gaussian process $G := \{G(t)\}_{t\in\mathbf{R}_+}$ in $\mathbf{R}^d$ define $X := Y + G$. It follows from Theorem 3.1 that $\dim_{\rm P} X([0\,,1]) = \dim_{\rm P} Y([0\,,1]) = \min(2\,,d)$ almost surely.

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