Recent Developments on Fractal Properties of Gaussian Random Fields

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Abstract We review some recent developments in studying fractal and analytic properties of Gaussian random fields. It is shown that various forms of strong local nondeterminism are useful for studying many fine properties of Gaussian random fields. A list of open questions is included.

1 Introduction

Fractal properties of Brownian motion and Lévy processes have been studied since the pioneering works of P. Lévy (1953) and S.J. Taylor (1953) and have become significant part of the theory on stochastic processes. We refer to Taylor (1986), Xiao (2004) and the references therein for further information.

In recent years, there has been an increased interest in investigating various properties of random fields. On one hand, random fields arise naturally in probability theory, stochastic partial differential equations and in studies of Markov processes. On the other hand, they are extensively applied as stochastic models in various scientific areas such as image processing, physics, biology, hydrology, geostatistics and spatial statistics. However, compared with the rich theory on fine properties of Brownian motion and Lévy processes, the progress in studying random fields has been relatively slow. One of the main difficulties is the lack of powerful technical tools such as Markov property and stopping times.

In this paper we survey some recent studies on sample path properties of Gaussian random fields. We will mainly focus on results which are either based on various properties of strong local nondeterminism or on new concepts in fractal geometry (such as packing dimension profiles).

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The rest of this paper is organized as follows. Section 2 is an introduction on Gaussian random fields, in which we recall the notions of strong local nondeterminism and provide some typical examples. In Section 3, we discuss analytic properties of Gaussian random fields, such as exact modulus of continuity, law of the iterated logarithm (LIL), Chung's LIL, existence and regularity of local times. Section 4 is on fractal properties of Gaussian random fields. We illustrate how anisotropy in the time-variable and/or space-variable may affect the fractal structures of the random fields. More specifically we provide Hausdorff and packing dimension results on the images, graph, inverse images, set of intersections, set of exceptional oscillations, of Gaussian random fields. There are many open problems on analytic and geometric properties of Gaussian random fields. We list some of them in Sections 3 and 4.

We end the Introduction with some general notation. Throughout this paper, we consider random fields $\{X(t), t \in \mathbb{R}^N\}$ which take values in $\mathbb{R}^d$, and we call them (N,d)-random fields. We use $|\cdot|$ and $\langle \cdot, \cdot \rangle$ to denote respectively the Euclidean norm and the inner product in $\mathbb{R}^N$ (or $\mathbb{R}^d$). The Lebesgue measure in $\mathbb{R}^N$ is denoted by λ_N. A point $t \in \mathbb{R}^N$ is written as $t = (t_1, \ldots, t_N)$, or $\langle t_j \rangle$ or as $\langle c \rangle$ if $t_1 = \cdots = t_N = c$. For any $s, t \in \mathbb{R}^N$ such that $s_j < t_j$ $(j = 1, \ldots, N)$, $[s,t] = \prod_{j=1}^N [s_j, t_j]$ is called a closed interval (or a rectangle). We will let $\mathscr{A}$ denote the class of all closed intervals in $\mathbb{R}^N$. For two functions f and g defined on $T \subseteq \mathbb{R}^N$, the notation $f(t) \asymp g(t)$ for $t \in T$ means that the function $f(t)/g(t)$ is bounded from below and above by positive constants that do not depend on $t \in T$.

We will use c to denote an unspecified positive and finite constant which may not be the same in each occurrence. More specific constants are numbered as $c_1, c_2, \ldots$.

2 Gaussian random fields

Two of the most important Gaussian random fields are respectively the Brownian sheet $W = \{W(t), t \in \mathbb{R}^N_+\}$ and fractional Brownian motion $B^H = \{B^H(t),\ t \in \mathbb{R}^N\}$ of index $H \in (0,1)$, and they have been under extensive investigations for several decades. Both of them are centered (N,d)-Gaussian random fields, the former has covariance function

$$\mathbb{E}[W_i(s) W_j(t)] = \delta_{ij} \prod_{k=1}^N s_k \wedge t_k, \quad \forall s, t \in \mathbb{R}^N_+, \tag{1}$$

and the latter has covariance function

$$\mathbb{E}[B_i^H(s) B_j^H(t)] = \frac{1}{2} \delta_{ij} \Big(|s|^{2H} + |t|^{2H} - |s-t|^{2H} \Big), \quad \forall s, t \in \mathbb{R}^N. \tag{2}$$

In the above $\delta_{ij} = 1$ if $i = j$ and 0 if $i \neq j$. The Brownian sheet W and fractional Brownian motion B^H are natural multiparameter extensions of Brownian motion in $\mathbb{R}^d$ and have played important roles in probability theory and in various applications. It is known that there are some fundamental differences between W and B^H.

For example, it follows from (1) that the increments of W over disjoint intervals of $\mathbb{R}_+^N$ are independent and, along each direction of the axis, W is a rescaled Brownian motion in $\mathbb{R}^d$. On the other hand, (2) implies that B^H is H-self-similar and has stationary increments. Moreover, B^H is isotropic in the sense that $B^H(s) - B^H(t)$ depends only on the Euclidean distance $|s-t|$. For further information on the Brownian sheet W and fractional Brownian motion, we refer to Adler (1981), Kahane (1985a), Khoshnevisan (2002), Samorodnitsky and Taqqu (1994).

Several classes of anisotropic Gaussian random fields have been introduced and studied for theoretical and application purposes. For instance, Kamont (1996) introduced fractional Brownian sheets and studied some of their regularity properties. Bonami and Estrade (2003), Biermé, Meerschaert and Scheffler (2007), Li and Xiao (2011), Xue and Xiao (2011) constructed several classes of anisotropic Gaussian random fields with stationary increments and certain operator-scaling properties. Anisotropic Gaussian random fields also arise naturally in stochastic partial differential equations [see, e.g., Dalang (1999), Øksendal and Zhang (2000), Mueller and Tribe (2002), Hu and Nualart (2009)], and as spatial or spatiotemporal models in statistics [e.g., Christakos (2000), Gneiting (2002), Stein (2005)].

It is known that, compared with isotropic Gaussian fields such as fractional Brownian motion, the probabilistic and geometric properties of anisotropic Gaussian random fields are much richer [see Ayache and Xiao (2005), Wu and Xiao (2007, 2009, 2011), Xiao (2009a), Xue and Xiao (2011)] and their estimation and prediction problems are more difficult to study.

There are three kinds of anisotropy, namely, time-variable anisotropy, space-variable anisotropy, and the anisotropy in both variables. Typical examples of time-anisotropic Gaussian random fields are fractional Brownian sheets introduced by Kamont (1996), the solution of the stochastic heat equation driven by space-time white noise [see Mueller and Tribe (2002)], and operator-scaling Gaussian random fields with stationary increments constructed by Biermé, et al. (2007). Examples of space-anisotropic Gaussian random fields include those in Adler (1981) and Xiao (1995), Gaussian fields with fractional Brownian motion components in Xiao (1997a) and the operator fractional Brownian motion studied by Mason and Xiao (2002), Didier and Pipiras (2011). A large class of (N,d) random fields which are anisotropic in both space and time variables have been constructed and studied by Li and Xiao (2011).

In the following two sections, we provide a brief discussion on the properties of anisotropy in the space and time variables of (N,d)-random fields.

2.1 Space-anisotropic Gaussian random fields

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be an (N,d) Gaussian random field defined by

$$X(t) = (X_1(t), \ldots, X_d(t)), \tag{3}$$

where the coordinate processes $X_1,\ldots,X_d$ are assumed to be stochastically continuous. For every $i=1,\ldots,d$, let

$$\sigma_i^2(t,h)=\mathbb{E}\Big[\big(X_i(t+h)-X_i(t)\big)^2\Big],\qquad t,h\in\mathbb{R}^N.$$

Then for every $t\in\mathbb{R}^N$, $\sigma_i(t,h)\to 0$ as $|h|\to 0$. Many probabilistic and geometric properties of X are determined by the asymptotic properties of $\sigma_i^2(t,h)$. If, as $h\to 0$, $\sigma_i^2(t,h)\to 0$ with different rates for $i=1,\ldots,d$, then the coordinate processes $X_1,\ldots,X_d$ have different asymptotic properties which can affect the properties of X in various ways. In this case, we say that X is anisotropic in the space-variable (or, simply, space-anisotropic).

An important class of space-anisotropic random fields are those satisfying the operator-self-similarity. An (N,d)-random field X is called *operator-self-similar in the space-variable* if there exists a $d\times d$ matrix $D=(d_{ij})$ such that for all constants $c>0$,

$$\{X(ct),t\in\mathbb{R}^N\}\stackrel{d}{=}\{c^D X(t),t\in\mathbb{R}^N\}.\tag{4}$$

In the above and in the sequel, $\stackrel{d}{=}$ denotes equality of all finite dimensional distributions and c^D is the linear operator on $\mathbb{R}^d$ defined by

$$c^D=\sum_{n=0}^{\infty}\frac{(\ln c)^n D^n}{n!}.$$

The linear operator D is called a space-variable self-similarity exponent [which may not be unique].

Mason and Xiao (2002) constructed a class of operator-self-similar Gaussian random fields with stationary increments, which are called operator-fractional Brownian motions, and they showed that the Hausdorff dimension of the image $X(E)$ is determined by the positive parts of the eigenvalues of D, the self-similarity exponent of X. To study the effect of space-anisotropy on fractal properties of X, we can first work with the special example of $X=\{X(t),t\in\mathbb{R}^N\}$, which is defined by (3), where $X_1,\ldots,X_d$ are independent N-parameter fractional Brownian motions in $\mathbb{R}$ with indices $\alpha_1,\ldots,\alpha_d$, respectively. Then the (N,d)-Gaussian field X is operator-self-similar with exponent $D=(d_{ij})$ which is the diagonal matrix with $d_{ii}=\alpha_i$ for $i=1,\ldots,d$. When $\alpha_1,\ldots,\alpha_d$ are not the same, X is anisotropic in the space-variable. We call X a Gaussian random field with independent fractional Brownian motion components. Didier and Pipiras (2011) consider more general framework and provide a characterization of all operator-fractional Brownian motions by means of their integral representations in the spectral and time domains.

2.2 Time-anisotropic Gaussian random fields

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be an (N,d)-random field defined by (3). We say that X is anisotropic in the time-variable (or time-anisotropic) if the coordinate processes are (approximately) identically distributed and for some $1 \le i \le d$, the random field $\{X_i(t), t \in \mathbb{R}^N\}$ has different distributional properties along different directions of $\mathbb{R}^N$.

Analogous to the space-anisotropy case, an (N,d)-random field $\{X(t), t \in \mathbb{R}^N\}$ is called *operator-self-similar in the time-variable* if there exists an $N \times N$ matrix E such that for all constants $c > 0$,

$$\big\{X(c^E t), t \in \mathbb{R}^N\big\} \stackrel{d}{=} \big\{cX(t), t \in \mathbb{R}^N\big\}. \tag{5}$$

The linear operator E is called a time-variable self-similarity exponent [which may not be unique].

A typical example of Gaussian random fields which are operator-self-similar in the time-variable is fractional Brownian sheets [cf. Kamont (1996)]. For a given vector $\mathbf{H} = (H_1, \ldots, H_N) \in (0,1)^N$, an $(N,1)$-fractional Brownian sheet $W_0^{\mathbf{H}} = \{W_0^{\mathbf{H}}(t), t \in \mathbb{R}^N\}$ with index $\mathbf{H}$ is a real-valued, centered Gaussian random field with covariance function given by

$$\mathbb{E}\big[W_0^{\mathbf{H}}(s) W_0^{\mathbf{H}}(t)\big] = \prod_{j=1}^N \frac{1}{2}\Big(|s_j|^{2H_j} + |t_j|^{2H_j} - |s_j - t_j|^{2H_j}\Big), \qquad s, t \in \mathbb{R}^N. \tag{6}$$

It follows from (6) that $W_0^{\mathbf{H}}$ is operator-self-similar in the time-variable with exponent $E = (a_{ij})$, which is the $N \times N$ diagonal matrix with $a_{ii} = (NH_i)^{-1}$ for all $1 \le i \le N$ and $a_{ij} = 0$ if $i \ne j$. Note that, in the direction of ith coordinate, $W_0^{\mathbf{H}}(s)$ is a re-scaled fractional Brownian motion of index H_i. Hence, if $H_1, \ldots, H_N$ are not the same, then $W_0^{\mathbf{H}}$ is a time-anisotropic Gaussian field. Other important examples of time-anisotropic random fields include the solution to stochastic heat equation driven by space-time white noise [Mueller and Tribe (2002), Hu and Nualart (2009)] and those with stationary increments constructed by Biermé, Meerschaert and Scheffler (2007), and Xiao (2009a).

Recently Li and Xiao (2011) have extended the notions of operator-self-similarity and operator-scaling to multivariate random fields by allowing scaling of the random field in both "time"-domain and state space by linear operators. For any given $N \times N$ matrix E and $d \times d$ matrix D, they construct a large class of (N,d) Gaussian or stable random fields $X = \{X(t), t \in \mathbb{R}^N\}$ such that for all constants $c > 0$

$$\big\{X(c^E t), t \in \mathbb{R}^N\big\} \stackrel{d}{=} \big\{c^D X(t), t \in \mathbb{R}^d\big\}. \tag{7}$$

If $E = I$, the identity matrix, then (7) reduces to (4). If $D = I$, then (7) reduces to (5).

Similarly to Mason and Xiao (2002), Xiao (2009a) proved that the Hausdorff and packing dimensions of the range, graph and level sets of a Gaussian random field X which satisfies (5) are determined by the real-parts of the eigenvalues of E. However, fractal properties of (N,d) Gaussian random fields which satisfy (7) have not been studied in general.

2.3 Assumptions

Now let us specify the class of Gaussian random fields to be considered in this paper.

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a Gaussian random field with values in $\mathbb{R}^d$ defined by (3). We assume that $X_1, \ldots, X_d$ are independent copies of a real-valued, centered Gaussian random field $X_0 = \{X_0(t), t \in \mathbb{R}^N\}$. Many sample path properties of X can be determined by the following function:

$$\sigma^2(s,t) = \mathbb{E}\big(X_0(s) - X_0(t)\big)^2, \qquad \forall s,t \in \mathbb{R}^N. \tag{8}$$

Let $I \in \mathscr{A}$ be a fixed closed interval and we will consider various sample path properties of $X(t)$ when $t \in I$. We say that X_0 is approximately isotropic on I if there is a function $g : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\sigma^2(s,t) \asymp g(|s-t|), \qquad \forall\ s,t \in I.$$

For simplicity we will mostly assume $I = [\varepsilon, 1]^N$, where $\varepsilon \in (0,1)$ is fixed, or $I = [0,1]^N$.

Let $\mathbf{H} = (H_1, \ldots, H_N) \in (0,1)^N$ be a fixed vector. Let ρ be the metric on $\mathbb{R}^N$ defined by

$$\rho(s,t) = \sum_{j=1}^{N} |s_j - t_j|^{H_j}, \qquad \forall s,t \in \mathbb{R}^N. \tag{9}$$

For any $r > 0$ and $t \in \mathbb{R}^N$, we denote by $B_\rho(t,r) = \{s \in \mathbb{R}^N : \rho(s,t) \le r\}$ the closed ball in the metric ρ.

As in Xiao (2009a), we will make use of the following general conditions on X_0.

(C1). There exists a positive constant $c \ge 1$ such that

$$c^{-1}\,\rho(s,t)^2 \le \sigma^2(s,t) \le c\,\rho(s,t)^2, \qquad \forall\ s,t \in I. \tag{10}$$

(C2). There exists a constant $c > 0$ such that for all $s,t \in I$,

$$\mathrm{Var}\big(X_0(t)\big|X_0(s)\big) \ge c\,\rho(s,\,t)^2.$$

Here and in the sequel, $\mathrm{Var}\big(Y\big|Z\big)$ denotes the conditional variance of Y given Z.

(C3). There exists a constant $c > 0$ such that for all integers $n \ge 1$ and all u, $t^1, \ldots, t^n \in I$ (and $u \neq 0$ if $0 \in I$),

$$\mathrm{Var}\Big(\, X_0(u)\,|\,X_0(t^1), \ldots, X_0(t^n)\Big) \ge c\, \sum_{j=1}^{N} \min_{0 \le k \le n} \big|u_j - t_j^k\big|^{2H_j}, \tag{11}$$

where $t_j^0 = 0$ for every $j = 1, \dots, N$.

(C4). There exists a constant $c > 0$ such that for all integers $n \geq 1$ and all u, $t^1, \dots, t^n \in I$ (and $u \neq 0$ if $0 \in I$),

$$\mathrm{Var}\big(X_0(u)\,|\,X_0(t^1), \dots, X_0(t^n)\big) \geq c \min_{0 \leq k \leq n} \rho(u, t^k)^2, \tag{12}$$

where $t^0 = 0$.

Remark 1. The following are some remarks about the above conditions.

- Under condition (C1), X_0 has a version which has continuous sample functions on I almost surely. Henceforth we will assume that the Gaussian random field X has continuous sample paths.
- Condition (C2) is referred to as "two point" local nondeterminism in the metric ρ [or, with indices $\mathbf{H} = (H_1, \dots, H_N) \in (0,1)^N$]. Together with (C1), it is useful for determining the fractal dimensions of many random sets generated by X.
- Condition (C3) is weaker than (C4). Following the terminology in Khoshnevisan and Xiao (2007), (C3) is called the property of sectorial local nondeterminism. Condition (C4) is called the strong local nondeterminism in the metric ρ. These conditions are important for establishing sharp results on modulus of continuity, Chung's law of the iterated logarithm, sharp Hölder conditions for the local times, exact Hausdorff measure functions, among others.
- Pitt (1978) proved that multiparameter fractional Brownian motion B^H satisfies (C4) with $\mathbf{H} = \langle H \rangle$. Wu and Xiao (2007) proved that a fractional Brownian sheet $W^{\mathbf{H}}$ satisfies Condition (C3). We refer to Xiao (2009a), Xue and Xiao (2011) for more examples of anisotropic Gaussian random fields which satisfy Condition (C4).

  Recently, Luan and Xiao (2012) have provided a general condition for a Gaussian random field X_0 with stationary increments to satisfy (C4) in terms of the spectral measure Δ of X_0. This condition can be applied even when Δ is singular, supported on a fractal set or on a discrete set such as $\mathbb{Z}^N$. Their theorem can be applied to prove that the solution of a fractional stochastic heat equation on the circle $\mathbb{S}_1$ [see Tindel, Tudor and Viens (2004), Nualart and Viens (2009)] has the property of strong local nondeterminism in the space variable (at fixed time t). Hence fine properties of the sample functions of the solution can be obtained by using the results discussed below. Similarly, we can show that the spherical fractional Brownian motion on $\mathbb{S}_1$ introduced by Istas (2005) is also strongly locally nondeterministic. These processes share local properties with ordinary fractional Brownian motion with appropriate indices.

3 Analytic Results

Sample functions of a Gaussian random field may present various fine properties such as continuity and differentiability. See Adler and Taylor (2007), Marcus and Rosen (2006), Talagrand (2006).

For an anisotropic Gaussian random field, its sample function may be differentiable in certain directions, but not differentiable in other directions [see Xue and Xiao (2011) for explicit criterion for Gaussian random fields with stationary increments] and may have rich (sometimes complicated) geometric structures. Our main objective is to characterize the analytic and geometric properties of a Gaussian random field C in terms of its parameter $\mathbf{H} = (H_1, \ldots, H_N)$, if X satisfies some of the Conditions (C1)–(C4).

Geometric properties of a Gaussian random field X are very closely related to the regularities (or irregularities) of the sample functions of X. In this section, we discuss analytic properties such as uniform and local moduli of continuity and local times of Gaussian random fields.

3.1 Exact Modulus of Continuity and LIL

Sample path continuity and Hölder regularity of Gaussian random fields have been studied by many authors. A powerful chaining argument leads to sharp upper bounds for uniform and local moduli of continuity of Gaussian processes in terms of metric entropy or majorizing measures. Here "sharp" means logarithmic correction factors can be obtained. See Talagrand (2006), Marcus and Rosen (2006) and Adler and Taylor (2007). Sharp lower bounds for local and uniform moduli of continuity of Gaussian processes are discussed in Marcus and Rosen (2006, Chapter 7). However, except for a few special cases such as certain one-parameter Gaussian processes [Marcus and Rosen (2006)], the Brownian sheet [Orey and Pruitt (1973)] and fractional Brownian motion [Benassi *et al.* (1997)], there have not been many explicit results on sharp lower bounds for uniform and local moduli of Gaussian random fields.

The following theorem on uniform modulus of continuity is proved in Meerschaert, Wang and Xiao (2011).

Theorem 1. *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be real-valued and centered Gaussian random field which satisfies Conditions (C1) and (C3). Then for every compact interval $I \subseteq \mathbb{R}^N$, there exists a positive and finite constant c_1, depending only on I and H_j, $(j = 1, \ldots, N)$ such that*

$$\lim_{\varepsilon \to 0+} \sup_{s,t \in I, \sigma(s,t) \le \varepsilon} \frac{|X(t) - X(s)|}{\sigma(s,t)\sqrt{\log(1+\sigma(s,t)^{-1})}} = c_1 \quad a.s., \tag{13}$$

where $\sigma(s,t)$ is as in (8).

Notice that the limit in (13) exists almost surely due to monotonicity. So the real issue is to prove that the limit is a (non-random) constant which is positive and finite. Condition (C1) allows us to apply standard method (e.g., the entropy method) to derive an upper bound for $\sup_{s,t\in I,\sigma(s,t)\le\varepsilon}|X(t)-X(s)|$. Thus the 0-1 law in Marcus and Rosen (2006, Chapter 7) implies that the limit is non-random and finite. The hard part is to prove $c_1>0$, this is where the sectorial local nondeterminism (C3) plays an important role.

For the local modulus of continuity, Meerschaert, Wang and Xiao (2011) proved the following law of the iterated logarithm.

Theorem 2. *Let $\{X(t), t\in\mathbb{R}^N\}$ be a real valued, centered Gaussian random field with stationary increments and $X(0)=0$. If X satisfies Condition (C1) for $I=[0,1]^N$, then there is a positive constant c_2 such that for every $t_0\in\mathbb{R}^N$ we have*

$$\limsup_{|\varepsilon|\to 0+}\sup_{s:|s_j|\le\varepsilon_j}\frac{|X(t_0+s)-X(t_0)|}{\sigma(s)\sqrt{\log\log\left(1+\prod_{j=1}^N|s_j|^{-H_j}\right)}}=c_2\quad a.s., \tag{14}$$

where $\sigma^2(s)=\mathbb{E}\big[X(s)^2\big]$.

For the local oscillation of $X(t)$ with $t\in B_\rho(t_0,r)$, Meerschaert, Wang and Xiao (2011) proved the following result.

Theorem 3. *Let $X=\{X(t), t\in\mathbb{R}^N\}$ be a real-valued, centered Gaussian random field with stationary increments and $X(0)=0$. If X satisfies Condition (C1) for $I=[0,1]^N$, then there is a positive and finite constant c_3 such that for every $t_0\in\mathbb{R}^N$ we have*

$$\lim_{r\to 0+}\sup_{s:\sigma(s)\le r}\frac{|X(t_0+s)-X(t_0)|}{\sigma(s)\sqrt{\log\log(1+\sigma(s)^{-1})}}=c_3\quad a.s. \tag{15}$$

Remark 2. Noticed that the logarithmic factors in Theorems 2 and 3 are quite different, since (C1) implies $\sigma(s)\asymp\sum_{j=1}^N|s_j|^{H_j}$ as $s\to 0$ in Theorem 3, and the corresponding term $\prod_{j=1}^N|s_j|^{-H_j}$ in Theorem 2 is much bigger. This is due to the fact that the supremum in (14) is taken over a larger domain.

Theorems 2 and 3 can not be applied directly to fractional Brownian sheet $W^{\mathbf{H}}$ because it does not have stationary increments in the ordinary sense. However, Meerschaert, Wang and Xiao (2011, Theorem 6.4) show that (14) and (15) still hold for all $t_0\in[a,\infty)^N$ (where $a>0$ is a constant). The oscillation behavior of $W^{\mathbf{H}}(t)$ at the origin $t_0=0$ is very different and is characterized by Wang (2007). Together these results reveal the subtlety of the influence of anisotropy on fine properties of random fields.

Several interesting questions can be raised. Comparing Theorem 1 and Theorems 2 and 3, one can show as in Orey and Taylor (1974) that there exists a random point t at which the local oscillation $\sup_{s:\sigma(s)\le r}|X(t+s)-X(t)|$ is unusually large, say, of the order $\sigma(s)\sqrt{\log(1+\sigma(s)^{-1})}$. Motivated by this, we define the following sets of "fast points"

$$F_1(\gamma) = \left\{ t \in I : \limsup_{|s| \to 0+} \frac{|X(t+s) - X(t)|}{\sigma(s) \sqrt{\log\left(1 + \prod_{j=1}^N |s_j|^{-H_j}\right)}} \ge \gamma \right\} \tag{16}$$

and

$$F_2(\gamma) = \left\{ t \in I : \limsup_{|s| \to 0+} \frac{|X(t+s) - X(t)|}{\sigma(s) \sqrt{\log(1 + \sigma(s)^{-1})}} \ge \gamma \right\}. \tag{17}$$

It follows from Theorems 2 and 3 and Fubini's theorem that both $F_1(\gamma)$ and $F_2(\gamma)$ have zero Lebesgue measure. It is interesting to study their Hausdorff and packing dimensions. In the case of $N = 1$ and X is Brownian motion, $F_1(\gamma) = F_2(\gamma)$ and its Hausdorff dimension was determined by Orey and Taylor (1974) and further refined by Kaufman (1975). Khoshnevisan, Peres and Xiao (2000) developed a general method for studying limsup type random fractals which is not only applicable to the set of fast points of Brownian motion, but also to many other random sets defined by exceptional oscillation or growth, including the set of fast points of the Brownian sheet [See also Dindar (2001) for another treatment of the set of fast points of two parameter Brownian sheet W] and fractional Brownian motion, thick points of the sojourn measure of Brownian motion [Dembo, et al. (2000)].

In the current random field setting, however, it is not known whether $F_1(\gamma)$ and $F_2(\gamma)$ have different fractal properties. Hence we formulate the following problem.

Problem 1. Determine the Hausdorff and packing dimensions of $F_1(\gamma)$ and $F_2(\gamma)$. For a give Borel set $E \subseteq \mathbb{R}^N$, when do we have $F_1(\gamma) \cap E \neq \varnothing$ and $F_2(\gamma) \cap E \neq \varnothing$?

Following Walsh (1982), a point t_0 is called a singularity of X if the law of the iterated logarithm fails at t_0. Hence the points in $F_1(\gamma)$ and $F_2(\gamma)$ are singularities of X. Walsh (1982) studied the propagation of the singularities of the Brownian sheet and proved the following theorem. See also Zimmerman (1972) for part (i).

Theorem 4. *Let $W = \{W(s,t), (s,t) \in \mathbb{R}^2_+\}$ be the real-valued Brownian sheet.*

(i). For each fixed $s \ge 0$,

$$\mathbb{P}\left\{ \limsup_{r \to 0} \frac{|W(s+r,t) - W(s,t)|}{\sqrt{2r \log\log(1/r)}} = \sqrt{t} \quad \text{for all } t \ge 0 \right\} = 1. \tag{18}$$

(ii). Let $t_0 > 0$ be fixed and let $S \ge 0$ be a random variable which is measurable with respect to $\sigma\{W(s,t) : s \ge 0, 0 \le t \le t_0\}$. Then for almost every $\omega \in \Omega$,

$$\begin{aligned} &\limsup_{r \to 0} \frac{|W(S(\omega), t_0 + r) - W(S(\omega), t_0)|}{\sqrt{2r \log\log(1/r)}} = \infty \iff \\ &\qquad \limsup_{r \to 0} \frac{|W(S(\omega), t + r) - W(S(\omega), t)|}{\sqrt{2r \log\log(1/r)}} = \infty \quad \text{for all } t \ge t_0. \end{aligned} \tag{19}$$

Part (i) says that, at every fixed time $s \ge 0$, the laws of iterated logarithm for the (rescaled) Brownian motion $W(\cdot, t)$ holds uniformly for all $t \ge 0$. This is a refinement of LIL for Brownian motion. Part (ii) is a converse of (i) and means that, if

$S(\omega)$ is a random singularity of the Brownian motion $W(\cdot, t_0)$, then every point on the vertical ray $\{(S(\omega), t) : t \geq t_0\}$ is a singularity of W. This reveals specifically that the singularities of W propagate parallel to the coordinate axes.

Blath and Martin (2008) have extended Theorem 4 to the two-parameter fractional Brownian sheet $W^{\mathbf{H}}$ with $H_1 = 1/2$ and $H_2 \in (0,1)$ (which they call semi-fractional Brownian sheet). The fact that, for any $t_2 > 0$, $\{W^{\mathbf{H}}(t_1, t_2), t_1 \geq 0\}$ is a Brownian motion plays a crucial role in their proofs.

Walsh (1982) asked whether an analogous property holds for other Gaussian random fields such as Lévy's multiparameter Brownian motion $B^{1/2}$. As far as I know, this problem has not been solved. Hence we formulate the following problem.

Problem 2. We say that t_0 is a singularity of Gaussian random field X if the limsup in (15) is infinity. How do the singularities of X propagate?

3.2 Chung's LIL and modulus of nondifferentiability

The results in Section 3.1 are about large oscillations. For small local oscillation of X at $t_0 \in \mathbb{R}^N$, Luan and Xiao (2010) proved the following Chung-type law of iterated logarithm. For earlier results on Chung's LIL for isotropic Gaussian random fields, as well as their connections to small ball probabilities we refer to the excellent survey of Li and Shao (2001).

Theorem 5. *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a Gaussian random field with stationary increments and $X(0) = 0$. If X satisfies Conditions (C1) and (C4) on an interval $I \subseteq \mathbb{R}^N$, then there exists a positive and finite constant c_4 such that for every $t_0 \in \mathbb{R}^N$,*

$$\liminf_{r \to 0} \frac{\max_{\rho(t,t_0) \leq r} |X(t) - X(t_0)|}{r(\log\log 1/r)^{-1/Q}} = c_4, \qquad a.s. \tag{20}$$

where $Q = \sum_{i=1}^N H_i^{-1}$.

When $N = 1$ this extends a result of Monrad and Rootzén (1995).

Theorem 5 assumes that X has stationary increments and satisfies (C4), thus is not applicable to other Gaussian random fields such as fractional Brownian sheets. Talagrand (1994) and Zhang (1996) proved the Chung's LIL for the Brownian sheet $W = \{W(s,t), (s,t) \in \mathbb{R}_+^2\}$ as "time" goes to ∞. Their arguments also prove the following Chun's LIL at $t_0 = 0$, which is quite different from (20).

Theorem 6. *There is a positive and finite constant c_5 such that*

$$\liminf_{r \to 0} \frac{(\log\log 1/r)^{1/2}}{r(\log\log\log 1/r)^{3/2}} \max_{0 \leq s,t \leq r} |W(s,t)| = c_5 \qquad a.s. \tag{21}$$

The problem for proving a Chung's law of the iterated logarithm for a general fractional Brownian sheet $W^{\mathbf{H}}$ with index $\mathbf{H} \in (0,1)^N$ has not been solved completely. For some interesting partial results, see Mason and Shi (2001). By applying the result of Monrad and Rootzén (1995) to the restriction of $W^{\mathbf{H}}$ to the direction of the j-th coordinate, say $\{W^{\mathbf{H}}(1,\ldots,1,t_j,1,\ldots,1), t_j \in \mathbb{R}\}$, one can see that the Chung's law of the iterated logarithm is analogous to that of a one-parameter fractional Brownian motion of Hurst index H_j.

It is also an open problem to establish a uniform version of (5) for Gaussian random fields. In the special case of fractional Brownian motion, we believe [cf. Xiao (1997b)] that there is a positive and finite constant c_6 such that

$$\liminf_{r\to 0} \sup_{t\in[0,1]^N} \frac{\max_{|s-t|\le r} |B^H(t)-B^H(s)|}{r^H (\log 1/r)^{-H/N}} = c_6, \qquad \text{a.s.} \tag{22}$$

Even though the lower bound can be easily proved [cf. Xiao (1997b)], the upper bound is more difficult. For the special case of $N=1$, (22) has been proved recently by Hwang, Wang and Xiao (2011).

Similarly to (16) and (17) one can define the set of exceptionally small oscillation

$$S(\gamma) = \left\{ t \in I : \liminf_{r\to 0+} \frac{\max_{\rho(t,t_0)\le r} |X(t)-X(t_0)|}{r(\log 1/r)^{-1/Q}} \le \gamma \right\} \tag{23}$$

and ask similar questions for $S(\gamma)$ as in Problem 1. The method of limsup random fractals developed by Khoshnevisan, Peres and Xiao (2000) should be useful in studying these problems.

3.3 Regularity of local times

The roughness or irregularity of sample functions of X can be reflected in the regularity (or smoothness) of the local times of X. This was first observed by Berman (1972) who developed Fourier analytic method for studying the existence and continuity of local times of Gaussian processes. Berman (1973) introduced the notion of "local nondeterminism" for Gaussian processes to overcome many difficulties caused by the lack of Markov property and to unify his methods for studying local times. Berman's work has been extended and strengthened in various ways. See Geman and Horowitz (1980) and Xiao (2007, 2009) for more information.

We recall briefly the definition of local times. Let $Y(t)$ be a Borel vector field on $\mathbb{R}^p$ with values in $\mathbb{R}^q$. For any Borel set $T \subseteq \mathbb{R}^p$, the occupation measure of Y on T is defined as the following measure on $\mathbb{R}^q$:

$$\mu_T(\cdot) = \lambda_p\big\{t \in T : Y(t) \in \cdot\big\}.$$

If μ_T is absolutely continuous with respect to the Lebesgue measure λ_q, we say that $Y(t)$ has *local times* on T, and define its local time, $L(\cdot,T)$, as the Radon–Nikodým derivative of μ_T with respect to λ_q, i.e.,

$$L(x,T) = \frac{d\mu_T}{d\lambda_q}(x), \qquad \forall x \in \mathbb{R}^q.$$

In the above, x is called the *space variable*, and T is the *time variable*. Note that if Y has local times on T then for every Borel set $S \subseteq T$, $L(x,S)$ also exists.

Suppose we fix a rectangle $T = \prod_{i=1}^p [a_i, a_i + h_i] \subseteq \mathbb{R}^p$, where $a \in \mathbb{R}^p$ and $h \in \mathbb{R}_+^p$. If we can choose a version of the local time, still denoted by $L(x, \prod_{i=1}^p [a_i, a_i + t_i])$, such that it is a continuous function of $(x, t_1, \cdots, t_p) \in \mathbb{R}^q \times \prod_{i=1}^p [0, h_i]$, Y is said to have a *jointly continuous local time* on T. When a local time is jointly continuous, $L(x, \cdot)$ can be extended to a finite Borel measure supported on the level set

$$Y_T^{-1}(x) = \{t \in T : Y(t) = x\}; \tag{24}$$

see Adler (1981) for details. This makes local times a useful tool in studying fractal properties of Y.

When $X = \{X(t), t \in \mathbb{R}^N\}$ is an (N,d)-Gaussian random field with approximately isotropic increments (e.g., fractional Brownian motion), Xiao (1997b) proved sharp local and uniform modulus of continuity for the local time $L(x, \cdot)$ in the set variable. For simplicity, we focus on fractional Brownian motion $B^H = \{B^H(t), t \in \mathbb{R}^N\}$ with values in $\mathbb{R}^d$. The following theorem was proved by Baraka and Mountford (2008), Baraka, Mountford and Xiao (2009). See also Chen, et al. (2011) for the case of $N = 1$, where large deviation results for the local times and intersection local times are proved.

Theorem 7. *Let $B^H = \{B^H(t),\ t \in \mathbb{R}^N\}$ be a fractional Brownian motion in $\mathbb{R}^d$ with index $H \in (0,1)$. If $N > Hd$, then there exists a positive and finite constant c_7 such that for every $a \in \mathbb{R}^N$,*

$$\limsup_{r \to 0} \frac{L^*(B(a,r))}{r^{N-Hd}(\log\log(1/r))^{Hd/N}} = c_7, \quad \text{a.s.} \tag{25}$$

and for every $T > 0$, there is a positive and finite constant c_8 such that

$$\limsup_{r \to 0} \sup_{a \in [-T,T]^N} \frac{L^*(B(a,r))}{r^{N-Hd}(\log(1/r))^{Hd/N}} = c_8, \quad \text{a.s.} \tag{26}$$

In the above, $L^*(B(a,r)) = \max_{x \in \mathbb{R}^d} L(x, B(a,r))$.

Eq. (25) gives the LIL for $L^*(B(a,r))$ and is used to derive an exact Hausdorff measure function for the level set of fractional Brownian motion in Baraka and Mountford (2011). There result significantly improves that in Xiao (1997b).

The existence and joint continuity of local times of a fractional Brownian sheet $W^{\mathbf{H}}$ with values in $\mathbb{R}^d$ and index $\mathbf{H} = (H_1, \ldots, H_N) \in (0,1)^N$ were studied by Xiao and Zhang (2002). Ayache, Wu and Xiao (2008) proved that the optimal condition for the joint continuity of the local times of $W^{\mathbf{H}}$ is $\sum_{j=1}^N H_j^{-1} > d$. Xiao (2009a) proved similar results for a class of Gaussian random fields with stationary increments which satisfy Conditions (C1) and (C3). Wu and Xiao (2011) provided a unified treatment by applying sectorial local nondeterminism to estimate high moments of local times and improved significantly the results in Ayache, Wu and Xiao (2008) and Xiao (2009a).

When X is anisotropic, the problems for finding sharp local and uniform modulus of continuity for $L(x,\cdot)$ and $L^*(\cdot) = \max_{x\in\mathbb{R}^d} L(x,\cdot)$ in the set variable are more complicated and have not been solved. In the following we state the main result in Wu and Xiao (2011).

First we give some notation. Henceforth we assume that $\mathbf{H} = (H_1,\ldots,H_N) \in (0,1)^N$ is fixed and

$$0 < H_1 \le \ldots \le H_N < 1. \tag{27}$$

When $\sum_{j=1}^N \frac{1}{H_j} > d$, there exists $\tau \in \{1,2,\ldots,N\}$ such that

$$\sum_{j=1}^{\tau-1} \frac{1}{H_j} \le d < \sum_{j=1}^{\tau} \frac{1}{H_j},$$

with the convention that $\sum_1^0(\cdot) \equiv 0$. We denote

$$\alpha := \sum_{j=1}^N \frac{1}{H_j} - d, \qquad \eta_\tau := \tau + H_\tau d - \sum_{j=1}^{\tau} \frac{H_\tau}{H_j} \tag{28}$$

and we will distinguish three cases:

Case 1. $\sum\limits_{j=1}^{\tau-1} \frac{1}{H_j} < d < \sum\limits_{j=1}^{\tau} \frac{1}{H_j}$.

Case 2. $\sum\limits_{j=1}^{\tau-1} \frac{1}{H_j} = d < \sum\limits_{j=1}^{\tau} \frac{1}{H_j}$ and $H_{\tau-1} = H_\tau$.

Case 3. $\sum\limits_{j=1}^{\tau-1} \frac{1}{H_j} = d < \sum\limits_{j=1}^{\tau} \frac{1}{H_j}$ and $H_{\tau-1} < H_\tau$.

The first result is on local Hölder condition for $L^*(\cdot)$.

Theorem 8. *Let $X = \{X(t),\, t \in \mathbb{R}^N\}$ be an anisotropic Gaussian random field with values in $\mathbb{R}^d$ which satisfies Conditions (C1) and (C3) on an interval $I \in \mathscr{A}$. Then there exist positive constants c_9 and c_{10} such that for every $a \in I$,*

$$\begin{aligned} &\limsup_{r\to 0} \frac{L^*(B_\rho(a,r))}{\varphi_1(r)} \le c_9, \quad \text{a.s.} \quad \textit{in Cases 1 and 2,} \\ &\limsup_{r\to 0} \frac{L^*(B_\rho(a,r))}{\varphi_2(r)} \le c_{10}, \quad \text{a.s.} \quad \textit{in Cases 3,} \end{aligned} \tag{29}$$

where

$$\begin{aligned} \varphi_1(r) &= r^\alpha \big(\log\log(1/r)\big)^{\eta_\tau}, \\ \varphi_2(r) &= r^\alpha \big(\log\log(1/r)\big)^{\eta_\tau} \log\log\log\big(1/r\big). \end{aligned} \tag{30}$$

The second result is on uniform Hölder condition for $L^*(\cdot)$.

Theorem 9. *Under the same conditions as in Theorem 8, we have*

$$\begin{aligned}
&\limsup_{r\to 0}\sup_{a\in I}\frac{L^*(B_\rho(a,r))}{\Phi_1(r)}\le c_{11}, \quad \text{a.s.} \quad \textit{in Cases 1 and 2,}\\
&\limsup_{r\to 0}\sup_{a\in I}\frac{L^*(B_\rho(a,r))}{\Phi_2(r)}\le c_{12}, \quad \text{a.s.} \quad \textit{in Case 3,}
\end{aligned} \tag{31}$$

where c_{11} and c_{12} are positive and finite constants and

$$\begin{aligned}
\Phi_1(r) &= r^{\alpha}\left(\log(1/r)\right)^{\eta_\tau},\\
\Phi_2(r) &= r^{\alpha}\left(\log(1/r)\right)^{\eta_\tau}\log\log\left(1/r\right).
\end{aligned} \tag{32}$$

In the special case of $H_1=\cdots=H_N:=H$, we have $\alpha=\frac{N}{H}-d$ and $\eta_\tau=Hd$. The above theorems give local and uniform Hölder conditions for $L^*(\cdot)$ in the Euclidean metric.

$$\limsup_{r\to 0}\frac{L^*(B(a,r))}{r^{N-Hd}(\log\log(1/r))^{Hd}}\le c_9, \quad \text{a.s.} \tag{33}$$

and

$$\limsup_{r\to 0}\sup_{a\in I}\frac{L^*(B_\rho(a,r))}{r^{N-Hd}(\log(1/r))^{Hd}}\le c_{11}, \quad \text{a.s.} \tag{34}$$

Notice that the powers of $\log\log 1/r$ in (25) and (33) are different. This is due to the different forms of strong local nondeterminism.

Unlike in Theorem 7, we do not know whether the results in Theorems 8 and 9 are sharp. Hence we raise the following question.

Problem 3. Under what conditions can one establish exact local and uniform moduli of continuity for the local time $L(x,\cdot)$ and $L^*(\cdot)$?

4 Fractal Properties

Now we turn to fractal properties of Gaussian random fields, which include Hausdorff and packing dimensions of random sets such as the images, graph, level sets, and the set of intersections. We also present uniform dimension results as well as exact Hausdorff and packing measure functions for the image sets. These latter results depend on properties of strong local nondeterminism. We refer to Falconer (2003) or Mattila (1995) for definitions and basic properties of Hausdorff and packing measures, and corresponding dimensions.

Given an (N,d) random field X, the following random sets generated by X are often random fractals.

(i). Range or image set $X(E)=\{X(t): t\in E\}$, where $E\subseteq\mathbb{R}^N$.
(ii). Graph $\mathrm{Gr}X(E)=\{(t,X(t)): t\in E\}$.
(iii). Level set $X^{-1}(x)=\{t\in E: X(t)=x\}$, where $x\in\mathbb{R}^d$.

(iv). Excursion set (or inverse image)

$$X^{-1}(F) = \big\{t \in \mathbb{R}^N : X(t) \in F\big\},$$

where $F \subseteq \mathbb{R}^d$.

(v). Set of k-multiple times

$$L_k = \big\{(t^1, \ldots, t^k) \in \mathbb{R}^{Nk}_{\neq} : X(t^1) = \cdots = X(t^k)\big\},$$

where $\mathbb{R}^{Nk}_{\neq} = \big\{(t^1, \ldots, t^k) \in \mathbb{R}^{Nk} : t^1, \ldots, t^k \text{ are distinct.}\big\}$.

(vi). Set of k-multiple points

$$M_k = \big\{x \in \mathbb{R}^d : \exists\, (t^1, \ldots, t^k) \in \mathbb{R}^{Nk}_{\neq} \text{ such that } \ x = X(t^1) = \cdots = X(t^k)\big\}.$$

4.1 Hausdorff dimension results

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be an (N,d)-Gaussian field defined by (3) such that the coordinate processes $X_1, \ldots, X_d$ are independent copies of a real-valued, centered Gaussian random field $X_0 = \{X_0(t), t \in \mathbb{R}^N\}$. Recall that we have assumed that $\mathbf{H} = (H_1, \ldots, H_N) \in (0,1)^N$ satisfies (27).

The Hausdorff and packing dimensions of the range, graph, level sets of fractional Brownian sheets and, more generally, anisotropic Gaussian random fields which satisfy Conditions (C1) and (C2) have been established in Ayache and Xiao (2005) and Xiao (2009a).

Theorem 10. *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be an (N,d)-Gaussian field defined in the above. Assume that X_0 satisfies Condition (C1) with $I = [0,1]^N$. Then, with probability 1,*

$$\dim_{\rm H} X\big([0,1]^N\big) = \dim_{\rm P} X\big([0,1]^N\big) = \min\left\{d; \quad \sum_{j=1}^N \frac{1}{H_j}\right\} \tag{35}$$

and

$$\begin{aligned}
&\dim_{\rm H} {\rm Gr} X\big([0,1]^N\big) = \dim_{\rm P} {\rm Gr} X\big([0,1]^N\big) \\
&\quad = \min\left\{\sum_{j=1}^k \frac{H_k}{H_j} + N - k + (1-H_k)d,\ 1 \le k \le N;\ \sum_{j=1}^N \frac{1}{H_j}\right\} \\
&\quad = \begin{cases} \sum_{j=1}^N \frac{1}{H_j}, & \text{if } \ \sum_{j=1}^N \frac{1}{H_j} \le d, \\ \sum_{j=1}^k \frac{H_k}{H_j} + N - k + (1-H_k)d, & \text{if } \ \sum_{j=1}^{k-1} \frac{1}{H_j} \le d < \sum_{j=1}^k \frac{1}{H_j}, \end{cases}
\end{aligned} \tag{36}$$

where $\sum_{j=1}^0 \frac{1}{H_j} := 0$.

Theorem 11. *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be an (N,d)-Gaussian random field defined as above. Assume that X_0 satisfies Conditions (C1) and (C2) on $I = [\varepsilon, 1]^N$. Then the following statements hold.*

(i) If $\sum_{j=1}^N \frac{1}{H_j} < d$, then for every $x \in \mathbb{R}^d$, $X^{-1}(x) \cap I = \emptyset$ a.s.
(ii) If $\sum_{j=1}^N \frac{1}{H_j} > d$, then for every $x \in \mathbb{R}^d$, with positive probability,

$$\begin{aligned} \dim_{\rm H}\big(X^{-1}(x)\cap I\big) &= \dim_{\rm P}\big(X^{-1}(x)\cap I\big) \\ &= \min\left\{ \sum_{j=1}^{k} \frac{H_k}{H_j} + N - k - H_k d,\ 1 \le k \le N \right\} \\ &= \sum_{j=1}^{k} \frac{H_k}{H_j} + N - k - H_k d, \quad \text{if } \sum_{j=1}^{k-1} \frac{1}{H_j} \le d < \sum_{j=1}^{k} \frac{1}{H_j}. \end{aligned} \tag{37}$$

For the inverse images, Xiao (2009a) and Biermé, Lacaux and Xiao (2009) provided conditions on F such that $\mathbb{P}\big\{X^{-1}(F) \cap I \neq \varnothing\big\} > 0$ and proved the following result.

Theorem 12. *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be an (N,d)-Gaussian random field as in Theorem 11 and let $F \subseteq \mathbb{R}^d$ be a Borel set such that* $\dim F \ge d - Q$. *Then*

$$\big\|\dim_{\rm H}\big(X^{-1}(F)\cap I\big)\big\|_{L^\infty(\mathbb{P})} = \min_{1\le k\le N} \left\{ \sum_{j=1}^{k} \frac{H_k}{H_j} + N - k - H_k\big(d - \dim_{\rm H} F\big) \right\}, \tag{38}$$

where, for any function $Y : \Omega \to \mathbb{R}_+$, $\|Y\|_{L^\infty(\mathbb{P})}$ is defined as

$$\|Y\|_{L^\infty(\mathbb{P})} = \sup\big\{\theta : Y \ge \theta \ \text{ on an event } E \text{ with } \mathbb{P}(E) > 0\big\}.$$

Under an extra condition on F, Theorem 2.5 in Biermé, Lacaux and Xiao (2009) shows that with positive probability,

$$\begin{aligned} \dim_{\rm H}\big(X^{-1}(F)\cap I\big) &= \min_{1\le k\le N} \left\{ \sum_{j=1}^{k} \frac{H_k}{H_j} + N - k - H_k(d - \dim_{\rm H} F) \right\} \\ &= \sum_{j=1}^{k} \frac{H_k}{H_j} + N - k - H_k(d - \dim_{\rm H} F), \quad \text{if } \sum_{j=1}^{k-1} \frac{1}{H_j} \le d - \dim_{\rm H} F < \sum_{j=1}^{k} \frac{1}{H_j}. \end{aligned} \tag{39}$$

There are several possible ways to strengthen and extend Theorems 10, 11 and 12, and to ask further questions about these random sets. For example, it would be interesting to determine the exact Hausdorff and packing measure functions for the range $X\big([0,1]^N\big)$, graph $\mathrm{Gr}X\big([0,1]^N\big)$ and the level set $X^{-1}(x)$; and to characterize the hitting probabilities of these random sets. Further information on exact Hausdorff and packing measure functions will be provided in Sections 4.5 and 4.6. Testard (1986), Xiao (1999, 2009a), Biermé, Lacaux and Xiao (2009) and Chen and Xiao (2011) have provided necessary conditions and sufficient conditions on

$E \subseteq \mathbb{R}^N$ or/and $F \subseteq R^d$ for $X\big([0,1]^N\big) \cap F \neq \varnothing$, or $\mathrm{Gr}X\big([0,1]^N\big) \cap (E \times F) \neq \varnothing$ with positive probability. See 4.7 below. However, except in a few special cases, the following questions are still open.

Problem 4. Find necessary and sufficient conditions on $F \subseteq \mathbb{R}^d$ or $E \subseteq \mathbb{R}^N$ for $X\big([0,1]^N\big) \cap F \neq \varnothing$, $\mathrm{Gr}X\big([0,1]^N\big) \cap (E \times F) \neq \varnothing$ with positive probability.

For results on the Brownian sheet, the hitting probabilities of the range and level sets have been completely characterized by Khoshnevisan and Shi (1999), Khoshnevisan and Xiao (2007). The corresponding problem for the graph set is more complicated and has only been solved for Brownian motion; see Khoshnevisan and Xiao (2012) and the references therein for further information.

In the following, we consider the natural questions to find the Hausdorff, Fourier and packing dimensions of the image set $X(E)$, where $E \subseteq \mathbb{R}^N$ is an arbitrary Borel set (typically, a fractal set). It is not hard to see that, due to the anisotropy of X, the Hausdorff dimension of $X(E)$ can not be determined by $\dim_{\mathrm{H}} E$ and the index $\mathbf{H}$ alone [see Wu and Xiao (2007)]. This is in contrast with the cases of fractional Brownian motion or the Brownian sheet.

To determine the Hausdorff dimension of $X(E)$ for an arbitrary Borel set E, Wu and Xiao (2007) and Xiao (2009a) make use of Hausdorff dimension $\dim_{\mathrm{H}}^{\rho}$ on the metric space $(\mathbb{R}^N, \rho)$, where ρ is defined in (9), and prove the following theorem.

Theorem 13. *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be an (N,d)-Gaussian field as in Theorem 10. Then for every Borel set $E \subseteq \mathbb{R}^N$,*

$$\dim_{\mathrm{H}} X(E) = \min\big\{d, \dim_{\mathrm{H}}^{\rho} E\big\}, \qquad a.s. \tag{40}$$

The Fourier and packing dimensions of $X(E)$ will be discussed in Sections 4.2 and 4.3 below. We end this section with the following remark.

Remark 3. Note that, the results in this section and in most of the subsequent sections are concerned with Gaussian random fields which may be time-anisotropic, but not space-anisotropic. Hausdorff dimensions for the range, graph, level sets and other properties of Gaussian random fields which are space-anisotropic have been considered by Cuzick (1978, 1982), Adler (1981) and Xiao (1995, 1997a). The results are different, and there are still many open questions. For example, for space-anisotropic Gaussian random fields, the Hausdorff dimension of $X^{-1}(F)$ and the Fourier dimensions [see section 4.2 below] of the images have not been determined. Moreover, for (N,d) random fields which are anisotropic in both time and space, little has been known about their fractal properties.

4.2 The Fourier dimension and Salem sets

Besides Hausdorff and packing dimensions, one can define the Fourier dimension of a set $F \subseteq \mathbb{R}^d$, which is related to the asymptotic behavior of the Fourier transforms of the probability measures carried by F.

Let us recall from Kahane (1985a) the definitions of Fourier dimension and Salem set. Given a constant $\beta \geq 0$, a Borel set $F \subseteq \mathbb{R}^d$ is said to be an M_β-set if there exists a probability measure ν on F such that

$$\left|\widehat{\nu}(\xi)\right| = o(|\xi|^{-\beta}) \quad \text{as } \xi \to \infty, \tag{41}$$

where $\widehat{\nu}$ denotes the Fourier transform of ν. The asymptotic behavior of $\widehat{\nu}(\xi)$ at infinity carries some information about the geometry of F. It can be verified that (i) if $\beta > d/2$ in (41), then $\widehat{\nu} \in L^2(\mathbb{R}^d)$ and, consequently, F has positive d-dimensional Lebesgue measure; (ii) if $\beta > d$, then $\widehat{\nu} \in L^1(\mathbb{R}^d)$. Hence ν has a continuous density function which implies that F has interior points.

For any Borel set $F \subseteq \mathbb{R}^d$, the Fourier dimension of F, denoted by $\dim_{\mathscr{F}} F$, is defined as

$$\dim_{\mathscr{F}} F = \sup\left\{\gamma \in [0,d] : \ F \text{ is an } \ M_{\gamma/2}\text{-set} \ \right\}. \tag{42}$$

It follows from Frostman's theorem that $\dim_{\mathscr{F}} F \leq \dim_{\mathrm{H}} F$ for all Borel sets $F \subseteq \mathbb{R}^d$. The strict inequality may hold. For example, the Fourier dimension of triadic Cantor set is 0, but its Hausdorff dimension is $\log 2/\log 3$. It has been known that the Hausdorff dimension $\dim_{\mathrm{H}} F$ describes a metric property of F, whereas the Fourier dimension measures an arithmetic property of F. As a further example of this aspect, we mention that every set $F \subseteq \mathbb{R}^d$ with positive Fourier dimension generates $\mathbb{R}^d$ as a group [cf. Kahane (1985a)].

A Borel set $F \subseteq \mathbb{R}^d$ is called a *Salem set* if $\dim_{\mathscr{F}} F = \dim_{\mathrm{H}} F$. Such sets are of importance in studying the problem of uniqueness and multiplicity for trigonometric series [cf. Zygmund (1959, Chapter 9) and Kahane and Salem (1994)] and the restriction problem for the Fourier transforms [cf. Mockenhaupt (2000)].

The images of many random fields are Salem sets. For fractional Brownian motion $B^H = \{B^H(t), t \in \mathbb{R}^N\}$ with values in $\mathbb{R}^d$, Kahane (1985a, 1985b) proved that, for every Borel set $E \subseteq \mathbb{R}^N$ with $\dim_{\mathrm{H}} E \leq Hd$, $B^H(E)$ is almost surely a Salem set with Fourier dimension $\frac{1}{H}\dim_{\mathrm{H}} E$.

The Fourier dimensions of the images of various Gaussian random fields have been studied by Khoshnevisan, Wu and Xiao (2006) for the Brownian sheet, Shieh and Xiao (2006) for a large class of approximately isotropic Gaussian random fields and by Wu and Xiao (2007) for fractional Brownian sheets. We mention that the argument in Wu and Xiao (2007) is based on the property of sectorial local nondeterminism and can be applied more generally.

It would be interesting to know whether other random sets such as the graph, level sets or inverse images of a Gaussian random field are Salem sets. Recently Fouché and Mukeru (2011) showed that the zero set of 1-dimensional Brownian motion is a Salem set of Fourier dimension 1/2 by studying the Fourier transform of the local times of Brownian motion. Their result is related to that of Kahane (1985b) for the images of stable Lévy processes because the zero set of Brownian motion equals, up to a countable set, the image of a stable subordinator of index $\frac{1}{2}$. However, for a Gaussian random field X, no direct connection between its level set $X^{-1}(x)$ and the image of a tractable random field has been established.

4.3 Packing dimension results

Packing measure and packing dimension were introduced by Tricot (1982) and Taylor and Tricot (1985) as dual concepts to Hausdorff measure and dimension. Since Hausdorff and packing dimensions of a set (or a measure) are determined by different geometric aspects of the set (or the measure), many random sets have different values for their Hausdorff and packing dimensions. To understand better the fractal nature of a set, it is important to determine both Hausdorff and packing dimensions of the set.

As we have seen in Section 4.1, for a Gaussian random field X which satisfies conditions (C1) and (C2), the Hausdorff and packing dimensions of $X([0,1]^N)$, $\mathrm{Gr}X([0,1]^N)$ and $X^{-1}(x)$ coincide. However, Xiao (2007, 2009b), Estrade, Wu and Xiao (2011) have shown that, for many Gaussian random fields, the Hausdorff and packing dimensions of these random sets may differ.

In this section, we mainly consider the packing dimension of $X(E)$, where X is a Gaussian field as in Section 4.1 and $E \subseteq \mathbb{R}^N$ is an arbitrary set.

In the special case of Brownian motion $W = \{W(t), t \in \mathbb{R}_+\}$ in $\mathbb{R}^d$, Perkins and Taylor (1987) proved that, if $d \geq 2$, then with probability 1

$$\dim_{\mathrm{P}} W(E) = 2\dim_{\mathrm{P}} E \quad \text{for every Borel set} \quad E \subseteq \mathbb{R}_+. \tag{43}$$

This not only determines the packing dimension of the image $W(E)$, but also says that the exceptional null probability event [on which (43) fails] does not depend on E. Hence (43) is called a *uniform dimension result*; see Section 4.4 for more information. However, when $d = 1$, Talagrand and Xiao (1996) constructed a compact set $E \subseteq \mathbb{R}_+$ such that $\dim_{\mathrm{P}} W(E) < 2\dim_{\mathrm{P}} E$ a.s.; they also obtained the best possible lower bound for $\dim_{\mathrm{P}} W(E)$. Xiao (1997c) solved the problem of finding $\dim_{\mathrm{P}} W(E)$ by proving

$$\dim_{\mathrm{P}} W(E) = 2\mathrm{Dim}_{1/2} E \quad \text{a.s.}, \tag{44}$$

where $\mathrm{Dim}_s E$ is the packing dimension profile of E defined by Falconer and Howroyd (1997) [which is defined by replacing ρ in (45) below by the Euclidean metric]. We mention that Khoshnevisan, Schilling and Xiao (2012) have recently introduced more general notion of packing dimension profiles and determined the packing dimension of the images of an arbitrary Lévy process.

Xiao (1997c) proved results analogous to (44) for fractional Brownian motion B^H and the Brownian sheet. Khoshnevisan and Xiao (2008) provided a connection between $B^H(E)$ and the packing dimension profile of Howroyd (2001), thus showed that the packing dimension profile of Falconer and Howroyd (1997) coincides with that of Howroyd (2001).

In order to determine the packing dimension of $X(E)$ for time-anisotropic Gaussian field X as in Section 4.1, Estrade, Wu and Xiao (2011) extended the packing dimensional profile of Falconer and Howroyd (1997) to the metric space $(\mathbb{R}^N, \rho)$ and define, for any finite Borel measure μ on $\mathbb{R}^N$, the s-dimensional packing dimension profile of μ in the metric ρ as

$$\mathrm{Dim}_s^\rho \mu = \sup\left\{\beta \ge 0:\ \liminf_{r\to 0} \frac{F_{s,\rho}^\mu(x,r)}{r^\beta} = 0 \ \text{ for } \mu\text{-a.a. } x \in \mathbb{R}^N\right\}, \tag{45}$$

where, for any $s > 0$, $F_{s,\rho}^\mu(x,r)$ is the s-dimensional potential of μ in metric ρ defined by

$$F_{s,\rho}^\mu(x,r) = \int_{\mathbb{R}^N} \min\left\{1,\ \frac{r^s}{\rho(x,y)^s}\right\} d\mu(y). \tag{46}$$

For any Borel set $E \subseteq \mathbb{R}^N$, the s-dimensional packing dimension profile of E in the metric ρ is defined by

$$\mathrm{Dim}_s^\rho E = \sup\left\{\mathrm{Dim}_s^\rho \mu:\ \mu \in \mathcal{M}_c^+(E)\right\}, \tag{47}$$

where $\mathcal{M}_c^+(E)$ denotes the family of finite Borel measures with compact support in E.

The following packing dimension analogue of Theorem 13 is a special case of Theorem 4.5 in Estrade, Wu and Xiao (2011).

Theorem 14. *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be an (N,d)-Gaussian field as in Theorem 10. Then for every Borel set $E \subseteq \mathbb{R}^N$,*

$$\dim_{\mathrm{P}} X(E) = \mathrm{Dim}_d^\rho E, \qquad a.s. \tag{48}$$

In the aforementioned references, the measure-theoretic approach to packing dimension and packing dimension profiles has been essential. Even though in this paper we focus on fractal properties of random sets, similar questions can be investigated for related random measures as well. We refer to Shieh and Xiao (2010), Falconer and Xiao (2011) for some recent results.

The packing dimensions of other random sets such as $\mathrm{Gr}X(E)$ and $X^{-1}(F)$ are not known in general. The former is related to the following problem.

Problem 5. What is the packing dimension of $X(E)$ if X is space-anisotropic? In particular, what is $\dim_{\mathrm{P}} \mathrm{Gr}B^H(E)$?

Motivated by Khoshnevisan, Schilling and Xiao (2012), we expect that an answer to the above question is to use the packing dimension profile associated with the kernels $\kappa = \{\kappa_r, r > 0\}$ defined by

$$\kappa_r(s,t) = \mathbb{P}\big\{\|X(s) - X(t)\| \le r\big\}, \quad \forall s,t \in \mathbb{R}^N. \tag{49}$$

When the coordinate processes of X are approximately independent and have approximately scaling properties, κ is comparable with the kernel

$$\widetilde{\kappa}_r(s,t) = \prod_{i=1}^d \min\left\{1, \frac{r}{|s_i - t_i|^{\alpha_i}}\right\}.$$

Details will be given elsewhere.

4.4 Uniform dimension results

We note that the exceptional null probability events in (40), (48) and (39) depend on $E \subseteq \mathbb{R}^N$ and $F \subseteq \mathbb{R}^d$, respectively. In many applications, we have a random time set $E(\omega)$ or $F(\omega) \subseteq \mathbb{R}^d$ and wish to know the fractal dimensions of $X(E(\omega),\omega)$ and $X^{-1}(F(\omega),\omega)$. For example, for any Borel set $F \subseteq \mathbb{R}^d$, we can write the intersection $X(\mathbb{R}_+)\cap F$ as $X(X^{-1}(F))$, the set M_k of k-multiple points of X as $X(L_k')$, where L_k' is the projection of L_k into $\mathbb{R}^N$. For such problems, the results of the form (40), (48) and (39) give no information.

Kaufman (1968) was the first to show that if W is the planar Brownian motion, then

$$\mathbb{P}\Big\{\dim_{\rm H} W(E) = 2\dim_{\rm H} E \ \text{ for all Borel sets } E \subseteq \mathbb{R}_+\Big\} = 1. \tag{50}$$

Since the exceptional null probability event in (50) does not depend on E, it is referred to as a *uniform dimension result*. For Brownian motion in $\mathbb{R}$, (50) does not hold. This can be seen by taking $E = W^{-1}(0)$.

Several authors, including J. Hawkes, W.E. Pruitt, E.A. Perkins and S.J. Taylor, have studied the problems on uniform Hausdorff and packing dimension results for the ranges and level sets of stable Lévy processes. See Taylor (1986) or Xiao (2004) for more information.

For approximately isotropic Gaussian random fields, Monrad and Pitt (1987) established a uniform Hausdorff dimension result for the images under the condition of strong local nondeterminism. In the special case of fractional Brownian motion, their result gives: If $N \le Hd$, then a.s.

$$\dim_{\rm H} B^H(E) = \frac{1}{H}\dim_{\rm H} E \ \text{ for all Borel sets } E \subseteq \mathbb{R}^N.$$

A similar result for the Brownian sheet was established by Mountford (1989) by using a very different method, which relies on special properties of the Brownian sheet. Khoshnevisan, Wu and Xiao (2006) gave an alternative proof for Mountford's result by applying the sectorial local nondetermininsm (C3), and their argument is similar in spirit to that in Monrad and Pitt (1987).

Recently, Wu and Xiao (2009) have shown that, while the anisotropy in the space-variable destroys the uniform dimension result for the images, the uniform Hausdorff dimension result still holds for the image sets of time-anisotropic Gaussian random fields.

Theorem 15. *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be an (N,d)-Gaussian random field defined by (3) whose coordinate processes are independent copies of X_0. If X_0 satisfies Conditions (C1) and (C3), and $\sum_{j=1}^N H_j^{-1} \le d$, then with probability 1*

$$\dim_{\rm H} X(E) = \dim_{\rm H}^{\rho} E \quad \textit{for all Borel sets } \ E \subseteq (0,\infty)^N, \tag{51}$$

where $\dim_{\rm H}^{\rho}$ is Hausdorff dimension on the metric space $(\mathbb{R}^N, \rho)$.

In light of Theorem 15 and the results in Section 4.2, one can ask naturally whether the Hausdorff dimension $\dim_{\rm H} X(E)$ in (51) can be replaced by the Fourier dimension $\dim_{\mathscr{F}} X(E)$. Such a uniform result would be useful when E is a random set. This question is open even for Brownian motion.

Next we turn to the uniform dimension problem on the inverse images. It follows from Monrad and Pitt (1987) that for fractional Brownian motion B^H with $N > Hd$,

$$\dim_{\rm H} \big(B^H\big)^{-1}(F) = N - Hd + H\dim_{\rm H} F \quad \text{ for all Borel sets } F \subseteq \mathbb{R}^d. \tag{52}$$

More generally, if $X = \{X(t), t \in \mathbb{R}^N\}$ is an (N,d)-Gaussian random field which satisfies conditions (C1) and (C3) with $H_1 = \cdots = H_N := H$ and $N > Hd$, then one can modify the proofs in Monrad and Pitt (1987) to prove that (52) still holds for X and for all $F \subseteq \mathscr{O}$, where

$$\mathscr{O} = \bigcup_{a<b:\, a,b\in\mathbb{Q}} \big\{x \in \mathbb{R}^d : L(x,[a,b]) > 0\big\}.$$

However, it is not known whether similar results still hold for Gaussian random fields which are anisotropic either in the space-variable or in the time-variable.

Problem 6. Do uniform Hausdorff and packing dimension results hold for the inverse images of time-anisotropic or space-anisotropic Gaussian random fields?

For the time-anisotropic Gaussian fields in Theorem 12 which also satisfy (C3), we can prove that, if $\sum_{\ell=1}^N H_\ell^{-1} > d$, then almost surely

$$\dim_{\rm H} X^{-1}(F) \le \min_{1\le k\le N} \left\{ \sum_{j=1}^k \frac{H_k}{H_j} + N - k - H_k\big(d - \dim_{\rm H} F\big)\right\} \tag{53}$$

holds for all Borel sets $F \subseteq \mathbb{R}^d$.

We end this section with the following problem which is related to Problem 4. Note that, when $E \subseteq [0,1]^N$, ${\rm Gr}X\big([0,1]^N\big) \cap (E \times F) \neq \varnothing$ is equivalent to $X(E) \cap F \neq \varnothing$. When this happens, it is of interest to determine the Hausdorff and packing dimensions of the random sets $X(E) \cap F$ and $E \cap X^{-1}(F)$.

In the previous sections, Theorems 11, 13 and 14 consider the special cases of $E = I$ or $F = \mathbb{R}^d$, respectively. When $E \subseteq \mathbb{R}_+$ and $F \subseteq \mathbb{R}^d$ are both arbitrary Borel sets, there have only been a few partial results. Some upper and lower bounds for the Hausdorff dimension $\dim_{\rm H}\big(X(E) \cap F\big)$ have been obtained by Kaufman (1972) for Brownian motion, Hawkes (1978) for stable Lévy processes and Testard (1986) for fractional Brownian motion. Recently, Khoshnevisan and Xiao (2011) have determined the Hausdorff dimension $\dim\big(W(E) \cap F\big)$, where W is a Brownian motion in $\mathbb{R}^d$. Similar problems for Gaussian random fields and the packing dimension of $X(E) \cap F$ (even when X is Brownian motion) are open. Regarding the latter problem, we expect that a new form of packing dimension profile may be needed in order to determine the packing dimension of $X(E) \cap F$.

4.5 Exact Hausdorff measure functions

In Section 4.1, Hausdorff and packing dimensions of the range $X\big([0,1]^N\big)$, graph $\mathrm{Gr}X\big([0,1]^N\big)$ and level sets are obtained for time-anisotropic Gaussian random fields. It is a natural question to determine exact Hausdorff and packing measure functions for these random sets. Recall that a measure function $\varphi : (0,1) \to \mathbb{R}_+$ is called an exact Hausdorff measure function for a set $F \subseteq \mathbb{R}^d$ if $0 < \varphi\text{-}m(F) < \infty$. Here φ-m denotes the φ-Hausdorff measure. In Section 4.6, we will use φ-p to denote the φ-packing measure. A measure function φ is called an exact packing measure function for F if $0 < \varphi\text{-}p(F) < \infty$.

Investigating exact Hausdorff and packing measure functions for the random sets generated by a random field X not only provides more precise information about the fractal properties of the sample functions of X, but also stimulates deep understanding of the probability properties such as small ball probabilities, large deviations and dependence structures of X. These latter questions have proved to be significant and sometimes challenging.

The problems on finding exact Hausdorff measure functions for the range and graph of the Brownian sheet and fractional Brownian motion have been considered by Ehm (1981), Talagrand (1995, 1998), Xiao (1997a, 1997b). Here is a brief summary of the known results on the ranges and graph sets.

(i) Let $W = \{W(t), t \in \mathbb{R}^N_+\}$ be the Brownian sheet in $\mathbb{R}^d$. Ehm (1981) proved the following results. If $2N < d$, then $\varphi_3(r) = r^{2N}\big(\log\log 1/r\big)^N$ ia an exact Hausdorff measure function for the range and graph of W. If $2N > d$, then $W([0,1]^N)$ a.s. has interior points and $\varphi_4(r) = r^{N+\frac{d}{2}}(\log\log 1/r)^{\frac{d}{2}}$ is an exact Hausdorff measure function for the graph of W.
When $2N = d$, the problems for finding exact Hausdorff measure functions for the range and graph of W are open.

(ii) Let $B^H = \{B^H(t), t \in \mathbb{R}^N\}$ be an (N,d)-fractional Brownian motion of index H. Talagrand (1995) proved that if $N < Hd$ then $\varphi_5(r) = r^{N/H}\log\log 1/r$ is an exact Hausdorff measure function for the range and graph of B^H. If $N > Hd$, Pitt (1978) showed that then $B^H([0,1]^N)$ a.s. has positive Lebesgue measure and interior points. Xiao (1997a) showed that $\varphi_6(r) = r^{N+(1-H)d}\big(\log\log 1/r\big)^{Hd/N}$ is an exact Hausdorff measure function for the graph of B^H.
In the case of $N = Hd$, Talagrand (1998) showed that $\varphi_7\text{-}m\big(B^H([0,1]^N)\big)$ is σ-finite almost surely, where $\varphi_7(r) = r^d \log(1/r)\log\log\log 1/r$. The same is also true for the Hausdorff measure of the graph set of B^H. However, the corresponding lower bound problems for the Hausdorff measure of the range and graph have remained open.

It is interesting to notice the subtle differences in the exact Hausdorff functions for the range and graph sets of fractional Brownian motion and the Brownian sheet, respectively. The differences are a reflection of the two different types of strong local nondeterminism [i.e., (C3) and (C4)] that they satisfy.

The exact Hausdorff measure of the level sets of a class of approximately isotopic Gaussian random fields was obtained by Xiao (1997b). In the case of fractional Brownian motion, Baraka and Mountford (2011) established the following result which improves Theorem 1.3 in Xiao (1997b) significantly.

Theorem 16. *Let $B^H = \{B^H(t), t \in \mathbb{R}^N\}$ be a d-dimensional fractional Brownian motion of index H. For every $I \in \mathscr{A}$, there exists a finite constant $c_{13} > 0$ (depending only on H, N, and d) such that, with probability 1,*

$$\varphi_8\text{-}m\Big((B^H)^{-1}(\{0\}) \cap I\Big) = c_{13}\, L(0,I), \tag{54}$$

where $L(0,I)$ is the local time of B^H at 0 and $\varphi_8(r) = r^{N-Hd}(\log\log 1/r)^{Hd/N}$.

For the Brownian sheet $W = \{W(t), t \in \mathbb{R}^N_+\}$ with values in $\mathbb{R}^d$, Lin (2001) proved that $\varphi_9(r) = r^{N-d/2}(\log\log 1/r)^{d/2}$ is an exact Hausdorff measure function for $W^{-1}(\{0\})$. His method is based on moment estimates of the local times of W. Notice that there is a subtle differences between $\varphi_8(r)$ and $\varphi_9(r)$.

Xiao (1997a) provided the exact Hausdorff measure functions for the ranges of a class of space-anisotropic Gaussian random fields. The following result for the range of a time-anisotropic Gaussian random field is proved by Luan and Xiao (2012).

Theorem 17. *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be an (N,d)-Gaussian random field defined by (3), where $X_1, \ldots, X_d$ are independent copies of a centered, real-valued Gaussian field X_0 with stationary increments and $X_0(0) = 0$. We assume that X_0 satisfies Conditions (C1) and (C4). If $d > \sum_{j=1}^N H_j^{-1}$, then we have*

$$0 < \varphi_{10}\text{-}m(X([0,1]^N)) < \infty \quad a.s., \tag{55}$$

where $\varphi_{10}(r) = r^{\sum_{j=1}^N H_j^{-1}} \log\log(1/r)$.

Many questions on exact Hausdorff measure functions for anisotropic Gaussian random fields remain unsolved. In particular, we ask

Problem 7. What are the exact Hausdorff measure functions for the graph and level sets of Gaussian random fields in Theorem 17?

4.6 Exact packing measure functions

The exact packing measure function for random sets was first considered by Taylor and Tricot (1985) who proved that $\psi_1(r) = r^2/(\log|\log r|)$ is an exact packing measure function for the range of Brownian motion in $\mathbb{R}^d$ ($d \geq 3$). The situation for $d = 2$ is very different. Le Gall and Taylor (1987) proved that the range of the planar Brownian motion $W^{(2)}$ does not have an exact packing measure function. More precisely they showed that, for any measure function of the form $\psi(r) = r^2 \log(1/r) h(r)$, where $h(r)$ is monotone increasing but $\log(1/r)h(r)$ is decreasing, almost surely,

$$\psi\text{-}p\big(W^{(2)}([0,1])\big) = \begin{cases} 0 \\ \infty \end{cases} \quad \text{according as} \quad \sum_{n=1}^{\infty} h\big(2^{-2^n}\big) \begin{cases} < \infty \\ = \infty. \end{cases} \tag{56}$$

Furthermore, Le Gall (1987a) proved that, for every integer $k \ge 2$, the set of k-multiple points $M_k^{(2)}$ of $W^{(2)}$ does not have an exact packing measure function. Recently, Mörters and Shieh (2009) proved a similar result for the set of double points of Brownian motion in $\mathbb{R}^3$ and established an integral test in terms of the intersection exponent $\xi_3(2,2)$ of two packets of two independent Brownian motions in $\mathbb{R}^3$.

So far no exact packing measure results have been established for Gaussian random fields other than those mentioned above and fractional Brownian motion considered by Xiao (1996, 2003).

Theorem 18. *Let $B^H = \{B^H(t),\, t \in \mathbb{R}^N\}$ be a fractional Brownian motion in $\mathbb{R}^d$ of index H. If $N < Hd$, then there exist positive constants c_{14} and c_{15} such that, with probability 1,*

$$c_{14} \le \psi_2\text{-}p\big(X([0,1]^N)\big) \le \psi_2\text{-}p\big(\mathrm{Gr}X([0,1]^N)\big) \le c_{15}, \tag{57}$$

where $\psi_2(s) = s^{N/H}/(\log\log 1/s)^{N/(2H)}$.

The proof of Theorem 18 in Xiao (2003) relies on the liminf properties of the occupation measures of B^H and the delayed hitting probability estimates. Such results have not been established for other Gaussian fields including the Brownian sheet. The main difficulty lies in dealing with their complicated dependence structures. I think it would be interesting to investigate the following problems.

Problem 8. Determine the exact packing measure functions for the range, graph set and level sets of the Brownian sheet and anisotropic Gaussian random fields.

4.7 Hitting probabilities and intersections of Gaussian random fields

Many authors have investigated intersections of the trajectories of stochastic processes. For Brownian motion, the questions have been studied by A. Dvoretzky, P. Erdös, S. Kakutani, S. J. Taylor, and J.-F. Le Gall. See Khoshnevisan (2003) for historical accounts and a very nice proof for the existence theorem using an elementary argument based on the self-similarity and Markov property of Brownian motion. The results on intersections of Brownian motion have been extended to Lévy processes, Gaussian processes and other processes. We refer to the survey papers of Taylor (1986) and Xiao (2004) for further information on intersections of Markov processes.

In this section, we give some recent results on intersections of two independent Gaussian random fields obtained in Chen and Xiao (2011). These results are established based on refining the hitting probabilities estimates for Gaussian random fields obtained in Xiao (1999, 2009a) and Biermé, Lacaux and Xiao (2009); see also Dalang, et al. (2007, 2009), Dalang and Sanz-Solé (2010) for related results. This approach is different from those based on intersection local times in Rosen (1984, 1987), Hu and Nualart (2005), Wu and Xiao (2010), where fractional Brownian motions are considered.

The following theorem from Chen and Xiao (2011) extends the results on hitting probabilities in the aforementioned references.

Theorem 19. *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be an (N,d)-Gaussian random field defined by (3) and assume that X_0 satisfies Conditions (C1) and (C2) on a closed interval I. If $E \subseteq I$ and $F \subseteq \mathbb{R}^d$ are Borel sets, then there is a constant $c_{16} \ge 1$, which depends on I, F and $\mathbf{H}$ only, such that*

$$c_{16}^{-1}\,\mathscr{C}_{\widetilde{\rho},d}(E\times F) \le \mathbb{P}\Big\{X(E)\cap F \neq \varnothing\Big\} \le c_{16}\,\mathscr{H}^{d}_{\widetilde{\rho}}(E\times F), \tag{58}$$

where $\mathscr{C}_{\widetilde{\rho},d}$ and $\mathscr{H}^{d}_{\widetilde{\rho}}$ denote respectively the d-dimensional capacity and Hausdorff measure in the metric space $(\mathbb{R}^{N+d}, \widetilde{\rho})$, and where

$$\widetilde{\rho}\big((s,x),(t,y)\big) = \max\big\{\rho(s,t),\ |x-y|\big\}, \qquad \forall (s,x),(t,y) \in \mathbb{R}^N\times\mathbb{R}^d.$$

As we have mentioned in Problem 4, except in the case of Brownian motion, it is an open problem to provide a necessary and sufficient condition for $\mathbb{P}\big\{X(E)\cap F \neq \varnothing\big\} > 0$.

Next we apply Theorem 19 to study intersections of two independent Gaussian random fields. Let $X^{\mathbf{H}} = \{X^{\mathbf{H}}(s), s\in\mathbb{R}^{N_1}\}$ and $X^{\mathbf{K}} = \{X^{\mathbf{K}}(t), t\in\mathbb{R}^{N_2}\}$ be two independent Gaussian random fields taking values in $\mathbb{R}^d$, defined as in (3). More specifically, we assume that $X^{\mathbf{H}}$ is defined as

$$X^{\mathbf{H}}(s) = (X^{\mathbf{H}}_1(s),\ldots,X^{\mathbf{H}}_d(s)), \qquad s\in\mathbb{R}^{N_1}, \tag{59}$$

where $X^{\mathbf{H}}_1,\ldots,X^{\mathbf{H}}_d$ are independent copies of real-valued, centered Gaussian random field $X^{\mathbf{H}}_0$. The Gaussian random field $X^{\mathbf{K}}$ is defined in the same way. Here $\mathbf{H}\in(0,1)^{N_1}$ and $\mathbf{K}\in(0,1)^{N_2}$ are constant vectors.

We say the two Gaussian fields $X^{\mathbf{H}}$ and $X^{\mathbf{K}}$ intersect if there exist $s\in\mathbb{R}^{N_1}$ and $t\in\mathbb{R}^{N_2}$ such that $X^{\mathbf{H}}(s) = X^{\mathbf{K}}(t)$. The following problems are concerned with existence of intersections.

(i). When do $X^{\mathbf{H}}$ and $X^{\mathbf{K}}$ intersect (with positive probability)?

(ii). Let $E_1\subseteq\mathbb{R}^{N_1}$ and $E_2\subseteq\mathbb{R}^{N_2}$ be arbitrary Borel sets. When do $X^{\mathbf{H}}$ and $X^{\mathbf{K}}$ intersect if we restrict the "time" $s\in E_1$ and $t\in E_2$? That is, when is

$$\mathbb{P}\Big\{X^{\mathbf{H}}(E_1)\cap X^{\mathbf{K}}(E_2)\neq\varnothing\Big\} > 0? \tag{60}$$

(iii). Given a Borel set $F\subseteq\mathbb{R}^d$, when does F contain intersection points of $X^{\mathbf{H}}(s)$ $(s\in E_1)$ and $X^{\mathbf{K}}(t)$, $(t\in E_2)$? That is, when is

$$\mathbb{P}\Big\{X^{\mathbf{H}}(E_1) \cap X^{\mathbf{K}}(E_2) \cap F \neq \varnothing\Big\} > 0? \tag{61}$$

Clearly, Question (i) is a special case of Question (ii), which is a special case of Question (iii). For answering Questions (i) and (ii), consider the Gaussian random field $Z = \{Z(s,t), (s,t) \in \mathbb{R}^{N_1+N_2}\}$ with values in $\mathbb{R}^d$ defined by

$$Z(s,t) \equiv X^H(s) - X^K(t), \qquad s \in \mathbb{R}^{N_1}, t \in \mathbb{R}^{N_2}. \tag{62}$$

Then (60) is equivalent to $\mathbb{P}\big(Z(E_1 \times E_2) \cap \{0\} \neq \varnothing\big) > 0$. Hence sufficient conditions and necessary conditions for this to hold can be obtained from hitting probability estimates for Gaussian field Z, which is done in Theorem 2.1 of Chen and Xiao (2011). Instead of giving more details, we content with the following simpler result which provides an answer to Question (i). In the following, we let $Q := \sum_{j=1}^{N_1} H_j^{-1} + \sum_{j=1}^{N_2} K_j^{-1}$.

Theorem 20. *Let $X^{\mathbf{H}} = \{X^{\mathbf{H}}(s), s \in \mathbb{R}^{N_1}\}$ and $X^{\mathbf{K}} = \{X^{\mathbf{K}}(t), t \in \mathbb{R}^{N_2}\}$ be two independent Gaussian random fields with values in $\mathbb{R}^d$ such that $X_0^{\mathbf{H}}$ and $X_0^{\mathbf{K}}$ satisfy (C1) and (C2) respectively on interval $I_1 \subseteq \mathbb{R}^{N_1}$ with indices $H = (H_1, \ldots, H_{N_1})$ and on interval $I_2 \subseteq \mathbb{R}^{N_2}$ with indices $K = (K_1, \ldots, K_{N_2})$.*

(i). If $d < Q$, then $\mathbb{P}\big\{X^H(I_1) \cap X^K(I_2) \neq \varnothing\big\} = 0$.
(ii). If $d > Q$, then $\mathbb{P}\big\{X^H(I_1) \cap X^K(I_2) \neq \varnothing\big\} > 0$.
(iii). If, in addition, we assume that $X_0^{\mathbf{H}}$ has stationary increments and satisfies (C4) on interval $I_1 \subseteq \mathbb{R}^{N_1}$, then $d = Q$ implies $\mathbb{P}\big\{X^H(I_1) \cap X^K(I_2) \neq \varnothing\big\} = 0$.

In order to answer Question (iii), we consider the Gaussian random field $Y = \{Y(s,t), (s,t) \in \mathbb{R}^{N_1+N_2}\}$ with values in $\mathbb{R}^{2d}$ defined by

$$Y(s,t) = \big(X^H(s), X^K(t)\big), \qquad \forall (s,t) \in \mathbb{R}^{N_1+N_2}.$$

Then (61) holds if and only if

$$\mathbb{P}\Big\{Y(E_1 \times E_2) \cap \widetilde{F} \neq \varnothing\Big\} > 0, \tag{63}$$

where $\widetilde{F} = \{(x,x) : x \in F\} \subseteq \mathbb{R}^{2d}$. This hitting probability is also estimated by Chen and Xiao (2011) in terms of Bessel-Riesz type capacity and Hausdorff measure of $E_1 \times E_2 \times F$, under appropriate metric on $\mathbb{R}^{N_1+N_2+d}$.

It follows from Theorem 20 that, when $\sum_{i=1}^{N_1} H_i^{-1} + \sum_{j=1}^{N_2} K_j^{-1} > d$, we have $\mathbb{P}\big(X^{\mathbf{H}}(I_1) \cap X^{\mathbf{K}}(I_2) \neq \varnothing\big) > 0$. It is of interest to determine the Hausdorff dimensions of the set of intersection times $L_2 := \{(s,t) \in I_1 \times I_2 : X^{\mathbf{H}}(s) = X^{\mathbf{K}}(t)\}$ and the set of intersections $M_2 = X^{\mathbf{H}}(I_1) \cap X^{\mathbf{K}}(I_2)$. Since L_2 is the level set of the Gaussian random field $Z(s,t) = X^{\mathbf{H}}(s) - X^{\mathbf{K}}(t)$, the Hausdorff and packing dimensions of L_2 can be obtained from Theorem 11. However, the Hausdorff and packing dimensions of M_2 has not been determined in its full generality.

In the following, we provide a partial answer for the intersection set of two independent fractional Brownian motions obtained by Wu and Xiao (2010).

Let $B^{\alpha_1} = \{B^{\alpha_1}(s), s \in \mathbb{R}^{N_1}\}$ and $B^{\alpha_2} = \{B^{\alpha_2}(t), t \in \mathbb{R}^{N_2}\}$ be two independent fractional Brownian motions with values in $\mathbb{R}^d$ and indices α_1 and α_2, respectively. Let

$$\begin{aligned} M_2 &= \big\{x \in \mathbb{R}^d : x = B^{\alpha_1}(s) = B^{\alpha_2}(t) \ \text{ for some } \ (s,t) \in \mathbb{R}^{N_1+N_2}\big\} \\ &= B^{\alpha_1}(\mathbb{R}^{N_1}) \cap B^{\alpha_2}(\mathbb{R}^{N_2}). \end{aligned}$$

Theorem 21. *If $\frac{N_1}{\alpha_1} + \frac{N_2}{\alpha_2} > d$, then with probability 1,*

$$\dim_{\rm H} M_2 = \dim_{\rm P} M_2 = \begin{cases} d & \text{if } N_1 > \alpha_1 d \text{ and } N_2 > \alpha_2 d, \\ \frac{N_2}{\alpha_2} & \text{if } N_1 > \alpha_1 d \text{ and } N_2 \le \alpha_2 d, \\ \frac{N_1}{\alpha_1} & \text{if } N_1 \le \alpha_1 d \text{ and } N_2 > \alpha_2 d, \\ \frac{N_1}{\alpha_1} + \frac{N_2}{\alpha_2} - d & \text{if } N_1 \le \alpha_1 d \text{ and } N_2 \le \alpha_2 d. \end{cases} \tag{64}$$

Besides intersections of independent Gaussian random fields, one can study analogous questions for self-intersections of a Gaussian random field. The arguments described above are still applicable for studying the existence of self-intersections. For related work on fractional Brownian motion and the Brownian sheet, we refer to Kôno (1978), Goldman (1981), Rosen (1984), Talagrand (1998) and Dalang et al. (2011). The first three papers provide sufficient conditions for the existence of k-multiple points and the last two papers show that the corresponding conditions are also necessary. We mention that the methods in Talagrand (1998) and Dalang et al. (2011) are very different and Dalang et al. (2011) only prove necessity for $k = 2$.

Finally we remark that, while the Hausdorff and packing dimensions of the set of k-multiple points of fractional Brownian motion and the Brownian sheet have been obtained [cf. Khoshnevisan, Wu and Xiao (2006)], no results on exact Hausdorff or packing measure functions have been obtained for any Gaussian random fields.

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References

1. Adler, R.J. (1981). *The Geometry of Random Fields*. Wiley, New York.
2. Adler, R.J. and Taylor, J.E. (2007). *Random Fields and Geometry*. Springer, New York.
3. Ayache, A., Wu, D. and Xiao, Y. (2008). Joint continuity of the local times of fractional Brownian sheets. *Ann. Inst. H. Poincaré Probab. Statist.* **44**, 727–748.
4. Ayache, A. and Xiao, Y. (2005). Asymptotic properties and Hausdorff dimensions of fractional Brownian sheets. *J. Fourier Anal. Appl.* **11**, 407–439.
5. Baraka, D. and Mountford, T.S. (2008), A law of iterated logarithm for fractional Brownian motions. *Séminaire de probabilités XLI,* 161–179, *Lecture Notes in Math.*, **1934**, Springer, Berlin.
6. Baraka, D. and Mountford, T.S. (2011). The exact Hausdorff measure of the zero set of fractional Brownian motion. *J. Theor. Probab.* **24**, 271–293.

7. Baraka, D., Mountford, T.S. and Xiao, Y. (2009). Hölder properties of local times for fractional Brownian motions. *Metrika* **69**, 125–152.
8. Benassi, A., Jaffard, S. and Roux, D. (1997). Elliptic Gaussian random processes. *Rev. Mat. Iberoamericana* **13**, 19–90.
9. Berman, S.M. (1972). Gaussian sample function: uniform dimension and Hölder conditions nowhere. *Nagoya Math. J.* **46**, 63–86.
10. Berman, S.M. (1973). Local nondeterminism and local times of Gaussian processes. *Indiana Univ. Math. J.* **23**, 69–94.
11. Berman, S.M. (1988). Spectral conditions for local nondeterminism. *Stoch. Process. Appl.* **27**, 73–84.
12. Biermé, H., Lacaux, C. and Xiao, Y. (2009). Hitting probabilities and the Hausdorff dimension of the inverse images of anisotropic Gaussian random fields. *Bull. London Math. Soc.* **41**, 253–273.
13. Biermé, H., Meerschaert, M.M. and Scheffler, H.-P. (2007). Operator scaling stable random fields. *Stoch. Process. Appl.* **117**, 313–332.
14. Blath, J. and Martin, A. (2008). Propagation of singularities in the semi-fractional Brownian sheet. *Stoch. Process. Appl.* **118**, 1264–1277.
15. Bonami, A. and Estrade, A. (2003). Anisotropic analysis of some Gaussian models. *J. Fourier Anal. Appl.* **9**, 215–236.
16. Chen, X., Li, W. V., Rosinski, J. and Shao, Q.-M. (2011). Large deviations for local times and intersection local times of fractional Brownian motions and Riemann-Liouville processes. *Ann. Probab.* **39**, 729–778.
17. Chen, Z. and Xiao, Y. (2011). On intersections of independent anisotropic Gaussian random fields. *Submitted*.
18. Christakos, M.J. (2000). *Modern Spatiotemporal Geostatistics,* Oxford University Press.
19. Cuzick, J. (1978). Some local properties of Gaussian vector fields. *Ann. Probab.* **6**, 984–994.
20. Cuzick, J. (1982). Multiple points of a Gaussian vector field. *Z. Wahrsch. verw. Gebiete.* **61**, 431–436.
21. Dalang, R.C. (1999). Extending martingale measure stochastic integral with applications to spatially homogeneous s.p.d.e.'s. *Electron. J. Probab.* **4**, no. 6, 1–29. Erratum in *Electron. J. Probab.* **6** (2001), no. 6, 1–5.
22. Dalang, R.C., Khoshnevisan, D. and Nualart, E. (2007), Hitting probabilities for systems of non-linear stochastic heat equations with additive noise. *Latin Amer. J. Probab. Statist. (Alea)*, **3**, 231–271.
23. Dalang, R. C., Khoshnevisan, D. and Nualart, E. (2009), Hitting probabilities for the non-linear stochastic heat equation with multiplicative noise. *Probab. Th. Rel. Fields*, **117**, 371–427.
24. Dalang, R.C., Khoshnevisan, D., Nualart, E., Wu, D. and Xiao, Y. (2011). Critical Brownian sheet does not have double points. *Ann. Probab.*, to appear.
25. Dalang, R.C. and Sanz-Solé, M. (2010), Criteria for hitting probabilities with applications to systems of stochastic wave equations. *Bernoulli* **16**, 1343-1368.
26. Dembo, A., Peres, P., Rosen, J. and Zeitouni, O. (2000). Thick points for spatial Brownian motion: multifractal analysis of occupation measure. *Ann. Probab.* **28**, 1–35.
27. Didier, G. and Pipiras, V. (2011). Integral representations of operator fractional Brownian motions. *Bernoulli* **17**, 1–33.
28. Dindar, Z. (2011). On the Hausdorff Dimension of the Set Generated by Exceptional Oscillations of a Two-Parameter Wiener Process. *J. Multivar. Anal.* **79**, 52–70.
29. Ehm, E. (1981). Sample function properties of multi-parameter stable processes. *Z. Wahrsch. verw Gebiete* **56**, 195–228.
30. Estrade, A., Wu, D. and Xiao, Y. (2011). Packing dimension results for anisotropic Gaussian random fields, *Commun. Stoch. Anal.* **5**, 41–64.
31. Falconer, K.J. (2003). *Fractal Geometry—Mathematical Foundations and Applications*, 2nd Ed. John Wiley.
32. Falconer, K.J. and Howroyd, J.D. (1997). Packing dimensions of projections and dimension profiles, *Math. Proc. Cambridge Philos. Soc.* **121**, 269–286.

33. Falconer, K.J. and Xiao, Y. (2011). Generalized dimensions of images of measures under Gaussian processes. *Submitted*.
34. Fouché, W.L. and Mukeru, S. (2011). Fourier structure of the zero set of Brownian motion. *Preprint*.
35. Geman, D. and Horowitz, J. (1980). Occupation densities. *Ann. Probab.* **8**, 1–67.
36. Gneiting, T. (2002). Nonseparable, stationary covariance functions for space-time data, *J. Amer. Statist. Assoc.* **97**, 590–600.
37. Goldman, A. (1981). Points multiples des trajectoires de processus gaussiens. *Z. Wahrsch. Verw. Gebiete* **57**, 481–494.
38. Hawkes, J. (1978). Measures of Hausdorff type and stable processes. *Mathematika* **25**, 202–212.
39. Hwang, K.-S., Wang, W. and Xiao, Y. (2012), The modulus of non-differentiability of a fractional Brownian motion. *Preprint*.
40. Hu, H. and Nualart, D. (2005). Renormalized self-intersection local time for fractional Brownian motion. *Ann. Probab.*, **33**, 948–983.
41. Hu, Y. and Nualart, D. (2009). Stochastic heat equation driven by fractional noise and local time. *Probab. Th. Rel. Fields* **143**, 285–328.
42. Istas, J. (2005), Spherical and hyperbolic fractional Brownian motion. *Electron. Comm. Probab.* **10**, 254–262.
43. Kahane, J.-P. (1985a), *Some Random Series of Functions*. 2nd edition, Cambridge University Press, Cambridge.
44. Kahane, J.-P. (1985b), Ensembles aleatoires et dimensions. In: *Recent Progress in Fourier Analysis (El Escorial, 1983)*, pp. 65–121, North-Holland, Amsterdam.
45. Kahane, J.-P. and Salem, R. (1994). *Ensembles Parfaits et Series Trigonometriques.* 2nd ed., Hermann, Paris.
46. Kamont, A. (1996). On the fractional anisotropic Wiener field. *Probab. Math. Statist.* **16**, 85–98.
47. Kaufman, R. (1968). Une propriété métrique du mouvement brownien. *C. R. Acad. Sci. Paris* **268**, 727–728.
48. Kaufman, R. (1972). Measures of Hausdorff-type, and Brownian motion. *Mathematika* **19**, 115–119.
49. Kaufman, R.(1975). Large increments of Brownian motion. *Nagoya Math. J.* **56**, 139–145.
50. Kaufman, R. (1985). Temps locaux et dimensions. *C. R. Acad. Sci. Paris Sér. I Math.* **300**, 281–282.
51. Khoshnevisan, D. (2002). *Multiparameter Processes: An Introduction to Random Fields*. Springer, New York.
52. Khoshnevisan, D. (2003). Intersections of Brownian motions. *Expos. Math.* **21**, 97–114.
53. Khoshnevisan, D., Peres, Y. and Xiao, Y. (2000). Limsup random fractals. *Electron. J. Probab.* **5** No. 4, 1–24.
54. Khoshnevisan, D., Schilling, R. and Xiao, Y. (2012). Packing dimension profiles and Lévy processes. *Bull. London Math. Soc.*, doi:10.1112/blms/bds022.
55. Khoshnevisan, D. and Shi, Z. (1999). Brownian sheet and capacity. *Ann. Probab.* **27**, 1135–1159.
56. Khoshnevisan, D., Wu, D. and Xiao, Y. (2006). Sectorial local non-determinism and the geometry of the Brownian sheet. *Electron. J. Probab.* **11** (2006), 817–843.
57. Khoshnevisan, D. and Xiao, Y. (2007). Images of the Brownian sheet. *Trans. Amer. Math. Soc.* **359**, 3125–3151.
58. Khoshnevisan, D. and Xiao, Y. (2008a). Packing dimension of the range of a Lévy process. *Proc. Amer. Math. Soc.* **136**, 2597–2607.
59. Khoshnevisan, D. and Xiao, Y. (2008b). Packing dimension profiles and fractional Brownian motion. *Math. Proc. Cambridge Philos. Soc.* **145**, 205–213.
60. Khoshnevisan, D. and Xiao, Y. (2012). Brownian motion and thermal capacity. *Submitted*.
61. Kôno, N. (1978). Double points of a Gaussian sample path. *Z. Wahrsch. Verw. Gebiete* **45**, 175-180.

62. Le Gall, J.-F. (1987a). The exact Hausdorff measure of Brownian multiple points. In: *Seminar on Stochastic Processes (Charlottesville, Va., 1986)*, (E. Cinlar, K. L. Chung, R. K. Getoor, eds.), pp. 107–137, Progr. Probab. Statist., **13**, Birkhäuser, Boston, MA.
63. Le Gall, J.-F. (1987b). Temps locaux d'intersection et points multiples des processus de Lévy. *Séminaire de Probabilités, XXI,* pp. 341–374, Lecture Notes in Math., **1247**, Springer-Verlag, Berlin.
64. Le Gall, J.-F. (1989). The exact Hausdorff measure of Brownian multiple points. II. In: *Seminar on Stochastic Processes (Gainesville, FL, 1988),* pp. 193–197, Progr. Probab., **17**, Birkhäuser Boston, MA.
65. Le Gall, J.-F. and Taylor, S.J. (1987). The packing measure of planar Brownian motion. In: *Seminar on Stochastic Processes (Charlottesville, Va., 1986)*, pp. 139–147, Progr. Probab. Statist., **13**, Birkhäuser Boston, MA.
66. Lévy, P. (1953). La mesure de Hausdorff de la courbe du mouvement brownien. *Giorn. Ist. Ital. Attuari* **16**, 1–37.
67. Li, W.V. and Shao, Q.-M. (2001). Gaussian processes: inequalities, small ball probabilities and applications. In *Stochastic Processes: Theory and Methods*. Handbook of Statistics, **19**, (C. R. Rao and D. Shanbhag, editors), pp. 533–597, North-Holland.
68. Li, Y. and Xiao, Y. (2011). Multivariate operator-self-similar random fields. *Stoch. Process. Appl.* **121**, 1178–1200.
69. Lin, H. (2001). The local times and Hausdorff measure for level sets of a Wiener sheet. *Sci. China Ser. A* **44**, 696–708.
70. Luan, N. and Xiao, Y. (2010). Chung's law of the iterated logarithm for anisotropic Gaussian random fields. *Statist. Probab. Lett.* **80**, 1886–1895.
71. Luan, N. and Xiao, Y. (2012). Spectral conditions for strong local nondeterminism and exact Hausdorff measure of ranges of Gaussian random fields. *J. Fourier Anal. Appl.* **18** (2012), 118–145.
72. Marcus, M.B. and Rosen, J. (2006). *Markov Processes, Gaussian Processes, and Local Times.* Cambridge University Press, Cambridge.
73. Mason, D.M. and Shi, Z. (2001). Small deviations for some multi-parameter Gaussian processes. *J. Theoret. Probab.* **14**, 213–239.
74. Mason, D.J. and Xiao, Y. (2002). Sample path properties of operator self-similar Gaussian random fields. *Th. Probab. Appl.* **46**, 58–78.
75. Mattila, P. (1995). *Geometry of Sets and Measures in Euclidean Spaces*. Cambridge Univ. Press, Cambridge.
76. Meerschaert, M.M., Wang, W. and Xiao, Y. (2011). Fernique-type inequalities and moduli of continuity of anisotropic Gaussian random fields. *Trans. Amer. Math. Soc.*, to appear.
77. Mockenhaupt, G. (2000). Salem sets and restriction properties of Fourier transforms. *Geom. Funct. Anal.* **10**, 1579-1587.
78. Monrad, D. and Pitt, L.D. (1987). Local nondeterminism and Hausdorff dimension. *Progress in Probability and statistics. Seminar on Stochastic Processes 1986*, (Cinlar, E., Chung, K. L., Getoor, R. K., Editors), pp. 163–189, Birkhäuser, Boston.
79. Monrad, D. and Rootzén, H. (1995). Small values of Gaussian processes and functional laws of the iterated logarithm. *Probab. Th. Rel. Fields* **101**, 173–192.
80. Mörters, P. and Shieh, N.-R. (2009). The exact packing measure of Brownian double points. *Probab. Th. Rel. Fields* **143**, 113–136.
81. Mountford, T.S. (1989). Uniform dimension results for the Brownian sheet. *Ann. Probab.* **17**, 1454–1462.
82. Mueller, C. and Tribe, R. (2002). Hitting probabilities of a random string. *Electron. J. Probab.* **7**, Paper No. 10, 1–29.
83. Nualart, E. and Viens, F. (2009). The fractional stochastic heat equation on the circle: time regularity and potential theory. *Stoch. Process. Appl.* **119**, 1505–1540.
84. Øksendal, B. and Zhang, T. (2000). Multiparameter fractional Brownian motion and quasi-linear stochastic partial differential equations. *Stoch. Stoch. Reports* **71**, 141–163.
85. Orey, S. and Pruitt, W.E. (1973). Sample functions of the N-parameter Wiener process. *Ann. Probab.* **1**, 138–163.

86. Orey, S. and Taylor, S.J. (1974). How often on a Brownian path does the law of the iterated logarithm fail? *Proc. London Math. Soc.* **28**, 174–192.
87. Perkins, E.A. (1981). The exact Hausdorff measure of the level sets of Brownian motion. *Z. Wahrsch. Verw Gebiete* **58**, 373–388.
88. Perkins, E.A. and Taylor, S.J. (1987). Uniform measure results for the image of subsets under Brownian motion. *Probab. Th. Rel. Fields* **76**, 257–289.
89. Rosen, J. (1984). Self-intersections of random fields. *Ann. Probab.* **12**, 108–119.
90. Rosen, J. (1987). The intersection local time of fractional Brownian motion in the plane. *J. Multivar. Anal.* **23**, 37–46.
91. Shieh, N.-R. and Xiao, Y. (2006). Images of Gaussian random fields: Salem sets and interior points. *Studia Math.* **176**, 37–60.
92. Shieh, N.-R. and Xiao, Y. (2010). Hausdorff and packing dimensions of the images of random fields, *Bernoulli* **16**, 926–952.
93. Stein, M.L. (2005). Space-time covariance functions, *J. Amer. Statist. Assoc.* **100**, 310–321.
94. Talagrand, M. (1995), Hausdorff measure of trajectories of multiparameter fractional Brownian motion. *Ann. Probab.* **23**, 767–775.
95. Talagrand, M. (1998). Multiple points of trajectories of multiparameter fractional Brownian motion. *Probab. Th. Rel. Fields* **112**, 545–563.
96. Talagrand, M. (2006). *Generic Chaining*. Springer-Verlag, New York.
97. Talagrand, M. and Xiao, Y. (1996). Fractional Brownian motion and packing dimension. *J. Theoret. Probab.* **9**, 579–593.
98. Taylor, S.J. (1953). The Hausdorff α-dimensional measure of Brownian paths in n-space. *Proc. Cambridge Philo. Soc.* **49**, 31–39.
99. Taylor, S.J. (1986). The measure theory of random fractals. *Math. Proc. Cambridge Philo. Soc.* **100**, 383–406.
100. Taylor, S.J. and Tricot, C. (1985). Packing measure and its evaluation for a Brownian path. *Trans. Amer. Math. Soc.* **288**, 679–699.
101. Testard, F. (1986). Polarité, points multiples et géométrie de certains processus gaussiens. *Publ. du Laboratoire de Statistique et Probabilités de l'U.P.S.* Toulouse, 01–86.
102. Tindel, S., Tudor, C.A. and Viens, F. (2004), Sharp Gaussian regularity on the circle, and applications to the fractional stochastic heat equation. *J. Funct. Anal.* **217**, 280–313.
103. Tricot, C. (1982). Two definitions of fractional dimension. *Math. Proc. Cambridge Philo. Soc.* **91**, 57–74.
104. Walsh, J.B. (1982). Propagation of singularities in the Brownian sheet. *Ann. Probab.* **10**, 279–288.
105. Wang, W. (2007). Almost-sure path properties of fractional Brownian sheet. *Ann. Inst. Henri Poincaré Probab. Statist.* **43**, 619–631.
106. Wu, D. and Xiao, Y. (2007). Geometric properties of the images fractional Brownian sheets. *J. Fourier Anal. Appl.* **13**, 1–37.
107. Wu, D. and Xiao, Y. (2009). Uniform Hausdorff dimension results for Gaussian random fields. *Sci. in China, Ser. A* **52**, 1478–1496.
108. Wu, D. and Xiao, Y. (2010). Regularity of intersection local times of fractional Brownian motions. *J. Theoret. Probab.* **23**, 972–1001.
109. Wu, D. and Xiao, Y. (2011). On local times of anisotropic Gaussian random fields. *Comm. Stoch. Anal.* **5**, 15–39.
110. Xiao, Y. (1995). Dimension results for Gaussian vector fields and index-α stable fields. *Ann. Probab.* **23**, 273–291.
111. Xiao, Y. (1996a). Hausdorff measure of the sample paths of Gaussian random fields. *Osaka J. Math.* **33**, 895–913.
112. Xiao, Y. (1996a). Packing measure of the sample paths of fractional Brownian motion. *Trans. Amer. Math. Soc.* **348**, 3193–3213.
113. Xiao, Y. (1997a), Hausdorff dimension of the graph of fractional Brownian motion. *Math. Proc. Cambridge Philo. Soc.* **122**, 565–576.
114. Xiao, Y. (1997b), Hölder conditions for the local times and the Hausdorff measure of the level sets of Gaussian random fields. *Probab. Th. Rel. Fields* **109**, 129–157.

115. Xiao, Y. (1997c). Packing dimension of the image of fractional Brownian motion, *Statist. Probab. Lett.* **333**, 379–387.
116. Xiao, Y. (1998). Hausdorff-type measures of the sample paths of fractional Brownian motion. *Stoch. Process. Appl.* **74**, 251–272.
117. Xiao, Y. (1999). Hitting probabilities and polar sets for fractional Brownian motion. *Stoch. Stoch. Reports* **66**, 121–151.
118. Xiao, Y. (2003). The packing measure of the trajectories of multiparameter fractional Brownian motion. *Math. Proc. Cambridge Philo. Soc.* **135**, 349–375.
119. Xiao, Y. (2004).Random fractals and Markov processes. In: *Fractal Geometry and Applications: A Jubilee of Benoit Mandelbrot*, (Eds: Michel L. Lapidus and Machiel van Frankenhuijsen), pp. 261–338, American Mathematical Society.
120. Xiao, Y. (2007). Strong local nondeterminism and the sample path properties of Gaussian random fields. In: *Asymptotic Theory in Probability and Statistics with Applications* (Tze Leung Lai, Qiman Shao, Lianfen Qian, editors), pp. 136–176, Higher Education Press, Beijing.
121. Xiao, Y. (2009a). Sample path properties of anisotropic Gaussian random fields. In: *A Minicourse on Stochastic Partial Differential Equations*, (Eds: D. Khoshnevisan and F. Rassoul-Agha), *Lecture Notes in Math.* **1962**, 145–212 (Springer, New York, 2009).
122. Xiao, Y. (2009b). A packing dimension theorem for Gaussian random fields. *Statist. Probab. Lett.* **79**, 88–97.
123. Xiao, Y. (2011). Properties of strong local nondeterminism and local times of stable random fields. In: *Seminar on Stochastic Analysis, Random Fields and Applications VI*, (Eds: R.C. Dalang, M. Dozzi and F. Russo), *Progr. Probab.* **63**, pp. 279–310. Birkhäuser, Basel.
124. Xiao, Y. and Zhang, T. (2002). Local times of fractional Brownian sheet. *Probab. Th. Rel. Fields* **124**, 204–226.
125. Xue, Y. and Xiao, Y. (2011). Fractal and smoothness properties of anisotropic Gaussian models. *Frontiers Math. China* **6**, 1217–1246.
126. Zhang, L.X. (1996). Two different kinds of liminfs on the LIL for two-parameter Wiener processes. *Stoch. Process. Appl.* **63**, 175–188.
127. Zimmerman, G. J. (1972). Some sample function properties of the two-parameter Gaussian process. *Ann. Math. Statist.* **43**, 1235–1246.
128. Zygmund, A. (1959). *Trigonometric Series, Vol. I.* Cambridge University Press, Cambridge.