

Geometric Properties of Fractional Brownian Sheets

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Abstract

Let $B^H = \{B^H(t), t \in \mathbb{R}_+^N\}$ be an (N, d) -fractional Brownian sheet with Hurst index $H = (H_1, \dots, H_N) \in (0, 1)^N$. The objective of the present paper is to characterize the anisotropic nature of B^H in terms of H . We prove the following results:

(1) B^H is sectorially locally nondeterministic.

(2) By introducing a notion of “dimension” for Borel measures and sets, which is suitable for describing the anisotropic nature of B^H , we determine $\dim_{\mathbb{H}} B^H(E)$ for an arbitrary Borel set $E \subset (0, \infty)^N$. Moreover, when $B^{(\alpha)}$ is an (N, d) -fractional Brownian sheet with index $\langle \alpha \rangle = (\alpha, \dots, \alpha)$ ($0 < \alpha < 1$), we prove the following uniform Hausdorff dimension result for its image sets: If $N \leq \alpha d$, then with probability one,

$$\dim_{\mathbb{H}} B^{(\alpha)}(E) = \frac{1}{\alpha} \dim_{\mathbb{H}} E \quad \text{for all Borel sets } E \subset (0, \infty)^N.$$

(3) We provide sufficient conditions for the image $B^H(E)$ to be a Salem set or to have interior points.

The results in (2) and (3) describe the geometric and Fourier analytic properties of B^H . They extend and improve the previous theorems of Mountford (1989), Khoshnevisan and Xiao (2004) and Khoshnevisan, Wu and Xiao (2005) for the Brownian sheet.

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1 Introduction

For a given vector $H = (H_1, \dots, H_N)$ ($0 < H_j < 1$ for $j = 1, \dots, N$), an $(N, 1)$ -fractional Brownian sheet $B_0^H = \{B_0^H(t), t \in \mathbb{R}_+^N\}$ with Hurst index H is a real-valued, centered Gaussian

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random field with covariance function given by

$$\mathbb{E}[B_0^H(s)B_0^H(t)] = \prod_{j=1}^N \frac{1}{2} \left(|s_j|^{2H_j} + |t_j|^{2H_j} - |s_j - t_j|^{2H_j} \right), \quad s, t \in \mathbb{R}^N. \quad (1.1)$$

It follows from (1.1) that $B_0^H(t) = 0$ a.s. for every $t \in \mathbb{R}^N$ with at least one zero coordinate. When $H_1 = \dots = H_N = \alpha \in (0, 1)$, we will write $H = \langle \alpha \rangle$.

It is known that B_0^H has the following two important stochastic integral representations:

- **Moving average representation**

$$B_0^H(t) = \kappa_H^{-1} \int_{-\infty}^{t_1} \dots \int_{-\infty}^{t_N} g(t, s) W(ds), \quad (1.2)$$

where $W = \{W(s), s \in \mathbb{R}^N\}$ is a standard real-valued Brownian sheet and

$$g(t, s) = \prod_{j=1}^N \left[((t_j - s_j)_+)^{H_j-1/2} - ((-s_j)_+)^{H_j-1/2} \right]$$

with $s_+ = \max\{s, 0\}$, and where κ_H is the normalizing constant given by

$$\kappa_H^2 = \int_{-\infty}^1 \dots \int_{-\infty}^1 \left[\prod_{j=1}^N g^2(\langle 1 \rangle, s) \right]^2 ds$$

so that $\mathbb{E}[(B_0^H(t))^2] = \prod_{j=1}^N |t_j|^{2H_j}$ for all $t \in \mathbb{R}^N$. Here $\langle 1 \rangle = (1, 1, \dots, 1) \in \mathbb{R}^N$.

- **Harmonizable representation**

$$B_0^H(t) = K_H^{-1} \int_{\mathbb{R}^N} \psi_t(\lambda) \widehat{W}(d\lambda), \quad (1.3)$$

where $\widehat{W}$ is the Fourier transform of the white noise in $\mathbb{R}^N$ and

$$\psi_t(\lambda) = \prod_{j=1}^N \frac{e^{it_j \lambda_j} - 1}{|\lambda_j|^{H_j + \frac{1}{2}}},$$

and where K_H is the normalizing constant so that $\mathbb{E}[(B_0^H(t))^2] = \prod_{j=1}^N |t_j|^{2H_j}$ for all $t \in \mathbb{R}^N$.

Let $B_1^H, \dots, B_d^H$ be d independent copies of B_0^H . Then the (N, d) -fractional Brownian sheet with Hurst index $H = (H_1, \dots, H_N)$ is the Gaussian random field $B^H = \{B^H(t) : t \in \mathbb{R}^N\}$ with values in $\mathbb{R}^d$ defined by

$$B^H(t) = (B_1^H(t), \dots, B_d^H(t)), \quad t \in \mathbb{R}^N. \quad (1.4)$$

Note that if $N = 1$, then B^H is a fractional Brownian motion in $\mathbb{R}^d$ with Hurst index $H_1 \in (0, 1)$; if $N > 1$ and $H = \langle 1/2 \rangle$, then B^H is the (N, d) -Brownian sheet, denoted by B . Hence B^H can be regarded as a natural generalization of one parameter fractional Brownian motion in $\mathbb{R}^d$ to (N, d) Gaussian random fields, as well as a generalization of the Brownian sheet.

Another well known generalization is the multiparameter fractional Brownian motion $X = \{X(t), t \in \mathbb{R}^N\}$, which is a centered (N, d) -Gaussian random field with covariance function

$$\mathbb{E}[X_i(s)X_j(t)] = \frac{1}{2}\delta_{ij}\left(|s|^{2H_1} + |t|^{2H_1} - |s - t|^{2H_1}\right), \quad \forall s, t \in \mathbb{R}^N, \quad (1.5)$$

where $0 < H_1 < 1$ is a constant and $\delta_{ij} = 1$ if $i = j$ and 0 if $i \neq j$, and where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^N$. The main difference between B^H and the multiparameter fractional Brownian motion is that the latter is isotropic and self-similar with stationary increments while B^H is anisotropic and does not have stationary increments. It follows from (1.1) that B^H is operator-self-similar in the sense that for all constants $c > 0$,

$$\{B^H(c^A t), t \in \mathbb{R}^N\} \stackrel{d}{=} \{c^N B^H(t), t \in \mathbb{R}^N\}, \quad (1.6)$$

where $A = (a_{ij})$ is the $N \times N$ diagonal matrix with $a_{ii} = 1/H_i$ for all $1 \leq i \leq N$ and $a_{ij} = 0$ if $i \neq j$, and $X \stackrel{d}{=} Y$ means that the two processes have the same finite dimensional distributions. The anisotropic property and the operator-self-similarity (1.6) of B^H make it a possible model for bone structure [Bonami and Estrade (2003)] and aquifer structure in hydrology [Benson et al. (2004)].

Many authors have studied various properties of fractional Brownian sheets. For example, Dunker (2000), Mason and Shi (2001), Belinski and Linde (2002), Kühn and Linde (2002) studied the small ball probabilities of an $(N, 1)$ -fractional Brownian sheet B_0^H . Mason and Shi (2001) also computed the Hausdorff dimension of some exceptional sets related to the oscillation of the sample paths of B_0^H . Ayache and Taqu (2003) derived an optimal wavelet series expansion for the fractional Brownian sheet B_0^H ; see also Kühn and Linde (2002), Dzharapidze and van Zanten (2005) for other optimal series expansions for B_0^H . Xiao and Zhang (2002) studied the existence of local times of an (N, d) -fractional Brownian sheet B^H and proved a sufficient condition for the joint continuity of the local times. Kamont (1996) and Ayache (2002) studied the box dimension and the Hausdorff dimension of the graph set of an $(N, 1)$ -fractional Brownian sheet B_0^H .

Recently, Ayache and Xiao (2005) have investigated the uniform and local asymptotic properties of B^H by using wavelet methods, and determined the Hausdorff and packing dimensions of the range $B^H([0, 1]^N)$, the graph $\text{Gr}B^H([0, 1]^N)$ and the level set $L_x = \{t \in (0, \infty)^N : B^H(t) = x\}$. Their results suggest that, due to the anisotropy of B^H in t , the sample paths of B^H are more irregular than those of the Brownian sheet or (N, d) -fractional Brownian motion. Hence it is of interest to further describe the anisotropic properties of B^H in terms of the Hurst index $H = (H_1, \dots, H_N)$.

The main objective of this paper is to investigate the geometric and Fourier analytic properties of the image $B^H(E)$ of a Borel set $E \subset (0, \infty)^N$. In Section 2, we prove that fractional Brownian sheets satisfy a type of “sectorial local nondeterminism” [see Theorem 2.1], extending a result of Khoshnevisan and Xiao (2004) for the Brownian sheet. This property plays an important role in this paper and it will be useful in studying other problems such as local times of fractional Brownian sheets.

In Section 3, we determine the Hausdorff dimension of the image $B^H(E)$. Unlike the well-known cases of fractional Brownian motion or the Brownian sheet, $\dim_{\text{H}} B^H(E)$ can not be determined by $\dim_{\text{H}} E$ alone due to the anisotropy of B^H [see Example 3.4]. To solve the

problem, we introduce a new concept of “dimension” [we call it *Hausdorff dimension contour*] for finite Borel measures and Borel sets; and we prove that $\dim_{\mathbb{H}} B^H(E)$ can be represented in terms of the Hausdorff dimension contour of E and the Hurst index H .

We believe that the concept of Hausdorff dimension contour is of independent interest because it carries more information about the geometric properties of Borel measures and sets than Hausdorff dimension does; and it is the appropriate notion for studying the image $B^H(E)$ and the local times of the fractional Brownian sheet B^H on E , as shown by the results in Sections 3, 4 and 5. Therefore it would be interesting to further study Hausdorff dimension contours and to develop ways to determine them for large classes of fractal sets.

For an (N, d) -fractional Brownian sheet $B^{(\alpha)}$ with index $H = \langle \alpha \rangle$, we also prove the following uniform Hausdorff dimension result for its images: If $N \leq \alpha d$, then with probability one,

$$\dim_{\mathbb{H}} B^{(\alpha)}(E) = \frac{1}{\alpha} \dim_{\mathbb{H}} E \quad \text{for all Borel sets } E \subset (0, \infty)^N. \quad (1.7)$$

This extends the results of Mountford (1989) and Khoshnevisan, Wu and Xiao (2005) for the Brownian sheet. Our proof is based on the sectorial local nondeterminism of $B^{(\alpha)}$ and is similar to that of Khoshnevisan, Wu and Xiao (2005).

Let μ be a probability measure carried by E and let $\nu = \mu_{B^H}$ be the image measure of μ under the mapping $t \mapsto B^H(t)$. In Section 4, we study the asymptotic properties of the Fourier transform $\widehat{\nu}(\xi)$ of ν as $\xi \rightarrow \infty$. In particular, we show that the image $B^H(E)$ is a Salem set whenever $s(H, E) \leq d$, see Section 3 for the definition of $s(H, E)$. These results extend those of Kahane (1985a, b) and Khoshnevisan, Wu and Xiao (2005) for fractional Brownian motion and the Brownian sheet, respectively.

In Section 5, we prove a sufficient condition for $B^H(E)$ to have interior points. This problem is closely related to the existence of a continuous local time of B^H on E [cf. Pitt (1978), Geman and Horowitz (1980), Kahane (1985a, b)]. Our Theorem 5.1 extends and improves significantly the previous result of Khoshnevisan and Xiao (2004) for the Brownian sheet.

We end the introduction with some notation. Throughout this paper, the underlying parameter space is $\mathbb{R}^N$ or $\mathbb{R}_+^N = [0, \infty)^N$. A typical parameter, $t \in \mathbb{R}^N$ is written as $t = (t_1, \dots, t_N)$, or as $\langle c \rangle$, if $t_1 = \dots = t_N = c$. For any $s, t \in \mathbb{R}^N$ such that $s_j < t_j$ ($j = 1, \dots, N$), we define the closed interval (or rectangle) $[s, t] = \prod_{j=1}^N [s_j, t_j]$. We use $\langle \cdot, \cdot \rangle$ and $|\cdot|$ to denote the ordinary scalar product and the Euclidean norm in $\mathbb{R}^m$ respectively, no matter the value of the integer m .

We will use c to denote an unspecified positive and finite constant which may not be the same in each occurrence. More specific constants in Section i are numbered as $c_{i,1}, c_{i,2}, \dots$, and so on.

2 Sectorial local nondeterminism

Recently Khoshnevisan and Xiao (2004) have proved that the Brownian sheet satisfies a type of “sectorial” local nondeterminism and applied this property to study geometric properties of the Brownian sheet; see also Khoshnevisan, Wu and Xiao (2005). In the following we prove that the (N, d) -fractional Brownian sheet $B^H = \{B^H(t), t \in \mathbb{R}^N\}$ satisfies a similar type of sectorial local nondeterminism [It is formulated for B_0^H since the coordinate processes of B^H

are independent copies of B_0^H . This property will play an important rôle in this paper, as well as in studying the local times, self-intersections and other sample path properties of fractional Brownian sheets.

Theorem 2.1 (Sectorial LND) *For any fixed positive number $\varepsilon \in (0, 1)$, there exists a positive constant $c_{2,1}$, depending on ε, H and N only, such that for all positive integers $n \geq 1$, and all $u, t^1, \dots, t^n \in [\varepsilon, \infty)^N$, we have*

$$\text{Var} \left(B_0^H(u) \mid B_0^H(t^1), \dots, B_0^H(t^n) \right) \geq c_{2,1} \sum_{j=1}^N \min_{0 \leq k \leq n} |u_j - t_j^k|^{2H_j}, \quad (2.1)$$

where $t^0 = 0$.

Remark 2.2 The proof of the sectorial local nondeterminism of the Brownian sheet in Khoshnevisan and Xiao (2004) is based on the properties of independent increments and Gaussian distributions. The method we use for proving Theorem 2.1 has to be different because fractional Brownian sheets have no independent increments. Our proof is mainly based on the harmonizable representation (1.3) of B_0^H and is reminiscent to Kahane (1985a, Chapter 18) or Xiao (2005).

Proof Let $\ell \in \{1, \dots, N\}$ be fixed and denote $r_\ell \equiv \min_{0 \leq k \leq n} |u_\ell - t_\ell^k|$. Firstly, we prove that there exists a positive constant c_ℓ such that the following inequality holds:

$$\text{Var} \left(B_0^H(u) \mid B_0^H(t^1), \dots, B_0^H(t^n) \right) \geq c_\ell r_\ell^{2H_\ell}. \quad (2.2)$$

Summing over ℓ from 1 to N in (2.2), we get (2.1).

In order to prove (2.2), we work in the Hilbert space setting and write the conditional variance in (2.2) as the square of the $L^2(\mathbb{P})$ -distance of $B_0^H(u)$ from the subspace generated by $\{B_0^H(t^1), \dots, B_0^H(t^n)\}$. Hence it suffices to show that for all $a_k \in \mathbb{R}$,

$$\mathbb{E} \left(B_0^H(u) - \sum_{k=1}^n a_k B_0^H(t^k) \right)^2 \geq c_\ell r_\ell^{2H_\ell}. \quad (2.3)$$

It follows from the harmonizable representation (1.3) of B_0^H that

$$\begin{aligned} \mathbb{E} \left(B_0^H(u) - \sum_{k=1}^n a_k B_0^H(t^k) \right)^2 &= K_H^{-2} \int_{\mathbb{R}^N} \left| \psi_u(\lambda) - \sum_{k=1}^n a_k \psi_{t^k}(\lambda) \right|^2 d\lambda \\ &= K_H^{-2} \int_{\mathbb{R}^N} \left| \prod_{j=1}^N (e^{iu_j \lambda_j} - 1) - \sum_{k=1}^n a_k \prod_{j=1}^N (e^{it_j^k \lambda_j} - 1) \right|^2 f_H(\lambda) d\lambda, \end{aligned} \quad (2.4)$$

where

$$f_H(\lambda) = \prod_{j=1}^N |\lambda_j|^{-2H_j-1}.$$

Now for every $j = 1, \dots, N$, we choose a bump function $\delta_j(\cdot) \in C^\infty(\mathbb{R})$ with values in $[0, 1]$ such that $\delta_j(0) = 1$ and strictly decreasing in $|\cdot|$ near 0 [e.g., on $(-\varepsilon, \varepsilon)$] and vanishes outside

the open interval $(-1, 1)$. Let $\widehat{\delta}_j$ be the Fourier transform of δ_j . It is known that $\widehat{\delta}_j(\lambda_j)$ is also in $C^\infty(\mathbb{R})$ and decays rapidly as $\lambda_j \rightarrow \infty$. Also, the Fourier inversion formula gives

$$\delta_j(s_j) = (2\pi)^{-1} \int_{\mathbb{R}} e^{-is_j \lambda_j} \widehat{\delta}_j(\lambda_j) d\lambda_j. \quad (2.5)$$

Let $\delta_\ell^{r_\ell}(s_\ell) = r_\ell^{-1} \delta_\ell(\frac{s_\ell}{r_\ell})$, then by (2.5) and a change of variables, we have

$$\delta_\ell^{r_\ell}(s_\ell) = (2\pi)^{-1} \int_{\mathbb{R}} e^{-is_\ell \lambda_\ell} \widehat{\delta}_\ell(r_\ell \lambda_\ell) d\lambda_\ell. \quad (2.6)$$

By the definition of r_ℓ , we have $\delta_\ell^{r_\ell}(u_\ell) = 0$ and $\delta_\ell^{r_\ell}(u_\ell - t_\ell^k) = 0$ for all $k = 1, \dots, n$. Hence it follows from (2.5) and (2.6) that

$$\begin{aligned} I &\equiv \int_{\mathbb{R}^N} \left(\prod_{j=1}^N (e^{iu_j \lambda_j} - 1) - \sum_{k=1}^n a_k \prod_{j=1}^N (e^{it_j^k \lambda_j} - 1) \right) \prod_{j=1}^N e^{-iu_j \lambda_j} \left(\prod_{j \neq \ell}^N \widehat{\delta}_j(\lambda_j) \right) \widehat{\delta}_\ell(r_\ell \lambda_\ell) d\lambda \\ &= (2\pi)^N \left[\left(\prod_{j \neq \ell} (\delta_j(0) - \delta_j(u_j)) \right) \left(\delta_\ell^{r_\ell}(0) - \delta_\ell^{r_\ell}(u_\ell) \right) \right] \\ &\quad - (2\pi)^N \left[\sum_{k=1}^n a_k \left(\prod_{j \neq \ell} (\delta_j(u_j - t_j^k) - \delta_j(u_j)) \right) \left(\delta_\ell^{r_\ell}(u_\ell - t_\ell^k) - \delta_\ell^{r_\ell}(u_\ell) \right) \right] \\ &\geq (2\pi)^N \prod_{j \neq \ell} (1 - \delta_j(\varepsilon)) r_\ell^{-1}. \end{aligned} \quad (2.7)$$

On the other hand, by the Cauchy-Schwarz inequality, (2.4) and (2.7), we have

$$\begin{aligned} I^2 &\leq \int_{\mathbb{R}^N} \left| \prod_{j=1}^N (e^{iu_j \lambda_j} - 1) - \sum_{k=1}^n a_k \prod_{j=1}^N (e^{it_j^k \lambda_j} - 1) \right|^2 f_H(\lambda) d\lambda \\ &\quad \times \int_{\mathbb{R}^N} \frac{1}{f_H(\lambda)} \left| \left(\prod_{j \neq \ell}^N \widehat{\delta}_j(\lambda_j) \right) \widehat{\delta}_\ell(r_\ell \lambda_\ell) \right|^2 d\lambda \\ &= K_H^2 \mathbb{E} \left(B_0^H(u) - \sum_{k=1}^n a_k B_0^H(t^k) \right)^2 \cdot r_\ell^{-2H_\ell-2} \int_{\mathbb{R}^N} \frac{1}{f_H(\lambda)} \prod_{j=1}^N |\widehat{\delta}_j(\lambda_j)|^2 d\lambda \\ &= c_{2,2} r_\ell^{-2H_\ell-2} \mathbb{E} \left(B_0^H(u) - \sum_{k=1}^n a_k B_0^H(t^k) \right)^2. \end{aligned} \quad (2.8)$$

Combining (2.7) and (2.8) yields (2.2). This finishes the proof of the theorem. $\square$

Given Gaussian random variables $Z_1, \dots, Z_n$, we denote by $\text{Cov}(Z_1, \dots, Z_n)$ their covariance matrix. The following formula relates the determinant of $\text{Cov}(Z_1, \dots, Z_n)$ with conditional variances:

$$\det \text{Cov}(Z_1, \dots, Z_n) = \text{Var}(Z_1) \prod_{k=2}^n \text{Var}(Z_k | Z_1, \dots, Z_{k-1}). \quad (2.9)$$

Hence the inequality (2.1) can be used to estimate the joint distribution of the Gaussian random variables $B_0^H(t^1), \dots, B_0^H(t^n)$. It is for this reason why sectorial local nondeterminism is essential in this paper and in studying local times of fractional Brownian sheets.

The following simple result will be needed in Section 5.

Lemma 2.3 *Let $n \geq 1$ be a fixed integer. Then for all $t^1, \dots, t^n \in [\varepsilon, \infty)^N$ such that $t_j^1, \dots, t_j^n$ are all distinct for some $j \in \{1, \dots, N\}$, the Gaussian random variables $B_0^H(t^1), \dots, B_0^H(t^n)$ are linearly independent.*

Proof Let $t^1, \dots, t^n \in [\varepsilon, \infty)^N$ be given as above. Then it follows from Theorem 2.1 and (2.9) that $\det \text{Cov}(B_0^H(t^1), \dots, B_0^H(t^n)) > 0$. This proves the lemma. $\square$

3 Dimension results for the images

In this section, we study the Hausdorff dimension of the image set $B^H(E)$ of an arbitrary Borel set $E \subset (0, \infty)^N$. When $E = [0, 1]^N$ or any Borel set with positive Lebesgue measure, this problem has been solved by Ayache and Xiao (2005). However, when $E \subset (0, \infty)^N$ is a fractal set, the Hausdorff dimension of $B^H(E)$ can not be determined by $\dim_H E$ and H alone, as shown by Example 3.4 below. This is in contrast with the cases of fractional Brownian motion or the Brownian sheet.

To solve this problem, we will introduce a new notion of dimension, namely, *Hausdorff dimension contour*, for finite Borel measures and Borel sets. It turns out the Hausdorff dimension contour of E is the natural object in determining the Hausdorff dimension and other geometric properties of $B^H(E)$ for all Borel sets E .

We start with the following Proposition 3.1 which determines $\dim_H B^H(E)$ when E belongs to a special class of Borel sets in $\mathbb{R}_+^N$.

Proposition 3.1 *Let $B^H = \{B^H(t), t \in \mathbb{R}^N\}$ be an (N, d) -fractional Brownian sheet with index $H = (H_1, \dots, H_N)$. Assume that E_j ($j = 1, \dots, N$) are Borel sets in $(0, \infty)$ satisfying the following property: $\exists \{j_1, \dots, j_{N-1}\} \subset \{1, \dots, N\}$ such that $\dim_H E_{j_k} = \dim_P E_{j_k}$ for $k = 1, \dots, N-1$. Let $E = E_1 \times \dots \times E_N \subset (0, \infty)^N$, then we have*

$$\dim_H B^H(E) = \min \left\{ d; \sum_{j=1}^N \frac{\dim_H E_j}{H_j} \right\}, \quad \text{a.s.} \quad (3.1)$$

In the above, $\dim_P F$ denotes the packing dimension of F which is defined as follows. For any $\varepsilon > 0$ and any bounded set $F \subseteq \mathbb{R}^d$, let

$$N(F, \varepsilon) = \text{smallest number of balls of radius } \varepsilon \text{ needed to cover } F.$$

Then the *upper box-counting dimension* of F is defined as

$$\overline{\dim}_B F = \limsup_{\varepsilon \rightarrow 0} \frac{\log N(F, \varepsilon)}{-\log \varepsilon}. \quad (3.2)$$

The packing dimension of F can be defined by

$$\dim_{\mathbb{P}} F = \inf \left\{ \sup_n \overline{\dim}_{\mathbb{B}} F_n : F \subseteq \bigcup_{n=1}^{\infty} F_n \right\}. \quad (3.3)$$

Further information on packing dimension can be found in Tricot (1982), Falconer (1990) and Mattila (1995).

For proving Proposition 3.1, we need the next two lemmas that are due to Ayache and Xiao (2005).

Lemma 3.2 *Let $B^H = \{B^H(t), t \in \mathbb{R}^N\}$ be an (N, d) -fractional Brownian sheet with index $H = (H_1, \dots, H_N)$. For all $T > 0$, there exist a random variable $A_1 = A_1(\omega) > 0$ of finite moments of any order and an event Ω_1^* of probability 1 such that for every $\omega \in \Omega_1^*$,*

$$\sup_{s, t \in [0, T]^N} \frac{|B^H(s, \omega) - B^H(t, \omega)|}{\sum_{j=1}^N |s_j - t_j|^{H_j} \sqrt{\log(3 + |s_j - t_j|^{-1})}} \leq A_1(\omega). \quad (3.4)$$

Lemma 3.3 *Let $B_0^H = \{B_0^H(t), t \in \mathbb{R}^N\}$ be an $(N, 1)$ -fractional Brownian sheet with index $H = (H_1, \dots, H_N)$, then for any $0 < \varepsilon < T$, there exist positive and finite constants $c_{3,1}$ and $c_{3,2}$ such that for all $s, t \in [\varepsilon, T]^N$,*

$$c_{3,1} \sum_{j=1}^N |s_j - t_j|^{2H_j} \leq \mathbb{E} \left[(B_0^H(s) - B_0^H(t))^2 \right] \leq c_{3,2} \sum_{j=1}^N |s_j - t_j|^{2H_j}. \quad (3.5)$$

Proof of Proposition 3.1 As usual, the proof of (3.1) is divided into proving the upper and lower bounds separately. We will show that the upper bound in (3.1) follows from the modulus of continuity of the fractional Brownian sheet and a covering argument, and the lower bound follows from Forstman's Theorem [see e.g., Kahane (1985a, Chapter 10)] and Lemma 3.3.

For simplicity of notation, we will only consider the case $N = 2$ and $\dim_{\mathbb{H}} E_1 = \dim_{\mathbb{P}} E_1$. The proof for the general case is similar.

Upper bound: By the σ -stability of $\dim_{\mathbb{H}}$ and (3.3), it is sufficient to prove that for every Borel set $E = E_1 \times E_2$,

$$\dim_{\mathbb{H}} B^H(E) \leq \min \left\{ d; \frac{\overline{\dim}_{\mathbb{B}} E_1}{H_1} + \frac{\dim_{\mathbb{H}} E_2}{H_2} \right\}, \quad \text{a.s.} \quad (3.6)$$

For any $\forall \gamma_1 > \overline{\dim}_{\mathbb{B}} E_1$, $\gamma_2 > \dim_{\mathbb{H}} E_2$, we choose and fix $\gamma_2' \in (\dim_{\mathbb{H}} E_2, \gamma_2)$. Then there exists an $r_0 > 0$ such that, for all $r \leq r_0$, E_1 can be covered by $N(E_1, r) \leq r^{-\gamma_1}$ many small intervals of length r ; and there exists a covering $\{U_n, n \geq 1\}$ of E_2 such that $r_n := |U_n| \leq r_0$ and

$$\sum_{n=1}^{\infty} r_n^{\gamma_2'} \leq 1. \quad (3.7)$$

For every $n \geq 1$ and any constant $\delta \in (0, 1)$ small enough, let $\{V_{n,m} : 1 \leq m \leq N_n\}$ be $N_n := N(E_1, r_n^{(H_2-\delta)/(H_1-\delta)})$ intervals of length $r_n^{(H_2-\delta)/(H_1-\delta)}$ which cover E_1 . Then the

rectangles $\{V_{n,m} \times U_n : n \geq 1, 1 \leq m \leq N_n\}$ form a covering of $E_1 \times E_2$, that is,

$$E \subset \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{N_n} V_{n,m} \times U_n,$$

and thus

$$B^H(E) \subset \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{N_n} B^H(V_{n,m} \times U_n). \quad (3.8)$$

It follows from Lemma 3.2 that, almost surely, $B^H(V_{n,m} \times U_n)$ can be covered by a ball of radius $c r_n^{H_2 - \delta}$. By this and (3.8), we have covered $B^H(E)$ a.s. by balls of radius $c r_n^{H_2 - \delta}$ ($n = 1, 2, \dots$). Moreover, recalling that $N_n \leq r_n^{-\gamma_1(H_2 - \delta)/(H_1 - \delta)}$, we have

$$\begin{aligned} \sum_{n=1}^{\infty} \sum_{m=1}^{N_n} \left(r_n^{H_2 - \delta} \right)^{\frac{\gamma_1}{H_1} + \frac{\gamma_2}{H_2}} &\leq \sum_{n=1}^{\infty} r_n^{-\gamma_1 \frac{H_2 - \delta}{H_1 - \delta}} \cdot r_n^{(H_2 - \delta) \left(\frac{\gamma_1}{H_1} + \frac{\gamma_2}{H_2} \right)} \\ &= \sum_{n=1}^{\infty} r_n^{\gamma_2 - \gamma_1 \left(\frac{H_2 - \delta}{H_1 - \delta} - \frac{H_2 - \delta}{H_1} \right) - \frac{\delta \gamma_2}{H_2}}. \end{aligned} \quad (3.9)$$

Now we choose $\delta > 0$ small enough so that

$$\gamma_2 - \gamma_1 \left(\frac{H_2 - \delta}{H_1 - \delta} - \frac{H_2 - \delta}{H_1} \right) - \frac{\delta \gamma_2}{H_2} > \gamma'_2.$$

Then (3.7) and (3.9) imply that

$$\sum_{n=1}^{\infty} \sum_{m=1}^{N_n} \left(r_n^{H_2 - \delta} \right)^{\frac{\gamma_1}{H_1} + \frac{\gamma_2}{H_2}} \leq 1. \quad (3.10)$$

It should be clear that (3.6) follows from (3.10).

Lower bound: Choose γ_1, γ_2 such that $0 < \gamma_1 < \dim_H E_1$, $0 < \gamma_2 < \dim_H E_2$ and $\frac{\gamma_1}{H_1} + \frac{\gamma_2}{H_2} < d$, then there exist probability measures σ_1 on E_1 and σ_2 on E_2 such that

$$\int_{E_1} \int_{E_1} \frac{\sigma_1(ds_1)\sigma_1(dt_1)}{|s_1 - t_1|^{\gamma_1}} < \infty, \quad \int_{E_2} \int_{E_2} \frac{\sigma_2(ds_2)\sigma_2(dt_2)}{|s_2 - t_2|^{\gamma_2}} < \infty. \quad (3.11)$$

Let $\sigma = \sigma_1 \times \sigma_2$. Then σ is a probability measure on E . By Lemma 3.3 and the fact that $\frac{\gamma_1}{H_1} + \frac{\gamma_2}{H_2} < d$, we have

$$\begin{aligned} &\mathbb{E} \int_E \int_E \frac{\sigma(ds)\sigma(dt)}{|B^H(s) - B^H(t)|^{\frac{\gamma_1}{H_1} + \frac{\gamma_2}{H_2}}} \\ &\leq c \int_E \int_E \frac{\sigma_1(ds_1)\sigma_1(dt_1)\sigma_2(ds_2)\sigma_2(dt_2)}{(|s_1 - t_1|^{2H_1} + |s_2 - t_2|^{2H_2})^{\frac{1}{2} \left(\frac{\gamma_1}{H_1} + \frac{\gamma_2}{H_2} \right)}} \\ &\leq c \int_{E_1} \int_{E_1} \sigma_1(ds_1)\sigma_1(dt_1) \int_{E_2} \int_{E_2} \frac{\sigma_2(ds_2)\sigma_2(dt_2)}{(|s_1 - t_1|^{H_1} + |s_2 - t_2|^{H_2})^{\frac{\gamma_1}{H_1} + \frac{\gamma_2}{H_2}}}. \end{aligned} \quad (3.12)$$

By an inequality for the weighted arithmetic mean and geometric mean with $\beta_1 = \frac{H_2\gamma_1}{H_2\gamma_1 + H_1\gamma_2}$ and $\beta_2 = 1 - \beta_1 = \frac{H_1\gamma_2}{H_2\gamma_1 + H_1\gamma_2}$, we have

$$\begin{aligned}
& |s_1 - t_1|^{H_1} + |s_2 - t_2|^{H_2} \\
& \geq \beta_1 |s_1 - t_1|^{H_1} + \beta_2 |s_2 - t_2|^{H_2} \\
& \geq \left(|s_1 - t_1|^{H_1}\right)^{\beta_1} \left(|s_2 - t_2|^{H_2}\right)^{\beta_2} \\
& = |s_1 - t_1|^{\frac{H_1 H_2 \gamma_1}{H_2 \gamma_1 + H_1 \gamma_2}} |s_2 - t_2|^{\frac{H_1 H_2 \gamma_2}{H_2 \gamma_1 + H_1 \gamma_2}}.
\end{aligned} \tag{3.13}$$

Therefore, the last denominator in (3.12) can be bounded from below by $|s_1 - t_1|^{\gamma_1} |s_2 - t_2|^{\gamma_2}$. It follows from this and (3.11), (3.12) that

$$\mathbb{E} \int_E \int_E \frac{\sigma(ds)\sigma(dt)}{|B^H(s) - B^H(t)|^{\frac{\gamma_1}{H_1} + \frac{\gamma_2}{H_2}}} < \infty.$$

This yields the lower bound in (3.1). $\square$

The following simple example illustrates that, in general, $\dim_H E$ alone is not enough to determine the Hausdorff dimension of $B^H(E)$.

Example 3.4 Let $B^H = \{B^H(t), t \in \mathbb{R}^2\}$ be a $(2, d)$ -fractional Brownian sheet with index $H = (H_1, H_2)$ and $H_1 < H_2$. Let $E = E_1 \times E_2$, $F = E_2 \times E_1$, where $E_1 \subset (0, \infty)$ satisfies $\dim_H E_1 = \dim_P E_1$ and $E_2 \subset (0, \infty)$ is arbitrary. It is well known that

$$\dim_H E = \dim_H E_1 + \dim_H E_2 = \dim_H F,$$

cf. Falconer (1990, p.94). However, by Proposition 3.1 we have

$$\begin{aligned}
\dim_H B^H(E) &= \min \left\{ d; \frac{\dim_H E_1}{H_1} + \frac{\dim_H E_2}{H_2} \right\}. \\
\dim_H B^H(F) &= \min \left\{ d; \frac{\dim_H E_2}{H_1} + \frac{\dim_H E_1}{H_2} \right\}.
\end{aligned}$$

We see that $\dim_H B^H(E) \neq \dim_H B^H(F)$ unless $\dim_H E_1 = \dim_H E_2$.

Example 3.4 shows that in order to determine $\dim_H B^H(E)$, we need to have more information about the geometry of E than its Hausdorff dimension. We have found it is more convenient to work with Borel measures carried by E .

Recall that the Hausdorff dimension of a Borel measure μ on $\mathbb{R}^N$ (or lower Hausdorff dimension as it is sometimes called) is defined by

$$\dim_H \mu = \inf \left\{ \dim_H F : \mu(F) > 0 \text{ and } F \subseteq \mathbb{R}^N \text{ is a Borel set} \right\}. \tag{3.14}$$

The packing dimension of μ , denoted by $\dim_P \mu$, is defined by replacing $\dim_H F$ in (3.14) by $\dim_P F$.

Hu and Taylor (1994) proved the following characterization of $\dim_{\mathbb{H}}\mu$: If μ is a finite Borel measure on $\mathbb{R}^N$ then

$$\dim_{\mathbb{H}}\mu = \sup \left\{ \gamma \geq 0 : \limsup_{r \rightarrow 0^+} r^{-\gamma} \mu(U(t, r)) = 0 \text{ for } \mu\text{-a.e. } t \in \mathbb{R}^N \right\}, \quad (3.15)$$

where $U(t, r) = \{s \in \mathbb{R}^N : |s - t| \leq r\}$. There is a similar identity for $\dim_{\mathbb{P}}\mu$ as well; see Hu and Taylor (1994, Corollary 4.2) and Falconer and Howroyd (1997) for further information. It is easy to see that for every Borel set $E \subset \mathbb{R}^N$, we have

$$\dim_{\mathbb{H}}E = \sup \left\{ \dim_{\mathbb{H}}\mu : \mu \in \mathcal{M}_c^+(E) \right\}, \quad (3.16)$$

where $\mathcal{M}_c^+(E)$ denotes the family of finite Borel measures on E with compact support in E .

From (3.15), we note that $\dim_{\mathbb{H}}\mu$ only describes the local behavior of μ in an isotropic way and is not quite informative if we want to characterize the anisotropic nature of μ , as what we are dealing with in this paper. To overcome this difficulty, we introduce the following new notion of “dimension” for $E \subset (0, \infty)^N$ that is natural for studying $B^H(E)$.

Definition 3.5 *Given a Borel probability measure μ on $\mathbb{R}^N$, we define the set $\Lambda_\mu \subseteq \mathbb{R}_+^N$ by*

$$\Lambda_\mu = \left\{ \lambda = (\lambda_1, \dots, \lambda_N) \in \mathbb{R}_+^N : \limsup_{r \rightarrow 0^+} \frac{\mu(R(t, r))}{r^{\langle \lambda, H^{-1} \rangle}} = 0, \text{ for } \mu\text{-a.e. } t \in \mathbb{R}^N \right\}, \quad (3.17)$$

where $R(t, r) = \prod_{j=1}^N [t_j - r^{1/H_j}, t_j + r^{1/H_j}]$ and $H^{-1} = (\frac{1}{H_1}, \dots, \frac{1}{H_N})$.

Some basic properties of Λ_μ are summarized in the following lemma. For simplicity of notation, we assume $H_1 = \min\{H_j : 1 \leq j \leq N\}$.

Lemma 3.6 *Λ_μ has the following properties:*

(i) *The set Λ_μ is bounded:*

$$\Lambda_\mu \subseteq \left\{ \lambda = (\lambda_1, \dots, \lambda_N) \in \mathbb{R}_+^N : \langle \lambda, H^{-1} \rangle \leq \frac{N}{H_1} \right\}. \quad (3.18)$$

(ii) *For every $\beta \in (0, 1]^N$ and $\lambda \in \Lambda_\mu$, we have $\beta \circ \lambda \in \Lambda_\mu$, where $\beta \circ \lambda = (\beta_1 \lambda_1, \dots, \beta_N \lambda_N)$ is the Hadamard product of β and λ .*

(iii) *The set Λ_μ is convex, i.e., $\forall \lambda, \eta \in \Lambda_\mu$ and $0 < b < 1$, $b\lambda + (1 - b)\eta \in \Lambda_\mu$.*

(iv) *For every $a \in (0, \infty)^N$, $\sup_{\lambda \in \Lambda_\mu} \langle a, \lambda \rangle$ is achieved on the boundary of Λ_μ .*

Because of (iv) and its importance in this paper, we call the boundary of Λ_μ , denoted by $\partial\Lambda_\mu$, the *Hausdorff dimension contour* of μ .

Proof Suppose $\lambda = (\lambda_1, \dots, \lambda_N) \in \Lambda_\mu$. Then

$$\limsup_{r \rightarrow 0^+} \frac{\mu(R(t, r))}{r^{\langle \lambda, H^{-1} \rangle}} = 0 \quad \text{for } \mu\text{-a.e. } t \in \mathbb{R}^N. \quad (3.19)$$

Fix a $t \in \mathbb{R}^N$ such that (3.19) holds. Since for any $a > 0$, the ball $U(t, a)$ centered at t with radius a can be covered by $R(t, a^{H_1})$. It follows from (3.19) that

$$\limsup_{r \rightarrow 0^+} \frac{\mu(U(t, a))}{a^{H_1 \langle \lambda, H^{-1} \rangle}} = 0 \quad \text{for } \mu\text{-a.e. } t \in \mathbb{R}^N. \quad (3.20)$$

It follows from (3.15) and (3.20) that $\dim_{\mathbb{H}} \mu \geq H_1 \langle \lambda, H^{-1} \rangle$. Hence we have $\langle \lambda, H^{-1} \rangle \leq N/H_1$. This proves (i).

Statements (ii) and (iii) follow directly from the definition of Λ_μ . To prove (iv), we note that for every $a \in (0, \infty)^N$, Property (i) implies $\sup_{\lambda \in \Lambda_\mu} \langle a, \lambda \rangle < \infty$. On the other hand, the function $\lambda \mapsto \langle a, \lambda \rangle$ is increasing in each coordinate λ_j . Hence $\sup_{\lambda \in \Lambda_\mu} \langle a, \lambda \rangle$ must be achieved on the boundary of Λ_μ . $\square$

As examples, we mention that if m is the Lebesgue measure on $\mathbb{R}_+^N$, then

$$\Lambda_m = \left\{ (\lambda_1, \dots, \lambda_N) \in \mathbb{R}_+^N : \sum_{j=1}^N \frac{\lambda_j}{H_j} < \sum_{j=1}^N \frac{1}{H_j} \right\} \quad (3.21)$$

and $\sup_{\lambda \in \Lambda_m} \langle H^{-1}, \lambda \rangle = \sum_{j=1}^N \frac{1}{H_j}$. More generally we can verify that, if $\mu = \sigma_1 \times \dots \times \sigma_N$, where σ_j ($j = 1, \dots, N$) are Borel probability measures on $\mathbb{R}$ such that $\dim_{\mathbb{H}} \sigma_{j_k} = \dim_{\mathbb{P}} \sigma_{j_k}$ for some $\{j_1, \dots, j_{N-1}\} \subset \{1, \dots, N\}$, then

$$\Lambda_\mu = \left\{ (\lambda_1, \dots, \lambda_N) \in \mathbb{R}_+^N : \sum_{j=1}^N \frac{\lambda_j}{H_j} < \sum_{j=1}^N \frac{\beta_j}{H_j} \right\},$$

where $\beta_j = \dim_{\mathbb{H}} \sigma_j$ for $j = 1, \dots, N$. In the special case of $H_1 = \dots = H_N = \alpha \in (0, 1)$, we derive from (3.15) that for every Borel measure μ ,

$$\Lambda_\mu = \left\{ (\lambda_1, \dots, \lambda_N) \in \mathbb{R}_+^N : \sum_{j=1}^N \lambda_j < \dim_{\mathbb{H}} \mu \right\}. \quad (3.22)$$

For any Borel measure μ on $\mathbb{R}_+^N$, its image measure under the mapping $t \mapsto B^H(t)$ is defined by

$$\mu_{B^H}(F) = \mu \left\{ t \in \mathbb{R}_+^N : B^H(t) \in F \right\} \quad \text{for all Borel sets } F \subset \mathbb{R}^d.$$

We will make use of the following result.

Proposition 3.7 *Let $B^H = \{B^H(t), t \in \mathbb{R}^N\}$ be an (N, d) -fractional Brownian sheet with index $H \in (0, 1)^N$. Then for every Borel probability measure μ on $\mathbb{R}_+^N$,*

$$\dim_{\mathbb{H}} \mu_{B^H} \leq s_\mu(H) \wedge d \quad \text{a.s.}, \quad (3.23)$$

where $s_\mu(H) = \sup_{\lambda \in \Lambda_\mu} \langle H^{-1}, \lambda \rangle$.

Remark 3.8 Applying a moment argument [see, e.g., Xiao (1996)] and the sectorial local nondeterminism of B^H , we can actually prove that the equality in (3.23) holds. Since this result is not needed in this paper and its proof is rather long, we omit it.

Proof To prove the upper bound in (3.23), note that, without loss of generality, we may and will assume the support of μ is contained in $[\varepsilon, T]^N$ for some $0 < \varepsilon < T < \infty$. Furthermore, if $s_\mu(H) \geq d$, then $\dim_{\mathbb{H}} \mu_{B^H} \leq s_\mu(H) \wedge d$ holds trivially. Therefore, we will also assume that $s_\mu(H) < d$.

Now, for any $0 < \gamma < \dim_{\mathbb{H}} \mu_{B^H}$, by (3.15) we have

$$\limsup_{\rho \rightarrow 0} \frac{\mu_{B^H}(U(u, \rho))}{\rho^\gamma} = 0 \quad \text{for } \mu_{B^H}\text{-a.e. } u \in \mathbb{R}^d. \quad (3.24)$$

This is equivalent to

$$\limsup_{\rho \rightarrow 0} \frac{1}{\rho^\gamma} \int_{[\varepsilon, T]^N} \mathbb{1}_{\{|B^H(s) - B^H(t)| \leq \rho\}} \mu(ds) = 0 \quad \text{for } \mu\text{-a.e. } t \in \mathbb{R}_+^N. \quad (3.25)$$

Note that $\forall \delta \in (0, 1)$, Lemma 3.2 implies that $\forall s, t \in [\varepsilon, T]^N$

$$|B^H(s) - B^H(t)| \leq c \sum_{j=1}^N |s_j - t_j|^{H_j(1-\delta)} \quad \text{a.s.} \quad (3.26)$$

It follows from (3.25) and (3.26) that almost surely

$$\limsup_{\rho \rightarrow 0} \frac{1}{\rho^\gamma} \mu\left(R(t, \rho^{\frac{1}{1-\delta}})\right) = 0 \quad \text{for } \mu\text{-a.e. } t \in \mathbb{R}_+^N. \quad (3.27)$$

We claim that $\gamma \leq s_\mu(H)$. In fact, if $\gamma > s_\mu(H)$, then there exists $\beta \notin \Lambda_\mu$ such that $\langle \beta, H^{-1} \rangle < \gamma$. Since $\beta \notin \Lambda_\mu$, there is a set $A \subset [\varepsilon, T]^N$ with positive μ -measure such that

$$\limsup_{r \rightarrow 0} \frac{\mu(R(t, r))}{r^{\langle \beta, H^{-1} \rangle}} > 0 \quad \text{for every } t \in A. \quad (3.28)$$

Now we choose $\delta > 0$ small enough such that $(1-\delta)\gamma - \langle \beta, H^{-1} \rangle > 0$, then (3.28) implies that for every $t \in A$,

$$\limsup_{\rho \rightarrow 0} \frac{1}{\rho^\gamma} \mu\left(R(t, \rho^{\frac{1}{1-\delta}})\right) = \limsup_{r \rightarrow 0} \frac{1}{r^{(1-\delta)\gamma - \langle \beta, H^{-1} \rangle}} \times \frac{\mu(R(t, r))}{r^{\langle \beta, H^{-1} \rangle}} = \infty. \quad (3.29)$$

This contradicts (3.27). Therefore, we have proved that $\gamma \leq s_\mu(H)$. Since $\gamma < \dim_{\mathbb{H}} \mu_{B^H}$ is arbitrary, we have

$$\dim_{\mathbb{H}} \mu_{B^H} \leq s_\mu(H) \wedge d, \quad \text{a.s.} \quad (3.30)$$

This finishes the proof of Proposition 3.7. $\square$

The following corollary follows directly from Proposition 3.7 and (3.21). It is related to Theorem 3.1 in Ayache and Xiao (2005).

Corollary 3.9 *Let m be the Lebesgue measure on $\mathbb{R}_+^N$, then*

$$\dim_{\mathbb{H}} m_{B^H} \leq \sum_{j=1}^N \frac{1}{H_j} \wedge d, \quad a.s. \quad (3.31)$$

For any Borel set $E \subset (0, \infty)^N$, we define

$$\Lambda(E) = \bigcup_{\mu \in \mathcal{M}_c^+(E)} \Lambda_\mu. \quad (3.32)$$

Recall that $\mathcal{M}_c^+(E)$ is the family of finite Borel measures with compact support in E . Similar to the case of Borel measures, we call the set $\bigcup_{\mu \in \mathcal{M}_c^+(E)} \partial \Lambda_\mu$ the Hausdorff dimension contour of E . It follows from Lemma 3.6 that, for every $a \in (0, \infty)^N$, the supremum $\sup_{\lambda \in \Lambda(E)} \langle \lambda, a \rangle$ is determined by the Hausdorff dimension contour of E .

The following is the main result of this section.

Theorem 3.10 *Let B^H be an (N, d) -fractional Brownian sheet with index $H \in (0, 1)^N$. Then, for any Borel set $E \subset (0, \infty)^N$,*

$$\dim_{\mathbb{H}} B^H(E) = s(H, E) \wedge d \quad a.s., \quad (3.33)$$

where $s(H, E) = \sup_{\lambda \in \Lambda(E)} \langle \lambda, H^{-1} \rangle$.

We need the following lemmas to prove Theorem 3.10.

Lemma 3.11 *Let $E \subset \mathbb{R}^N$ be an analytic set and let $f : \mathbb{R}^N \rightarrow \mathbb{R}^d$ be a continuous function. If $0 \leq \tau < \dim_{\mathbb{H}} f(E)$, then there exists a compact set $E_0 \subseteq E$ such that $\tau < \dim_{\mathbb{H}} f(E_0)$.*

Proof The proof is the same as that of Lemma 4.1 in Xiao (1997), with packing dimension replaced by Hausdorff dimension. $\square$

Lemma 3.12 *Let $E \subset (0, \infty)^N$ be an analytic set. Then for any continuous function $f : \mathbb{R}^N \rightarrow \mathbb{R}^d$,*

$$\dim_{\mathbb{H}} f(E) = \sup \{ \dim_{\mathbb{H}} \mu_f : \mu \in \mathcal{M}_c^+(E) \}. \quad (3.34)$$

Proof For any $\mu \in \mathcal{M}_c^+(E)$, we have $\mu_f \in \mathcal{M}_c^+(f(E))$. By (3.16), we have

$$\dim_{\mathbb{H}} f(E) = \sup \{ \dim_{\mathbb{H}} \nu : \nu \in \mathcal{M}_c^+(f(E)) \}, \quad (3.35)$$

which implies that

$$\dim_{\mathbb{H}} f(E) \geq \sup \{ \dim_{\mathbb{H}} \mu_f : \mu \in \mathcal{M}_c^+(E) \}. \quad (3.36)$$

To prove the reverse inequality, let $\tau < \dim_{\mathbb{H}} f(E)$. By Lemma 3.11, there exists a compact set $E_0 \subset E$ such that $\dim_{\mathbb{H}} f(E_0) > \tau$. Hence, by (3.35), there exists a finite Borel measure

$\nu \in \mathcal{M}_c^+(f(E_0))$ such that $\dim_{\mathbb{H}} \nu > \tau$. It follows from Theorem 1.20 in Mattila (1995) that there exists $\mu \in \mathcal{M}_c^+(E_0)$ such that $\nu = \mu_f$, which implies $\sup\{\dim_{\mathbb{H}} \mu_f : \mu \in \mathcal{M}_c^+(E)\} > \tau$. Since $\tau < \dim_{\mathbb{H}} f(E)$ is arbitrary, we have

$$\dim_{\mathbb{H}} f(E) \leq \sup\{\dim_{\mathbb{H}} \mu_f : \mu \in \mathcal{M}_c^+(E)\}. \quad (3.37)$$

Equation (3.34) now follows from (3.36) and (3.37). $\square$

Proof of Theorem 3.10. First we prove the lower bound:

$$\dim_{\mathbb{H}} B^H(E) \geq s(H, E) \wedge d \quad \text{a.s.} \quad (3.38)$$

For any $0 < \gamma < s(H, E) \wedge d$, there exists a Borel measure μ with compact support in E such that $\gamma < s_{\mu}(H) \wedge d$. Hence we can find $\lambda' = (\lambda'_1, \dots, \lambda'_N) \in \Lambda_{\mu}$ such that $\gamma < \sum_{j=1}^N \frac{\lambda'_j}{H_j} \wedge d$ and

$$\limsup_{r \rightarrow 0} \frac{\mu(R(t, r))}{r^{\langle \lambda', H^{-1} \rangle}} = 0 \quad \text{for } \mu\text{-a.e. } t \in \mathbb{R}_+^N. \quad (3.39)$$

For $\varepsilon > 0$ we define

$$E_{\varepsilon} = \left\{ t \in E : \mu(R(t, r)) \leq r^{\langle \lambda', H^{-1} \rangle} \text{ for all } 0 < r \leq \varepsilon \right\}. \quad (3.40)$$

Then (3.39) implies that $\mu(E_{\varepsilon}) > 0$ if ε is small enough. In order to prove (3.38), it suffices to show $\dim_{\mathbb{H}} B^H(E_{\varepsilon}) \geq \gamma$ a.s.

The proof of the latter is standard: We only need to show

$$\mathbb{E} \int_{E_{\varepsilon}} \int_{E_{\varepsilon}} \frac{\mu(ds) \mu(dt)}{|B^H(t) - B^H(s)|^{\gamma}} < \infty. \quad (3.41)$$

By Lemma 3.3 and the fact that $\gamma < d$, we have

$$\mathbb{E} \int_{E_{\varepsilon}} \int_{E_{\varepsilon}} \frac{\mu(ds) \mu(dt)}{|B^H(t) - B^H(s)|^{\gamma}} \leq c \int_{E_{\varepsilon}} \int_{E_{\varepsilon}} \frac{\mu(ds) \mu(dt)}{(\sum_{j=1}^N |s_j - t_j|^{2H_j})^{\gamma/2}}. \quad (3.42)$$

Let $t \in E_{\varepsilon}$ be fixed and let n_0 be the smallest integer n such that $2^{-n} \leq \varepsilon$. For every $n \geq n_0$, denote

$$D_n = \left\{ s \in \mathbb{R}_+^N : 2^{-(n+1)/H_j} < |s_j - t_j| \leq 2^{-n/H_j} \text{ for all } 1 \leq j \leq N \right\}.$$

Then by (3.40) we have

$$\begin{aligned} \int_{E_{\varepsilon}} \frac{\mu(ds)}{(\sum_{j=1}^N |s_j - t_j|^{2H_j})^{\gamma/2}} &\leq c + \sum_{n=n_0}^{\infty} \int_{D_n} \frac{\mu(ds)}{(\sum_{j=1}^N |s_j - t_j|^{2H_j})^{\gamma/2}} \\ &\leq c + c \sum_{n=n_0}^{\infty} 2^{-n(\sum_{j=1}^N \frac{\lambda'_j}{H_j} - \gamma)} \\ &\leq c_{3,10}. \end{aligned} \quad (3.43)$$

This and (3.42) yield (3.41). So we have proven (3.38).

Now we prove the upper bound in (3.33). It follows from Proposition 3.7 that for any $\mu \in \mathcal{M}_c^+(E)$ we have

$$\dim_{\mathbb{H}} \mu_{B^H} \leq s_{\mu}(H) \wedge d \quad \text{a.s.} \quad (3.44)$$

Hence by (3.32) and (3.34) we derive

$$\dim_{\mathbb{H}} B^H(E) \leq s(H, E) \wedge d \quad \text{a.s.} \quad (3.45)$$

Combining (3.38) and (3.45) finishes the proof. $\square$

Corollary 3.13 *Let $B^{(\alpha)} = \{B^{(\alpha)}(t), t \in \mathbb{R}^N\}$ be an (N, d) -fractional Brownian sheet with Hurst index $H = \langle \alpha \rangle$ and $E \subset (0, \infty)^N$ be a Borel set. Then with probability 1,*

$$\dim_{\mathbb{H}} B^{(\alpha)}(E) = \min \left\{ d, \frac{1}{\alpha} \dim_{\mathbb{H}} E \right\}. \quad (3.46)$$

Proof This is a direct consequence of (3.22) and Theorem 3.10. $\square$

The following theorem gives us a uniform Hausdorff dimension result for the image sets of $B^{(\alpha)}$. It extends the results of Mountford (1989) and Khoshnevisan, Wu and Xiao (2005) for the Brownian sheet.

Theorem 3.14 *If $N \leq \alpha d$, then with probability 1*

$$\dim_{\mathbb{H}} B^{(\alpha)}(E) = \frac{1}{\alpha} \dim_{\mathbb{H}} E \text{ for all Borel sets } E \subset (0, \infty)^N. \quad (3.47)$$

Our proof of Theorem 3.14 is reminiscent to that of Khoshnevisan, Wu and Xiao (2005) for the Brownian sheet. The key step is provided by the following lemma, which will be proven by using the sectorial local nondeterminism of $B^{(\alpha)}$.

Lemma 3.15 *Assume $N \leq \alpha d$ and let $\delta > 0$ and $0 < 2\alpha - \delta < \beta < 2\alpha$ be given constants. Then with probability 1, for all integers n large enough, there do not exist more than $2^{n\delta d}$ distinct points of the form $t^j = 4^{-n} k^j$, where $k^j \in \{1, 2, \dots, 4^n\}^N$, such that*

$$\left| B^{(\alpha)}(t^i) - B^{(\alpha)}(t^j) \right| < 3 \cdot 2^{-n\beta} \quad \text{for } i \neq j. \quad (3.48)$$

Proof Let A_n be the event that there do exist more than $2^{n\delta d}$ distinct points of the form $4^{-n} k^j$ such that (3.48) holds. Let N_n be the number of n -tuples of distinct $t^1, \dots, t^n$ such that (3.48) holds. Then

$$A_n \subseteq \left\{ N_n \geq \binom{[2^{n\delta d} + 1]}{n} \right\}.$$

So,

$$\mathbb{P}(A_n) \leq \frac{\mathbb{E}(N_n)}{\binom{[2^{n\delta d} + 1]}{n}}. \quad (3.49)$$

In order to estimate $\mathbb{E}(N_n)$, we write it as

$$\begin{aligned}\mathbb{E}(N_n) &= \mathbb{E} \left[\underbrace{\sum_{t^1} \sum_{t^2} \cdots \sum_{t^n}}_{\text{distinct}} \mathbf{1}_{\{(3.48) \text{ holds}\}} \right] \\ &= \sum_{\underbrace{t^1, t^2, \dots, t^n}_{\text{distinct}}} \mathbb{P} \left\{ \left| B^{(\alpha)}(t^i) - B^{(\alpha)}(t^j) \right| < 3 \cdot 2^{-n\beta}, \quad \forall i \neq j \leq n \right\}.\end{aligned}\tag{3.50}$$

Now we fix $n-1$ distinct points $t^1, \dots, t^{n-1}$ and estimate the following sum first:

$$\sum_{t^n} \mathbb{P} \left\{ \left| B^{(\alpha)}(t^i) - B^{(\alpha)}(t^j) \right| < 3 \cdot 2^{-n\beta}, \quad \forall i \neq j \leq n \right\}.\tag{3.51}$$

Note that for fixed $t^1 = k^1 4^{-n}, \dots, t^{n-1} = k^{n-1} 4^{-n}$, there are at most $(n-1)^N$ points $\tau^u = (\tau_1^u, \dots, \tau_N^u)$ defined by

$$\tau_\ell^u = t_\ell^j \quad \text{for some } j = 1, \dots, n-1.$$

We denote the collection of τ^u 's by $\Gamma_n = \{\tau^u\}$. Clearly, $t^1, \dots, t^{n-1}$ are all included in Γ_n .

It follows from Theorem 2.1 that, for every $t^n \notin \Gamma_n$, there exists $\tau^{u_n} \in \Gamma_n$ such that

$$\text{Var} \left(B_0^{(\alpha)}(t^n) | B_0^{(\alpha)}(t^1), \dots, B_0^{(\alpha)}(t^{n-1}) \right) \geq c_{3,5} |t^n - \tau^{u_n}|^{2\alpha}.\tag{3.52}$$

In this case, since $B_1^{(\alpha)}, \dots, B_d^{(\alpha)}$ are the independent copies of $B_0^{(\alpha)}$, a standard conditioning argument and (3.52) yield

$$\begin{aligned}& \mathbb{P} \left\{ \left| B^{(\alpha)}(t^i) - B^{(\alpha)}(t^j) \right| < 3 \cdot 2^{-n\beta}, \quad \forall i \neq j \leq n \right\} \\ & \leq \mathbb{P} \left\{ \left| B^{(\alpha)}(t^i) - B^{(\alpha)}(t^j) \right| < 3 \cdot 2^{-n\beta}, \quad \forall i \neq j \leq n-1 \right\} \cdot \left(\frac{3 \cdot 2^{-n\beta}}{c_{3,5}^{1/2} |t^n - \tau^{u_n}|^\alpha} \right)^d.\end{aligned}\tag{3.53}$$

If $t^n \in \Gamma_n$, then we use the trivial bound

$$\begin{aligned}& \mathbb{P} \left\{ \left| B^{(\alpha)}(t^i) - B^{(\alpha)}(t^j) \right| < 3 \cdot 2^{-n\beta}, \quad \forall i \neq j \leq n \right\} \\ & \leq \mathbb{P} \left\{ \left| B^{(\alpha)}(t^i) - B^{(\alpha)}(t^j) \right| < 3 \cdot 2^{-n\beta}, \quad \forall i \neq j \leq n-1 \right\}.\end{aligned}\tag{3.54}$$

Hence, by combining (3.53) and (3.54), we obtain

$$\begin{aligned}& \sum_{t^n} \mathbb{P} \left\{ \left| B^{(\alpha)}(t^i) - B^{(\alpha)}(t^j) \right| < 3 \cdot 2^{-n\beta}, \quad \forall i \neq j \leq n \right\} \\ & \leq \mathbb{P} \left\{ \left| B^{(\alpha)}(t^i) - B^{(\alpha)}(t^j) \right| < 3 \cdot 2^{-n\beta}, \quad \forall i \neq j \leq n-1 \right\} \\ & \quad \times \left[\sum_{t^n \notin \Gamma_n} c_{3,6} \left(\frac{3 \cdot 2^{-n\beta}}{|t^n - \tau^{u_n}|^\alpha} \right)^d + (n-1)^N \right].\end{aligned}\tag{3.55}$$

Note that

$$\begin{aligned}
\sum_{t^n \notin \Gamma_n} \left(\frac{3 \cdot 2^{-n\beta}}{|t^n - \tau^{u_n}|^\alpha} \right)^d &\leq \sum_{\tau^u \in \Gamma_n} \sum_{t^n \neq \tau^u} \left(\frac{3 \cdot 2^{-n\beta}}{|t^n - \tau^u|^\alpha} \right)^d \\
&\leq \sum_{\tau^u \in \Gamma_n} 3^d \cdot 2^{-n\beta d} \sum_{t^n \neq \tau^u} \frac{1}{|t^n - \tau^u|^{\alpha d}} \\
&\leq c_{3,7} (n-1)^{N+1} 2^{n(2\alpha-\beta)d},
\end{aligned} \tag{3.56}$$

where, in deriving the last inequality, we have used the fact that if $N \leq \alpha d$ then for all fixed τ^u ,

$$\sum_{t^n \neq \tau^u} \frac{1}{|t^n - \tau^u|^{\alpha d}} \leq c \cdot 2^{2\alpha n d} n.$$

Plug (3.56) into (3.55), we get

$$\begin{aligned}
&\sum_{t^n} \mathbb{P} \left\{ \left| B^{(\alpha)}(t^i) - B^{(\alpha)}(t^j) \right| < 3 \cdot 2^{-n\beta}, \quad \forall i \neq j \leq n \right\} \\
&\leq \mathbb{P} \left\{ \left| B^{(\alpha)}(t^i) - B^{(\alpha)}(t^j) \right| < 3 \cdot 2^{-n\beta}, \quad \forall i \neq j \leq n-1 \right\} \\
&\quad \times c_{3,8} (n-1)^{N+1} 2^{n(2\alpha-\beta)d}.
\end{aligned} \tag{3.57}$$

Therefore, by iteration, we obtain

$$\begin{aligned}
&\underbrace{\sum_{t^1} \sum_{t^2} \cdots \sum_{t^n}}_{\text{distinct}} \mathbb{P} \left\{ \left| B^{(\alpha)}(t^i) - B^{(\alpha)}(t^j) \right| < 3 \cdot 2^{-n\beta}, \quad \forall i \neq j \leq n \right\} \\
&\leq c_{3,9}^n [(n-1)!]^{N+1} 2^{n^2(2\alpha-\beta)d},
\end{aligned} \tag{3.58}$$

which implies

$$\mathbb{E}(N_n) \leq c_{3,9}^n (n-1)^{n(N+1)} 2^{n^2(2\alpha-\beta)d}. \tag{3.59}$$

By (3.49), (3.58) and the elementary inequality

$$\binom{[2^{n\delta d} + 1]}{n} \geq \left(\frac{2^{n\delta d} + 1}{n} \right)^n \geq \frac{2^{n^2\delta d}}{n^n},$$

we obtain

$$\mathbb{P}(A_n) \leq c_{3,10}^n (n-1)^{n(N+2)} 2^{n^2(2\alpha-\beta-\delta)d}. \tag{3.60}$$

Since $0 < 2\alpha - \beta < \delta$, by (3.60), we get $\sum_n \mathbb{P}(A_n) < \infty$. Hence the Borel-Cantelli Lemma implies that $\mathbb{P}(\overline{\lim}_n A_n) = 0$. This completes the proof of our lemma. $\square$

For $n = 1, 2, \dots$ and $k = (k_1, \dots, k_N)$, where each $k_i \in \{1, 2, \dots, 4^n\}$, define

$$I_k^n = \{t \in [0, 1]^N : (k_i - 1)4^{-n} \leq t_i \leq k_i 4^{-n} \text{ for all } i = 1, \dots, N\}. \tag{3.61}$$

The following lemma is a consequence of Lemma 3.15 and the modulus of continuity of the fractional Brownian sheet [cf. Lemma 3.2].

Lemma 3.16 *Let $\delta > 0$ and $0 < 2\alpha - \delta < \beta < 2\alpha$. Then with probability 1, for all large enough n , there exists no ball $O \subset \mathbb{R}^d$ of radius $2^{-n\beta}$ for which the inverse image*

$$(B^{(\alpha)})^{-1}(O) = \{t \in \mathbb{R}_+^N : B^{(\alpha)}(t) \in O\}$$

intersects more than $2^{n\delta d}$ cubes I_k^n .

Now we are ready to prove Theorem 3.14.

Proof of Theorem 3.14 For simplicity, we shall only prove (3.47) for all Borel sets $E \subseteq (0, 1]^N$. By Lemma 3.2, we know that almost surely $B^{(\alpha)}(t)$ satisfies a uniform Hölder condition on $[0, 1]^N$ of any order smaller than α . It follows from Theorem 6 in Chapter 10 of Kahane (1985a) [or Proposition 2.3 in Falconer (1990)] that

$$\mathbb{P}\left\{\dim_{\mathbb{H}} B^{(\alpha)}(E) \leq \frac{1}{\alpha} \dim_{\mathbb{H}} E \quad \text{for every Borel set } E \subset [0, 1]^N\right\} = 1.$$

To prove the lower bound we need only to show that almost surely for every compact set $F \subseteq \mathbb{R}^d$,

$$\dim_{\mathbb{H}} \{t \in (0, 1]^N : B^{(\alpha)}(t) \in F\} \leq \alpha \dim_{\mathbb{H}} F. \quad (3.62)$$

This follows from Lemma 3.16 and a simple covering argument as in Khoshnevisan, Wu and Xiao (2005). Therefore, the proof of Theorem 3.14 is finished. $\square$

4 Salem set

Let us recall from Kahane (1985a, b) the definitions of Fourier dimension and Salem set. Given a constant $\beta \geq 0$, a Borel set $F \subset \mathbb{R}^d$ is said to be an M_β -set if there exists a probability measure ν on F such that

$$\widehat{\nu}(\xi) = o(|\xi|^{-\beta}) \quad \text{as } \xi \rightarrow \infty. \quad (4.1)$$

Note that if $\beta > d/2$, then (4.1) implies that $\widehat{\nu} \in L^2(\mathbb{R}^d)$ and, consequently, F has positive d -dimensional Lebesgue measure. For any Borel set $F \subset \mathbb{R}^d$, its Fourier dimension $\dim_{\mathbb{F}} F$ is defined by

$$\dim_{\mathbb{F}} F = \sup \left\{ \gamma \geq 0 : F \text{ is an } M_{\gamma/2}\text{-set} \right\}. \quad (4.2)$$

It follows from Frostman's theorem and the following formula for the γ -energy of ν :

$$I_\gamma(\nu) = \int_{\mathbb{R}^d} |\widehat{\nu}(\xi)|^2 |\xi|^{\gamma-d} d\xi \quad (4.3)$$

[see Kahane (1985a, Ch. 10)] that $\dim_{\mathbb{F}} F \leq \dim_{\mathbb{H}} F$ for all Borel sets $F \subset \mathbb{R}^d$.

We say that a Borel set $F \subset \mathbb{R}^d$ is a *Salem set* if $\dim_{\mathbb{F}} F = \dim_{\mathbb{H}} F$. Such sets are of importance in studying the problem of uniqueness and multiplicity for trigonometric series; see Zygmund (1959, Chapter 9) and Kahane and Salem (1994) for further information.

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a centered Gaussian random field with values in $\mathbb{R}^d$. When X is an (N, d) -fractional Brownian motion of index $\gamma \in (0, 1)$, Kahane (1985a, b) studied the asymptotic properties of the Fourier transforms of the image measures of X and proved that, for every Borel set $E \subset \mathbb{R}^N$ with $\dim_{\mathbb{H}} E \leq \gamma d$, $X(E)$ is a Salem set almost surely. Kahane

(1993) further raised the question of studying the Fourier dimensions of other random sets. Recently, Shieh and Xiao (2004) extended Kahane's results to a large class of Gaussian random fields with stationary increments and Khoshnevisan, Wu and Xiao (2005) proved similar results for the Brownian sheet.

In this section, we study the asymptotic properties of the Fourier transforms of the image measures of the (N, d) -fractional Brownian sheet B^H . The main result of this section is Theorem 4.2 below, whose proof depends crucially on the ideas of sectorial local nondeterminism and Hausdorff dimension contour. Moreover, by combining Theorems 3.10 and 4.2 we show that, for every Borel set $E \subset (0, \infty)^N$, $B^H(E)$ is almost surely a Salem set whenever $s(H, E) \leq d$. Recall that $s(H, E)$ is defined in Theorem 3.10.

Let B_0^H be an $(N, 1)$ -fractional Brownian sheet with index $H = (H_1, \dots, H_N)$. Let $0 < \varepsilon < T$ be fixed. For all $n \geq 2$, $t^1, \dots, t^n, s^1, \dots, s^n \in E \subset [\varepsilon, T]^N$, denote $\mathbf{s} = (s^1, \dots, s^n)$, $\mathbf{t} = (t^1, \dots, t^n)$ and

$$\Psi(\mathbf{s}, \mathbf{t}) = \mathbb{E} \left[\sum_{k=1}^n (B_0^H(t^k) - B_0^H(s^k)) \right]^2. \quad (4.4)$$

For $\mathbf{s} \in E^n$ and $r > 0$, we define

$$F(\mathbf{s}, r) = \bigcup_{i_1=1}^n \cdots \bigcup_{i_N=1}^n \bigcap_{j=1}^N \left\{ u \in E : |u_j - s_j^{i_j}| \leq r^{1/H_j} \right\}.$$

This is a union of at most n^N rectangles of side-lengths $2r^{1/H_1}, \dots, 2r^{1/H_N}$, centered at $(s_1^{i_1}, \dots, s_N^{i_N})$. Let

$$G(\mathbf{s}, r) = \{ \mathbf{t} = (t^1, \dots, t^n) : t^k \in F(\mathbf{s}, r) \text{ for } 1 \leq k \leq n \}. \quad (4.5)$$

The following lemma is essential for the proof of Theorem 4.2.

Lemma 4.1 *There exists a positive constant $c_{4,1}$, depending on ε, T, H, N only, such that for all $r \in (0, \varepsilon]$ and all $\mathbf{s}, \mathbf{t} \in E^n$ with $\mathbf{t} \notin G(\mathbf{s}, r)$, we have $\Psi(\mathbf{s}, \mathbf{t}) \geq c_{4,1} r^2$.*

Proof Since $\mathbf{t} \notin G(\mathbf{s}, r)$, there exist $k_0 \in \{1, \dots, n\}$ and $j_0 \in \{1, \dots, N\}$ such that $|t_{j_0}^{k_0} - s_{j_0}^{k_0}| > r^{1/H_{j_0}}$ for all $k = 1, \dots, n$. It follows from (1.3) that

$$\Psi(\mathbf{s}, \mathbf{t}) = K_H^{-2} \int_{\mathbb{R}^N} \left| \sum_{k=1}^n \left(\prod_{j=1}^N (e^{it_j^k \lambda_j} - 1) - \prod_{j=1}^N (e^{is_j^k \lambda_j} - 1) \right) \right|^2 f_H(\lambda) d\lambda, \quad (4.6)$$

where

$$f_H(\lambda) = \prod_{j=1}^N |\lambda_j|^{-2H_j-1}.$$

Let $\delta_j(\cdot) \in C^\infty(\mathbb{R})$ ($1 \leq j \leq N$) be the bump functions in the proof of Theorem 2.1. We define

$$\delta_{j_0}^r(u_{j_0}) = r^{-1/H_{j_0}} \delta_{j_0}(r^{-1/H_{j_0}} u_{j_0})$$

and

$$\delta_j^\varepsilon(u_j) = \varepsilon^{-1} \delta_j(\varepsilon^{-1} u_j), \quad \text{if } j \neq j_0.$$

Then, by using the Fourier inversion formula again, we have

$$\delta_{j_0}^r(u_{j_0}) = (2\pi)^{-1} \int_{\mathbb{R}} e^{-iu_{j_0} \lambda_{j_0}} \widehat{\delta}_{j_0}(r^{1/H_{j_0}} \lambda_{j_0}) d\lambda_{j_0}$$

and similar identities holds for $\delta_j^\varepsilon(u_j)$ with $j \neq j_0$.

Since $r \in (0, \varepsilon)$, we have $\delta_{j_0}^r(t_{j_0}^{k_0}) = 0$ and $\delta_j^\varepsilon(t_j^{k_0}) = 0$ for all $j \neq j_0$. Similarly, $\delta_{j_0}^r(t_{j_0}^{k_0} - s_{j_0}^k) = 0$ for all $k = 1, \dots, n$. Hence,

$$\begin{aligned} J &\equiv \int_{\mathbb{R}^N} \left[\sum_{k=1}^n \left(\prod_{j=1}^N (e^{it_j^k \lambda_j} - 1) - \prod_{j=1}^N (e^{is_j^k \lambda_j} - 1) \right) \right] \\ &\quad \times \prod_{j=1}^N e^{-it_j^{k_0} \lambda_j} \left(\prod_{j \neq j_0}^N \widehat{\delta}_j(\varepsilon \lambda_j) \right) \widehat{\delta}_{j_0}(r^{1/H_{j_0}} \lambda_{j_0}) d\lambda \\ &= (2\pi)^N \sum_{k=1}^n \left(\prod_{j \neq j_0}^N (\delta_j^\varepsilon(t_j^{k_0} - t_j^k) - \delta_j^\varepsilon(t_j^{k_0})) \right) \left(\delta_{j_0}^r(t_{j_0}^{k_0} - t_{j_0}^k) - \delta_{j_0}^r(t_{j_0}^{k_0}) \right) \\ &\quad - (2\pi)^N \sum_{k=1}^n \left(\prod_{j \neq j_0}^N (\delta_j^\varepsilon(t_j^{k_0} - s_j^k) - \delta_j^\varepsilon(t_j^{k_0})) \right) \left(\delta_{j_0}^r(t_{j_0}^{k_0} - s_{j_0}^k) - \delta_{j_0}^r(t_{j_0}^{k_0}) \right) \\ &\geq (2\pi)^N \varepsilon^{-(N-1)} r^{-1/H_{j_0}}. \end{aligned} \tag{4.7}$$

In the above, all the terms in the first sum are non-negative and the second sum equals 0.

On the other hand, by the Cauchy-Schwarz inequality, (4.6) and (4.7), we get

$$\begin{aligned} J^2 &\leq K_H^2 \Psi(\mathbf{t}, \mathbf{s}) \int_{\mathbb{R}^N} \frac{1}{f_H(\lambda)} \prod_{j \neq j_0}^N \left| \widehat{\delta}_j(\varepsilon \lambda_j) \right|^2 \left| \widehat{\delta}_{j_0}(r^{1/H_{j_0}} \lambda_{j_0}) \right|^2 d\lambda \\ &= K_H^2 \Psi(\mathbf{t}, \mathbf{s}) \varepsilon^{-2(N-1)-2 \sum_{j \neq j_0} H_j} r^{-2-2/H_{j_0}} \prod_{j=1}^N \int_{\mathbb{R}} |\lambda_j|^{2H_j+1} \left| \widehat{\delta}_j(\lambda_j) \right|^2 d\lambda_j \\ &= c_{4,2} r^{-2-2/H_{j_0}} \Psi(\mathbf{t}, \mathbf{s}). \end{aligned} \tag{4.8}$$

Square the both sides of (4.7) and combine it with (4.8), the lemma follows. $\square$

For any Borel probability measure μ on $\mathbb{R}_+^N$, let $\nu = \mu_{B^H}$ be the image measure of μ under B^H . The Fourier transform of ν can be written as

$$\widehat{\nu}(\xi) = \int_{\mathbb{R}_+^N} e^{i\langle \xi, B^H(t) \rangle} \mu(dt). \tag{4.9}$$

The following theorem describes the asymptotic behavior of $\widehat{\nu}(\xi)$ as $\xi \rightarrow \infty$. Compared to the results for the fractional Brownian motion and the Brownian sheet mentioned above, the behavior of $\widehat{\nu}(\xi)$ is more delicate.

Theorem 4.2 Let $B^H = \{B^H(t), t \in \mathbb{R}^N\}$ be an (N, d) -fractional Brownian sheet. Assume that, for every $j = 1, \dots, N$, the function $\tau_j : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is non-decreasing such that $\tau_j(0) = 0$ and $\tau_j(2r) \leq c_{4,3} \tau_j(r)$ for all $r \geq 0$ [i.e., τ_j satisfies the doubling property]. If μ is a Borel probability measure on $[\varepsilon, T]^N$ such that

$$\mu(R(t, r)) \leq c_{4,4} \prod_{j=1}^N \tau_j(r^{1/H_j}), \quad \forall t \in \mathbb{R}_+^N, \quad (4.10)$$

where $R(t, r) = \prod_{j=1}^N [t_j - r^{1/H_j}, t_j + r^{1/H_j}]$. Then there exists a positive and finite constant ϱ such that almost surely,

$$\limsup_{|\xi| \rightarrow \infty} \frac{|\widehat{\nu}(\xi)|}{\sqrt{\left(\prod_{j=1}^N \tau_j(|\xi|^{-\frac{1}{H_j}})\right) \log^\varrho |\xi|}} < \infty. \quad (4.11)$$

Proof The argument is similar to that of Kahane (1985a). First note that by considering the restriction of μ on subsets of its support and the linearity of the Fourier transform, we see that, without loss of generality, we may and will assume μ is supported on a Borel set $E \subset [\varepsilon, T]^N$ with $\text{diam} E < \varepsilon^{1/H_1}$ [we have assumed that $H_1 = \min\{H_j : 1 \leq j \leq N\}$]. The reason for this reduction will become clear below.

For any positive integer $n \geq 1$, (4.9) yields

$$\begin{aligned} \mathbb{E}(|\widehat{\nu}(\xi)|^{2n}) &= \mathbb{E} \int_{\mathbb{R}_+^{nN}} \int_{\mathbb{R}_+^{nN}} e^{i\langle \xi, \sum_{k=1}^n (B^H(t^k) - B^H(s^k)) \rangle} \mu^n(ds) \mu^n(dt) \\ &= \int_{\mathbb{R}_+^{nN}} \int_{\mathbb{R}_+^{nN}} \exp\left(-\frac{1}{2}|\xi|^2 \Psi(\mathbf{s}, \mathbf{t})\right) \mu^n(ds) \mu^n(dt), \end{aligned} \quad (4.12)$$

where $\mu^n(ds) = \mu(ds^1) \cdots \mu(ds^n)$.

Let $\mathbf{s} \in [\varepsilon, T]^{nN}$ be fixed and we write

$$\begin{aligned} \int_{\mathbb{R}_+^{nN}} \exp\left(-\frac{1}{2}|\xi|^2 \Psi(\mathbf{s}, \mathbf{t})\right) \mu^n(dt) &= \int_{G(\mathbf{s}, r)} \exp\left(-\frac{1}{2}|\xi|^2 \Psi(\mathbf{s}, \mathbf{t})\right) \mu^n(dt) \\ &\quad + \sum_{m=1}^{\infty} \int_{G(\mathbf{s}, r2^m) \setminus G(\mathbf{s}, r2^{m-1})} \exp\left(-\frac{1}{2}|\xi|^2 \Psi(\mathbf{s}, \mathbf{t})\right) \mu^n(dt). \end{aligned} \quad (4.13)$$

Since μ is supported on E with $\text{diam} E < \varepsilon^{1/H_1}$, the above summation is taken over the integers m such that $r2^m \leq \varepsilon$. Hence we can apply Lemma 4.1 to estimate the integrands.

By (4.10), we always have

$$\int_{G(\mathbf{s}, r)} \exp\left(-\frac{1}{2}|\xi|^2 \Psi(\mathbf{s}, \mathbf{t})\right) \mu^n(dt) \leq \left(c_{4,4} n^N \prod_{j=1}^N \tau_j(r^{1/H_j})\right)^n. \quad (4.14)$$

Given $\xi \in \mathbb{R}^d \setminus \{0\}$, we take $r = |\xi|^{-1}$. It follows from Lemma 4.1, the doubling property of functions τ_j , and (4.10) that

$$\begin{aligned} & \int_{G(\mathbf{s}, r2^m) \setminus G(\mathbf{s}, r2^{m-1})} \exp\left(-\frac{1}{2}|\xi|^2 \Psi(\mathbf{s}, \mathbf{t})\right) \mu^n(d\mathbf{t}) \\ & \leq \exp\left(-\frac{1}{2}c_{4,1}|\xi|^2(r2^{m-1})^2\right) \cdot \left(c_{4,4}n^N \prod_{j=1}^N \tau_j(2^m r^{1/H_j})\right)^n \\ & \leq \left(c_{4,4}n^N \prod_{j=1}^N \tau_j(r^{1/H_j})\right)^n \exp\left(-c_{4,5}2^{2m}\right) \cdot c_{4,3}^{Nmn}. \end{aligned} \quad (4.15)$$

We will also need to use the following elementary inequality:

$$1 + \sum_{m=1}^{\infty} \exp\left(-c_{4,5}2^{2m}\right) \cdot c_{4,3}^{Nmn} \leq c_{4,6}^n n^{\rho n}, \quad (4.16)$$

where $\rho = N/(2 \log c_{4,3})$.

Combining (4.14), (4.15) and (4.16), we derive an upper bound for the integral in (4.13):

$$\int_{\mathbb{R}_+^{nN}} \exp\left(-\frac{1}{2}|\xi|^2 \Psi(\mathbf{s}, \mathbf{t})\right) \mu^n(d\mathbf{t}) \leq c_{4,7}^n \cdot n^{(N+\rho)n} \cdot \left(\prod_{j=1}^N \tau_j(|\xi|^{-\frac{1}{H_j}})\right)^n. \quad (4.17)$$

Integrate the both sides of (4.17) in $\mu^n(ds)$, we get

$$\mathbb{E}(|\widehat{\nu}(\xi)|^{2n}) \leq c_{4,7}^n \cdot n^{\varrho n} \cdot \left(\prod_{j=1}^N \tau_j(|\xi|^{-\frac{1}{H_j}})\right)^n, \quad (4.18)$$

where $\varrho = N + \rho$.

The same argument as in Kahane (1985a, pp. 254–255) using (4.18) and the Borel-Cantelli lemma implies that almost surely

$$\limsup_{z \in \mathbb{Z}^d, |z| \rightarrow \infty} \frac{|\widehat{\nu}(z)|}{\sqrt{\left(\prod_{j=1}^N \tau_j(|z|^{-\frac{1}{H_j}})\right) \log^{\varrho} |z|}} < \infty. \quad (4.19)$$

Therefore (4.11) follows from (4.19) and Lemma 1 of Kahane (1985a, p.252). This finishes the proof of Theorem 4.2. $\square$

Now we determine the Fourier dimension of the image set $B^H(E)$.

Theorem 4.3 *Let $B^H = \{B^H(t), t \in \mathbb{R}^N\}$ be an (N, d) -fractional Brownian sheet with Hurst index $H \in (0, 1)^N$. Then for every Borel set $E \subset (0, \infty)^N$ with $s(H, E) \leq d$, $B^H(E)$ is almost surely a Salem set with Fourier dimension $s(H, E)$.*

Proof It follows from (3.33) and the fact that $\dim_{\mathbb{F}} F \leq \dim_{\mathbb{H}} F$ for all Borel sets $F \subset \mathbb{R}^d$ that for every Borel set $E \subset (0, \infty)^N$ satisfying $s(H, E) \leq d$, we have $\dim_{\mathbb{F}} B^H(E) \leq \dim_{\mathbb{H}} B^H(E) = s(H, E)$ a.s.

To prove the reverse inequality, it suffices to show that if $s(H, E) \leq d$ then for all $\gamma \in (0, s(H, E))$ we have $\dim_{\mathbb{F}} B^H \geq \gamma$ a.s.

Note that for any $0 < \gamma < s(H, E)$, there exists a Borel probability measure μ with compact support in E such that $\gamma < s_{\mu}(H)$. Hence we can find $\lambda' = (\lambda'_1, \dots, \lambda'_N) \in \Lambda_{\mu}$ such that $\gamma < \sum_{j=1}^N \frac{\lambda'_j}{H_j}$ and (3.39) holds. Let μ_{ε} be the restriction of μ to the set E_{ε} defined by (3.40). Then μ_{ε} satisfies the condition (4.10) with $\tau_j(r) = r^{\lambda'_j}$ ($j = 1, \dots, N$).

Let ν be the image measure of μ_{ε} under B^H . Then by Theorem 4.2 we have almost surely,

$$|\widehat{\nu}(\xi)| = O\left(\sqrt{|\xi|^{-\sum_{j=1}^N \frac{\lambda'_j}{H_j} \log^{\varrho} |\xi|}}\right), \quad \text{as } \xi \rightarrow \infty. \quad (4.20)$$

This and (4.2) imply that $\dim_{\mathbb{F}} B^H(E) \geq \sum_{j=1}^N \frac{\lambda'_j}{H_j}$ a.s., which yields $\dim_{\mathbb{F}} B^H(E) \geq \gamma$ a.s. Therefore $\dim_{\mathbb{F}} B^H(E) = \dim_{\mathbb{H}} B^H(E)$, i.e., $B^H(E)$ is a Salem set. $\square$

Applying Theorems 4.2 and 4.3 to $B^{(\alpha)}$, we have the following corollary.

Corollary 4.4 *Let $B^{(\alpha)} = \{B^{(\alpha)}(t), t \in \mathbb{R}^N\}$ be an (N, d) -fractional Brownian sheet, and let $\tau : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a non-decreasing function satisfying $\tau(0) = 0$ and the doubling property. If μ is a probability measure on $[\varepsilon, T]^N$ such that*

$$\mu(B(x, r)) \leq c_{4,8} \tau(2r), \quad \forall x \in \mathbb{R}_+^N, \quad r \geq 0. \quad (4.21)$$

Then there exists a positive and finite constant ϱ such that

$$\limsup_{|\xi| \rightarrow \infty} \frac{|\widehat{\nu}(\xi)|}{\sqrt{\tau(|\xi|^{-\frac{1}{\alpha}} \log^{\varrho} |\xi|)}} < \infty, \quad \text{a.s.} \quad (4.22)$$

Moreover, for every Borel set $E \subset (0, \infty)^N$ with $\dim_{\mathbb{H}} E \leq \alpha d$, $B^{(\alpha)}(E)$ is almost surely a Salem set with Fourier dimension $\frac{1}{\alpha} \dim_{\mathbb{H}} E$.

5 Interior points

By using the Fourier analytic argument of Kahane (1985a), it is easy to show the following: If a Borel set $E \subset (0, \infty)^N$ carries a probability measure μ such that

$$\int_E \int_E \frac{1}{\left(\sum_{j=1}^N |s_j - t_j|^{2H_j}\right)^{d/2}} \mu(ds) \mu(dt) < \infty, \quad (5.1)$$

then almost surely, $B^H(E)$ has positive d -dimensional Lebesgue measure. In particular, it follows from the proof of Theorem 3.10 that, if $E \subset (0, \infty)^N$ satisfies $s(H, E) > d$, then $B^H(E)$ has positive d -dimensional Lebesgue measure. It is a natural question to further ask when $B^H(E)$ has interior points. This question for Brownian motion was first considered

by Kaufman (1975), and then extended by Pitt (1978) and Kahane (1985a, b) to fractional Brownian motion and by Khoshnevisan and Xiao (2004) to the Brownian sheet. Recently, Shieh and Xiao (2004) proved similar results under more general conditions for a large class of Gaussian random fields.

In the following, we prove that a condition similar to that in Shieh and Xiao (2004) is sufficient for $B^H(E)$ to have interior points almost surely. This theorem extends and improves the result of Khoshnevisan and Xiao (2004) mentioned above.

Theorem 5.1 *Let $B^H = \{B^H(t), t \in \mathbb{R}^N\}$ be an (N, d) -fractional Brownian sheet with index $H \in (0, 1)^N$. If a Borel set $E \subset (0, \infty)^N$ carries a probability measure μ such that*

$$\sup_{t \in \mathbb{R}_+^N} \int_{\mathbb{R}_+^N} \frac{1}{\left(\sum_{j=1}^N |s_j - t_j|^{2H_j}\right)^{d/2}} \log_+^{(N+1)\gamma} \left(\frac{1}{\sum_{j=1}^N |s_j - t_j|^{2H_j}} \right) \mu(ds) \leq c_{5,1} \quad (5.2)$$

for some finite constants $c_{5,1} > 0$ and $\gamma > N$, where $\log_+ x = \max\{1, \log x\}$. Then $B^H(E)$ has interior points almost surely.

From Theorem 5.1 we derive the following corollaries.

Corollary 5.2 *If $E \subset (0, \infty)^N$ is a Borel set with $s(H, E) > d$, then $B^H(E)$ a.s. has interior points.*

Proof It follows from the proof of Theorem 3.10 that, if $s(H, E) > d$, then there is a Borel probability measure μ on E satisfying (5.2). Hence the conclusion follows from Theorem 5.1. $\square$

Corollary 5.3 *Let $B^{(\alpha)} = \{B^{(\alpha)}(t), t \in \mathbb{R}^N\}$ be an (N, d) -fractional Brownian sheet with Hurst index $H = \langle \alpha \rangle$. If a Borel set $E \subset (0, \infty)^N$ carries a probability measure μ such that*

$$\sup_{t \in \mathbb{R}_+^N} \int_{\mathbb{R}_+^N} \frac{1}{|s - t|^{\alpha d}} \log_+^{(N+1)\gamma} \left(\frac{1}{|s - t|} \right) \mu(ds) \leq c_{5,2} \quad (5.3)$$

for some finite constants $c_{5,2} > 0$ and $\gamma > N$. Then $B^{(\alpha)}(E)$ has interior points almost surely.

The existence of interior points in $B^H(E)$ is related to the regularity of the local times of B^H on E . Let us recall briefly the definition of local time of B^H on E . For any Borel probability measure μ on E , let μ_{B^H} be the image measure of μ . If μ_{B^H} is absolutely continuous with respect to the Lebesgue measure m_d in $\mathbb{R}^d$, then B^H is said to have a local time on E . The local time $l_\mu(x)$ is defined to be the Radon–Nikodým derivative $d\mu_{B^H}/dm_d(x)$ and it satisfies the following occupation density formula: for all Borel measurable functions $f : \mathbb{R}^d \rightarrow \mathbb{R}_+$,

$$\int_E f(B^H(s)) \mu(ds) = \int_{\mathbb{R}^d} f(x) l_\mu(x) dx. \quad (5.4)$$

It is known from Geman and Horowitz (1980) and Kahane (1985a) that, when (5.1) holds, then $l_\mu(x) \in L^2(\mathbb{R}^d)$ a.s.

In order to prove Theorem 5.1, we will need to use the following continuity lemma of Garsia (1972).

Lemma 5.4 Assume that $p(u)$ and $\Psi(u)$ are two positive increasing functions on $[0, \infty)$, $p(u) \downarrow 0$ as $u \downarrow 0$, $\Psi(u)$ is convex and $\Psi(u) \uparrow \infty$ as $u \uparrow \infty$. Let D denote an open hypercube in $\mathbb{R}^d$. If the function $f(x) : D \mapsto \mathbb{R}$ is measurable and

$$A := A(D, f) = \int_D \int_D \Psi \left(\frac{|f(x) - f(y)|}{p(|x - y|/\sqrt{d})} \right) dx dy < \infty, \quad (5.5)$$

then after modifying $f(x)$ on a set of Lebesgue measure 0, we have

$$|f(x) - f(y)| \leq 8 \int_0^{|x-y|} \Psi^{-1} \left(\frac{A}{u^{2d}} \right) dp(u) \quad \text{for all } x, y \in D. \quad (5.6)$$

We take the function $p(u)$ in Garsia's lemma as follows: Let γ be the constant in (5.2) and define

$$p(u) = \begin{cases} 0, & \text{if } u = 0, \\ \log^{-\gamma} \left(\frac{e}{u} \right), & \text{if } 0 < u \leq 1, \\ \gamma u - \gamma + 1, & \text{if } u > 1. \end{cases} \quad (5.7)$$

Clearly, the function $p(u)$ is strictly increasing on $[0, \infty)$ and $p(u) \downarrow 0$ as $u \downarrow 0$.

Proof of Theorem 5.1 First note that, since μ is a Borel probability measure on E , without loss of generality, we can assume that E is compact. Hence there are constants $0 < \varepsilon < T < \infty$ such that $E \subseteq [\varepsilon, T]^N$. Since $B^H(E)$ is a compact subset of $\mathbb{R}^d$, (5.4) implies that $\{x : l_\mu(x) > 0\}$ is a subset of $B^H(E)$. Hence, in order to prove our theorem, it is sufficient to prove that the local time $l_\mu(x)$ has a version which is continuous in x ; see Pitt (1978, p.324) or Geman and Horowitz (1980, p.12). This will be proved by deriving moment estimates for the local time l_μ and by applying Garsia's continuity lemma.

Secondly, as in Khoshnevisan and Xiao (2004), we may and will assume that the Borel probability measure μ in (5.2) has the following property: For any constant $c > 0$ and $\ell = 1, \dots, N$,

$$\mu \{t = (t_1, \dots, t_N) \in E : t_\ell = c\} = 0. \quad (5.8)$$

Otherwise, we can replace B^H by an $(N-1, d)$ -fractional Brownian sheet $\tilde{B}^H$ and prove the desired conclusion for $\tilde{B}^H(E_\ell)$, where E_ℓ is the set in (5.8) with positive μ -measure.

Consider the set

$$\tilde{E}_n = \left\{ \tilde{\mathbf{t}} = (t^1, \dots, t^n) \in E^n : t_\ell^j = t_\ell^i \text{ for some } i \neq j \text{ and } 1 \leq \ell \leq N \right\}. \quad (5.9)$$

It follows from (5.8) and the Fubini–Tonelli theorem that $\mu^n(\tilde{E}_n) = 0$.

The following lemma provides estimates on high moments of the local time, which is the key for finishing the proof of Theorem 5.1.

Lemma 5.5 Let μ be a Borel probability measure on $E \subset [\varepsilon, T]^N$ satisfying (5.2) and (5.8) and let $p(u)$ be defined by (5.7). Then for every hypercube $D \subset \mathbb{R}^d$ there exists a finite constant $c_{5,3} > 0$, depending on N , d , γ , μ and D only, such that for all even integers $n \geq 2$,

$$\mathbb{E} \int_D \int_D \left(\frac{l_\mu(x) - l_\mu(y)}{p(|x - y|/\sqrt{d})} \right)^n dx dy \leq c_{5,3}^n (n!)^N \log^{n(N+1)\gamma} n. \quad (5.10)$$

We now continue with the proof of Theorem 5.1 and defer the proof of Lemma 5.5 to the end of this section.

Let $\Psi(u) = u \exp(u^\theta)$, where $\theta \in (\frac{1}{\gamma}, \frac{1}{N})$ is a constant. Then Ψ is increasing and convex on $(0, \infty)$. It follows from Jensen's inequality, the Fubini–Tonelli theorem and Lemma 5.5 that for all closed hypercubes $D \subset \mathbb{R}^d$ and all integers n with $\theta + 1/n < 1$,

$$\begin{aligned} & \mathbb{E} \int_D \int_D \left(\frac{|l_\mu(x) - l_\mu(y)|}{p(|x - y|/\sqrt{d})} \right)^{n\theta+1} dx dy \\ & \leq (m_d(D))^{2-\theta-1/n} \left\{ \mathbb{E} \int_D \int_D \left(\frac{|l_\mu(x) - l_\mu(y)|}{p(|x - y|/\sqrt{d})} \right)^n dx dy \right\}^{\theta+1/n} \\ & \leq c^n (n!)^{N(\theta+1/n)} (\log n)^{n(N+1)\gamma(\theta+1/n)} \\ & \leq c_{5,4}^n (n!)^{N\theta} \log^{n(N+1)\gamma\theta} n, \end{aligned} \tag{5.11}$$

where $c_{5,4}$ is a finite constant depending on N, d, θ, D and $c_{5,3}$ only.

Expanding $\Psi(u)$ into a power series and applying the inequality (5.11), we derive

$$\begin{aligned} & \mathbb{E} \int_D \int_D \Psi \left(\frac{|l_\mu(x) - l_\mu(y)|}{p(|x - y|/\sqrt{d})} \right) dx dy \\ & = \sum_{n=0}^{\infty} \frac{1}{n!} \mathbb{E} \int_D \int_D \left(\frac{|l_\mu(x) - l_\mu(y)|}{p(|x - y|/\sqrt{d})} \right)^{n\theta+1} dx dy \\ & \leq c_{5,5} < \infty, \end{aligned} \tag{5.12}$$

the last inequality follows from the fact that $N\theta < 1$. Hence Garsia's lemma implies that there are positive and finite random variables A_1 and A_2 such that for almost all $x, y \in D$ with $|x - y| \leq e^{-1}$,

$$\begin{aligned} |l_\mu(x) - l_\mu(y)| & \leq \int_0^{|x-y|} \Psi^{-1} \left(\frac{A_1}{u^{2d}} \right) dp(u) \\ & \leq A_2 \left[\log(1/|x - y|) \right]^{-(\gamma-1/\theta)}. \end{aligned}$$

Note that, by our choice of θ , we have $\theta > 1/\gamma$ and hence B^H has almost surely a local time $l_\mu(x)$ on E that is continuous for all $x \in D$. Finally, by taking a sequence of closed hypercubes $\{D_n, n \geq 1\}$ such that $\mathbb{R}^d = \cup_{n=1}^{\infty} D_n$, we have proved that almost surely $l_\mu(x)$ is continuous for all $x \in \mathbb{R}^d$. This completes the proof of Theorem 5.1. $\square$

It remains to prove Lemma 5.5. Our proof relies on the sectorial local nondeterminism of B^H and on an argument which improves those in Khoshnevisan and Xiao (2004) and Shieh and Xiao (2004).

We will need several lemmas. Lemma 5.6 is essentially due to Cuzick and DuPreez (1982), where the extra condition on g is dropped in Khoshnevisan and Xiao (2004). Lemma 5.7 is a slight modification of Lemma 4 in Cuzick and DuPreez (1982).

Lemma 5.6 *Let Z_k ($k = 1, \dots, n$) be linearly independent centered Gaussian variables. If $g : \mathbb{R} \rightarrow \mathbb{R}_+$ is a Borel measurable function, then*

$$\int_{\mathbb{R}^n} g(v_1) e^{-\frac{1}{2}\text{Var}(\langle v, Z \rangle)} dv = \frac{(2\pi)^{(n-1)/2}}{Q^{1/2}} \int_{-\infty}^{\infty} g(z/\sigma_1) e^{-\frac{1}{2}z^2} dz, \quad (5.13)$$

where $\sigma_1^2 = \text{Var}(Z_1 | Z_2, \dots, Z_n)$ and $Q = \det \text{Cov}(Z_1, \dots, Z_n)$ is the determinant of the covariance matrix of $Z_1, \dots, Z_n$.

Lemma 5.7 *If $\alpha \geq e^2/2$, then*

$$\int_1^{\infty} \log^{\alpha} x \exp\left(-\frac{x^2}{2}\right) dx \leq \sqrt{\pi} \log^{\alpha} \alpha. \quad (5.14)$$

Consider the non-decreasing function $\Lambda(u) = 2 \min\{1, u\}$ on $[0, \infty)$. Later we will make use of the elementary inequality

$$|e^{iu} - 1| \leq \Lambda(|u|), \quad \forall u \in \mathbb{R}. \quad (5.15)$$

Lemma 5.8 *Assume $h(y)$ is any positive and non-decreasing function on $[0, \infty)$ such that $h(0) = 0$, $y^n/h^n(y)$ is non-decreasing on $[0, 1]$, and $\int_1^{\infty} h^{-2}(y) dy < \infty$. Then there exists a constant $c_{5,6}$ such that for all integers $n \geq 1$ and $v \in (0, \infty)$,*

$$\int_0^{\infty} \frac{\Lambda^n(vy)}{h^n(y)} dy \leq c_{5,6}^n h_+^{-n}\left(\frac{1}{v}\right), \quad (5.16)$$

where $h_+(y) = \min\{1, h(y)\}$ so that $h_+^{-n}(y) = \max\{1, h^{-n}(y)\}$.

Proof The proof is the same as that of Lemma 3 in Cuzick and DuPreez (1982). $\square$

The following result is about the function $p(u)$ defined by (5.7).

Lemma 5.9 *Let $p(u)$ be defined as in (5.7). Then for all $\sigma > 0$ and integers $n \geq 1$,*

$$\int_0^{\infty} p_+^{-n}\left(\frac{\sigma}{v}\right) \exp\left(-\frac{v^2}{2}\right) dv \leq c_{5,7}^n \left[\log^{n\gamma} n + \log_+^{n\gamma}\left(\frac{e}{\sigma}\right) \right]. \quad (5.17)$$

Proof Since

$$p_+^{-n}(x) = \begin{cases} \log^{n\gamma}\left(\frac{e}{x}\right), & \text{if } 0 < x < 1, \\ 1 & \text{if } x \geq 1 \end{cases}$$

and $\log_+^{\alpha}(xy) \leq 2^{\alpha}(\log_+^{\alpha} x + \log_+^{\alpha} y)$ for all $\alpha \geq 0$, we deduce that the integral in (5.17) is at most

$$\begin{aligned} & \int_{\sigma/v \geq 1} \exp\left(-\frac{v^2}{2}\right) dv + 2^{n\gamma} \int_{\sigma/v < 1} \log_+^{n\gamma}(v) \exp\left(-\frac{v^2}{2}\right) dv \\ & + 2^{n\gamma} \int_{\sigma/v < 1} \log_+^{n\gamma}\left(\frac{e}{\sigma}\right) \exp\left(-\frac{v^2}{2}\right) dv. \end{aligned} \quad (5.18)$$

It follows from (5.18) and Lemma 5.7 that

$$\begin{aligned} \int_0^\infty p_+^{-n} \left(\frac{\sigma}{v} \right) \exp\left(-\frac{v^2}{2}\right) dv &\leq c^n \left[\log^{n\gamma}(n\gamma) + \log_+^{n\gamma} \left(\frac{e}{\sigma} \right) \right] \\ &\leq c_{5,7}^n \left[\log^{n\gamma} n + \log_+^{n\gamma} \left(\frac{e}{\sigma} \right) \right]. \end{aligned} \quad (5.19)$$

This completes the proof of Lemma 5.9. $\square$

Now we proceed to prove Lemma 5.5.

Proof of Lemma 5.5 By (25.7) in Geman and Horowitz (1980), we have that for every $x, y \in \mathbb{R}^d$, and all even integers $n \geq 2$,

$$\begin{aligned} \mathbb{E}[(l_\mu(x) - l_\mu(y))^n] &= (2\pi)^{-nd} \int_{E^n} \int_{\mathbb{R}^{nd}} \prod_{k=1}^n \left[e^{-i\langle u^k, x \rangle} - e^{-i\langle u^k, y \rangle} \right] \\ &\quad \times \exp \left[-\frac{1}{2} \text{Var} \left(\sum_{k=1}^n \langle u^k, B^H(t^k) \rangle \right) \right] d\mathbf{u} \mu^n(d\mathbf{t}). \end{aligned} \quad (5.20)$$

In the above, $\mathbf{u} = (u^1, \dots, u^n)$, $u^k \in \mathbb{R}^d$ for each $k = 1, \dots, n$ and we will write it coordinate-wise as $u^k = (u_1^k, \dots, u_d^k)$.

Note that for $u^1, \dots, u^n$, $y \in \mathbb{R}^d$, the triangle inequality implies

$$\begin{aligned} \prod_{k=1}^n \left| \exp(-i\langle u^k, y \rangle) - 1 \right| &\leq \prod_{k=1}^n \left| \sum_{j=1}^d \left[\exp\left(-i \sum_{\ell=0}^j u_\ell^k y_\ell\right) - \exp\left(-i \sum_{\ell=0}^{j-1} u_\ell^k y_\ell\right) \right] \right| \\ &\leq \prod_{k=1}^n \left[\sum_{j=1}^d \left| \exp(-i u_j^k y_j) - 1 \right| \right] \\ &= \sum' \prod_{k=1}^n \left| \exp(-i u_{j_k}^k y_{j_k}) - 1 \right|, \end{aligned} \quad (5.21)$$

where $y_0 = u_0^j = 0$ in the first inequality and the last summation $\sum'$ is taken over all sequences $(j_1, \dots, j_n) \in \{1, \dots, d\}^n$.

It follows from (5.20), (5.21), (5.15) and the Fubini–Tonelli theorem that for any fixed hypercube $D \subset \mathbb{R}^d$ and any even integer $n \geq 2$, we have

$$\begin{aligned} \mathbb{E} \int_D \int_D \left(\frac{l_\mu(x) - l_\mu(y)}{p(|x - y|/\sqrt{d})} \right)^n dx dy &\leq \sum' \int_D \int_D \int_{E^n} \int_{\mathbb{R}^{nd}} \prod_{k=1}^n \frac{|\exp(i u_{j_k}^k (y_{j_k} - x_{j_k})) - 1|}{p(|y - x|/\sqrt{d})} \\ &\quad \times \exp \left[-\frac{1}{2} \text{Var} \left(\sum_{k=1}^n \langle u^k, B^H(t^k) \rangle \right) \right] d\mathbf{u} \mu^n(d\mathbf{t}) dx dy \\ &\leq m_d(D) \sum' \int_{D \oplus D} \int_{E^n} \int_{\mathbb{R}^{nd}} \prod_{k=1}^n \frac{\Lambda(|u_{j_k}^k y_{j_k}|)}{p(|y|/\sqrt{d})} \\ &\quad \times \exp \left[-\frac{1}{2} \text{Var} \left(\sum_{k=1}^n \langle u^k, B^H(t^k) \rangle \right) \right] d\mathbf{u} \mu^n(d\mathbf{t}) dy. \end{aligned} \quad (5.22)$$

In the above, we have made a change of variables and $D \ominus D = \{x - y : x, y \in D\}$. By our assumptions on μ , we see that the integral in (5.22) with respect to μ^n can be taken over the set $E^n \setminus \tilde{E}_n$, where $\tilde{E}_n$ is defined by (5.9).

Now we fix $\mathbf{t} \in E^n \setminus \tilde{E}_n$, a sequence $\mathbf{j} = (j_1, \dots, j_n) \in \{1, \dots, d\}^n$ and define $\mathcal{M}_n(\mathbf{t}) \equiv \mathcal{M}_n(\mathbf{j}, \mathbf{t})$ by

$$\mathcal{M}_n(\mathbf{t}) = \int_{D \ominus D} \int_{\mathbb{R}^{nd}} \prod_{k=1}^n \frac{\Lambda(|u_{j_k}^k y_{j_k}|)}{p(|y|/\sqrt{d})} \exp\left[-\frac{1}{2} \text{Var}\left(\sum_{k=1}^n \langle u^k, B^H(t^k) \rangle\right)\right] d\mathbf{u} dy. \quad (5.23)$$

Then the last integral in (5.22) corresponding to the sequence $\mathbf{j} = (j_1, \dots, j_n)$ can be written as

$$\mathcal{N}_{\mathbf{j}} \equiv \int_{E^n \setminus \tilde{E}_n} \mathcal{M}_n(\mathbf{t}) \mu^n(dt). \quad (5.24)$$

We will estimate the above integral by integrating in the order $\mu(dt^n), \mu(dt^{n-1}), \dots, \mu(dt^1)$. For this purpose, we need to derive an upper bound for $\mathcal{M}_n(\mathbf{t})$. Observe that for any positive numbers $\beta_1, \dots, \beta_n$ satisfying $\sum_{k=1}^n \beta_k = n$, we can write

$$\mathcal{M}_n(\mathbf{t}) = \int_{D \ominus D} \int_{\mathbb{R}^{nd}} \prod_{k=1}^n \frac{\Lambda(|u_{j_k}^k y_{j_k}|)}{p^{\beta_k}(|y|/\sqrt{d})} \exp\left[-\frac{1}{2} \text{Var}\left(\sum_{k=1}^n \langle u^k, B^H(t^k) \rangle\right)\right] d\mathbf{u} dy. \quad (5.25)$$

Later it will be clear that the flexibility in choosing β_k in (5.25) is essential to our proof. More precisely, by choosing the constants β_k ($1 \leq k \leq n$) appropriately, we minimize the effect of “bad points” [see (5.37) below].

For any n points $t^1, \dots, t^n \in E^n \setminus \tilde{E}_n$, Lemma 2.3 implies that the Gaussian random variables $B_j^H(t^k)$ ($j = 1, \dots, d$, $k = 1, \dots, n$) are linearly independent. By applying the generalized Hölder’s inequality, Lemma 5.6 and a change of variables, we see that $\mathcal{M}_n(\mathbf{t})$ is bounded by

$$\begin{aligned} & \prod_{k=1}^n \left\{ \int_{D \ominus D} \int_{\mathbb{R}^{nd}} \frac{\Lambda^n(|u_{j_k}^k y_{j_k}|)}{p^{n\beta_k}(|y|/\sqrt{d})} \exp\left[-\frac{1}{2} \text{Var}\left(\sum_{k=1}^n \sum_{\ell=1}^d u_{\ell}^k B_{\ell}^H(t^k)\right)\right] d\mathbf{u} dy \right\}^{1/n} \\ &= \frac{c_{5,8}^{nd}}{[\det \text{Cov}(B_0^H(t^1), \dots, B_0^H(t^n))]^{d/2}} \\ & \quad \times \prod_{k=1}^n \left\{ \int_{D \ominus D} \int_{\mathbb{R}} \frac{\Lambda^n(|u_{j_k}^k y_{j_k}|/\sigma_k)}{p^{n\beta_k}(|y|/\sqrt{d})} \exp\left(-\frac{(u_{j_k}^k)^2}{2}\right) du_{j_k}^k dy \right\}^{1/n}, \end{aligned} \quad (5.26)$$

where, for every $1 \leq k \leq n$, $\sigma_k^2 \equiv \sigma_k^2(\mathbf{t})$ is the conditional variance of $B_{j_k}^H(t^k)$ given $B_{\ell}^H(t^m)$ ($\ell \neq j_k$ and $1 \leq m \leq n$, or $\ell = j_k$ and $m \neq k$).

Denote the n integrals in the last product of (5.26) by $\mathcal{J}_1, \dots, \mathcal{J}_n$, respectively. In order to estimate them, we will apply the sectorial local nondeterminism of B_0^H . Since $B_1^H, \dots, B_d^H$ are independent copies of B_0^H , we have

$$\sigma_k^2(\mathbf{t}) = \text{Var}\left(B_0^H(t^k) \mid \{B_0^H(t^m)\}_{m \neq k}\right). \quad (5.27)$$

It follows from Theorem 2.1 that for every $1 \leq k \leq n$,

$$\sigma_k^2(\mathbf{t}) \geq c_{2,1} \sum_{\ell=1}^N \min_{m \neq k} |t_{\ell}^m - t_{\ell}^k|^{2H_{\ell}}. \quad (5.28)$$

In order to estimate the sum in (5.28) as a function of t^n , we introduce N permutations $\Gamma_1, \dots, \Gamma_N$ of $\{1, \dots, n-1\}$ such that for every $\ell = 1, \dots, N$,

$$t_\ell^{\Gamma_\ell(1)} < t_\ell^{\Gamma_\ell(2)} < \dots < t_\ell^{\Gamma_\ell(n-1)}. \quad (5.29)$$

This is possible since t_ℓ^k ($1 \leq k \leq n-1$, $1 \leq \ell \leq N$) are distinct. For convenience, we denote $t_\ell^{\Gamma_\ell(0)} = \varepsilon$ and $t_\ell^{\Gamma_\ell(n)} = T$ for all $1 \leq \ell \leq N$.

For every sequence $(i_1, \dots, i_N) \in \{1, \dots, n-1\}^N$, let $\tau_{i_1, \dots, i_N} = (t_1^{\Gamma_1(i_1)}, \dots, t_N^{\Gamma_N(i_N)})$ be the “center” of the rectangle

$$I_{i_1, \dots, i_N} = \prod_{\ell=1}^N \left[t_\ell^{\Gamma_\ell(i_\ell)} - \frac{1}{2}(t_\ell^{\Gamma_\ell(i_\ell)} - t_\ell^{\Gamma_\ell(i_\ell-1)}), t_\ell^{\Gamma_\ell(i_\ell)} + \frac{1}{2}(t_\ell^{\Gamma_\ell(i_\ell+1)} - t_\ell^{\Gamma_\ell(i_\ell)}) \right) \quad (5.30)$$

with the convention that the left-end point of the interval is ε whenever $i_\ell = 1$; and the interval is closed and its right-end is T whenever $i_\ell = n-1$. Thus the rectangles $\{I_{i_1, \dots, i_N}\}$ form a partition of $[\varepsilon, T]^N$.

For every $t^n \in E$, let $I_{i_1, \dots, i_N}$ be the unique rectangle containing t^n . Then (5.28) yields the following lower bound for $\sigma_n^2(\mathbf{t})$:

$$\sigma_n^2(\mathbf{t}) \geq c_{2,1} \sum_{\ell=1}^N \left| t_\ell^n - t_\ell^{\Gamma_\ell(i_\ell)} \right|^{2H_\ell}. \quad (5.31)$$

For every $k = 1, \dots, n-1$, we say that $I_{i_1, \dots, i_N}$ *cannot see* t^k *from direction* ℓ if

$$t_\ell^k \notin \left[t_\ell^{\Gamma_\ell(i_\ell)} - \frac{1}{2}(t_\ell^{\Gamma_\ell(i_\ell)} - t_\ell^{\Gamma_\ell(i_\ell-1)}), t_\ell^{\Gamma_\ell(i_\ell)} + \frac{1}{2}(t_\ell^{\Gamma_\ell(i_\ell+1)} - t_\ell^{\Gamma_\ell(i_\ell)}) \right]. \quad (5.32)$$

We emphasize that if $I_{i_1, \dots, i_N}$ cannot see t^k from all N directions, then

$$|t_\ell^k - t_\ell^n| \geq \frac{1}{2} \min_{m \neq k, n} |t_\ell^k - t_\ell^m| \quad \text{for all } 1 \leq \ell \leq N. \quad (5.33)$$

Thus t^n does not contribute to the sum in (5.28). More precisely, the latter means that

$$\sigma_k^2(\mathbf{t}) \geq c_{5,9} \sum_{\ell=1}^N \min_{m \neq k, n} |t_\ell^k - t_\ell^m|^{2H_\ell}. \quad (5.34)$$

The right hand side of (5.34) only depends on $t^1, \dots, t^{n-1}$, which will be denoted by $\tilde{\sigma}_k^2(\mathbf{t})$. Because of this, t^k is called a “good” point for $I_{i_1, \dots, i_N}$ when (5.32) holds for every $\ell = 1, \dots, N$.

Let $1 \leq k \leq n-1$. If $I_{i_1, \dots, i_N}$ sees the point t^k from a direction and $t^k \neq \tau_{i_1, \dots, i_N}$, then it is impossible to control $\sigma_k^2(\mathbf{t})$ from below as in (5.31) or (5.34). We say that t^k is a “bad” point for $I_{i_1, \dots, i_N}$ [Note that by definition $n \notin \Theta_{i_1, \dots, i_N}^n$]. It is important to note that, because of (5.29), the rectangle $I_{i_1, \dots, i_N}$ can only have at most N bad points t^k ($1 \leq k \leq n-1$), i.e., at most one in each direction. We denote the set of bad points for $I_{i_1, \dots, i_N}$ by

$$\Theta_{i_1, \dots, i_N}^n = \{1 \leq k \leq n-1 : t^k \text{ is a bad point for } I_{i_1, \dots, i_N}\}$$

and denote its cardinality by $\#(\Theta_{i_1, \dots, i_N}^n)$. Then $\#(\Theta_{i_1, \dots, i_N}^n) \leq N$.

Now we choose the constants $\beta_1, \dots, \beta_n$ [they depend on the sequence $(i_1, \dots, i_N)$] as follows: $\beta_k = 0$ if t^k is a bad point for $I_{i_1, \dots, i_N}$; $\beta_k = 1$ if t^k is a good point for $I_{i_1, \dots, i_N}$ and

$$\beta_n = 1 + \#(\Theta_{i_1, \dots, i_N}^n).$$

Clearly, $\beta_n \leq N + 1$.

By Lemma 5.8 and Lemma 5.9, we have

$$\begin{aligned} \mathcal{J}_n &= \int_{D \ominus D} \int_{\mathbb{R}} \frac{\Lambda^n(|u_{j_n}^n y_{j_n}| / \sigma_n(\mathbf{t}))}{p^{n\beta_n}(|y|/\sqrt{d})} \exp\left(-\frac{(u_{j_n}^n)^2}{2}\right) du_{j_n}^n dy \\ &\leq c \int_{\mathbb{R}} \exp\left(-\frac{v^2}{2}\right) dv \int_0^\infty \frac{\Lambda^n(v y_{j_n} / \sigma_n(\mathbf{t}))}{p^{n\beta_n}(y_{j_n}/\sqrt{d})} dy_{j_n} \\ &\leq c_{5,10}^n \int_{\mathbb{R}} p_+^{-n\beta_n}\left(\frac{\sigma_n(\mathbf{t})}{v}\right) \exp\left(-\frac{v^2}{2}\right) dv \\ &\leq c_{5,11}^n \left[\log^{(N+1)\gamma} n + \log_+^{(N+1)\gamma} \left(\frac{e}{\sigma_n(\mathbf{t})} \right) \right]. \end{aligned} \quad (5.35)$$

In the above, we have also used the fact that $p(|y|/\sqrt{d}) \geq p(|y_j|/\sqrt{d})$ for all $j = 1, \dots, d$.

If t^k is a good point for $I_{i_1, \dots, i_N}$, then by the monotonicity of the function Λ we have

$$\begin{aligned} \mathcal{J}_k &= \int_{D \ominus D} \int_{\mathbb{R}} \frac{\Lambda^n(|u_{j_k}^k y_{j_k}| / \sigma_k(\mathbf{t}))}{p^{n\beta_k}(|y|/\sqrt{d})} \exp\left(-\frac{(u_{j_k}^k)^2}{2}\right) du_{j_k}^k dy \\ &\leq \int_{D \ominus D} \int_{\mathbb{R}} \frac{\Lambda^n(u_{j_k}^k y_{j_k} / \tilde{\sigma}_k(\mathbf{t}))}{p^n(|y|/\sqrt{d})} \exp\left(-\frac{(u_{j_k}^k)^2}{2}\right) du_{j_k}^k dy. \end{aligned} \quad (5.36)$$

When t^k is a bad points for $I_{i_1, \dots, i_N}$, we use the inequality $\Lambda(u) \leq 2$ to obtain

$$\mathcal{J}_k \leq 2^n \int_{D \ominus D} \int_{\mathbb{R}} \exp\left(-\frac{(u_{j_k}^k)^2}{2}\right) du_{j_k}^k dy \leq c_{5,12}^n. \quad (5.37)$$

Since there are at most N bad points for $I_{i_1, \dots, i_N}$, their total contribution to $\mathcal{M}_n(\mathbf{t})$ is bounded by a constant $c_{5,13}$, which depends on D , d and N only.

Combining (5.26), (5.35), (5.36) and (5.37), we derive that

$$\begin{aligned} \mathcal{M}_n(\mathbf{t}) &\leq \frac{c_{5,14}^n}{[\det \text{Cov}(B_0^H(t^1), \dots, B_0^H(t^n))]^{d/2}} \left[\log^{(N+1)\gamma} n + \log_+^{(N+1)\gamma} \left(\frac{e}{\sigma_n(\mathbf{t})} \right) \right] \\ &\quad \times \prod_{k \notin \Theta_{i_1, \dots, i_N}^n} \left\{ \int_{D \ominus D} \int_{\mathbb{R}} \frac{\Lambda^n(u_{j_k}^k y_{j_k} / \tilde{\sigma}_k(\mathbf{t}))}{p^n(|y|/\sqrt{d})} \exp\left(-\frac{(u_{j_k}^k)^2}{2}\right) du_{j_k}^k dy \right\}^{1/n} \\ &= \frac{c_{5,15}^n}{[\det \text{Cov}(B_0^H(t^1), \dots, B_0^H(t^{n-1}))]^{d/2}} \\ &\quad \times \prod_{k \notin \Theta_{i_1, \dots, i_N}^n} \left\{ \int_{D \ominus D} \int_{\mathbb{R}} \frac{\Lambda^n(u_{j_k}^k y_{j_k} / \tilde{\sigma}_k(\mathbf{t}))}{p^n(|y|/\sqrt{d})} \exp\left(-\frac{(u_{j_k}^k)^2}{2}\right) du_{j_k}^k dy \right\}^{1/n} \\ &\quad \times \frac{1}{\sigma_n(\mathbf{t})^d} \left[\log^{(N+1)\gamma} n + \log_+^{(N+1)\gamma} \left(\frac{e}{\sigma_n(\mathbf{t})} \right) \right]. \end{aligned} \quad (5.38)$$

Note that Condition (5.2) implies

$$\begin{aligned}
& \int_{I_{i_1, \dots, i_N}} \frac{1}{\sigma_n(\mathbf{t})^d} \left[\log^{(N+1)\gamma} n + \log_+^{(N+1)\gamma} \left(\frac{e}{\sigma_n(\mathbf{t})} \right) \right] \mu(dt^n) \\
& \leq c_{5,16} \int_{I_{i_1, \dots, i_N}} \frac{1}{\left(\sum_{\ell=1}^N |t_\ell^n - t_\ell^{\Gamma_\ell(i_\ell)}|^{2H_\ell} \right)^{d/2}} \\
& \quad \times \left[\log^{(N+1)\gamma} n + \log^{(N+1)\gamma} \left(\frac{1}{\sum_{\ell=1}^N |t_\ell^n - t_\ell^{\Gamma_\ell(i_\ell)}|^{2H_\ell}} \right) \right] \mu(dt^n) \\
& \leq c_{5,17} \log^{(N+1)\gamma} n.
\end{aligned} \tag{5.39}$$

Integrating $\mathcal{M}_n(\mathbf{t})$ as a function of t_n with respect to μ on $I_{i_1, \dots, i_N}$ and using (5.38) and (5.39), we obtain

$$\begin{aligned}
\int_{I_{i_1, \dots, i_N}} \mathcal{M}_n(\mathbf{t}) \mu(dt^n) & \leq \frac{c_{5,18}^n \log^{(N+1)\gamma} n}{[\det \text{Cov}(B_0^H(t^1), \dots, B_0^H(t^{n-1}))]^{d/2}} \\
& \quad \times \prod_{k \notin \Theta_{i_1, \dots, i_N}^n} \left\{ \int_{D \ominus D} \int_{\mathbb{R}} \frac{\Lambda^n(u_{jk}^k y_{jk} / \tilde{\sigma}_k(\mathbf{t}))}{p^n(|y|/\sqrt{d})} \exp\left(-\frac{(u_{jk}^k)^2}{2}\right) du_{jk}^k dy \right\}^{1/n}.
\end{aligned} \tag{5.40}$$

It is important that the right hand side of (5.40) depends on $t^1, \dots, t^{n-1}$ only and is similar to (5.26).

Summing (5.40) over all the sequences $(i_1, \dots, i_N) \in \{1, \dots, n-1\}^N$, we derive that the integral $\mathcal{N}_j$ in (5.24) is bounded by

$$\begin{aligned}
& c_{5,18}^n \log^{(N+1)\gamma} n \sum_{i_1, \dots, i_N} \int_{E^{n-1}} \frac{1}{[\det \text{Cov}(B_0^H(t^1), \dots, B_0^H(t^{n-1}))]^{d/2}} \\
& \quad \times \prod_{k \notin \Theta_{i_1, \dots, i_N}^n} \left\{ \int_{D \ominus D} \int_{\mathbb{R}} \frac{\Lambda^n(u_{jk}^k y_{jk} / \tilde{\sigma}_k(\mathbf{t}))}{p^n(|y|/\sqrt{d})} \exp\left(-\frac{(u_{jk}^k)^2}{2}\right) du_{jk}^k dy \right\}^{1/n} \mu(dt^{n-1}) \dots \mu(dt^1).
\end{aligned} \tag{5.41}$$

Note that, for different sequences $(i_1, \dots, i_N)$, the index sets $\Theta_{i_1, \dots, i_N}^n$ may be the same. We say that a set $\Theta^n \subseteq \{1, \dots, n-1\}$ is *admissible* if it is the set of bad points for some $I_{i_1, \dots, i_N}$. It can be seen that every admissible set Θ^n has the following properties:

- (i) $\#(\Theta^n) \leq N$ [recall that there are at most N bad points for each $I_{i_1, \dots, i_N}$];
- (ii) Denote by $\chi(\Theta^n)$ the number of sequences $(i_1, \dots, i_N)$ such that $\Theta_{i_1, \dots, i_N}^n = \Theta^n$. If $\#(\Theta^n) = p$, then $\chi(\Theta^n) \leq c n^{N-p}$.

It follows from (i), (ii) and an elementary combinatorics argument that

$$\sum_{\Theta^n} \chi(\Theta^n) = \sum_{p=1}^N \sum_{\#(\Theta^n)=p} \chi(\Theta^n) \leq c_{5,19} n^N, \tag{5.42}$$

where the first summation is taken over all admissible sets $\Theta^n \subseteq \{1, \dots, n-1\}$.

By regrouping $\Theta_{i_1, \dots, i_N}^n$ in (5.41), we can rewrite

$$\begin{aligned} \mathcal{N}_{\mathbf{j}} &\leq c_{5,20}^n \log^{(N+1)\gamma} n \sum_{\Theta^n} \chi(\Theta^n) \int_{E^{n-1}} \frac{1}{[\det \text{Cov}(B_0^H(t^1), \dots, B_0^H(t^{n-1}))]^{d/2}} \\ &\quad \times \prod_{k \notin \Theta^n} \left\{ \int_{D \ominus D} \int_{\mathbb{R}} \frac{\Lambda^n(u_{j_k}^k y_{j_k} / \tilde{\sigma}_k(\mathbf{t}))}{p^n(|y|/\sqrt{d})} \exp\left(-\frac{(u_{j_k}^k)^2}{2}\right) du_{j_k}^k dy \right\}^{1/n} \mu(dt^{n-1}) \dots \mu(dt^1), \end{aligned} \quad (5.43)$$

where, as in (5.42), the summation is taken over all admissible sets $\Theta^n \subseteq \{1, \dots, n-1\}$.

We now carry out the procedure iteratively. In order to simplify the computation, we will make some further reductions:

- (iii) Increasing the number of elements in Θ^n changes only the integrals in (5.43) by a constant factor [recall (5.37) and the fact that we have used $\#(\Theta^n) \leq N$ in deriving the first inequality in (5.38)]. Hence we may just consider the admissible sets Θ^n with $\#(\Theta^n) = N$ and
- (iv) Since $\det \text{Cov}(B_0(t^1), \dots, B_0(t^{n-1}))$ is symmetric in $t^1, \dots, t^{n-1}$, we can further assume $\Theta^n = \{1, \dots, N\}$.

Based on the above observations we can repeat the preceding argument and integrate $\mu(dt^{n-1})$ [we define $n - N - 1$ constants β_k , $k \in \{N+1, \dots, n-1\}$ accordingly] and then, in the same way, continue to integrate $\mu(dt^{n-2}), \dots, \mu(dt^1)$ respectively. We obtain

$$\begin{aligned} \mathcal{N}_{\mathbf{j}} &\leq c_{5,21}^n \log^{n(N+1)\gamma} n \sum_{\Theta^n} \dots \sum_{\Theta^1} \chi(\Theta^n) \dots \chi(\Theta^1) \\ &\leq c_{5,22}^n (n!)^N \log^{n(N+1)\gamma} n, \end{aligned} \quad (5.44)$$

where last inequality follows from (5.42) and, moreover, the positive constant $c_{5,22}$ is independent of $\mathbf{j}$.

By combining (5.22), (5.23), (5.24) and (5.44) we derive that

$$\mathbb{E} \int_D \int_D \left(\frac{l_\mu(x) - l_\mu(y)}{p(|x-y|/\sqrt{d})} \right)^n dx dy \leq c_{5,23}^n (n!)^N \log^{n(N+1)\gamma} n. \quad (5.45)$$

This finishes the proof of Lemma 5.5. □

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