Packing Dimension, Hausdorff Dimension and Cartesian Product Sets *

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Abstract

We show that the dimension adim introduced by R. Kaufman (1987) coincides with the packing dimension Dim, but the dimension aDim introduced by Hu and Taylor (1994) is different from the Hausdorff dimension. These results answer questions raised by Hu and Taylor.

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1 Introduction

Hausdorff dimension and packing dimension are two of the most important fractal dimensions used in analyzing fractal sets (see [3] [13] [14] [15] [16] [17] and references therein). There has been a lot of interest in studying the Hausdorff dimension and packing dimension of cartesian product sets (cf. [1] [2] [10] [15]). In [15], Tricot proved that for any $E, \quad F \subseteq \mathbf{R^N}$,

$$\dim E + \dim F \leq \dim(E \times F) \leq \dim E + \mathrm{Dim} F$$

$$\leq \mathrm{Dim}(E \times F) \leq \mathrm{Dim} E + \mathrm{Dim} F \ , \tag{1.1}$$

where $\dim E$ and $\mathrm{Dim} E$ are the Hausdorff dimension and the packing dimension of E respectively. By using (1.1), Kaufman ([9]) introduced a dimension adim defined by

$$\mathrm{adim} E = \sup\{\dim(E \times F) - \dim F \ , \quad F \in \mathcal{B}(\mathbf{R})\} \ ,$$

where $\mathcal{B}(\mathbf{R})$ is the Borel σ-algebra in $\mathbf{R}$. Similarly, Hu and Taylor ([7]) defined another dimension aDim by

$$\mathrm{aDim} E = \inf\{\mathrm{Dim}(E \times F) - \mathrm{Dim} F \ , \quad F \in \mathcal{B}(\mathbf{R})\}.$$

Clearly,

$$\dim E \leq \mathrm{adim} E \leq \mathrm{Dim} E, \quad \dim E \leq \mathrm{aDim} E \leq \mathrm{Dim} E.$$

Hu and Taylor ([7]) asked whether for $E \subseteq \mathbf{R}$ the followings are true:

$$\mathrm{adim} E = \mathrm{Dim} E, \tag{1.2}$$

$$\mathrm{aDim} E = \dim E. \tag{1.3}$$

The objective of this paper is to answer these questions. In section 3, We prove that for every Borel set $E \subseteq \mathbf{R}$, (1.2) holds. Section 4 deals with aDim. Theorem 4.1 proves the following result which strengthens the third inequality in (1.1): for any compact set $E \subseteq \mathbf{R}$ and any Borel (or analytic) set $F \subseteq \mathbf{R}$,

$$\widehat{\delta}(E) + \mathrm{Dim} F \leq \mathrm{Dim}(E \times F) \ . \tag{1.4}$$

where $\widehat{\delta}(E)$ is the modified lower box-counting dimension of E (cf. [3] [15]). Inequality (1.4) implies that for every compact set $E \subseteq \mathbf{R}$, $\mathrm{aDim} E \geq \widehat{\delta}(E)$.

Therefore, for any compact set E with $\dim E < \widehat{\delta}(E)$ (see [15] for an example), (1.3) does not hold. Theorem 4.2 proves that for any bounded set $E \subseteq \mathbf{R}$,

$$\mathrm{aDim} E \leq \delta(E) \ .$$

and we give an example showing that the strict inequality can hold.

2 Preliminaries

First we recall briefly the definitions of Hausdorff dimension, packing dimension and modified box-counting dimension. Let Φ be the class of functions $\phi : (0,\delta) \to (0,1)$ which are right continuous, monotone increasing with $\phi(0+) = 0$ and such that there exists a finite constant $K > 0$ for which

$$\frac{\phi(2s)}{\phi(s)} \leq K, \quad \text{for } 0 < s < \frac{1}{2}\delta \ .$$

For $\phi \in \Phi$, $E \subseteq \mathbf{R^N}$, the ϕ-Hausdorff measure of E is defined by

$$\phi\text{-}m(E) = \lim_{\epsilon \to 0} \ \inf\{\sum_i \phi(2r_i) \ , \quad E \subseteq \cup_i B(x_i, r_i) \ , \ r_i \leq \epsilon\} \ ,$$

where $B(x,r)$ denotes the open ball of radius r centered at x. It is known that ϕ-m is a metric outer measure , and so every Borel set is ϕ-m measurable (see [11] for a proof). If $\phi(s) = s^\alpha$, then s^α-$m(E)$ is called the α-dimensional Hausdorff measure of E. The Hausdorff dimension of E is defined by

$$\begin{aligned} \dim E &= \inf\{\alpha > 0 : \quad s^\alpha\text{-}m(E) = 0\} \\ &= \sup\{\alpha > 0 : \quad s^\alpha\text{-}m(E) = +\infty\} \ . \end{aligned}$$

For more properties of Hausdorff measure and Hausdorff dimension, we refer to [3].

In [16], Taylor and Tricot defined the set function ϕ-$P(E)$ on R^N by

$$\phi\text{-}P(E) = \lim_{\epsilon \to 0} \ \sup\{\sum_i \phi(2r_i) : \overline{B}(x_i, r_i) \text{ are disjoint, } x_i \in E, \ r_i < \epsilon\} \ .$$

ϕ-P is not an outer measure because it fails to be countably subadditive. However, ϕ-P is a premeasure, so one can obtain a metric outer measure ϕ-p on $\mathbf{R^N}$ by defining

$$\phi\text{-}p(E) = \inf\{\sum_n \phi\text{-}P(E_n) : \quad E \subseteq \cup_n E_n\} \ .$$

ϕ-$p(E)$ is called the ϕ-packing measure of E. If $\phi(s) = s^\alpha$, then s^α-$p(E)$ is sometimes called the α-dimensional packing measure of E. The packing dimension of E is defined by

$$\begin{aligned}\mathrm{Dim}E &= \inf\{\alpha > 0 : \quad s^\alpha\text{-}p(E) = 0\} \\ &= \sup\{\alpha > 0 : \quad s^\alpha\text{-}p(E) = +\infty\} \ .\end{aligned}$$

For any $\epsilon > 0$ and any bounded set $E \subseteq \mathbf{R^N}$, let

$$N_1(E,\epsilon) = \text{ smallest number of balls of radius } \epsilon \text{ needed to cover } E$$

and

$$N_2(E,\epsilon) = \text{ largest number of disjoint balls of radius } \epsilon \text{ with centers in } \ E \ .$$

Then we have

$$N_2(E,\epsilon) \le N_1(E,\epsilon) \le N_2(E,\frac{1}{2}\epsilon) \ .$$

To simplify the notations, we write $N(E,\epsilon)$ for $N_1(E,\epsilon)$ or $N_2(E,\epsilon)$ and define

$$\Delta(E) = \limsup_{\epsilon \to 0} \frac{\log N(E,\epsilon)}{-\log \epsilon} \ , \tag{2.1}$$

$$\delta(E) = \liminf_{\epsilon \to 0} \frac{\log N(E,\epsilon)}{-\log \epsilon} \ . \tag{2.2}$$

$\Delta(E)$ and $\delta(E)$ are called the upper and lower box-counting dimension of E ([3]) or the upper and lower entropy index of E (cf. [14]). The indices Δ and δ are not σ-stable (cf. [15] [3]). We can obtain σ-stable indices $\widehat{\Delta}$ and $\widehat{\delta}$ by letting

$$\widehat{\Delta}(E) = \inf\{\sup \Delta(E_n) : \quad E \subseteq \cup_n E_n\} \ , \tag{2.3}$$

$$\widehat{\delta}(E) = \inf\{\sup \delta(E_n) : \quad E \subseteq \cup_n E_n\} \ . \tag{2.4}$$

According to [3], we call $\widehat{\Delta}(E)$ and $\widehat{\delta}(E)$ the <u>modified</u> upper box-counting dimension of E and the <u>modified</u> lower box-counting dimension of E respectively. It is easy to see that dimE $\le \delta$(E) and $\widehat{\delta}(E) \le \widehat{\Delta}(E)$. In [15], Tricot proved that $\mathrm{Dim}E = \widehat{\Delta}(E)$. Hence, for any set $E \subseteq \mathbf{R^N}$,

$$0 \le \mathrm{dim}E \le \widehat{\delta}(E) \le \widehat{\Delta}(E) = \mathrm{Dim}E \le N \ .$$

(see [13] for another proof of $\mathrm{dim}E \le \mathrm{Dim}E$).

Let μ be a Borel measure on $\mathbf{R}^{\mathbf{N}}$. For any $\phi \in \Phi$ and any $x \in \mathbf{R}^{\mathbf{N}}$, the upper and lower ϕ-densities of μ at x are defined by

$$\overline{D}_{\mu}^{\phi}(x) = \limsup_{r \to 0} \frac{\mu(B(x,r) \cap E)}{\phi(2r)} \ ,$$

$$\underline{D}_{\mu}^{\phi}(x) = \liminf_{r \to 0} \frac{\mu(B(x,r) \cap E)}{\phi(2r)} \ .$$

If $\phi(s) = s^{\alpha}$, then we write $\overline{D}_{\mu}^{\phi}$ and $\underline{D}_{\mu}^{\phi}$ as $\overline{D}_{\mu}^{\alpha}$ and $\underline{D}_{\mu}^{\alpha}$ respectively.

The following lemma is a special case of the density theorems for Hausdorff measure and packing measure (see [12] [13] [16]).

Lemma 2.1 *Let μ be a probability Borel measure on $\mathbf{R}^{\mathbf{N}}$. Consider $E \subseteq \mathbf{R}^{\mathbf{N}}$ with $\mu(E) > 0$.*

(i) If there exists a constant $c_1 > 0$ such that

$$\sup_{x \in E} \overline{D}_{\mu}^{\alpha}(x) \leq c_1 \ ,$$

then $\dim E \geq \alpha$.

(ii) If there exists a constant $c_2 > 0$ such that

$$\sup_{x \in E} \underline{D}_{\mu}^{\alpha}(x) \leq c_2 \ ,$$

then $\mathrm{Dim} E \geq \alpha$.

Lemma 2.2 *For any sets $E, \ \ F \subseteq \mathbf{R}^{\mathbf{N}}$,*

$$\dim E + \dim F \leq \dim(E \times F) \leq \dim E + \mathrm{Dim} F$$

$$\leq \mathrm{Dim}(E \times F) \leq \mathrm{Dim} E + \mathrm{Dim} F.$$

Remark. The first inequality was proved by Marstrand ([10]) for any $E, \ F \subseteq \mathbf{R}$ and was studied earlier by Besicovitch and Moran ([1]) and Eggleston ([2]) under some extra hypothesis on E and F. The inequalities involving Dim were first proved by Tricot ([15]). The proof of the third inequality given in [15] is insufficient. But it can be proved for any Borel (or analytic) sets E and F by using Lemma 2.1 or a result of Haase ([5]). For a proof of the general (non-analytic) case, see [6].

The following lemma is proved in [4] (see also [8]).

Lemma 2.3 *Let $E \subseteq \mathbf{R}^{\mathbf{N}}$ be any analytic set. Then for any $\gamma < \mathrm{Dim}E$, there exists a compact set K such that $K \subseteq E$ and $\mathrm{Dim}K > \gamma$.*

Recall from [15] that Δ is uniform on E means that there exists a constant c such that for any $x \in \overline{E}$,

$$\lim_{r\to 0} \Delta(E \cap B(x,r)) = c \ .$$

Lemma 2.4 is proved in [15] (see also [3] Lemma 3.6).

Lemma 2.4 *If E is compact and Δ is uniform on E, then $\Delta(E) = \mathrm{Dim}E$. More generally, if $\Delta(E \cap U) \geq \gamma$ for every open ste U that intersects E, then $\mathrm{Dim}E \geq \gamma$.*

3 Packing dimension and Cartesian product sets

In [9], Kaufman introduced the dimension adim by using the Hausdorff dimension of cartesian product sets,

$$\mathrm{adim}E = \sup\{\dim(E \times F) - \dim F, \quad F \in \mathcal{B}(\mathbf{R})\}.$$

Clearly $\dim E \leq \mathrm{adim}E \leq \mathrm{Dim}E$. Hu and Taylor ([7]) asked whether for any set $E \subseteq \mathbf{R}$,

$$\mathrm{adim}E = \mathrm{Dim}E \ . \tag{3.1}$$

In this section, we prove that, for every Borel set $E \subseteq \mathbf{R}$, (3.1) holds.

A compact set E_γ is called a Cantor-type set if $E_\gamma = \cap_{n=1}^{\infty} E_n$, where

$$E_n = \cup_{i_1=1}^{N_0} \cdots \cup_{i_n=1}^{N_{i_1\cdots i_{n-1}}} I_{i_1\cdots i_n} \quad (n = 1,\ 2,\ \cdots)$$

is a decreasing sequence of compact sets and for each $n \geq 1$, $I_{i_1\cdots i_n}$ ($i_n = 1, \cdots, N_{i_1\cdots i_{n-1}}$) are disjoint closed subintervals of $I_{i_1\cdots i_{n-1}}$.

Lemma 3.1 *Let $E \subseteq \mathbf{R}$ be compact. For any $\gamma < \mathrm{Dim}E$, there exists a Cantor-type set $E_\gamma = \cap_{n=1}^{\infty} E_n$ with $E_\gamma \subseteq E$ and satisfies the following properties*

(i) $E_1 = \cup_{i_1=1}^{N_0} I_{i_1}$, *where*

$$N_0 = \left[\frac{1}{\eta_0^{\gamma}}\right] + 1 \quad (0 < \eta_0 < (\frac{1}{2})^{\frac{1}{\gamma}}) \ , \tag{3.2}$$

$[x]$ is the integer part of x , and $\{I_{i_1}\}$ $(i_1 = 1, \cdots, N_0)$ are closed intervals of length $2\delta_0 \le \frac{1}{2}\eta_0$ with gap between any two of them greater than η_0.

(ii) For $n \ge 2$, $E_n = \cup_{i_1=1}^{N_0} \cdots \cup_{i_n=1}^{N_{i_1\cdots i_{n-1}}} I_{i_1\cdots i_n}$, where

$$N_{i_1\cdots i_{n-1}} = \left[\frac{1}{\eta^{\gamma}_{i_1\cdots i_{n-1}}}\right] + 1 \ , \tag{3.3}$$

$$|I_{i_1\cdots i_n}| = 2\delta_{i_1\cdots i_{n-1}} \le \frac{1}{2}\eta_{i_1\cdots i_{n-1}} \ .$$

*Each interval $I_{i_1\cdots i_{n-1}}$ of E_{n-1} contains $N_{i_1\cdots i_{n-1}}$ closed subintervals $I_{i_1\cdots i_n}$ $(i_n = 1, 2, \cdots,$
$N_{i_1\cdots i_{n-1}})$ with gaps greater than $\eta_{i_1\cdots i_{n-1}}$.*

(iii) There exists a Borel probability measure σ on $\mathbf{R}$ with $\sigma(E_\gamma) = 1$ such that, for each interval I_{i_1} $(i_1 = 1, \cdots, N_0)$,

$$\sigma(I_{i_1}) = N_0^{-1} \ ,$$

and for each interval $I_{i_1\cdots i_n}$ in E_n,

$$\sigma(I_{i_1\cdots i_n}) = \sigma(I_{i_1\cdots i_{n-1}})N^{-1}_{i_1\cdots i_{n-1}} \ . \tag{3.4}$$

(iv) For every open set $V \subseteq \mathbf{R}$ that intersects E_γ, we have $\mathrm{Dim}(E_\gamma \cap V) \ge$ *γ.*

Proof. (i) - (iii) were proved in [18]. We include the main steps of the proof for the convenience of readers. Fix $\gamma < \gamma' < \mathrm{Dim}E$ and let

$$\mathcal{F} = \{I : I \text{ is a rational closed interval with } \mathrm{Dim}(I \cap E) \le \gamma'\}.$$

Then $\mathcal{F}$ is countable and by the σ-stability of Dim, we have

$$\mathrm{Dim}F = \mathrm{Dim}E,$$

where $F = E \backslash \cup_{I\in\mathcal{F}} I$. Observe that for any $x \in F$ and for any $r > 0$,

$$\mathrm{Dim}(F \cap [x - r, x + r]) > \gamma'. \tag{3.5}$$

Since $\Delta(F) \ge \mathrm{Dim}F > \gamma'$, there exists an $0 < \eta_0 < (\frac{1}{2})^{\frac{1}{\gamma}}$ such that

$$N(F, \eta_0) > (\frac{1}{\eta_0})^{\gamma}.$$

Let $x_1, \cdots, x_{N_0}$ be $N_0 = [(\frac{1}{\eta_0})^\gamma] + 1$ points in F with $|x_j - x_k| \geq 2\eta_0$ for $j \neq k$. If we choose $\delta_0 < \frac{1}{4}\eta_0$, then the intervals $I_{i_1} = [x_{i_1} - \delta_0, x_{i_1} + \delta_0]$ $(i_1 = 1, \cdots, N_0)$ are disjoint and by (3.5)

$$\text{Dim}(F \cap I_{i_1}) > \gamma' \quad \text{for every} \quad 1 \leq i_1 \leq N_0 \ . \tag{3.6}$$

Let $E_1 = \cup_{i_1=1}^{N_0} I_{i_1}$; then (ii) can be proved by using (3.6) and induction, (iii) follows from the mass distribution principle (see [3]) and (iv) follows from Lemma 2.4. □

Remark. By the construction of E_γ, once $\eta_{i_1 \cdots i_{n-1}}$ has been chosen, we can choose $\eta_{i_1 \cdots i_{n-1} i_n}$ as small as we please. In particular, we can choose η's satisfying

$$\frac{\eta_{i_1 \cdots i_{n-1}}}{\eta_{i_1 \cdots i_{n-1} i_n}^{1-\gamma}} \geq n + 1 \ . \tag{3.7}$$

Now we prove the main result of this section.

Theorem 3.1 *Let $E \subseteq \mathbf{R}$ be a Borel set. Then*

$$\text{Dim}E = \sup\{\dim(E \times F) - \dim F, \quad F \in \mathcal{B}(\mathbf{R})\}.$$

Proof. By Lemma 2.2, it is clear that

$$\text{Dim}E \geq \sup\{\dim(E \times F) - \dim F, \quad F \in \mathcal{B}(\mathbf{R})\} \ .$$

To prove the reverse inequality, it is sufficient to show that for any $\gamma <$ $\text{Dim}E$, there exists a set $F \subseteq [0, 1]$ such that

$$\dim F \leq 1 - \gamma, \tag{3.8}$$

$$\dim(E \times F) \geq 1. \tag{3.9}$$

By Lemma 2.3, we can and will assume that $E \subseteq \mathbf{R}$ is compact. Fix a $\gamma < \text{Dim}E$, let E_γ be the Cantor-type set in Lemma 3.1 with η's satisfying (3.7) and let σ be the Borel probability measure in Lemma 3.1 (iii). Now we construct a decreasing sequence of closed subsets $F_n \subseteq [0, 1]$ inductively and then define $F = \cap_{n=1}^\infty F_n$. Naturally, the construction of $\{F_n\}$ depends on the structure of $\{E_n\}$. To simplify the notations, from now on, we will not distinguish a positive number from its integer part.

The basic principle for the construction of F_n is, for each $I_{i_1 \cdots i_{n-1}}$ in E_{n-1}, we construct $F_{n, i_1 \cdots i_{n-1}}$ and then let F_n to be the union of them. In this way, we can make $\dim(E \times F)$ large and keep $\dim F$ small.

For $n = 1$, let

$$b_1 = \frac{1}{{\eta_0}^{1-\gamma}} \ , \tag{3.10}$$

where η_0 is the constant in (3.2), and let $F_1 = \cup_{j_1=1}^{b_1} J_{j_1}$, where J_{j_1} ($j_1 = 1, \cdots, b_1$) are closed subintervals of $[0, 1]$ of length η_0 with gaps at least $\tilde{\eta_0}$ and

$$\tilde{\eta_0} \approx \frac{1}{2}\eta_0^{1-\gamma},$$

where $a \approx b$ means $\frac{1}{2}b \le a \le b$. F_1 is well defined since

$$b_1(\tilde{\eta_0} + \eta_0) \le \frac{1}{2} + \eta_0^{\gamma} < 1.$$

By (3.10), we have

$$\sum_{j_1=1}^{b_1} |J_{j_1}|^{1-\gamma} = 1.$$

For $n = 2$, by Lemma 3.1 (ii), $E_2 = \cup_{i_1=1}^{N_0} \cup_{i_2=1}^{N_{i_1}} I_{i_1 i_2}$. In order to construct F_2, we construct F_{2,i_1} for $i_1 = 1, \cdots, N_0$, and then define $F_2 = \cup_{i_1=1}^{N_0} F_{2,i_1}$.

For each fixed i_1, to construct F_{2,i_1}, let

$$b_{2,i_1} = \frac{\eta_0}{{\eta_{i_1}}^{1-\gamma}} \ .$$

In each interval J_{j_1} of F_1, we construct b_{2,i_1} closed subintervals $J_{j_1 j_2}^{(i_1)}$ of length η_{i_1} with gaps at least $\tilde{\eta}_{i_1}$ and

$$\tilde{\eta}_{i_1} \approx \frac{1}{2}\eta_{i_1}^{1-\gamma} \ .$$

This is possible since

$$b_{2,i_1}(\tilde{\eta}_{i_1} + \eta_{i_1}) \le (\frac{1}{2} + \eta_{i_1}^{\gamma}) \cdot \eta_0 < \eta_0.$$

We set

$$F_{2,i_1} = \cup_{j_1=1}^{b_1} \cup_{j_2=1}^{b_{2,i_1}} J_{j_1 j_2}^{(i_1)}$$

and

$$F_2 = \cup_{i_1=1}^{N_0} F_{2,i_1} \ .$$

Then

$$\begin{aligned} \sum_{i_1=1}^{N_0} \sum_{j_1=1}^{b_1} \sum_{j_2=1}^{b_{2,i_1}} |\ J_{j_1 j_2}^{(i_1)}\ |^{1-\gamma} &= \sum_{i_1=1}^{N_0} b_1 b_{2,i_1} \eta_{i_1}^{1-\gamma} \\ &= \sum_{i_1=1}^{N_0} \frac{1}{\eta_0^{1-\gamma}} \cdot \frac{\eta_0}{\eta_{i_1}^{1-\gamma}} \cdot \eta_{i_1}^{1-\gamma} \\ &= 1 \ . \end{aligned}$$

Suppose now that

$$F_{n-1} = \cup_{i_1=1}^{N_0} \cdots \cup_{i_{n-2}=1}^{N_{i_1\cdots i_{n-3}}} F_{n-1,i_1\cdots i_{n-2}}$$

has been constructed with

$$F_{n-1,i_1\cdots i_{n-2}} = \cup_{j_1=1}^{b_1} \cdots \cup_{j_{n-1}=1}^{b_{n-1,i_1\cdots i_{n-2}}} J_{j_1\cdots j_{n-1}}^{(i_1\cdots i_{n-2})} \ ,$$

where $|J_{j_1\cdots j_{n-1}}^{(i_1\cdots i_{n-2})}| = \eta_{i_1\cdots i_{n-2}}$, and

$$\sum_{i_1=1}^{N_0} \cdots \sum_{i_{n-2}=1}^{N_{i_1\cdots i_{n-3}}} b_1 b_{2,i_1} \cdots b_{n-1,i_1\cdots i_{n-2}} \cdot \eta_{i_1\cdots i_{n-2}}^{1-\gamma} = 1. \tag{3.11}$$

We will construct F_n in the same way as that for $n = 2$. By Lemma 3.1 (ii),

$$E_n = \cup_{i_1=1}^{N_0} \cdots \cup_{i_n=1}^{N_{i_1\cdots i_{n-1}}} I_{i_1\cdots i_n} \ .$$

For each fixed $i_1 \cdots i_{n-1}$, we define

$$b_{n,i_1\cdots i_{n-1}} = \frac{\eta_{i_1\cdots i_{n-2}}}{\eta_{i_1\cdots i_{n-1}}^{1-\gamma}} \ . \tag{3.12}$$

In each interval $J_{j_1\cdots j_{n-1}}^{(i_1\cdots i_{n-2})}$ of $F_{n-1,i_1\cdots i_{n-2}}$, we construct $b_{n,i_1\cdots i_{n-1}}$ closed subintervals $J_{j_1\cdots j_n}^{(i_1\cdots i_{n-1})}$ of length $\eta_{i_1\cdots i_{n-1}}$ and gaps at least $\tilde{\eta}_{i_1\cdots i_{n-1}}$ with

$$\tilde{\eta}_{i_1\cdots i_{n-1}} \approx \frac{1}{2}\eta_{i_1\cdots i_{n-1}}^{1-\gamma}. \tag{3.13}$$

This is possible since

$$\begin{aligned} b_{n,i_1\cdots i_{n-1}}(\tilde{\eta}_{i_1\cdots i_{n-1}} + \eta_{i_1\cdots i_{n-1}}) \ &\leq \ (\frac{1}{2} + \eta_{i_1\cdots i_{n-1}}^{\gamma}) \cdot \eta_{i_1\cdots i_{n-2}} \\ &< \ \eta_{i_1\cdots i_{n-2}} \ . \end{aligned}$$

Let $F_{n,i_1\cdots i_{n-1}}$ be the union of all the intervals $J_{j_1\cdots j_n}^{(i_1\cdots i_{n-1})}$,

$$F_{n,i_1\cdots i_{n-1}} = \cup_{j_1=1}^{b_1} \cup_{j_2=1}^{b_{2,i_1}} \cdots \cup_{j_n=1}^{b_{n,i_1\cdots i_{n-1}}} J_{j_1\cdots j_n}^{(i_1\cdots i_{n-1})} \ .$$

As $i_1 \cdots i_{n-1}$ varies, we obtain a sequence $\{F_{n,i_1\cdots i_{n-1}}\}$ of compact sets. Let

$$F_n = \cup_{i_1=1}^{N_0} \cdots \cup_{i_{n-1}=1}^{N_{i_1\cdots i_{n-2}}} F_{n,i_1\cdots i_{n-1}} \ .$$

Then by (3.11) and (3.12), we have

$$\begin{aligned} & \sum_{i_1=1}^{N_0} \cdots \sum_{i_{n-1}=1}^{N_{i_1\cdots i_{n-2}}} b_1 b_{2,i_1} \cdots b_{n,i_1\cdots i_{n-1}} \cdot \eta_{i_1\cdots i_{n-1}}^{1-\gamma} \\ = \ & \sum_{i_1} \cdots \sum_{i_{n-2}} b_1 b_{2,i_1} \cdots b_{n-1,i_1\cdots i_{n-2}} \cdot \eta_{i_1\cdots i_{n-2}}^{1-\gamma} \\ = \ & 1 \ . \end{aligned} \tag{3.14}$$

By induction, we have constructed a decreasing sequence of closed sets $\{F_n\}$ satisfying (3.13) and (3.14). Set $F = \cap_{n=1}^{\infty} F_n$; then $F \subseteq [0,1]$ is a compact set.

Now we verify that F satisfies (3.8) and (3.9). Since for each $n \geq 1$, $F \subseteq F_n$, equation (3.14) implies that

$$\dim F \leq 1 - \gamma.$$

This proves (3.8). To prove (3.9), we observe that for each $n \geq 1, E_n \times F_n \supseteq G_n$, where $G_1 = E_1 \times F_1$ and for $n \geq 2$,

$$G_n = \cup_{i_1=1}^{N_0} \cdots \cup_{i_n=1}^{N_{i_1 \cdots i_{n-1}}} \cup_{j_1=1}^{b_1} \cup_{j_2=1}^{b_{2,i_1}} \cdots \cup_{j_n=1}^{b_{n,i_1 \cdots i_{n-1}}} I_{i_1 \cdots i_n} \times J_{j_1 \cdots j_n}^{(i_1 \cdots i_{n-1})}.$$

Let $G = \cap_{n=1}^{\infty} G_n$; then G is compact and $G \subseteq E_\gamma \times F$. So it is sufficient to prove that

$$\dim G \geq 1. \tag{3.15}$$

To this end, we first define a Borel measure μ on $\mathbf{R}^2$ with $\mu(G) = 1$ and then use Lemma 2.1 to prove (3.15). For each rectangle $I_{i_1} \times J_{j_1}$ in G_1, we define

$$\mu(I_{i_1} \times J_{j_1}) = \frac{\sigma(I_{i_1})}{b_1} = \eta_0.$$

For each rectangle $I_{i_1 i_2} \times J_{j_1 j_2}^{(i_1)}$ in G_2, we define

$$\begin{aligned} \mu(I_{i_1 i_2} \times J_{j_1 j_2}^{(i_1)}) &= \frac{\sigma(I_{i_1 i_2})}{b_1 b_{2,i_1}} \\ &= (\eta_0 \eta_{i_1})^\gamma \cdot \eta_0^{-\gamma} \eta_{i_1}^{1-\gamma} \\ &= \eta_{i_1} . \end{aligned}$$

Similarly, for each rectangle $I_{i_1 \cdots i_n} \times J_{j_1 \cdots j_n}^{(i_1 \cdots i_{n-1})}$ in G_n, we define

$$\begin{aligned} \mu(I_{i_1 \cdots i_n} \times J_{j_1 \cdots j_n}^{(i_1 \cdots i_{n-1})}) &= \frac{\sigma(I_{i_1 \cdots i_n})}{b_1 b_{2,i_1} \cdots b_{n,i_1 \cdots i_{n-1}}} \\ &= \eta_{i_1 \cdots i_{n-1}} . \end{aligned} \tag{3.16}$$

Finally, for each $n \geq 1$, we define $\mu(\mathbf{R}^2 \backslash G_n) = 0$. Then μ can be extended to a Borel measure on $\mathbf{R}^2$ with $\mu(G) = 1$.

For any $x \in G$, there exist two sequences $\mathbf{i} = i_1 i_2 \cdots i_n \cdots$ and $\mathbf{j} = j_1 j_2 \cdots j_n \cdots$, such that

$$\{x\} = \cap_{n=1}^{\infty} I_{i_1 \cdots i_n} \times J_{j_1 \cdots j_n}^{(i_1 \cdots i_{n-1})}.$$

For any $r > 0$, there exists an n such that

$$\eta_{i_1\cdots i_{n-1}} \le r < \eta_{i_1\cdots i_{n-2}}.$$

Consider first the case $\eta_{i_1\cdots i_{n-1}} \le r < \tilde{\eta}_{i_1\cdots i_{n-1}}$. Since the gaps between any two of $I_{i_1\cdots i_n}$ are at least $\eta_{i_1\cdots i_{n-1}}$, $B(x,r)$ can intersect at most $\frac{r}{\eta_{i_1\cdots i_{n-1}}}$ rectangles $I_{i_1\cdots i_n} \times J^{(i_1\cdots i_{n-1})}_{j_1\cdots j_n}$, so by (3.16), we have

$$\begin{aligned} \mu(B(x,r)) &\le \mu(I_{i_1\cdots i_n} \times J^{(i_1\cdots i_{n-1})}_{j_1\cdots j_n}) \cdot \frac{r}{\eta_{i_1\cdots i_{n-1}}} \\ &= r . \end{aligned} \tag{3.17}$$

Consider the second case $\tilde{\eta}_{i_1\cdots i_{n-1}} \le r < \eta_{i_1\cdots i_{n-2}}$. Since the gaps between any two intervals $J^{(i_1\cdots i_{n-1})}_{j_1\cdots j_n}$ are at least $\tilde{\eta}_{i_1\cdots i_{n-1}}$, then $B(x,r)$ can intersect at most $\frac{r}{\tilde{\eta}_{i_1\cdots i_{n-1}}} \cdot \eta^{-\gamma}_{i_1\cdots i_{n-1}}$ rectangles $I_{i_1\cdots i_n} \times J^{(i_1\cdots i_{n-1})}_{j_1\cdots j_n}$. So by (3.13) and (3.16), we have

$$\begin{aligned} \mu(B(x,r)) &\le \frac{r}{\tilde{\eta}_{i_1\cdots i_{n-1}}} \cdot \eta^{-\gamma}_{i_1\cdots i_{n-1}} \cdot \mu(I_{i_1\cdots i_n} \times J^{(i_1\cdots i_{n-1})}_{j_1\cdots j_n}) \\ &\le 4r . \end{aligned} \tag{3.18}$$

Combining (3.17) and (3.18), we have that for any $x \in G$,

$$\limsup_{r\to 0} \frac{\mu(B(x,r))}{2r} \le 2.$$

Hence by Lemma 2.1, we have $\dim G \ge 1$. This proves (3.15) and hence Theorem 3.1. □

Remark. In (1.1), by taking F to be any set with $\dim F = \mathrm{Dim} F$, we obtain that, for any Borel set $E \subseteq \mathbf{R}$,

$$\mathrm{Dim} E = \sup\{\ \mathrm{Dim}(E\times F) - \mathrm{Dim} F\ ,\quad F \subseteq \mathbf{R}\}\ ,$$

$$\dim E = \inf\{\ \dim(E\times F) - \dim F\ ,\quad F \subseteq \mathbf{R}\}\ .$$

4 The dimension $\widehat{\delta}$ and Cartesian product sets

We start with the following theorem which strengthens the third inequality in (1.1).

Theorem 4.1 *Let $E \subseteq \mathbf{R}$ be any Borel set and let $F \subseteq \mathbf{R}$ be compact. Then*

$$\mathrm{Dim} E + \widehat{\delta}(F) \le \mathrm{Dim}(E \times F)\ . \tag{4.1}$$

[1] *Proof.* It is sufficient to prove that for any $\gamma < \mathrm{Dim}E, \quad \beta < \widehat{\delta}(F)$

$$\mathrm{Dim}(E \times F) \geq \gamma + \beta \ . \tag{4.2}$$

By Lemma 2.3, we can assume that E is compact. Then by Lemma 3.1, there exists a Cantor-type set $E_\gamma \subseteq E$ such that $\mathrm{Dim}(E_\gamma \cap V) \geq \gamma$ for every open set V that intersects E_γ.

For any $\beta < \widehat{\delta}(E)$, let

$$\mathcal{G} = \Big\{ J : \ J \ \text{ is a rational open interval with } \ \widehat{\delta}(F \cap J) \leq \beta \Big\} \ .$$

Then $\mathcal{G}$ is countable and by the σ-stability of $\widehat{\delta}$, we have

$$\widehat{\delta}(F) = \widehat{\delta}\Big(F \backslash \cup_{J \in \mathcal{G}} J \Big) \ .$$

Set $F_\beta = F \backslash \cup_{J \in \mathcal{G}} J$; then $F_\beta \subseteq F$ is compact and for any open set W that intersects F_β, we have

$$\widehat{\delta}(F_\beta \cap W) > \beta \ . \tag{4.3}$$

For any $\epsilon > 0$, by (2.3), we may find $\{G_n\}$ with

$$E_\gamma \times F_\beta \subseteq \cup_{n=1}^{\infty} G_n \ ,$$

and for every n,

$$\Delta(G_n) \leq \mathrm{Dim}(E_\gamma \times F_\beta) + \epsilon \leq \mathrm{Dim}(E \times F) + \epsilon \ . \tag{4.4}$$

Since $\Delta(G_n) = \Delta(\overline{G}_n)$, we may take G_n to be closed and $G_n \cap (E_\gamma \times F_\beta) \neq \emptyset$. By Baire's category theorem, there exist n and an open set U that intersects $E_\gamma \times F_\beta$ such that $U \cap (E_\gamma \times F_\beta) \subseteq G_n$. We may find open sets $V, \ W$ such that $V \times W \subseteq U$ and $V \times W$ intersects $E_\gamma \times F_\beta$, so

$$(E_\gamma \cap V) \times (F_\beta \cap W) \subseteq G_n \ . \tag{4.5}$$

By (4.3), (4.4) and (4.5), we have

$$\begin{aligned} \gamma + \beta &\leq \mathrm{Dim}(E_\gamma \cap V) + \widehat{\delta}(F_\beta \cap W) \\ &\leq \Delta(E_\gamma \cap V) + \delta(F_\beta \cap W) \qquad \text{(cf. [15])} \\ &\leq \Delta\Big((E_\gamma \cap V) \times (F_\beta \cap W) \Big) \\ &\leq \Delta(G_n) \\ &\leq \mathrm{Dim}(E \times F) + \epsilon \ . \end{aligned}$$

[1] This proof of using Baire's category theorem was kindly suggested to me by the referee. My previous proof is constructive and is much longer.

Since $\epsilon > 0$ is arbitrary, this proves (4.2) and hence (4.1). □

Remark. For any Borel(or analytic) set $F \subseteq \mathbf{R}$ and compact set $E \subseteq \mathbf{R}$, we have

$$\widehat{\delta}(E) + \mathrm{Dim}F \leq \mathrm{Dim}(E \times F) \ .$$

This implies that for any compact set E,

$$\widehat{\delta}(E) \leq \mathrm{aDim}E \ .$$

Therefore, for any compact set $E \subseteq \mathbf{R}$ with $\dim E < \widehat{\delta}(E)$ (see [15] for an example), we have $\dim E < \mathrm{aDim}E$.

The next theorem gives an upper bound for aDim.

Theorem 4.2 *For any bounded set $E \subseteq \mathbf{R}$,*

$$\mathrm{aDim}E \leq \delta(E) \ . \tag{4.6}$$

Proof. In order to prove (4.6), we show that for any $\gamma > \delta(E)$, there exists a Borel set $F \subseteq \mathbf{R}$ such that

$$\mathrm{Dim}(E \times F) \leq 1 \ , \tag{4.7}$$

$$\mathrm{Dim}F \geq 1 - \gamma \ . \tag{4.8}$$

Since $\gamma > \delta(E)$, by (2.2), there exists a decreasing sequence of positive numbers $\{\epsilon_n\}$ such that

$$\epsilon_1^\gamma < \frac{1}{2} \ , \qquad \left(\frac{\epsilon_n}{\epsilon_{n-1}}\right)^\gamma < \frac{1}{2} \quad \text{for } n \geq 2 \ , \tag{4.9}$$

and

$$N(E,\ \epsilon_n) < \frac{1}{\epsilon_n^\gamma} \ . \tag{4.10}$$

Now we construct F_n inductively, and then define $F = \cap_{n=1}^\infty F_n$. For $n = 1$, let

$$b_1 = \frac{1}{\epsilon_1^{1-\gamma}} \ ,$$

and let $F_1 = \cup_{j_1=1}^{b_1} J_{j_1}$, where J_{j_1} $(j_1 = 1,\ 2,\ \cdots, b_1)$ are closed subintervals of $[0, 1]$ of length ϵ_1. We arrange these intervals so that they are equally spaced with gaps ϵ_1 and that they are contained in an interval of length $2\epsilon_1^\gamma$. This is possible since

$$b_1 \cdot 2\epsilon_1 = 2\epsilon_1^\gamma < 1 \ .$$

Suppose now that F_{n-1} has been constructed as a union of $\frac{1}{\epsilon_{n-1}^{1-\gamma}}$ disjoint closed intervals $J_{j_1\cdots j_{n-1}}$ with $|J_{j_1\cdots j_{n-1}}| = \epsilon_{n-1}$. Let

$$b_n = \left(\frac{\epsilon_{n-1}}{\epsilon_n}\right)^{1-\gamma} .$$

In each interval $J_{j_1\cdots j_{n-1}}$ of F_{n-1}, we construct b_n closed subintervals $J_{j_1\cdots j_n}$ $(j_n = 1, \cdots, b_n)$ of length ϵ_n in such a way that these intervals are equally spaced with gaps ϵ_n and they are contained in an interval of length $2\epsilon_{n-1}^{1-\gamma}\cdot\epsilon_n^{\gamma}$. This is possible since by (4.9), we have

$$\begin{aligned} b_n \cdot 2\epsilon_n &= 2\epsilon_{n-1}^{1-\gamma}\cdot\epsilon_n^{\gamma} \\ &= 2\left(\frac{\epsilon_n}{\epsilon_{n-1}}\right)^{\gamma}\cdot\epsilon_{n-1} \\ &< \epsilon_{n-1} . \end{aligned}$$

Let F_n be the union of these $\frac{1}{\epsilon_n^{1-\gamma}}$ closed intervals $J_{j_1\cdots j_n}$,

$$F_n = \cup_{j_1=1}^{b_1}\cdots\cup_{j_n=1}^{b_n} J_{j_1\cdots j_n} .$$

By induction, we have constructed a decreasing sequence of closed sets $\{F_n\}$ $(n = 1, 2, \cdots)$. Let $F = \cap_{n=1}^{\infty}F_n$; then $F \subseteq [0,1]$ is compact.

We observe that for $\epsilon = \frac{1}{2}\epsilon_n$,

$$N(F, \epsilon) = \frac{1}{\epsilon_n^{1-\gamma}} .$$

By (2.1), we have

$$\Delta(F) \geq 1-\gamma .$$

(In fact, we have $\Delta(F) = 1-\gamma$). On the other hand, Δ is uniform on F. Therefore by Lemma 2.4, we have

$$\mathrm{Dim}F = \Delta(F) \geq 1-\gamma .$$

This proves (4.8). For any $0 < \epsilon < (\frac{1}{2})^{\frac{1}{\gamma}}$, there exists an $n \geq 1$ such that

$$\epsilon_n \leq \epsilon < \epsilon_{n-1} .$$

Consider first the case $\epsilon_n \leq \epsilon < 2\epsilon_{n-1}^{1-\gamma}\cdot\epsilon_n^{\gamma}$. Then by (4.10), we have

$$\begin{aligned} N(E\times F, \epsilon) &\leq N(E,\epsilon_n)\cdot\frac{1}{\epsilon_{n-1}^{1-\gamma}}\cdot K_1\frac{2\epsilon_{n-1}^{1-\gamma}\epsilon_n^{\gamma}}{\epsilon} \\ &\leq \frac{2K_1}{\epsilon} , \end{aligned} \tag{4.11}$$

where $K_1 > 0$ is an absolute constant.

Consider the second case $2\epsilon_{n-1}^{1-\gamma} \cdot \epsilon_n^{\gamma} \leq \epsilon < \epsilon_{n-1}$. Then by (4.10), we have

$$\begin{aligned} N(E \times F,\ \epsilon) &\leq N(E,\ \epsilon_{n-1}) \cdot K_1 \frac{\epsilon_{n-1}}{\epsilon} \cdot \frac{1}{\epsilon_{n-1}^{1-\gamma}} \\ &\leq \frac{K_1}{\epsilon}\ . \end{aligned} \tag{4.12}$$

Combining (4.11) and (4.12), we obtain

$$\Delta(E \times F) \leq 1\ .$$

Therefore $\mathrm{Dim}(E \times F) \leq 1$. This completes the proof of Theorem 4.2. □

Remark. The inequality in (4.6) can be strict. For example, let $E = \{0,\ \frac{1}{n},\ n \geq 1\}$; then $\delta(E) = \frac{1}{2}$ (cf. [3]), but $\mathrm{aDim}E = 0$.

After having obtained the results in this paper, I was informed by Prof. Falconer and Prof. Taylor that Dr. C. J. Bishop and Dr. Y. Peres also obtained similar results independently.

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